I

A NEW DEAL FOR THE
ELEMENTARY FOUNDATIONS OF RELATIVITY

My aim today is to present an elementary procedure for laying the foundations of relativity. This new approach will have the advantage of isolating the precise point where classical and relativistic conceptions diverge from each other.

Of course Einstein himself and his followers have not failed to fix, and even to embody in one or more mathematical statements, a certain set of assumptions, which are to be replaced by another relativistic set (extremely close to the former under ordinary conditions). The most popular and fascinating modification of the whole point of view is, beyond doubt, the one, first adopted by Einstein, which consists in a systematic replacement of ordinary translations by the corresponding Lorentz transformation. The formal background of this has now become quite familiar to anyone who has had, in school or from books, an introduction to this modern line of thought. But I shall avoid presupposing any previous knowledge of this argument and beg you to accept, as a historical hint, the statement that, in all the elementary or advanced expositions that I know, restricted relativity is built up by simply passing, somewhat abruptly, from one set of formulæ to another. Usually the writer contents himself with the remark that the new formulæ contain the old as a limiting case.

1This lecture and the two following ones were delivered at the Rice Institute in September, 1936, by Professor T. Levi-Civita, of the University of Rome.
I regard it, however, as worth while to give attention to a careful survey of the traditional assumptions used in establishing the first principles of kinematics, especially those having regard to time, meanwhile searching for the essential point where a certain classical idea and the corresponding intuitive postulate have to be abandoned and replaced by another, as intuitive as possible, but implying relativistic and not classical consequences.

This new deal for the elements of relativity claims to be something more than an axiomatic anatomy of its logical ingredients; for the following reason. For the purposes of physics, astronomy, and engineering, a thorough training in classical mechanics cannot be omitted, and, as statical and dynamical sensibilities are valuable, they must still be developed in students of natural philosophy and engineering. It is therefore a sound policy, not only from a methodological point of view but also from a pedagogical one, to connect relativity with the foundations of school training, emphasizing the common geometrical basis and the common reckoning and measuring of time for separate observers and thus fixing the attention on the exact moment in which the two lines of thought cease to be the same. This occurs when individual notions of time, acquired by moving observers, come into comparison. The traditional kinematics accepts the simpler of a certain pair of alternatives, thus establishing absolute time. The other alternative is far more general, and needs to be duly specialized in order to supply the relativistic scheme.

Now to proceed with a rapid development of this program.

I. THE ORDINARY NOTION OF TIME FOR A GIVEN OBSERVER

I take for granted that the familiar Euclidean geometry holds for the ordinary astronomical space S. The "fixed
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stars," between which no appreciable changes of relative position were detected until recent times, used to offer a natural frame of reference, a fixed point being a point which preserved its position with respect to the "fixed stars." Now we may conceive of replacing the old Newtonian frame by a statistical one (in a well defined sense) and thus of being allowed to attribute a concrete meaning to fixed objects or observers in cosmic space $S$. With this understanding I proceed as if I had to explain in a classroom the elements of ordinary kinematics. First of all comes the notion of time and the measure of time. Without speculating on its philosophical meaning, there is undoubtedly agreement in ascribing a concrete (I do not say the only concrete) origin of time to our physiological or psychological sensations. To avoid vagueness, let us first fix our attention on a unique observer $O$.

Successions of sensations, in particular those arising from rhythmic phenomena, like the pulse, the alternation of day and night, of seasons, and so forth, afford the first rough estimates of time intervals. Ancient tools, such as the clepsydra, or more complicated machines, essentially based on vibrations (free or properly adapted), lead to more refined evaluations, and finally to a time-scale, like that of a good clock, corresponding to a continuous parameter $t$. Thus we get the time $t$ of $O$.

2. MANY OBSERVERS AT REST
IN THE ORDINARY ASTRONOMICAL SPACE $S$

If there are many observers $O, O', \cdots$, each develops in the above manner his own time, whence arise as many variables $t, t', \cdots$, which are a priori independent of each other. However, there is no possible ambiguity if, in accordance with our most deeply-rooted intuition, in establishing the
correlations among $t, t', \cdots$, we suppose that $O, O', \cdots$, are all immersed in the physical space (an ordinary Euclidean space $S$), and are at rest in it: at rest, of course, with respect to some frame of reference having a definite meaning, e.g., in the old style, the fixed stars, or in more modern style, their statistical substitute.

Assuming that there is perfect homogeneity in the behavior of local phenomena about the various observers $O, O', \cdots$, we must admit that the phenomena from which the times $t, t', \cdots$, have been inferred go on in the same way. This implies that the instruments (clocks), by means of which $O, O', \cdots$, deduce their time-measures, have the same behavior at all these places; therefore intervals $\Delta t, \Delta t', \cdots$, corresponding to the duration of analogous phenomena (that is, identical except for position) are to be considered equal. Thus, if we choose any special arrangement to characterize the unit of time for $O$, the same arrangement will fix the unit of time for $O'$, and we shall have for any other corresponding duration $\Delta t = \Delta t'$: $t$ and $t'$ will accordingly differ only by a constant $T$, which may of course depend on $O$ and $O'$.

It remains to be seen whether, and why (by convenient devices) $T$ may be determined. Some further obvious assumptions are needed concerning the exchange of signals between two observers.

3. LIGHT AND ITS PROPAGATION

We assume as a matter of fact the mere circumstance that a flash of light may be sent from one place to another; its propagation through the Euclidean space $S$ under consideration, being rectilinear with a constant speed $c$. If our two observers $O$ and $O'$, whom we have supposed to be at rest in $S$, are separated by a distance $l$, a light flash will
take a time $l/c$ to travel in either sense between $O$ and $O'$. By employing such signals, it is seen at once how to correlate $t$ and $t'$. Suppose that a signal starts from $O$ at the instant $t_0$, of the time $t$, and arrives at $O'$ at the instant $t_1'$ of the time $t'$. Owing to the identity of durations, between any $t$ and the corresponding $t'$ there is a constant difference:

$$t' = t + T.$$ 

Therefore the arrival at $O'$ takes place for $O$ at the instant $t_1$ defined by

$$t_1' - T,$$

and the interval of time required by the light to travel is

$$t_1' - T - t_0.$$ 

Equating this to $l/c$ we have the desired expression for $T$ by means of experimental data ($t_1'$, $t_0$, $l$ and $c$). Of course the compatibility of the assumptions requires that for any choice of $t_0$, we get always the same $T$. This being granted, we may obviously reduce $T$ to zero by a simple displacement of the origin of time of one of the observers, or of both in obvious correlation.

We may think of such a regulation of clocks as being performed at every point (assuming that the observer is at rest at this point) of the space $S$; and so we pass from the individual times $t$, $t'$, $\cdots$ to a unique time $t$, which is not yet the absolute time of the traditional kinematics, but has many of its characteristics: just those (as we shall ascertain) which subsist also on a more general footing, especially from the relativistic point of view. If we wish to give a name to this kind of time, we may call it pantopic, that is, valid for any place in the manner just described.

The absolute time of the old conception will now be introduced and illustrated. Thus we shall grasp the crucial moment at which relativity departs from the classical path.
4. ABSOLUTE TIME

Let $O$ designate as before an observer at rest in $S$, $t$ being his time (which may be adopted for every other fixed observer). We may describe the motion of any other point $P$, assigning at every instant the position of $P$ with respect to $O$, more precisely with respect to a Cartesian system $Oxyz$. According to the first elements of analytic geometry this amounts to regarding the coördinates $x$, $y$, $z$ of $P$ as functions of $t$. Now, if $O'$ with coördinates $\varphi$, $\psi$, $\chi$, is another point of $S$ (possibly moving with respect to $Oxyz$), we may introduce a trihedron $O'x'y'z'$ having axes $O'x'$, $O'y'$, $O'z'$ parallel to $Ox$, $Oy$, $Oz$ respectively. If $x'$, $y'$, $z'$ are the coördinates of $P$ with respect to this second frame of reference we obviously have the ordinary formulæ of transformation

$$x = x' + \varphi, \quad y = y' + \psi, \quad z = z' + \chi. \quad (1)$$

When $O'$, like $O$, was an invariable point of $S$, we recognized, as a result of almost irrefutable assumptions, that its time $t'$ is, or may be reduced so as to be, equal to $t$. Note that, among the circumstances justifying the relation $\Delta t' = \Delta t$, we essentially employed the homogeneity of space and of the behavior of phenomena around the two observers. Obviously such identity of behavior is no longer necessarily experienced if, $O$ being at rest, $O'$ is moving.

The classical kinematics completes its temporal postulates by assuming that, in establishing the time $t'$ of $O'$, as described, the possible motion of $O'$ has no influence: therefore a unique variable $t$ may be assumed, which plays the rôle of time for $O'$ as well as for $O$. This is the exact significance of introducing the absolute time $t$, that is, a time valid for any observer, whatever his motion may be.
5. GALILEAN PRINCIPLE OF COMPOSITION OF VELOCITIES

From the preceding, the mathematical deduction of the Galilean composition of velocities is immediate—as given in elementary textbooks. One has only to differentiate the formulæ (1) with respect to $t$, where, in the more general case, all may depend upon $t$. We get accordingly

$$\frac{dx}{dt} = \frac{dx'}{dt} + \frac{dx}{dt}, \quad \frac{dy}{dt} = \frac{dy'}{dt} + \frac{dy}{dt}, \quad \frac{dz}{dt} = \frac{dz'}{dt} + \frac{dz}{dt}.$$  

By definition, the absolute velocity of $P$ with respect to $O$ (which means with respect to the frame $Oxyz$) is nothing but the vector $v$ having components $dx/dt, dy/dt, dz/dt$; and the analogous velocity (frame-velocity) of $O'$ is the vector $w$ with components $dx'/dt, dy'/dt, dz'/dt$. If we introduce furthermore the relative velocity $v'$ of the same $P$ with respect to $O'$ (i.e., with respect to the second trihedron $O'x'y'z'$), namely the vector defined by the components $dx'/dt, dy'/dt, dz'/dt$ (the absolute time $t$ serving for $O'$ as well as for $O$, in the classical conception), we may condense the formulæ (2) in the unique vectorial relation

$$v = v' + w,$$

which states the rule of composition of velocities and is expressed in words as follows:

absolute velocity = relative velocity + frame-velocity;

a result deeply rooted in the traditional intuition of ordinary motions, e.g., when we compare velocities of some kind as they are estimated by observers at rest or by those in motion.
6. APPLICATION OF THE PRECEDING RULE
TO THE PROPAGATION OF LIGHT AT THE EARTH'S SURFACE—
THE MICHELSON EXPERIMENT

If we agree to regard the propagation of optical signals through $S$ (the absolute Euclidean space of classical mechanics and astronomy) as a particular rectilinear movement with constant vector-velocity $c$, the Galilean principle of composition will still be available. In passing from an observer $O$, at rest with respect to the fixed stars—or, as we may say in the old terminology, with respect to ether—to another observer $O'$, moving in any manner, we shall have from (3)

$\mathbf{c} = \mathbf{c}' + \mathbf{w},$

$c'$ being the light velocity as it appears to $O'$: in particular $O'$ may be identified with an observer on our earth. This is the case we shall consider. Owing to the additive term $\mathbf{w}$, $c'$ turns out to be in general different from $c$, and furthermore light is no longer propagated isotropically for the various directions issuing from $O'$. The difference is very small because, as firmly established by astronomical observations, the velocity $\mathbf{w}$ (properly the absolute value of the velocity) of the earth is roughly 30 km/sec, while $c$ is 300,000, that is $10^4$ times as great.

Considering in particular the case of a light-ray traveling in the direction of $\mathbf{w}$, first in the same sense and second in the opposite one, we get from (4) the scalar relation

$\mathbf{c} = \mathbf{c}' \pm \mathbf{w},$

or

$\mathbf{c}' = \mathbf{c} \mp \mathbf{w}.$

This difference in the velocity of propagation, though relatively very small, should produce, in a famous experiment
conceived and executed for the first time by Michelson in 1881, a displacement of diffraction fringes well within the limits of sensitiveness of optical instruments. As a matter of fact, not only the original experiment of Michelson but later ones, performed with every care by him and other physicists, have all given negative results, that is, almost complete absence of the expected displacement, at any rate an amount much less—only a few hundredths of that predicted by the above representation of the phenomenon.

Hence the conclusion, afforded by experimental evidence, that in reality the propagation of light behaves as if

\[ c' = c \]

either rigorously or with a closer approximation than would be predicted by the classical rule (4) for the composition of velocities.

7. NECESSARY REJECTION OF SOME OF THE ASSUMPTIONS WHOSE SIMULTANEOUS ACCEPTANCE IS CONTRADICTED BY THE MICHELSON EXPERIMENT

For more than twenty years the negative result, which we have just considered, has puzzled theoretical physicists. Obviously one at least of the premises, on which the expected but not realized optical effect rested, must be wrong. But which of these premises, all apparently so spontaneous and so immediately suggested by the simplest intuition, and confirmed by daily experience? Since both geometrical and kinematical principles were supposed to be beyond discussion, it was thought at first that the weak point might be the rigorous identification of the propagation of light-signals with points (particles) moving with constant velocity \( c \): in which case the Galilean rule of composition would no longer be a necessary consequence of the assumptions, and the
contradiction would disappear. This explanation, very reasonable from a logical and intuitive standpoint, was worked out on different lines by Stokes and Ritz, among others; but these theories are very complicated, and though restoring adherence to the facts as far as the Michelson experiment is concerned, they introduced discrepancies with respect to other phenomena and were never regarded as definitive.

A pragmatic explanation was found by Lorentz, as everyone knows, by means of the electro-magnetic theory of light. From this standpoint Maxwell's equations hold rigorously; on the contrary the uniform translation of rigid bodies is only an approximation (sufficient for ordinary phenomena) to be accompanied, for more refined purposes as in Michelson's experiment, by a certain very small contraction of body lengths in the direction of motion and, furthermore, by a similar contraction of time intervals, as calculated by co-moving observers.

Clearly this manner of presenting a law of motion, as originating (through *ad hoc* accommodation) from a familiar but only approximate law, did not seem satisfactory from a philosophical point of view. The last step, as is very well known, was accomplished by Einstein, who showed that the very meaning of the modified translation is nothing but the mathematical expression of his principle of relativity.

We wish, however, to obtain a deeper and more general insight into the very nature of time for a moving observer, without previously introducing rigid bodies and analyzing their translational movements. Of course, to get relativity itself, something more, especially suggested by it, must be postulated. This will be in fact a certain intrinsic character of events, which seems not to have been explicitly stated, though obviously involved in the theory.
8. ABANDONMENT OF ABSOLUTE TIME—
GENERAL FORMULA FOR THE COMPOSITION OF VELOCITIES

$O'$ being an observer, in general moving with respect to $O$, we still have the transformation formulæ (1) to relate the absolute and the relative motion of $P$, having the co-ordinates $x, y, z$ referred to the first frame with origin $O$, and the coördinates $x', y', z'$, referred to the second frame with origin in $O'$.

Suppose that, $t$ being still the time of $O$, there is no absolute time, so that the moving point $O'$ possesses a proper time $t'$, which is not necessarily identical with $t$, nor differs from $t$ by a constant. Ignoring for the moment the specific relations between $t'$ and $t$, we may still consider, as before, the formulæ (1) of transformation of parallel axes from the origin $O$ to $O'$, interpreting them as connecting two aspects of the motion of the same point $P$, having the coördinates $x, y, z$, with respect to the origin $O$, and the coördinates $x', y', z'$, with respect to the other origin $O'$.

In the first aspect the rôle of time is played by $t$, in the second by $t'$. Obviously we may differentiate the equation (1) with respect to $t$, noting that

$$\frac{dx'}{dt} = \frac{dx'}{dt'} \frac{dt'}{dt},$$

and similarly for $y'$ and $z'$.

Writing for brevity

$$\frac{dt'}{dt} = k,$$

we have

$$\frac{dx}{dt} = k \frac{dx'}{dt'}, \quad \frac{dy}{dt} = k \frac{dy'}{dt'}, \quad \frac{dz}{dt} = k \frac{dz'}{dt'} + \frac{dx}{dt};$$
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$dx/dt$, $dy/dt$, $dz/dt$ are, as before, the components of the absolute velocity $v$; $dx'/dt'$, $dy'/dt'$, $dz'/dt'$ (in which $t'$ takes the place of $t$) are correspondingly the components of the relative velocity $v'$, i.e., of the velocity referred to the parallel axes drawn through the point $O'$, which is supposed to move in any manner whatever. The formulæ just written give the vector relation

$$v = kv' + w,$$

which obviously differs from (3) only by the scalar factor $k$, and holds without any hypothesis.

Admitting as usual absolute time we should have $dt' = dt$, and then, from (5), $k = 1$; (6) reduces thus to (3).

Our purpose is simply to recognize what determination of $k$ characterizes relativity. The task will be very easy, since we (in this year of 1936) know beforehand where to go: the situation would of course be different had not Einstein previously discovered relativity! On the way, however, we have ascertained that one may establish a more general kinematics without introducing absolute time and that, even from this standpoint, there is still a very simple rule for the composition of velocities.

Now let us seek out a substitute for the usual postulate $k = 1$, and especially for the alternative adopted in relativity, one which is capable of explaining the failure of the Michelson experiment.

9. ELEMENTARY EVENTS

We shall give this name to events which, though really occupying a small space and a small interval of time, may be concentrated in a unique point $P$ of the space $S$, and a unique instant $t$ (of the pantopic time, common to all observers at rest in $S$).
From the mathematical aspect an elementary event $E$ is merely a point of the four-dimensional manifold corresponding to the couple $(P, t)$ consisting of a point $P$ of $S$, defined by its three coördinates, and an instant $t$. We may cite a great many facts which, with obvious abstractions, reduce to elementary events. For instance, the closing or opening of an electric circuit at a well determined place; the lighting or extinguishing of a lamp; the firing of a gun; someone's death; a collision, and so on.

10. AMOUNT OF UNLIKENESS BETWEEN TWO ELEMENTARY EVENTS $E_1$ AND $E_2$

If two elementary events $E_1$ and $E_2$ both happen in the same place $P$, but at two different instants $t_1$ and $t_2$, it is natural (disregarding qualitative diversity) to adopt the difference

\[ \Delta t = t_2 - t_1, \]

or better, its absolute value $|\Delta t|$, or, what amounts to the same, a proportional quantity, as an adequate measure of the deviation between one event and the other.

Passing to the general case of two events $E_1(P_1, t_1)$ and $E_2(P_2, t_2)$, which occur not only at different instants but also at different places, we may of course adopt several devices to estimate the distinction.

First we may disregard the diversity of position, and fix our attention on $|\Delta t|$ alone. This preponderance bestowed on time leads substantially to the traditional conception upon which the ordinary kinematics and, more generally, all classical physics, is founded. But it is equally permissible to pay attention also to local diversity. By mathematical analogies we should be led to apply Gauss's criterion, which is the following: for any two entities (e.g., geometrical
figures) $F_1, F_2,$ of the same sort, characterized by the values of $n$ parameters, say $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ respectively, an adequate numerical estimate of the total deviation is furnished by the quadratic form

$$\sum_{i=1}^{n} (x_i - y_i)^2,$$

or, possibly, by some other positive definite function of the arguments $x_i - y_i$.

Precisely on this model rests the introduction of the expression of constraint in analytical mechanics. The general idea is obviously suggested by the image of two geometric points, in ordinary space, which, unless coincident, appear always essentially distinct.

But there exist also entities of a different kind, endowed with one or more dominating characters, for which the equality of these principal features ensures a sufficient, or even striking equivalence, while the identity is only an irrelevant circumstance. A good example is furnished by a periodic function $f(x)$. Let us consider two values $x_1, x_2$ of the variable $x$, and suppose that it is only the corresponding values of the function, $f(x_1), f(x_2)$, which have an actual interest. The diversity between $x_1$ and $x_2$ then has to be estimated in terms of the values $f(x_1)$ and $f(x_2)$. Therefore, if $x_1$ and $x_2$ differ only by integral multiples of $\omega$ ($\omega$ being the period of the function $f$), their deviation must be regarded as zero, and in general it will depend only upon the remainder after dividing $x_2 - x_1$ by $\omega$. It is not worth while to inquire now whether the absolute value of this remainder, or its square, or some other function, is the most convenient expression for the deviation. What alone matters is the realization that the deviation must be ap-
preciated in close connection with the specific nature of things.

II. PROVISIONAL INTERVENTION OF AN AUXILIARY OBSERVER.
OPTICAL CONTEMPORANEITY.
ELIMINATION OF SINGLE OBSERVERS

Suppose that an observer \( O \) becomes acquainted with two events \( E_1 \) and \( E_2 \), in general occurring at different places \( P_1, P_2 \) and different instants \( t_1, t_2 \), by the transmission of signals emitted from \( P_1 \) and \( P_2 \) at the instants in which the corresponding events happened.

By the postulates of the propagation of light, recalled in §4, these signals will reach \( O \) at the instants \( t_1 + r_1/c \), \( t_2 + r_2/c \), where \( r_1, r_2 \), designate the distances \( OP_1 \) and \( OP_2 \).

If \( O \) does not belong to the straight line \( P_1P_2 \) the signals arrive at \( O \) from different directions. The same is true (the directions being opposite) if \( O \) lies on the straight line, inside the segment \( P_1P_2 \). This is enough to point out the diversity between the two events: for the moment we need not go further in searching for a quantitative representation.

If on the other hand \( O \) lies on the straight line \( P_1P_2 \) outside the segment, then light-signals, emitted towards \( O \) from the two stations \( P_1, P_2 \), reach \( O \) in the same direction and sense. The reception of both signals will occur at the same instant if, and only if,

\[
t_1 + r_1/c = t_2 + r_2/c.
\]

In this case the two events appear to be contemporaneous to the observer. We shall then say that they are synaesthetic for \( O \), that is perceived together by him.

Calling \( l \) the length of the segment \( P_1P_2 \) we have obviously

\[
r_2 - r_1 = \pm l,
\]
with the upper or the lower sign according as $O$ lies (outside of the segment $P_1P_2$) beyond $P_1$ or beyond $P_2$.

The preceding condition of synaesthesia, with respect to $O$, may thus be written

\[(8) \Delta t \pm \frac{l}{c} = 0,\]

where $\Delta t$ already defined by (7), is merely the difference in times, $t_2 - t_1$.

Let us now consider the general case, in which, $O$ lying outside of the segment $P_1P_2$, the condition (8) is not satisfied, the binomials

\[
\Delta t + \frac{l}{c}, \Delta t - \frac{l}{c},
\]

being accordingly both different from zero. We shall try to derive from them some quantitative estimate of the deviation between $E_1, E_2$, without intervention of the observer $O$, who may henceforth be located anywhere, compatibly with his being able to receive at the same instant the signals of the two events, for which it is only required that $O$ lies on the straight line $P_1P_2$ outside of the segment.

To this end we first multiply by $c$ in order to get lengths and fix our attention on the absolute values of the two binomials

\[(9) \begin{cases} \delta_1 = c\Delta t + l, \\ \delta_2 = c\Delta t - l. \end{cases}\]

Nobody will contest the statement that the more both
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differ simultaneously from zero the larger the diversity between $E_1$ and $E_2$ is to be considered.

12. INTERVAL BETWEEN TWO EVENTS.

EVENTS INFINITELY NEAR.

$d^2$ OF EINSTEIN-MINKOWSKI

Let us try to go further, introducing a criterion of measure. As already remarked, $\delta_1=0$, or $\delta_2=0$ represent, considered separately, the conditions under which the two events $E_1$, $E_2$ appear synaesthetic to observers located outside $P_1P_2$, either beyond $P_1$ or beyond $P_2$. Seeking now a definition of the diversity between the two events, in which both $\delta_1$ and $\delta_2$ are considered on an equal footing, we have to introduce some kind of mean: but what kind? Not the arithmetical one, whose absolute value

$$\frac{1}{2}|\delta_1+\delta_2|=c|\Delta t|$$

is merely (except for the constant factor $c$) the usual time difference, independent of the difference of position in space.

In seeking a mean, in which both elements are incorporated, the contemporaneity of optical perceptions being assumed as a test of equivalence (diversity then becoming zero), we are spontaneously led to think of the geometric mean, instead of the arithmetic; or, even more simply, the product

$$\delta_1\delta_2.$$ 

Indeed this quantity enjoys the double property of being symmetric with respect to $\delta_1$ and $\delta_2$ (vanishing when and only when either $\delta_1$ or $\delta_2$ does) and of increasing in absolute value with both $|\delta_1|$ and $|\delta_2|$.

All of this shows the plausibility of adopting as a measure of the diversity of two events $E_1$, $E_2$ the product $\delta_1\delta_2$, to be called the world interval. By (9), its numerical expression is
In particular, if the two events $E_1(P, t)$ and $E_2(P', t+dt)$ are very near (in both time and space), writing $dl$ for $l$, the interval $I$ becomes the quadratic indefinite form $ds^2$ of Einstein-Minkowski:

\[ ds^2 = c^2 dt^2 - dl^2. \]

We have thus reached a genesis and a concrete interpretation of $ds^2$ which, though perhaps a little less familiar than some of those used in geometry and in mechanics, has still the advantage of implying only the simplest notions of geometry and of uniform rectilinear motion, without requiring, from the very beginning, the more complex analysis of rigid motions and therefore the motions of not only one point but of a system of points. This analysis may be deferred to a riper stage, the methodological foundation of relativity having been already established by a clear-cut, primary assumption. The above considerations lead us to regard intervals between two events as concrete physical entities, like distances and durations, which, however, are only particular cases.

The measures of these intervals are accordingly expressed by the formula (10), or even (11), if we are considering especially two events infinitely near.

Once having admitted the intrinsic character of the $ds^2$ corresponding to a given interval, it follows as a necessary consequence that this $ds^2$ is not affected by, i.e., is an invariant with respect to, our choice of any independent space-time variables. But a further tempting step is the attribution to it (as, for example, to force in classical mechanics) of intrinsic features: i.e., of existence and determination, completely independent of any observer, be he at rest or
moving at will through S. It is here that relativity begins to separate itself from traditional thought. According to the latter, it is admitted that the same variable $t$ may represent time for any two observers $O$ and $O'$, irrespective of their rest or motion; the kinematics of relativity assumes, on the contrary, interval and not time as autonomous, i.e., independent of the observer.

It is hardly necessary to remark that, for the major part of terrestrial and astronomical events, we have to do with intervals for which $ct$ is much larger than $l$, so that the invariance of interval is approximately the same as the invariance of $\Delta t$, so far as observers at rest in the $S$-space are concerned; but it is no longer the same for other more refined phenomena, especially optical ones: and it is these that called forth the theory of relativity and have provided very brilliant, although minute, experimental tests of the theory.

13. GENERAL CORRELATION OF TIMES

It is obviously impossible to go further in stating the immense influence of the preceding postulate on the development of all mechanics and physics. I must confine myself now to a very simple example, and to a general remark which makes clearer the essence of the Lorentz transformation.

First as an immediate consequence of the postulate, we may recognize what is (instead of $\Delta t = \Delta t'$) the relation between the times of two observers $O$ and $O'$, the former at rest in $S$, the latter moving in any manner.

The transit of $O'$ through a position $P$ at a certain instant $t$ is obviously an event $E$; its transit at a successive instant $t + dt$, through another position $P'$, infinitely near to $P$, is a second event $E'$, infinitely near to $E$. The interval of these two events is expressed, according to (11), by

$$ds^2 = c^2 dt^2 - dl^2,$$
where \( dl = PP' \). Since, actually, the ratio \( dl/dt \) is, by definition, the absolute value of the velocity of \( O' \) at the instant \( t \), we may write

\[
(12) \quad ds^2 = c^2(1 - v^2/c^2)dt^2.
\]

Now let us apply the postulate affirming that the interval of any two events—in our case, the transits of \( O' \) through \( P \) at the instant \( t \) and through \( P' \) at the instant \( t + dt \)—may be estimated indifferently by the fixed observer \( O \), giving the result (12), or by the moving observer \( O' \). Adopting this plan, it will be remembered that the two events \( E \) and \( E' \) happen at the same place, namely at \( O' \) (passing first through \( P \) and then through \( P' \)). Therefore the \( ds \) in question is, except for the factor \( c \), merely an elementary interval \( dt' \) of the time \( t' \), as it runs for \( O' \). Accordingly

\[
(13) \quad dt' = \frac{1}{c}ds = dt\sqrt{1 - v^2/c^2},
\]

and the coefficient \( k \), which we have introduced generally to correlate the proper time \( t' \) of a moving observer \( O' \) with the standard time \( t \), appears to be

\[
k = \frac{1}{\sqrt{1 - v^2/c^2}}.
\]

Obviously we must suppose \( v < c \) in order to have real, finite, non-zero, values of \( k \). In other words, the velocity \( c \) of light is to be regarded as a limit, never attained in the motion of real bodies.

For \( v < c \) the relation (13) shows that, for a moving observer, the intervals \( dt' \) are smaller than the corresponding ones \( dt \); that is, the clocks of \( O' \) go slower (contraction of times).
Permit me to finish with a striking, though almost trivial remark. It refers to the simplest form of motion of a body, kinematically conceived as a system of several points.

From the classical point of view, if we designate the actual positions (i.e., positions at a generic instant \( t \)) of these points by \( P \), and their initial positions by \( P_o \), we have clearly, for any \( t \), a one-to-one correspondence between the \( P_o \)'s and the \( P \)'s. The same is true if we replace the initial positions \( P_o \) by another manifold of points \( M \) in one-to-one correspondence with the \( P_o \)'s, and therefore capable, like the \( P_o \)'s, of individualizing the single moving particles. Accordingly, with self-evident abridged notation, we may write

\[
P = P(M \mid t),
\]

from which we get the vector-velocity \( v \) at any instant by partial differentiation with respect to \( t \) (\( M \) being regarded as constant):

\[
v = \frac{\partial P(M \mid t)}{\partial t}.
\]

From the classical point of view we have to consider \( t \) as absolute time, valid for any moving point. On the other hand, by the new relativistic postulate of the invariance of intervals, we have, for every moving point actually occupying the position \( P \), a proper time \( t' \), whose elementary intervals \( dt' \) are shorter than the corresponding interval \( dt \) according to the formula (13), where \( v \) is the length of the vector (15).

Fixing the attention on a well determined point, \( M \), as already stated, must be regarded as constant, and \( v \) is, by (15), a well determined vector function of \( t \) alone. Equation (13) may then be integrated, giving

\[
t' = \int \frac{dt}{\sqrt{1 - \frac{v^2}{c^2}}} + \text{const.}
\]
As the last term is intended to be "constant with respect to \( t \)," it may depend upon \( M \), and the fourth relation above, to be associated with ordinary equations of motion (14), may be written

\[
(16) \quad t' = \int^t_0 \frac{dt}{\sqrt{1-v^2/c^2}} + \chi(M),
\]

\( \chi \) being an arbitrary scalar function of \( M \) alone. The conclusion is as follows: while, in ordinary kinematics, the motion of a continuous system is defined by (14), \( t \) appearing in the right hand side as a parameter, the necessary association of (16) to represent the phenomenon of motion from a relativistic point of view, shows that, analytically, we have to do with a fourfold (instead of ternary) transformation between two quadruples \((x, y, z, t)\) and \((x', y', z', t')\), \( x, y, z \), indicating the coördinates (e.g., Cartesian) of \( M \) and \( x', y', z' \) those of \( P \).

Among these general transformations, the simplest, from a relativistic point of view, are to be considered those for which not only the amount, but also the formal expression of the interval is preserved. They form a linear\(^1 \) group named after Lorentz. It is precisely the group in four variables for which the quadratic form

\[
ds^2 = c^2 dt^2 - dl^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)
\]

is left invariant, i.e., transformed (when expressed in terms of the new variables \( x', y', z', t' \)) into the corresponding form

\[
c^2 dt'^2 - (dx'^2 + dy'^2 + dz'^2).
\]

\(^1\)If we add the qualitative condition that to finite values of \( x, y, z, t \) correspond finite values of \( x', y', z', t' \), and vice versa. Cf. a note by Dr. Clarice Munari in the *Rend. della R. Acc. dei Lincei*, ser. 5, XXIII, 1 Sem. 1914, pp. 781–787.
From this point on, everything goes on as it is set forth in good treatises on relativity. I trust, however, that I have contributed, by means of this preliminary introduction, a methodological improvement deserving of some didactic interest.