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THE THREE-STATE-POLARITY-COINCIDENCE
CORRELATION DETECTOR

by

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ABSTRACT

The three-state-polarity-coincidence correlator (PCC'), a two-input detection device with dead zone clippers, has been investigated in detail. A criterion is devised to compare two detectors. Comparisons are made between the PCC' and several other types of detectors. The superiority of the PCC' in the detection of a weak signal embedded in an additive noise over the PCC and other detectors is demonstrated. It is shown, particularly for certain Gaussian inputs, that there exists a maximum relative efficiency with respect to the dead zone width of clippers or the clipping level "b". It is concluded that the PCC' can always be used to replace the PCC in a weak signal detection.
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I. Introduction

The purpose of this thesis is to analyze the three-state-polarity coincidence correlator (abbreviated as PCC') and to compare its performance as a detector with the two-state-polarity-coincidence correlator (abbreviated as PCC) and a few types of other detectors. The analysis is concentrated on the problem of detecting a weak signal embedded in an additive noise. Thus, the signal to noise ratio (designated as SNR) under consideration is much less than unity. The input distributions are not necessarily assumed to be Gaussian.

In order to compare the performances of various detectors, a measure of comparison is devised: the efficiency of a detector \( U \) with respect to another detector \( V \), designated as \( e_{uv} \), is defined as the ratio of \( N_v \) to \( N_u \) under the condition that both detectors achieve the same performances, where \( N_v \) and \( N_u \) are the numbers of samples per sequence of the detector \( V \) and the detector \( U \) respectively. Therefore, if the efficiency \( e_{uv} \) is greater than unity, i.e., when \( N_v \) is greater than \( N_u \), then it is said that the detector \( U \) is more efficient than the detector \( V \), or the detector \( U \) is superior to the detector \( V \). The details of the comparison are discussed in Section III.

Since the evaluation of the efficiency requires integration of the noise distribution function (see Section II), a specific example of the calculation of the efficiency with Gaussian inputs has been made by computer (IBM 1620) for various sample sizes, and the results are given in Section VI. It was surmised that PCC' would perform more efficiently than PCC for most input distributions. This thesis shows that this
conjecture is indeed true. The PCC' will be shown in particular to have improved performance over the PCC for the Gaussian inputs. However the PCC', like the PCC, is still inferior in performance to the conventional correlation detector as well as Neyman-Pearson detector for the Gaussian inputs.

The analysis also includes the important (and practical) case of finite sample sizes and is presented in Section V.
II. Background

In 1949, Lee, Cheathan and Wiesner [8] demonstrated that the presence of a signal in noise may be determined by cross correlation. Since then this detection technique has been widely employed in radio-astronomical [6,7] and underwater sound applications [11]. The treatment of correlation detectors was extended to operation on continuous waveforms rather than sampled values of these wave forms by Fano and Devenport [3,1] in 1951. Afterwards Farren and Hill [4] and others have discussed in detail the output SNR's of correlation detectors and have made use of the normalized autocorrelation function to analyze a polarity-coincidence correlator. In the paper of Wolff, Thomas and Williams [12], the non-parametric property of the polarity-coincidence correlator (PCC) was discussed. However, in that paper, the discussion was confined to the two-state polarity-coincidence correlator, and the effect of finite sample sizes was not considered. Ekre [2] has examined the performance of the PCC for the various input spectra, but the comparisons were not made with other detectors on the basis of probability of detection.
III. Analysis of PCC'

(A) Introduction

There are several ways available to analyze the polarity coincidence correlation detector. One way is to find the loss in attainable output SNR due to clipping and/or sampling. Another way, which will be used in this thesis, is to obtain the probability of detection, and then to compare the sample sizes of the two detectors for equal error probability.

The following assumptions are made:

1. The inputs are assumed to have been sampled so that only the sampled values are available to the input of the detector.

2. Noise is represented by two independent stationary random processes $n^1_j = N^1(\xi,t)$ and $n^2_j = N^2(\xi,t)$

3. The signal is represented by a stationary random process $s^j = S(\xi,t)$, which is independent of both $n^1_j$ and $n^2_j$.

4. When the signal is not present, the two inputs are given by $q^1_j = n^1_j$, $q^2_j = n^2_j$, where the index "j" designates the time that the inputs are sampled, and the superscripts "1" and "2" indicate the corresponding input channels.

5. The signal $s^j$ is common to both channels, so that $q^1_j = s^j + n^1_j$, $q^2_j = s^j + n^2_j$ when the signal is present (see Fig. 1).

The restrictions made on the probability distribution functions $F_n(\cdot)$ of the noises $n^1_j$ and $n^2_j$ and $F_s(\cdot)$ of the signal $s^j$ are:

1. The random variables have zero mean values and $F_n(0) = F_s(0) = 1/2$;
Fig. 1 The Block Diagram of PCC'
(2) \( F_n(-u) = 1 - F_n(u) \), \( F_S(-u) = 1 - F_S(u) \);

This implies that if their first and second derivatives exist, then the following relations are established:

\[
F_n'(-u) = F_n'(u) \quad \text{and} \quad F_S'(-u) = F_S'(u)
\]

where \( F_n'(u) \) and \( F_S'(u) \) are the density functions for the continuous distribution. The differentiation is with respect to \( u \).

(3) \( \int u^2 \, dF_n(u) < \infty \), \( \int u^2 \, dF_S(u) < \infty \);

(4) \( F_n(u) \) and \( F_S(u) \) are absolutely continuous.

(b) **Description of the PCC'**

The PCC' is a three-state-polarity-coincidence correlation detector which calculates the statistic

\[
T_1 = \sum_{j=1}^{N_1} \text{sgn}(q_j^1) \text{sgn}(q_j^2)
\]

where

\[
q_j^1 = s_j + n_j^1
\]

\[
q_j^2 = s_j + n_j^2
\]

\[
\text{sgn}(q_j) = \begin{cases} 
+1 & q_j > b \\
0 & -b < q_j < b \\
-1 & q_j < -b 
\end{cases}
\]

\( N_1 \): the number of sample sizes per sequence.

Thus for the sampled PCC', the input waveforms are sampled at regular intervals \( T_s \) to give a pulse train of rectangular pulses of
constant duration \( T_p \leq T_s \). The amplitude of a pulse is +1 when the input waveform at the sampling point is greater than a certain positive number "b", 0 when it lies between -b and +b, and -1 when it is less than -b. The two pulse trains are then multiplied in a coincidence circuit whose output is +1 when the two pulses are of same polarity, -1 when the two pulses are of opposite polarity, and 0 when either pulse is 0. The output of the coincidence circuit is then integrated in an RC integrator or equivalent digital integrator. A signal is considered to be present if \( T_1 \), the output of the integrator, exceeds a threshold value \( t_1 \). The determination of threshold value \( t_1 \) will be discussed in part (c).

(8). **Analysis**

Let \( H \) be the event that a signal is not present, and \( K \) be the event that a signal is present. There are two parameters of most interest in any statistical test:

(a) The false alarm probability "\( \alpha \)" which is defined as the probability of stating \( K \) when \( H \) is true.

(b) The probability of detection "\( \beta \)" which is defined as the probability of stating \( K \) when \( K \) is indeed true.

In order to obtain the expressions for "\( \alpha \)" and "\( \beta \)", the distribution function of the statistic \( T_1 \) has to be determined. However, the term \( T_1 \) which is given in the equation (1) is clearly trinomially distributed. For the purpose of analysis, large sample sizes are considered and the original trinomial distribution is replaced by its normal
approximation by applying the central limit theorem. Hence the probability of the summand in (1) taking the value +1, 0, -1 is first computed in order to evaluate the expectations and variances of the statistic $T_1$ so that the central limit theorem can be applied.

(1) When the signal is not present, "H":

\[ P_H[\text{sgn}(q^1_j) \text{ sgn}(q^2_j) = +1] = P[\text{sgn}(n^1_j) \text{ sgn}(n^2_j) = +1] \]
\[ = P(n^1_j > b \& n^2_j > b) + P(n^1_j < -b \& n^2_j < -b) \]
\[ = P^2(n_j > b) + P^2(n_j < -b) \]
\[ = 2P^2(n_j < -b) \]
\[ = 2F_n^2(-b) \]  

(2) When a signal is present, "K":

\[ P_K[\text{sgn}(q^1_j) \text{ sgn}(q^2_j) = +1] = P(q^1_j > b \& q^2_j > b) + P(q^1_j < -b \& q^2_j < -b) \]
\[ = \int P(n^1_j > -u+b \& n^2_j > -u+b) \text{ dF}_S(u) \]
\[ + \int P(n^1_j < -u-b \& n^2_j < -u-b) \text{ dF}_S(u) \]
\[ = \int [1-F_n(-u+b)]^2 \text{ dF}_S(u) + \int [F_n(-u-b)]^2 \text{ dF}_S(u) \]
\[ p_1 \equiv \int [F_n^2(u-b) + F_n^2(-u-b)] \, dF_s(u) \equiv p_1 \]  

The functions \( F_n(u-b) \) and \( F_n(-u-b) \) are then expanded in Taylor's series as functions of \( u \) [with "b" fixed] about the point \( u = 0 \) to give

\[
F_n(u-b) = F_n(-b) + F'_n(-b)u + (1/2) F''_n(-b)u^2 + \ldots
\]

\[
F_n(-u-b) = F_n(-b) - F'_n(-b)u + (1/2) F''_n(-b)u^2 + \ldots
\]

By substituting these two expressions into (5) and dropping those terms containing \( u^3 \) or higher, (5) then is simplified to (6):

\[
p_1 \equiv 2 \int [F_n^2(-b) + F_n^2(-b) u^2 + F_n(-b) F''_n(-b)u^2] \, dF_s(u)
\]

\[
= 2F_n^2(-b) + 2[F'_n(-b) + F_n(-b)F''_n(-b)] \, d^2
\]

\[ \ldots \]  

where \( d^2 \) is the variance of the signal distribution.

Similarly, the probability of a summand \( \text{sgn}(q_j) \, \text{sgn}(q'_j) \) taking the value \(-1\) is

\[
p_2 \equiv P_k [\text{sgn}(q_j^1) \, \text{sgn}(q_j^2) = -1]
\]

\[
= 2 \int F_n(u-b) F_n(-u-b) \, dF_s(u)
\]

\[
= 2 \int [F_n^2(-b) - F_n^2(-b) u^2 + F_n(-b) F''_n(-b)u^2] \, dF_s(u)
\]

\[
= 2F_n^2(-b) + 2[F'_n(-b) F''_n(-b) - F_n^2(-b)] \, d^2
\]

It is obvious for the case \( K \) that when the noises are removed, \( p_2 \) is always equal to zero, and \( p_1 \) has some non-zero positive value as long as the constant \( b \) has been appropriately chosen. So \( p_1 \) is greater than \( p_2 \). It is expected, however, that \( p_1 \) is greater than \( p_2 \) for any non-zero signal \( s_j \) common to both inputs even when the
noises are present. This can be shown in the following way:

Since the mean value of \( F_n(\cdot) \) is zero, it can be written as

\[
F_n(u-b) = 1/2 + R \quad \text{with } R \geq 0 \quad \text{for } u \geq b
\]

\[
F_n(-u-b) = 1/2 - T \quad \text{with } T \geq 0 \quad \text{for } u \geq -b
\]

Thus from (5) one has

\[
p_1 = \int \left[ \frac{1}{2} + R + R^2 - T + T^2 \right] dF_s(u)
\]

\[
= \frac{1}{2} + \int \left[ R + R^2 - T + T^2 \right] dF_s(u)
\]

(9)

and from (7):

\[
p_2 = 2 \int \left( \frac{1}{2} + R \right) \left( \frac{1}{2} - T \right) dF_s(u)
\]

\[
= \frac{1}{2} + \int (R - T - 2TR) dF_s(u)
\]

(10)

(9) - (10):

\[
p_1 - p_2 = \int (R + T)^2 dF_s(u) > 0
\]

therefore

\[
p_1 > p_2
\]

Since there is no restriction on SNR's of the inputs, this result assures one that the presence of the input signal can always be detected by a PCC' no matter how weak a signal is present.

Now the expectations and variances of \( T_1 \) are evaluated below:

\[
E_R [T_1] = 0
\]

\[
\text{Var}_R [T_1] = E_R [T_1^2] - E_R [T_1]^2 = 4N_1 F_n^2 (-b)
\]

\[
E_K [T_1] = N_1(p_1 - p_2) = 4N_1 F_n' (-b) d^2
\]

\[
\text{Var}_K [T_1] = N_1(p_1 + p_2 - (p_1 - p_2)^2)
\]

\[
= 4N_1 \left[ F_n^2 (-b) + F_n (-b)F_n'(-b)d^2 - 4F_n^4(-b)d^4 \right]
\]
Thus, for very large $N_1$, the quantities

$$T'_1 = \frac{T_1 - \mu_1}{\sigma_1}$$

and

$$T''_1 = \frac{T_1 - 4N_1 F_n^2(-b) d^2}{[4N_1(F_n^2(-b) + F_n(-b) F_n''(-b) d^2 - 4F_n^4(-b) d^4)]^{1/2}}$$

are normally distributed with zero mean and variance 1.

The corresponding false alarm probability $\alpha$ and the probability of detection $\beta$ are then given:

$$\alpha = P_H(T_1 > t_1) = 1 - \Phi \left[ \frac{t_1 - E_H[T_1]}{Var_H[T_1]} \right]^{1/2}$$

$$\beta = P_K(T_1 > t_1) = 1 - \Phi \left[ \frac{t_1 - E_K[T_1]}{Var_K[T_1]} \right]^{1/2}$$

where $t_1$ is the threshold value of the detector.
The function \( \Phi(\cdot) \) is called the normal distribution function, and is defined as:

\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2} y^2} \, dy
\]  

(15)

Since it is a monotonically increasing as well as continuous function, its inverse function is uniquely determined. Consequently, by (13), the threshold value \( t_1 \) can be expressed as a function of \( \alpha \) as:

\[
t_1 = 2 \frac{F_n(-b)}{\sqrt{N_1}} \Phi^{-1}(1 - \alpha)
\]  

(16)

which is substituted into (14) to yield the probability of detection:

\[
\beta = 1 - \Phi \left\{ \frac{\frac{F_n(-b)}{\sqrt{N_1}} \Phi^{-1}(1 - \alpha) - \frac{2\sqrt{N_1} F_n^2(-b) d^2}{[F_n^2(-b) + F_n(-b) F_n''(-b) d^2 - 4F_n^4(-b)d^4]^{1/2}}} \right\}
\]  

(17)

(D). Discussion

Equation (13) shows that as long as \( b \) is so chosen that \( F_n(-b) \approx 1/2 \), \( \alpha \) is quasi-independent of the signal and noise probability distribution. Then the PCC', like the PCC, may be regarded as a nonparametric detector in a restricted sense. That is, the false alarm rate of PCC' does not depend on the noise distribution at its inputs. The detection of signal can always be performed without a priori knowledge of the noise distribution.

It is expected in (14) and (17) that for the particular value of \( \alpha \) (fixed by choosing a certain value \( t_1 \)), \( \beta \) varies as the input distributions change, even though the input SNR is kept constant.
However, if the input distributions change only in variance but not in form, it is shown in the following that an increasing SNR cause $\beta$ to increase too.

From (14), $\beta$ can be rewritten as

$$\beta = 1 - \varphi \left[ \frac{2 {F_n}(-b) \varphi^{-1} (1 - \alpha) \sqrt{N_1} (p_1 - p_2)}{(p_1 + p_2 - (p_1 - p_2)^2)^{1/2}} \right]$$

(18)

Differentiate it with respect to $p_1$,

$$\frac{\partial \beta}{\partial p_1} = M \times \frac{2 {F_n}(-b) \varphi^{-1} (1 - \alpha)(1 - 2(p_1 - p_2)) + \sqrt{N_1} (p_1 + 3 p_2)}{(p_1 + p_2 - (p_1 - p_2)^2)^{3/2}}$$

where $M$ is a positive constant. Since

$p_1 > p_2$

and

$$0 < p_1 - p_2 = \int (R + T)^2 \, dF_g(u) < \frac{1}{2}$$

so the numerator of $\frac{\partial \beta}{\partial p_1}$ is

$$2 {F_n}(-b) \varphi^{-1} (1 - \alpha)(1 - 2(p_1 - p_2)) + \sqrt{N_1} (p_1 + 3 p_2) > 0$$

Hence for any $N_1$, $\frac{\partial \beta}{\partial p_1}$ is always positive.

Consider (9)

$$p_1 = \int [R + R^2 - T + T^2] \, dF_g(u) + \frac{1}{2}$$

It is observed from Fig. 2 that the integrand of $p_1$ is a non-decreasing function for positive argument, as is (5). The following shows this:
Fig. 2

\[ R^2 = (R^2 + T^2 + R - T) \]

Fig. 3

\[ |a| > 1 \]
\[ G(u) = F_n^2(u-b) + (1-F_n(u+b))^2 \]

then

\[ G(au) \geq G(u) \quad \text{for} \quad |a| > 1 \]

Now let

\[ F_n'(u) = F_n(au) \]

then

\[ \text{Var}[N'] < \text{Var}[N] \quad \text{(Fig. 3)} \]

Thus

\[ p_1' = \int G_n'(u) \, dF_s(u) \]

\[ = \int G_n(u) \, dF_s(u) \]

\[ \geq p_1 \]

Similarly, if

\[ F_s'(u) = F_s(au) \quad |a| < 1 \]

then

\[ \text{Var}[S'] > \text{Var}[S] \]

and

\[ p_1' = \int G_n(u) \, dF_s'(u) \]

\[ = \int G_n(u) \, dF_s(au) \]

\[ = \int G_n \left( \frac{u}{a} \right) \, dF_s(u) \]

\[ \geq p_1 \]

Therefore \( p_1 \) is a non-decreasing function of the input SNR

(defined as \( \text{Var}(s)/\text{Var}(n) \)), and it follows that an increasing SNR

causes \( \beta \) to increase.
IV. Comparisons

In order to compare different detectors, a criterion must be devised. Let the symbol "E" be the event that a detector makes an error, the symbol "T" be the output of the detector at the termination of a sequence of $N$ samples, and the symbol "t" be the corresponding threshold value. In other words, a signal is assumed to be present if $T$ is greater than $t$. The probability of error is defined as:

$$P(T > t|H) P(H) + P(T \leq t|K) P(K) = P(E)$$

where

$$P(H) + P(K) = 1. \quad (19)$$

From the definition of $\alpha$ and $\beta$ ((13) and (14)), $P(E)$ may be rewritten as:

$$P(E) = \alpha P(H) + (1 - \beta) P(K) \quad (20)$$

Thus in order to minimize the probability of error, it is necessary to increase $\beta$ or decrease $\alpha$ or do both.

The criterion used to compare two detectors is the ratio of the sample sizes required by the detectors to achieve the same values of $\alpha$ and $\beta$. This implies that the detectors are performed under the same value of error probability. A detector $U$ is considered to be better than a detector $V$ if the ratio $N_V/N_U$ is greater than unity for the same values of $\alpha$, $\beta$. This ratio is called the relative efficiency and is designated by the symbol $e_{uv}$.
\[ e_{uv} = \frac{N_v}{N_u} \quad (21) \]

If letting both \( N_u \) and \( N_v \) approach sufficiently large values, then

\[ e_{uv} = \lim_{N \to \infty} \frac{N_v}{N_u} \quad (22) \]

is called asymptotic relative efficiency of \( U \) with respect to \( V \).

The first comparison of detector will be made for Gaussian inputs. Thus it is necessary to evaluate \( \beta \) for PCC' from (17) for Gaussian inputs. With

\[ F_n(-b) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-b} e^{-x^2/2\nu^2} \, dx = \Phi(-\frac{b}{\nu}) \]

\[ F_n'(-b) = \frac{1}{\sqrt{2\pi}} e^{-b^2/2\nu^2} \quad (23) \]

\[ F_n''(-b) = \frac{b}{3\nu^3} e^{-b^2/2\nu^2} \]

the equation (17) yields,

\[ \beta_1 = 1 - \bar{\phi} \left[ \frac{\bar{\phi}(-\frac{b}{\nu})^{-1}(1-\alpha) - \frac{d^2}{4\nu^2} e^{-b^2/\nu^2}}{\bar{\phi}^2(-\frac{b}{\nu}) + \frac{bd^2}{2\pi^3\nu^3} \bar{\phi}(-\frac{b}{\nu}) e^{-b^2/\nu^2} - \frac{d^4}{4\nu^4} e^{-2b^2/\nu^2}} \right]^{1/2} \]

\[ \beta_1 = \beta_2 = \beta \]

Now compare the PCC' with PCC by letting

\[ \alpha_1 = \alpha_2 = \alpha \quad , \quad \beta_1 = \beta_2 = \beta \]
to obtain the relative efficiency $e_{1,2}$, where the subscripts 1 and 2 stand for the PCC' and PCC respectively.

$$1 - \tilde{\phi} \left\{ \frac{-\frac{b}{v}}{[\tilde{\phi}(-\frac{b}{v}) + \frac{bd^2}{\sqrt{2\pi}v^3} \tilde{\phi}(-\frac{b}{v}) e^{-\frac{b^2}{\omega^2}}]} - \sqrt{\frac{N_1}{\pi v^2}} \frac{d^2}{2} e^{-\frac{b^2}{\omega^2}} \right\} = 1 - \tilde{\phi} \left\{ \tilde{\phi}^{-1}(1-\alpha) - \frac{2d^2}{\pi v^2} \sqrt{N_2} \right\}$$

\[ (25) \]

where the last term of the denominator in the left side of the equation is neglected since the SNR is assumed vanishingly small.

From (25), one has

$$\frac{\tilde{\phi}(-\frac{b}{v}) \tilde{\phi}^{-1}(1-\alpha) - \sqrt{\frac{N_1}{\pi v^2}} \frac{d^2}{2} e^{-\frac{b^2}{\omega^2}}}{[\tilde{\phi}(-\frac{b}{v}) + \frac{bd^2}{\sqrt{2\pi}v^3} \tilde{\phi}(-\frac{b}{v}) e^{-\frac{b^2}{\omega^2}}]} = \tilde{\phi}^{-1}(1-\alpha) - \frac{2d^2}{\pi v^2} \sqrt{N_2}$$

Thus the asymptotic relative efficiency $\bar{e}_{1,2}$ is

$$\bar{e}_{1,2} = \lim_{N \to +\infty} \frac{N_2}{N_1} = \frac{1}{4 e^{2\frac{b^2}{\omega^2} / v^2} \tilde{\phi}(-\frac{b}{v}) [\tilde{\phi}(-\frac{b}{v}) + \frac{bd^2}{\sqrt{2\pi}v^3} e^{-\frac{b^2}{\omega^2}}]} = f\left(\frac{b}{v}, \frac{d}{v}\right)$$

\[ (26) \]

If $d^2/v^2 \ll 1$, then

$$\bar{e}_{1,2} = \frac{1}{4 e^{2\frac{b^2}{\omega^2} / v^2} \tilde{\phi}^2(-b/v)} = f\left(\frac{b}{v}\right)$$

\[ (27) \]

For instance, let $b/v = 0.5$, then

$$\bar{e}_{1,2} = 1.59$$

It is seen, therefore, that for small SNR, large sample sizes and
for \( b/v = 0.5 \), the PCC requires approximately one and a half as many samples to achieve the same performance as the PCC', when the inputs are Gaussian distributed.

Similarly, \( \overline{e}_{1,3} \) and \( \overline{e}_{1,4} \) are obtained, where the subscripts 3 and 4 stand for the Neyman-Pearson detector and correlation detector respectively [ref. Appendix A.]:

\[
\overline{e}_{1,3} = \frac{1}{4\pi^2 e^{2b^2/v^2} \hat{v}^2(-b/v)}
\]

\[
= 0.161 \quad \text{for } \frac{b}{v} = 0.5
\]

\[
\overline{e}_{1,4} = \frac{1}{\pi^2 e^{2b^2/v^2} \hat{v}^2(-b/v)}
\]

\[
= 0.644 \quad \text{for } \frac{b}{v} = 0.5
\]

Although it appears that PCC' is still inferior both to the Neyman-Pearson detector and the correlation detector just as the PCC is, these two efficiencies \( \overline{e}_{1,3} \) and \( \overline{e}_{1,4} \) are observed to be greater than \( \overline{e}_{2,3} \) and \( \overline{e}_{2,4} \) [ref. Appendix A.] which are equal to 0.101 and 0.405 respectively.

Now if the inputs no longer have Gaussian distributions, using (17), however, the general expression for \( \overline{e}_{1,2} \) may be easily obtained. Let (17) = (48), one has
Solving for the ratio of $N_1$ and $N_2$, it yields for $\frac{d^2}{v^2} \ll 1$

$$-e_{1,2} = \frac{F_n^4(-b)}{4 F_n^4(0) F_n^2(-b)}$$

provided $F_n'(0)$ exists.

This expression is a function of $b/v$ as well as the probability distribution laws. Consider a double exponential distribution (Fig. 4) in which the PCC has been shown to be remarkably superior to the Neyman-Pearson detector and correlation detector.

$$F_n'(x) = \frac{1}{\sqrt{2} v} e^{-\sqrt{2}|x|/v}, \quad F_n'(0) = \frac{1}{\sqrt{2} v} e^{-\sqrt{2} b/v}$$

$$F_n(x) = \frac{1}{\sqrt{2} v} \int_{-\infty}^{x} e^{-\sqrt{2}|x|/v} dx, \quad F_n(-b) = \frac{1}{2} e^{-\sqrt{2} b/v}$$

Substitute these equations into (31):

$$-e_{1,2} = e^{-2\sqrt{2} b/v}$$

$$= 0.76 \quad \text{for} \quad \frac{b}{v} = 0.1$$
Fig. 4 Double Exponential Density Function

\[ f_n(u) = \frac{1}{\sqrt{2\pi}v} e^{-\sqrt{2}|u|/v} \]
(32) shows that the maximum value of $e_{1,2}$ is 1, hence the PCC in this case is at most as efficient as the PCC.

For the same inputs, one has

$$e_{1,4} = \frac{4}{e^{2\sqrt{2} \frac{b}{v}}}$$

$$= 3.04 \quad \text{for} \quad \frac{b}{v} = 0.1$$

From (31), it is easy to obtain a criterion for any input distribution to ensure the asymptotic relative efficiency $e_{1,2}$ to be greater than 1.

$$F_n'(-b) \geq 2 F_n(0) F_n(-b)$$

or, since $F_n(-b) > 0$,

therefore

$$\frac{F_n'(-b)^2}{2F_n(-b)} \geq F_n(0)^2$$

provided $F_n'(0)$ exists.
V. Discussion on the finite sample sizes

All the previous developments are based upon the central limit theorem. This theorem is a proposition of an essentially asymptotic character, and its use for finite size samples is open to question. This section presents an evaluation of the error committed in using the central limit theorem for finite sample sizes. The theorem states the distribution function $F(u)$ of the variable

$$
u = \frac{T - E[T]}{\sqrt{\text{Var}[T]}}$$

where

$$T = \sum_{j=1}^{N} \text{sgn}(q_j^1) \text{sgn}(q_j^2)$$

$$= z_1 + z_2 + \ldots$$

and $z_j = \text{sgn}(q_j^1) \text{sgn}(q_j^2)$

approaches the limit

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx$$

as $N$ becomes infinite when the condition

$$W = \frac{E[|T|^2 + x]}{\text{Var}[T]} \left(1 + \frac{x}{2}\right) \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty$$
where
\[ E[|T|^2 + r] = \sum_{j=1}^{N} E[|Z_j|^2 + r] \]
is the absolute moment of \((2+r)\)th order.
(Covariance is assumed zero).
\[ r: \text{some positive number.} \]
holds.

For practical purposes, it is very important to estimate the error committed by replacing \(F(u)\) by its limit (i.e., the normal distribution) when \(N\) is a finite but very large number.

Liapunov [9] investigated the problem in detail and established the following results.

Assuming the existence of the absolute moments of the third order \(E[|Z_j|^3], \ j = 1, 2, \ldots, N\). If \(N\) is so large that
\[ \frac{\sum_{j=1}^{N} E[|Z_j|^3]}{\text{Var}[T]^{3/2}} < \frac{1}{20} \] \hspace{1cm} (35)
then letting
\[ F(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-x^2/2} \, dx + R \] \hspace{1cm} (36)
the absolute error will be
\[ |R| < 1.6 \omega \left[ (\log_e \frac{1}{3\omega})^{1/2} + 1.1 \right] + \omega^2 \log_e \frac{1}{3\omega} + (5/3)\omega^2 e^{-\omega} \] \hspace{1cm} (37)
This upper bound for the error term is probably too high, yet it seems to be the best available.

Taking this error estimation into consideration, (13), (14) and (48) may be rewritten as

\[
\alpha_1 = 1 - \left\{ \Phi \left[ \frac{t_1}{2F_n(-b)\sqrt{N_1}} \right] + R_1 \right\} \tag{38}
\]

\[
\beta_1 = 1 - \left\{ \Phi \left[ \frac{2\Phi(-b)\Phi^{-1}(1-R_1-\alpha_1) - \sqrt{N_1}(p_1-p_2)}{\sqrt{p_1 + p_2 - (p_1 - p_2)^2}} \right] + R_1 \right\} \tag{39}
\]

\[
\alpha_2 = 1 - \left\{ \Phi \left[ \frac{t_2}{\sqrt{N_2}} \right] + R_2 \right\} \tag{40}
\]

\[
\beta_2 = 1 - \left\{ \Phi \left[ \frac{\Phi^{-1}(1-R_2-\alpha_2) - \sqrt{N_2}(2p_+ - 1)}{2\sqrt{p_+(1-p_+)}} \right] + R_2 \right\} \tag{41}
\]

Let

\[2F_n(-b) = a\]

\[p_1 - p_2 = b\]

\[(p_1 + p_2 - (p_1 - p_2))^{1/2} = c\]

\[2p_+ - 1 = d\]

\[2(p_+ (1-p_+))^{1/2} = e\]

\[\alpha_1 = \alpha_2 = \alpha, \quad \beta_1 = \beta_2 = \beta, \]

and define \(R^* = R'_2 - R'_1\), then
\[
\tilde{\phi} \left[ \frac{a \tilde{\phi}^{-1}(1-R_1-\alpha)-b \sqrt{N_1}}{C} \right] = \tilde{\phi} \left[ \frac{d \tilde{\phi}^{-1}(1-R_2-\alpha) - d \sqrt{N_2}}{e} \right] + R^* \quad (42)
\]

solving for \( (N_1)^{1/2} \):

\[
(N_1)^{1/2} = \frac{a}{b} \tilde{\phi}^{-1}(1-R_1-\alpha) - \frac{c}{d} \left[ \tilde{\phi} \left[ e \tilde{\phi}^{-1}(1-R_2-\alpha) - \frac{d}{e} \sqrt{N_2} \right] + R^* \right]
\]

\[\ldots\] \quad (43)

If \( N_2 \) is held fixed, then \( N_1 \) is only a function of \( R_1, R_2, R^* \), or

\[ N_1 = f(R_1, R_2, R^*) \]

Hence the percentage error in the square root of efficiency can be obtained by inserting the error bound for \( R_1, R_2, R^* \) from (37) into (43) to get the lower and the upper bounds of \( (N_1)^{1/2} \) which are denoted by \( (N_{10})^{1/2} \) and \( (N_{hi})^{1/2} \) respectively. The result is given by

\[
\eta = \frac{(N_1)^{1/2}[(N_{hi})^{1/2} - (N_{10})^{1/2}]}{(N_{10}N_{hi})^{1/2}} \quad (44)
\]

In order to illustrate it, choose (Fig. 5)

\[ \frac{d^2}{v^2} = 0.01, \quad b/v = 0.6, \quad \alpha = 0.09 \]

then

\[ p_1 = 0.194, \quad p_2 = 0.189, \quad p_+ = 0.503 \]

and

\[ (c_{1,2})^{1/2} = 1.278 \quad \text{for} \quad N_1 = 100000 \quad N_2 = 160000 \]
Thus from (35), the upper bound of \( w \) is found to be

\[
\omega_1 = 1/102
\]

\[
\omega_2 = 1/400
\]

the corresponding error bounds are calculated by using (37):

\[
|R_1| = 0.0455
\]

\[
|R_2| = 0.0132
\]

substitute these error bounds into (43) yields

\[
(N_{10})^{1/2} = 215
\]

\[
(N_{hi})^{1/2} = 360
\]

with

\[
(N_1)^{1/2} = 316
\]

it follows from (44) that

\[
\eta = 60\%
\]

Such high variation for the square root of efficiency \((\varepsilon_{1,2})\) is probably due to the high error bound which is obtained by the equation (37).
VI. Computer results

This section presents the results of computer calculation about the various probabilities $p_1, p_2, p_+$ and the relative efficiencies compare the PCC' with the PCC with Gaussian inputs.

The values of $p_1, p_2$ and $p_+$ are obtained by the evaluation of the integrals of (5), (7) and (46) using numerical analysis method rather than applying a Taylor's series expansion. The error committed due to finite sample sizes is not taken into consideration. The curves for the various results are shown in Fig. 5 and Fig. 6.

From Fig. 5, it is found that the efficiencies decrease monotonically and approach limits $\bar{e}_{1,2}$ as the sample sizes increase. However, it is observed that the efficiencies are relatively independent of the sample sizes. Thus they are not important in setting the sample sizes.

The limit $\bar{e}_{1,2}$ depends on the clipping level "b". And a maximum value exists for certain value of "b" which is equal to 0.625 for the present case. (Strictly speaking, the limit $\bar{e}_{1,2}$ depends on the ratio of the clipping level to the standard deviation of input noise distribution, $b/v$). The maximum value of $(\bar{e}_{1,2})$ is also observed to increase as the input SNR decreases. This fact tells one that the PCC' can perform more efficiently for small SNR than the PCC.
Fig. 5 Relative Efficiency of PCC' w.r.t. PCC for Gaussian Inputs

Fig. 6 Asymptotic Relative Efficiency of PCC' w.r.t. PCC vs. b for Gaussian Inputs
VII. Conclusions and Results

The three-state-polarity-coincidence correlation detector PCC' has been analyzed in detail. It is shown that when it is operating with a fixed input distribution, the probability of detection $\beta$ is a non-decreasing function of the input signal to noise ratio SNR. As long as (34) criterion holds, the PCC' is shown to be more efficient than the PCC. One particular example showing the superiority of the PCC' over the PCC is a case with Gaussian inputs.

Computer calculation of the relative efficiency $e_{1,2}$ of the PCC' with respect to the PCC was made. The results reveal that for a given ratio $b/v$, the efficiency decreases monotonically as the sample sizes increase. It also shows that there exists a maximum relative efficiency with respect to clipping level "b". Furthermore, when the ratio of clipping level $b$ to the standard deviation of noises $v$ is fixed, then the relative efficiency increases as the signal to noise ratio decreases. It is for these reasons that the PCC' can always be used to replace the PCC in the weak signal detection.

The probability of detection $\beta$ for the Neyman-Pearson detector is incorrect as given in Wolff's paper. The correction is given in Appendix A and shows that the relative efficiency $e_{2,3}$ is reduced by a factor of 1/2.
Appendix A. Analysis of PCC, Neyman-Pearson detector and Correlation detector

(a) PCC

The PCC is a detector which calculates the following statistic:

$$T_2 = \sum_{j=1}^{N_2} \text{sgn}(q_j^1) \text{sgn}(q_j^2)$$

where

$$q_j^1 = s_j + n_j^1$$

$$q_j^2 = s_j + n_j^2$$

$$\text{sgn}(q_j) = \begin{cases} +1 & \text{for } q_j > 0 \\ 0 & \text{for } q_j = 0 \\ -1 & \text{for } q_j < 0 \end{cases}$$

$N_2$: The number of samples per sequence.

The expectations and variances of $T_2$ are given by

$$E_H[T_2] = 0$$

$$\text{Var}_H[T_2] = N_2$$

$$E_K[T_2] = N_2(2p_+ - 1)$$

$$\text{Var}_K[T_2] = 4N_2p_+ (1-p_+)$$

By paralleling the development of the equation (5), the probability of a summand $\text{sgn}(q_j^1) \text{sgn}(q_j^2)$ taking $+1$ is:

$$p_+ = P_K[\text{sgn}(q_j^1) \text{sgn}(q_j^2) = +1]$$
\[ = \int [1 - 2 F_n(u) + 2F_n^2(u)] \, dF_s(u) \]
\[ = \frac{1}{2} + 2F_n'(0) \, d^2 \]
\[ = \frac{1}{2} + \frac{1}{\pi} \frac{d^2}{\nu^2} \quad \text{for Gaussian inputs} \quad (46) \]

The false alarm rate \( \alpha_2 \) and the probability of detection \( \beta_2 \) are

\[ \alpha_2 = P_H (T_2 > t_2) = 1 - \Phi \left( \frac{t_2}{\sqrt{N_2}} \right) \quad (47) \]

\[ \beta_2 = P_K (T_2 > t_2) = 1 - \Phi \left( \frac{1}{2} \Phi^{-1}(1-\alpha) - \sqrt{\frac{2F_n'(0) \, d^2}{\nu^2}} \left( \frac{1}{4} - 4 F_n'(0) \, d^2 \right)^{1/2} \right) \quad (48) \]

\[ = 1 - \Phi \left( \Phi^{-1}(1-\alpha_2) - \frac{2}{\nu^2} \sqrt{N_2} \frac{d^2}{\nu^2} \right) \quad \text{for Gaussian inputs} \quad (49) \]

(b) Neyman-Pearson detector

This detector calculates the statistic

\[ T_3 = \sum_{j=1}^{N_3} (q_j^1 + q_j^2)^2 \quad (50) \]

The expectations and variances of \( T_3 \) are

\[ E_H[T_3] = 2 \nu^2 N_3 \]
\[ \text{Var}_H[T_3] = 4 \nu^4 N_3 \]
\[ E_K[T_3] = 2(\nu^2 + 2d^2) N_3 \]
\[ \text{Var}_K[T_3] = 4 N_2 \nu^2 (\nu^2 + 4d^2) \]

where \( d^2 \) and \( \nu^2 \) are the variances of the signal and noise distri-
bution respectively.

Once more paralleling the development in Section III, the $\alpha_3$ and $\beta_3$ are

$$\alpha_3 = 1 - \frac{t_3 - 2\nu^2 N_3}{2 \nu^2 \sqrt{N_3}}$$  \hspace{1cm} (51)

$$\beta_3 = 1 - \frac{\psi^{-1} (1-\alpha) - \frac{2d^2}{\nu^2} N_3^{1/2}}{[1 - 4\frac{d^2}{\nu^2}]^{1/2}}$$  \hspace{1cm} (52)

*: These expressions are different from those in Wolff's developments.

(c) Correlation Detector

This detector calculates the statistic

$$T_4 = \sum_{j=1}^{N_4} q_j^1 q_j^2$$

The expectation and variances of $T_4$ are

$$E_H[T_4] = 0$$

$$\text{Var}_H[T_4] = \nu^4 N_4$$

$$E_K[T_4] = d^2 N_4$$

$$\text{Var}_K[T_4] = (\nu^4 + 2\nu^2 d^2) N_4$$

With a similar development, one has

$$\alpha_4 = 1 - \frac{t_4}{\nu^2 \sqrt{N_4}}$$  \hspace{1cm} (53)
(d) Comparisons

Paralleling the treatment in Section IV, the comparisons made for these three detectors are obtained.

(i) For Gaussian inputs
\[ \bar{e}_{2,3} = 0.101 \]
\[ \bar{e}_{2,4} = 0.405 \]
\[ \bar{e}_{3,4} = 4 \]
where the indices 2, 3 and 4 stand for PCC, Neyman-Pearson detector and correlation detector respectively.

(ii) For the double exponential distribution
\[ \bar{e}_{2,3} = 7/2 \]
\[ \bar{e}_{2,4} = 4 \]
Appendix B. Bibliography


