RICE UNIVERSITY

A NEW STIFFNESS MATRIX FOR THE ANALYSIS OF THE BENDING OF PLATES USING RECTANGULAR FINITE ELEMENTS

by

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A new displacement function is obtained for the analysis of the bending of plates using rectangular finite elements which satisfies the conditions of compatibility across element boundaries as well as throughout the elements.

The improvement in accuracy and convergence is studied by comparing the results of some problems with those obtained by Zienkiewicz and Cheung using a different displacement function. Also, some results are compared with known exact solutions and with the results obtained by other methods.

Finally, a displacement function is proposed for the analysis of the bending of plates bounded by coordinate lines in polar coordinates, using elements consisting of annular sectors.
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NOTATION

A  Compatibility matrix
B  Matrix defined by  \( e = B \Delta \)
D  Flexural rigidity of a plate element
E  Matrix which characterizes the geometric and elastic properties of the isotropic plate element
F  Nodal force matrix
K  Stiffness matrix for the entire plate
M  Internal moments matrix
\( M_{x',y'} \)  Bending moments per unit length of section of a plate perpendicular to \( x \) and \( y \) axes, respectively
\( M_{xy} \)  Twisting moment per unit length of section of a plate perpendicular to the \( x \) axis
P  Equivalent forces matrix
Q  Single load
R  Smaller radius of curvature for the annular sector element
a,b  Dimensions of the rectangular plate element
c  Radial dimension for the annular sector element
d  Nodal displacement matrix for the entire plate
k  Stiffness matrix for the plate element
q  Intensity of a continuously distributed load
r,\( \phi \)  Polar coordinates
w  Displacement function for the plate element
x,y  Rectangular coordinates
\( x_1', \ x_2', \ y_1', \ y_2' \)  Coordinates of sides of a rectangle where a uniform load is distributed
$\Delta$  Nodal displacement matrix for the plate element

$\varepsilon$  Plate distortions matrix

$\nu$  Poisson's ratio
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CHAPTER I

INTRODUCTION

The finite element method, originally developed by Turner, Clough et al.,\(^{(1)}\) has been regarded as one of the most versatile and effective numerical methods for the analysis of plane stress and plane strain problems. Melosh\(^{(2)}\) has shown the extension of the method to the analysis of three-dimensional solids. Application of the method to bending of plates has been successfully presented by some investigators. Clough\(^{(3)}\) has suggested displacement functions for a triangular plate element in bending. Melosh\(^{(4)}\) has developed a stiffness matrix for an isotropic rectangular plate element of uniform thickness for the analysis of thin plates of variable thickness in bending. Zienkiewicz and Cheung\(^{(5)}\) have used displacement functions for the analysis of plate bending using rectangular and quadrilateral elements. Melosh\(^{(6)}\) has presented the direct stiffness method as a variational approach and has obtained a displacement function for a rectangular plate element.

In the analysis of a plate using the finite element method, the plate is subdivided into a number of elements. The original continuous system is replaced by these elements connected together at their corners (fig. 1). Although equilibrium is satisfied only at the corners of the element or nodal points, continuity of deformation as well as displacement can be maintained at these corners and also all along the edges, by constructing a suitable displacement function, \(w = w(x,y)\), for each element.
FIG. 1

A PLATE SUBDIVIDED INTO A NUMBER OF ELEMENTS

FIG. 2

A RECTANGULAR ELEMENT
Clough\(^{(3)}\) and Zienkiewicz and Cheung\(^{(5)}\) have formulated stiffness matrices for the solution of plate problems using as displacement functions algebraic polynomials in x and y. They establish, however, continuity of deformation and displacement only at the nodal points. Melosh\(^{(4)}\) has presented a stiffness matrix for a rectangular element by using the expression for the bending strain energy instead of constructing a displacement function. Although it gives good results, the convergence is not monotonic and it does not maintain displacement continuity. Melosh\(^{(6)}\) and Clough\(^{(3)}\) have suggested displacement functions which satisfy all compatibility requirements.

Herein a displacement function is obtained for rectangular elements which satisfies the conditions of compatibility across element boundaries, as well as throughout the elements.

A displacement function is also proposed for the analysis of the bending of plates bounded by coordinate lines in polar coordinates, using elements consisting of annular sectors.
CHAPTER II

DESCRIPTION OF THE METHOD

Consider the plate of fig. 1 and take a rectangular element $E_i$ (fig. 2). For each nodal point $j$ ($j=1,2,3,4$) three displacements are considered: vertical deflection ($A_j$), rotation about the $x$ axis ($\theta_{xj}$) and rotation about the $y$ axis ($\theta_{yj}$). The vertical deflection, $w$, is considered to be positive upward and rotations obey the right-hand-screw sign convention. Therefore, with each element $E_i$ is associated a $12 \times 1$ nodal displacement matrix

$$
\Delta = \begin{bmatrix}
A_1 \\
\theta_{x1} \\
\theta_{y1} \\
A_2 \\
\theta_{x2} \\
\theta_{y2} \\
A_3 \\
\theta_{x3} \\
\theta_{y3} \\
A_4 \\
\theta_{x4} \\
\theta_{y4}
\end{bmatrix}
$$

\[ \text{(2.1)} \]
The nodal forces required to produce these displacements define the stiffness-matrix relationship

\[
F = \mathbf{k} \Delta 
\]

where \( k \) is a 12 x 12 matrix which describes the stiffness of the element \( E_1 \).

Under the action of these external forces the middle surface of the plate undergoes twisting and bending distortions characterized by

\[
\varepsilon = \begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\sqrt{2} \varepsilon_{xy}
\end{bmatrix}
\]

where \( \varepsilon_{xx} = \frac{\partial^2 w}{\partial x^2}, \varepsilon_{yy} = \frac{\partial^2 w}{\partial y^2} \) and \( \varepsilon_{xy} = \frac{\partial^2 w}{\partial x \partial y} \).
\( \varepsilon \) can be obtained once the displacement function \( w \) is constructed.

The displacement function is a linear function of the nodal displacements. These nodal displacements are regarded as the generalized coordinates.

Thus, the matrix \( \varepsilon \) can be expressed by

\[
\varepsilon = BA
\] ..............(2.5)

where \( B \) denotes a matrix function of \( x \) and \( y \).

In addition, the internal moments

\[
M = \begin{bmatrix}
M_x \\
M_y \\
\sqrt{2} M_{xy}
\end{bmatrix}
\] ..............(2.6)

can be obtained from

\[
M = -E \varepsilon
\] ..............(2.7)

where

\[
E = D \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & 1-\nu
\end{bmatrix}
\] ..............(2.8)

characterizes the geometric and elastic properties of the isotropic plate.
For an orthotropic plate it is necessary only to choose appropriately the matrix \( E \).

By using the principle of virtual work, for a generalized virtual displacement \( \delta \Delta \), equation (2.5) becomes

\[
\delta \varepsilon = B \delta \Delta
\] ..............(2.9)

and the external work \( \delta \Delta^T F \) performed by the nodal forces \( F \) and the internal work

\[
- \int_0^a \int_0^b \delta \varepsilon^T M \, dx \, dy
\]
done by the moments $M$, through the specified virtual displacement $\delta \Delta$, can be equated as

$$\delta \Delta^T F = - \int_0^a \int_0^b \delta \epsilon^T M \, dx \, dy \quad \ldots \ldots (2.10)$$

or

$$\delta \Delta^T F = \int_0^a \int_0^b \delta \Delta^T B^T E B \Delta \, dx \, dy \quad \ldots \ldots (2.11)$$

Therefore,

$$F = \left( \int_0^a \int_0^b B^T E B \, dx \, dy \right) \Delta \quad \ldots \ldots (2.12)$$

which defines the stiffness matrix

$$k = \int_0^a \int_0^b B^T E B \, dx \, dy \quad \ldots \ldots (2.13)$$

Once the stiffness matrix $k$ for each element is determined, the matrix $K$ for the entire structure can be established by means of the equation

$$K = A^T k A \quad \ldots \ldots (2.14)$$

where $A$ is defined by the compatibility relation between the nodal displacement matrix for the elements $E_i$ and the displacement matrix $d$ for the entire plate ($\Delta = A d$).

Alternatively, a book-keeping approach is used and the $K$ matrix is easily constructed by simply summing up the appropriate stiffness matrices $k$ for the various finite elements.
CHAPTER III

CONSTRUCTION OF THE DISPLACEMENT FUNCTION

It is possible to construct many different displacement functions which satisfy the compatibility conditions across the element boundary as well as continuity over the element. Such displacement functions generally induce constraints leading to excessive stiffnesses of the elements. Therefore, the most suitable displacement function is the one based on a deformation pattern which gives the least stiff individual element. Although the displacement function here presented does not represent this limiting case, it does provide a lower bound to the exact solution of plate bending problems.

Consider the displacement \( w \) as a linear function of the nodal displacements

\[
\mathbf{w} = \bar{\mathbf{w}} \Delta
\]

where \( \bar{\mathbf{w}} \) is a \( 1 \times 12 \) matrix with its elements \( \bar{w}_j \) \( (j=1,2,\ldots,12) \) functions of \( x \) and \( y \),

\[
\bar{\mathbf{w}} = \begin{bmatrix}
\bar{w}_1 & \bar{w}_2 & \bar{w}_3 & \bar{w}_4 & \bar{w}_5 & \bar{w}_6 & \bar{w}_7 & \bar{w}_8 & \bar{w}_9 & \bar{w}_{10} & \bar{w}_{11} & \bar{w}_{12}
\end{bmatrix}
\]

To determine the elements \( \bar{w}_j \) of the matrix \( \bar{\mathbf{w}} \), first consider the rectangular element \( E_1 \) (fig. 3) clamped at sides 2-4, 3-4 and guided at sides 1-2, 1-3. A unit vertical displacement at corner 1 is imposed and displacements at the other corners and rotations at corner 1 about \( x \) and \( y \) axes are prevented. A displacement function \( \bar{\mathbf{w}}_1 \) for this plate element is required such that the following continuity requirements are satisfied:
FIG. 3
DEFLECTED SHAPE FOR A RECTANGULAR ELEMENT WHEN $\Delta_1 = 1$

FIG. 4
DEFLECTED SHAPE FOR A RECTANGULAR ELEMENT
WHEN $\theta_{x_1} = 1$
a) No rotation along edges 1-2 and 3-4 about the x axis

\[ \frac{\partial w}{\partial y} (x,0) = \frac{\partial w}{\partial y} (x,b) = 0 \]

b) No rotations along edges 1-3 and 2-4 about the y axis

\[ \frac{\partial w}{\partial x} (0,y) = \frac{\partial w}{\partial x} (a,y) = 0 \]

c) No vertical deflections along edges 2-4 and 3-4

\[ \bar{w}_1 (a,y) = \bar{w}_1 (x,b) = 0 \]

Various polynomial functions can be constructed which satisfy these deformation requirements. However, the polynomial function which is completely determined without further conditions is the one which, for any section parallel to a coordinate axis, expresses the deflection of the plate as a third degree polynomial and, in particular, for each deflected edge (1-2,1-3) represents the deflection of a clamped-guided beam with its maximum deflection equal to one. Consequently, it is the polynomial function of lowest degree satisfying the prescribed requirements. It is trivial to verify that the displacement function

\[ \bar{w}_1 = \frac{1}{a^3 b^3} \left( 2 x^3 + a^3 - 3 a x^2 \right) \left( 2 y^3 + b^3 - 3 b y^2 \right) \]

satisfies all the continuity requirements, including the unit vertical displacement at corner 1.

Next, a unit rotation about the x axis is imposed at corner 1, supposing no displacement at any other corner, no vertical deflection and no rotation about the y axis at corner 1 (fig. 4). A displacement
function \( \bar{w}_2 \) is required such that:

a) No rotation along edges 1-3, 2-4 about the y axis

\[
\frac{\partial \bar{w}_2}{\partial x} (0, y) = \frac{\partial \bar{w}_2}{\partial x} (a, y) = 0
\]

b) No rotation along the edge 3-4 about the x axis

\[
\frac{\partial \bar{w}_2}{\partial y} (x, b) = 0
\]

c) No vertical deflections along edges 3-4, 2-4, 1-2

\[
\bar{w}_2 (x, 0) = \bar{w}_2 (a, y) = \bar{w}_2 (x, b) = 0
\]

Considering for any section in the x direction a clamped-guided beam and in the y direction a clamped-hinged beam, the following displacement function

\[
\bar{w}_2 = \frac{1}{a^3 b^2} \left( 2 x^3 + a^3 - 3 a x^2 \right) \left( y^3 + b^2 y - 2 b y^2 \right)
\]

satisfies all the required conditions.

The displacement functions \( \bar{w}_3, \bar{w}_4, \bar{w}_5, \bar{w}_6, \bar{w}_7, \bar{w}_8, \bar{w}_9, \bar{w}_{10}, \bar{w}_{11} \) and \( \bar{w}_{12} \) are obtained in a similar manner, when a unit rotation is given at corner 1 about the y axis, and a unit vertical deflection and unit rotations are given at each corner 2, 3, 4, separately. These displacement functions are given below:

\[
\bar{w}_3 = \frac{1}{a^3 b^3} \left( 2 a x^2 - a^2 x - x^3 \right) \left( 2 y^3 + b^3 - 3 b y^2 \right)
\]

\[
\bar{w}_4 = \frac{1}{a^3 b^3} \left( 3 a x^2 - 2 x^3 \right) \left( 2 y^3 + b^3 - 3 b y^2 \right)
\]
\[ \bar{w}_5 = \frac{1}{a \cdot b^2} (2x^3 - 3ax^2)(2by^2 - b^2y - y^3) \]
\[ \bar{w}_6 = \frac{1}{a \cdot b^3} (ax^2 - x^3)(2y^3 + b^3 - 3by^2) \]
\[ \bar{w}_7 = \frac{1}{a \cdot b^3} (2x^3 + a^3 - 3ax^2)(3by^2 - 2y^3) \]
\[ \bar{w}_8 = \frac{1}{a \cdot b^2} (2x^3 + a^3 - 3ax^2)(y^3 - by^2) \]
\[ \bar{w}_9 = \frac{1}{a \cdot b^3} (2ax^2 - a^2x - x^3)(3by^2 - 2y^3) \]
\[ \bar{w}_{10} = \frac{1}{a \cdot b^3} (3ax^2 - 2x^3)(3by^2 - 2y^3) \]
\[ \bar{w}_{11} = \frac{1}{a \cdot b^2} (3ax^2 - 2x^3)(y^3 - by^2) \]

finally
\[ \bar{w}_{12} = \frac{1}{a \cdot b^3} (x^3 - ax^2)(2y^3 - 3by^2) \]

Consequently, the displacement function \( w \) for the element \( E_i \) can be written as

\[
\begin{align*}
W &= 0 \cdot \Omega \cdot (\epsilon^3 - \xi^2 + 1)(2\eta^3 - 3\eta^2 + 1) + \theta_{\epsilon_1} \cdot \Omega \cdot (2\xi^3 - 3\xi^2 + 1)(\eta^2 - 2\eta + 1) - \\
& \quad - \theta_{\xi_1} \cdot \Omega \cdot (\xi^2 - 2\xi + 1)(2\eta^3 - 3\eta^2 + 1) - \Delta_2 \cdot (2\xi^3 - 3\xi^2)(2\eta^3 - 3\eta^2 + 1) - \\
& \quad - \theta_{\eta_1} \cdot \Omega \cdot (\xi^3 - 3\xi^2)(\eta^2 - 2\eta + 1) - \theta_{\xi_2} \cdot \Omega \cdot (\xi^2 - 5)(2\eta^3 - 3\eta^2 + 1) - \\
& \quad - \Delta_3 \cdot (2\xi^3 - 3\xi^2 + 1)(2\eta^3 - 3\eta^2) + \theta_{\xi_3} \cdot \Omega \cdot (2\xi^3 - 3\xi^2 + 1)(\eta^2 - \eta) + \\
& \quad + \theta_{\eta_3} \cdot \Omega \cdot (\xi^2 - 2\xi + 1)(2\eta^3 - 3\eta^2) + \Delta_4 \cdot (2\xi^3 - 3\xi^2)(2\eta^3 - 3\eta^2) - \\
& \quad - \theta_{\xi_4} \cdot \Omega \cdot (2\xi^3 - 3\xi^2)(\eta^2 - \eta) + \theta_{\eta_4} \cdot \Omega \cdot (\xi^2 - 5)(2\eta^3 - 3\eta^2)
\end{align*}
\]
where

\[ \xi = \frac{X}{a} \quad \text{and} \quad \eta = \frac{Y}{b}. \]
CHAPTER IV

STIFFNESS MATRIX AND EQUIVALENT FORCES

Once the displacement function \( w \) is constructed, the stiffness matrix \( k \) for each element can be easily computed by using equation (2.13)

\[
k = \int_0^a \int_0^b B^T E B \, dx \, dy
\]

where \( B \) is given by

\[
B = \begin{bmatrix}
\bar{v}_{xx} \\
\bar{v}_{yy} \\
\sqrt{2} \bar{v}_{xy}
\end{bmatrix}
\]

Because of the geometrical symmetry of the plate element and taking into account the symmetry of the matrix, only 24 elements of the matrix have to be found, instead of 144. These elements are given below:

\[
k(1,1) = \frac{D}{a^3 b^3} \left( \frac{156}{35} b^4 \bar{v}_{xx} + \frac{156}{35} a^4 \bar{v}_{yy} + \frac{72}{25} b^2 a^2 \bar{v}_{xy} \right)
\]

where \( a \) and \( b \) are the dimensions of the element and \( D \) the flexural rigidity of the plate element.

\[
k(1,2) = \frac{D}{a^3 b^2} \left( \frac{126}{210} b^4 \bar{v}_{xx} + \frac{26}{35} a^4 \bar{v}_{yy} + \frac{12}{50} a^2 b^2 + \frac{6v}{5} a b^2 \right)
\]

where \( v \) is the Poisson's ratio.

\[
k(1,3) = -\frac{D}{a^2 b^3} \left( \frac{18}{35} b^4 \bar{v}_{xx} + \frac{132}{210} a^4 \bar{v}_{yy} + \frac{12}{50} a^2 b^2 + \frac{6v}{5} a b^2 \right)
\]
\[ k(1,4) = - \frac{D}{a^3b^3} \left( \frac{156}{35} b^4 - \frac{54}{35} a^4 + \frac{72}{25} a^2 b^2 \right) \]

\[ k(1,5) = - \frac{D}{a^3 b^2} \left( \frac{132}{210} b^4 - \frac{27}{35} a^4 + \frac{12}{50} a^2 b^2 + \frac{6}{5} a^2 b^2 \right) \]

\[ k(1,6) = - \frac{D}{a^2 b^3} \left( \frac{78}{35} b^4 - \frac{156}{120} a^4 + \frac{12}{50} a^2 b^2 \right) \]

\[ k(1,7) = \frac{D}{a^3 b^3} \left( \frac{54}{35} b^4 - \frac{156}{35} a^4 - \frac{72}{25} a^2 b^2 \right) \]

\[ k(1,8) = - \frac{D}{a^2 b^2} \left( \frac{156}{420} b^4 - \frac{78}{35} a^4 - \frac{12}{50} a^2 b^2 \right) \]

\[ k(1,9) = - \frac{D}{a^2 b^3} \left( \frac{27}{35} b^4 - \frac{132}{210} a^4 - \frac{12}{50} a^2 b^2 - \frac{6}{5} a^2 b^2 \right) \]

\[ k(1,10) = - \frac{D}{a^3 b^3} \left( \frac{54}{35} b^4 + \frac{54}{35} a^4 - \frac{72}{25} a^2 b^2 \right) \]

\[ k(1,11) = \frac{D}{a^2 b^2} \left( \frac{156}{420} b^4 + \frac{27}{35} a^4 - \frac{12}{50} a^2 b^2 \right) \]

\[ k(1,12) = - \frac{D}{a^3 b^3} \left( \frac{27}{35} b^4 + \frac{156}{420} a^4 - \frac{12}{50} a^2 b^2 \right) \]

\[ k(2,2) = \frac{D}{a^3 b} \left( \frac{12}{105} b^4 + \frac{52}{35} a^4 + \frac{24}{75} a^2 b^2 \right) \]

\[ k(2,3) = - \frac{D}{a^2 b^2} \left( \frac{66}{210} b^4 + \frac{66}{210} a^4 + \frac{1}{50} a^2 b^2 + \frac{6}{5} a^2 b^2 \right) \]

\[ k(2,4) = k(1,5) \]

\[ k(2,5) = - \frac{D}{a^3 b^3} \left( \frac{12}{105} b^4 - \frac{36}{70} a^4 + \frac{24}{75} a^2 b^2 \right) \]

\[ k(2,6) = - \frac{D}{a^2 b^2} \left( \frac{66}{210} b^4 - \frac{78}{420} a^4 + \frac{1}{50} a^2 b^2 + \frac{6}{10} a^2 b^2 \right) \]

\[ k(2,7) = - k(1,8) \]

\[ k(2,8) = - \frac{D}{a^3 b^3} \left( \frac{12}{140} b^4 - \frac{26}{35} a^4 + \frac{10}{150} a^2 b^2 \right) \]

\[ k(2,9) = - \frac{D}{a^2 b^2} \left( \frac{78}{420} b^4 - \frac{66}{210} a^4 - \frac{1}{50} a^2 b^2 - \frac{6}{10} a^2 b^2 \right) \]
\[ k(2,10) = - k(1,11) \]
\[ k(2,11) = \frac{D}{a^3 b} \left( \frac{12}{140} b^4 + \frac{18}{70} a^4 + \frac{12}{150} a^2 b^2 \right) \]
\[ k(2,12) = - \frac{D}{a^2 b^2} \left( \frac{78}{1420} b^4 + \frac{78}{420} a^4 - \frac{1}{50} a^2 b^2 \right) \]
\[ k(3,3) = \frac{D}{a b^3} \left( \frac{52}{35} b^4 + \frac{12}{105} a^4 + \frac{24}{75} a^2 b^2 \right) \]
\[ k(3,4) = - k(1,6) \]
\[ k(3,5) = - k(2,6) \]
\[ k(3,6) = \frac{D}{a b^3} \left( \frac{26}{35} b^4 - \frac{12}{140} a^4 - \frac{12}{150} a^2 b^2 \right) \]
\[ k(3,7) = k(1,9) \]
\[ k(3,8) = - k(2,9) \]
\[ k(3,9) = \frac{D}{a b^3} \left( \frac{36}{70} b^4 - \frac{12}{105} a^4 - \frac{24}{75} a^2 b^2 \right) \]
\[ k(3,10) = - k(1,12) \]
\[ k(3,11) = k(2,12) \]
\[ k(3,12) = \frac{D}{a b^3} \left( \frac{18}{70} b^4 + \frac{12}{140} a^4 + \frac{12}{150} a^2 b^2 \right) \]

\[ k(4,4) = k(1,1) \]
\[ k(4,5) = k(1,2) \]
\[ k(4,6) = - k(1,3) \]
\[ k(4,7) = k(1,10) \]
\[ k(4,8) = k(1,11) \]
\[ k(4,9) = - k(1,12) \]
\[ k(4,10) = k(1,7) \]
\[ k(4,11) = k(1,8) \]
\[ k(4,12) = - k(1,9) \]
\[ k(5,5) = k(2,2) \]
\[ k(5,6) = - k(2,3) \]
\[ k(5,7) = - k(1,11) \]
\[ k(5,8) = k(2,11) \]
\[ k(5,9) = - k(2,12) \]
\[ k(5,10) = - k(4,11) \]
\[ k(5,11) = k(2,8) \]
\[ k(5,12) = - k(2,9) \]
\[ k(6,6) = k(3,3) \]
\[ k(6,7) = - k(3,10) \]
\[ k(6,8) = - k(2,12) \]
\[ k(6,9) = k(3,12) \]
\[ k(6,10) = - k(3,7) \]
\[ k(6,11) = k(2,9) \]
\[ k(6,12) = k(3,9) \]
\[ k(7,7) = k(1,1) \]
\[ k(7,8) = - k(1,2) \]
\[ k(7,9) = k(1,3) \]
\[ k(7,10) = k(1,4) \ , \ k(7,11) = -k(1,5) \ , \ k(7,12) = k(1,6) ; \]

\[ k(8,8) = k(2,2) \ , \ k(8,9) = -k(2,3) \ , \ k(8,10) = -k(1,5) ; \]

\[ k(8,11) = k(2,5) \ , \ k(8,12) = -k(2,6) \ , \ k(9,9) = k(3,3) ; \]

\[ k(9,10) = -k(1,6) \ , \ k(9,11) = k(2,6) \ , \ k(9,12) = k(3,6) ; \]

\[ k(10,10) = k(1,1) \ , \ k(10,11) = -k(1,2) \ , \ k(10,12) = -k(1,3) ; \]

\[ k(11,11) = k(2,2) \ , \ k(11,12) = k(2,3) \ , \ k(12,12) = k(3,3) . \]

Other elements of the matrix are obtained by symmetry.

As was stated before, the matrix \( K \) for the entire structure is obtained by summing up appropriately stiffness matrices \( k \) for the various finite elements.

Any column in this matrix represents the forces exerted at all nodal points when a corresponding unit displacement, in the direction of one of the coordinate axes, is applied at a nodal point. Where two or more elements have a common nodal point, forces are added. Therefore, \( K \) is a \( 3N \times 3N \) matrix where \( N \) is the number of nodal points. When a unit displacement is given at a joint \( j \), the maximum number of joints affected by this displacement is nine, because a nodal point can belong to a maximum of only four different finite elements.

When the plate is combined with other structural elements such as beams, columns and other types of members, the corresponding contribution for the strain energy of these elements may be included in equation (2.10).

The equivalent forces \( P \) can be obtained by concentrating the external forces at the nodal points. Distributed as well as concentrated loads are considered.
These equivalent forces or nodal forces are represented by a column matrix

\[
P = \begin{bmatrix}
F_1 \\
M_{x1} \\
M_{y1} \\
F_2 \\
M_{x2} \\
M_{y2} \\
F_3 \\
M_{x3} \\
M_{y3} \\
F_4 \\
M_{x4} \\
M_{y4}
\end{bmatrix}
\]

which satisfies the relation

\[P = K d\]

where \(d\) is the displacement matrix for the entire plate. A matrix \(P\) for each external load is determined and the total load matrix is obtained by superposition.

Let \(q\) be a uniformly distributed load over a rectangle whose sides have coordinates \(x_1, x_2, y_1\) and \(y_2\) (fig. 5). Through a generalized virtual displacement \(\delta \Delta\), the work \(\delta \Delta^T P\) performed by the equivalent forces \(P\) must
be equal to the work 
\[
\int_{x_1}^{x_2} \int_{y_1}^{y_2} \delta w T \ q \ dx \ dy
\]
produced by the uniformly distributed load \(q\), or
\[
\delta \Delta^T P = q \int_{x_1}^{x_2} \int_{y_1}^{y_2} \delta w T \ dx \ dy
\]
where \(\delta w = \overline{w} \ \delta \Delta\), from equation (3.1).

Whence,
\[
\delta \Delta^T P = \delta \Delta^T q \int_{x_1}^{x_2} \int_{y_1}^{y_2} \overline{w}^T \ dx \ dy
\]
Finally
\[
P = q \int_{x_1}^{x_2} \int_{y_1}^{y_2} \overline{w}^T \ dx \ dy
\]

The results are given below:
\[
F_1 = \frac{a}{b^3} \left[ \frac{1}{3} (x_2^4 - x_1^4) - a(x_2^3 - x_1^3) + a^3(x_2 - x_1) \right]
\]
\[
\left[ \frac{1}{3} (y_2^4 - y_1^4) - b(y_2^3 - y_1^3) + b^3(y_2 - y_1) \right]
\]
\[
M_{x_1} = -\frac{a}{b^3} \left[ \frac{2}{3} (x_2^4 - x_1^4) - a(x_2^3 - x_1^3) + a^3(x_2 - x_1) \right]
\]
\[
\left[ \frac{2b}{3} (x_2^3 - x_1^3) - \frac{b^2}{2} (y_2^2 - y_1^2) - \frac{1}{4} (y_2^4 - y_1^4) \right]
\]
\[
M_{y_1} = \frac{a}{b^3} \left[ \frac{2}{3} (x_2^3 - x_1^3) - \frac{a^2}{2} (x_2^2 - x_1^2) - \frac{1}{4} (x_2^4 - x_1^4) \right]
\]
\[
\left[ \frac{1}{3} (y_2^4 - y_1^4) - b(y_2^3 - y_1^3) + b^3(y_2 - y_1) \right]
\]
\[ F_2 = \frac{a}{b^3} a^3 \left[ a \left( x_2^3 - x_1^3 \right) - \frac{1}{2} \left( x_2^4 - x_1^4 \right) \right] \\cdot \]
\[ \left[ \frac{1}{2} \left( y_2^4 - y_1^4 \right) - b \left( y_2^3 - y_1^3 \right) + b^3 \left( y_2 - y_1 \right) \right] \]

\[ M_{x2} = -\frac{g}{b^3} a^3 \left[ a \left( x_2^3 - x_1^3 \right) - \frac{1}{2} \left( x_2^4 - x_1^4 \right) \right] \\cdot \]
\[ \left[ \frac{gb}{3} \left( y_2^3 - y_1^3 \right) - \frac{b^2}{2} \left( y_2^2 - y_1^2 \right) - \frac{1}{4} \left( y_2^4 - y_1^4 \right) \right] \]

\[ M_{y2} = -\frac{g}{b^3} a^3 \left[ \frac{1}{4} \left( x_2^4 - x_1^4 \right) - \frac{a}{3} \left( x_2^3 - x_1^3 \right) \right] \\cdot \]
\[ \left[ \frac{1}{2} \left( y_2^4 - y_1^4 \right) - b \left( y_2^3 - y_1^3 \right) + b^3 \left( y_2 - y_1 \right) \right] \]

\[ F_3 = \frac{a}{b^3} a^3 \left[ \frac{1}{2} \left( x_2^4 - x_1^4 \right) - a \left( x_2^3 - x_1^3 \right) \right] + a^3 \left( x_2 - x_1 \right) \\cdot \]
\[ \left[ b \left( y_2^3 - y_1^3 \right) - \frac{1}{2} \left( y_2^4 - y_1^4 \right) \right] \]

\[ M_{x3} = \frac{g}{b^3} a^3 \left[ \frac{1}{2} \left( x_2^4 - x_1^4 \right) - a \left( x_2^3 - x_1^3 \right) \right] \\cdot \]
\[ \left[ \frac{1}{4} \left( y_2^4 - y_1^4 \right) - \frac{b}{3} \left( y_2^3 - y_1^3 \right) \right] \]

\[ M_{y3} = \frac{g}{b^3} a^3 \left[ \frac{2a}{3} \left( x_2^3 - x_1^3 \right) - \frac{a^2}{2} \left( x_2^2 - x_1^2 \right) - \frac{1}{4} \left( x_2^4 - x_1^4 \right) \right] \\cdot \]
\[ \left[ b \left( y_2^3 - y_1^3 \right) - \frac{1}{2} \left( y_2^4 - y_1^4 \right) \right] \]

\[ F_4 = \frac{a}{b^3} a^3 \left[ a \left( x_2^3 - x_1^3 \right) - \frac{1}{2} \left( x_2^4 - x_1^4 \right) \right] \\cdot \]
\[ \left[ b \left( y_2^3 - y_1^3 \right) - \frac{1}{2} \left( y_2^4 - y_1^4 \right) \right] \]

\[ M_{x4} = \frac{g}{b^3} a^3 \left[ a \left( x_2^3 - x_1^3 \right) - \frac{1}{2} \left( x_2^4 - x_1^4 \right) \right] \\cdot \]
\[ \left[ \frac{1}{4} \left( y_2^4 - y_1^4 \right) - \frac{b}{3} \left( y_2^3 - y_1^3 \right) \right] \]
\[ M_{yl} = - \frac{a}{b^3 a^2} \left[ \frac{1}{4} (x_2^4 - x_1^4) - \frac{a}{3} (x_2^3 - x_1^3) \right] \]

\[ + \left[ b(y_2^3 - y_1^3) - \frac{1}{2} (y_2^4 - y_1^4) \right] \]

For a concentrated load \( Q \) with coordinates \( x \) and \( y \) (fig. 5),

the expression for the equivalent load becomes

\[ P = Q w^T \]

which gives

\[
\begin{pmatrix}
(2x^3 + a^2 - 3ax^2)(2y^3 + b^3 - 3by^2) \\
-a(2x^3 + a^2 - 3ax^2)(2by^2 - b^2 y - y^3) \\
-a(2x^2 - a^2 x - x^3)(2y^3 + b^3 - 3by^2) \\
(3ax^2 - 2x^3)(2y^3 + b^3 - 3by^2) \\
-b(3ax^2 - 2x^3)(2by^2 - b^2 y - y^3) \\
-a(x^3 - ax^2)(2y^3 + b^3 - 3by^2) \\
(2x^3 + a^2 - 3ax^2)(3by^2 - 3by^2) \\
b(2x^3 + a^2 - 3ax^2)(y^3 - by^2) \\
a(2ax^2 - a^2 x - x^3)(3by^2 - 2y^3) \\
(3ax^2 - 2x^3)(3by^2 - 2y^3) \\
b(3ax^2 - 2x^3)(y^3 - by^2) \\
-a(x^3 - ax^2)(3by^2 - 2y^3)
\end{pmatrix}
\]

In order to determine the nodal point displacements \( d \), a system of simultaneous equations

\[ P = K d \]
FIG. 5

A RECTANGULAR ELEMENT WITH A UNIFORMLY DISTRIBUTED LOAD AND A CONCENTRATED LOAD
has to be solved. The Gauss-Seidel iteration method with an over-
relaxation factor was used.

The stresses are given by the equations

\[
M_x = -D(w_{xx} + \nu w_{yy})
\]
\[
M_y = -D(w_{yy} + \nu w_{xx})
\]
\[
M_{xy} = D(1 - \nu)w_{xy}
\]

and they were calculated at the center of each finite element. For
\[x = \frac{a}{2}\] and \[y = \frac{b}{a}\], the expressions for the stresses are:

\[
M_x = -\frac{D}{2} \left[ \frac{1}{a} \left( \theta_{y1} - \theta_{y2} + \theta_{y3} - \theta_{y4} \right) - \frac{\nu}{b} \left( \theta_x1 + \theta_x2 - \theta_x3 - \theta_x4 \right) \right]
\]
\[
M_y = -\frac{D}{2} \left[ -\frac{1}{b} \left( \theta_{x1} + \theta_{x2} - \theta_{x3} - \theta_{x4} \right) + \frac{\nu}{a} \left( \theta_y1 - \theta_y2 + \theta_y3 - \theta_y4 \right) \right]
\]
\[
M_{xy} = D(1-\nu) \left[ \frac{2}{ab} \left( \Delta_1 - \Delta_2 - \Delta_3 + \Delta_4 \right) + \frac{3}{8a} \left( \theta_x1 - \theta_x2 + \theta_x3 - \theta_x4 \right) \right.
\]
\[ \left. - \frac{3}{8b} \left( \theta_y1 + \theta_y2 - \theta_y3 - \theta_y4 \right) \right]
\]
CHAPTER V

NUMERICAL EXAMPLES

In order to verify the accuracy and convergence provided by this new stiffness matrix, several problems have been solved and the results compared with known exact solutions. Also, some results have been compared with those obtained by different approximate methods. For all examples, an isotropic material was used.

The numbering of points where deflections, rotations and moments were calculated is shown in Fig. 6 and Fig. 7.

Problem 1: Square clamped plate under uniform load

The results of deflections for three different meshes (2x2, 4x4, 8x8) and rotations for two meshes (4x4, 8x8) are compared with the exact solution in Table I. These deflections and rotations are in error by amounts varying from 1$ to 3$, for the 8x8 mesh. Even for the 4x4 mesh the percentage errors are relatively small, varying from 3$ to 7$.

The large relative errors shown in Table I are at points where the deflections are very small.

Notice that the value obtained for the maximum deflection using the 2x2 mesh is larger than the exact value, while deflections at other points are smaller than the exact values.

Fig. 8 compares the maximum vertical deflection, using meshes of 2x2, 4x4 and 8x8 elements, with results obtained by Zienkiewicz and Cheung using a different finite element formulation and by a higher order
finite difference method. It can be seen that the results here obtained and those given either by Zienkiewicz and Cheung or by the higher order finite difference method represent bounds for the exact solution.

Some points were chosen to assess the accuracy for the moments. These values are shown in Table II. Also in Table II, the maximum positive moment is compared with the exact solution given by Timoshenko and the solution obtained by Zienkiewicz and Cheung.

In a finite element the accuracy can vary from point to point within the element as is shown in Table II by the comparison of the maximum positive moment with the moment at point 6.

Problem 2: Rectangular clamped plate under uniform load

In Table IIIa the results of deflections for three different meshes 2x2, 4x4 and 8x8 are compared with the exact solutions. Rotations \( w_y \) at some points for the 4x4 and 8x8 meshes are also presented in Table IIIb. The deflections and rotations are in error by amounts varying from 1% to 3%, for the 8x8 mesh. They show the same order of accuracy as for the square clamped plate. In this case more points (1, 2 and 12) are shown having deflections for the 2x2 mesh larger than the corresponding exact solution.

Again the 4x4 mesh gives very good results for the deflections, except at the points where the deflections are small.

The moments \( M_x \) and \( M_y \) for the 8x8 mesh, at the center of some elements are compared with the exact solution in Table IV.

The meshes used in this case are relatively coarse in comparison with the meshes used in the square clamped plate.
Problem 3: Square simply supported plate under uniform load

The comparison of results of deflections for three meshes (2x2, 4x4, 8x8) and rotations for two meshes (4x4, 8x8) with the exact solution are presented in Table V. The deflections are in error by amounts varying from 4% to 6% and the rotations are in error by approximately 4%, for the 8x8 mesh.

The comparison of results of moments $M_x$ is given in Table VI. They are in error by approximately 6.6%.

Problem 4: Rectangular simply supported plate under uniform load

Table VII presents the comparison of deflections for three meshes (2x2, 4x4, 8x8) with the exact solution. Rotations $w_y$ are compared in Table VIII.

Notice that in both cases of the simply supported plate, all deflections are smaller than the corresponding exact solution.

The deflections are in error varying from 5% to 6% and the rotations are in error by approximately 4%, for the 8x8 mesh.

Some moments $M_x$ and $M_y$ for the 8x8 mesh are compared with the exact solution in Table IX. The order of accuracy is the same as for the square simply supported plate.

Problem 5: Cantilevered square plate under uniform load (Fig. 7)

Table X compares the results of deflections with those obtained with the beam theory, Rayleigh-Ritz method and point-matching solution. This same table compares the results of moments $M_x$ and $M_y$ with those obtained by Zienkiewicz and Cheung and the point-matching solution. The results here obtained are the closest ones to the point-matching solution.
FIG. 6
NUMBERING OF POINTS FOR THE CLAMPED AND SIMPLY SUPPORTED PLATES

FIG. 7
NUMBERING OF POINTS FOR THE CANTILEVERED SQUARE PLATE
TABLE I

COMPARISON OF RESULTS OF DEFLECTIONS AND ROTATIONS
FOR A SQUARE CLAMPED PLATE UNDER UNIFORM LOAD

\( v = .3 \)

<table>
<thead>
<tr>
<th>Nodal Point</th>
<th>Deflection Multiplier ((qL^4/D))</th>
<th>Exact Solution</th>
<th>Percentage Error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( 2 \times 2 )</td>
<td>( 4 \times 4 )</td>
<td>( 8 \times 8 )</td>
</tr>
<tr>
<td>1</td>
<td>.001325</td>
<td>.001211</td>
<td>.001227</td>
</tr>
<tr>
<td>2</td>
<td>.001117</td>
<td>.001085</td>
<td>.001096</td>
</tr>
<tr>
<td>3</td>
<td>.000662</td>
<td>.000735</td>
<td>.000737</td>
</tr>
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<td>.000207</td>
<td>.000256</td>
<td>.000272</td>
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<td>.000047</td>
<td>.000063</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Nodal Point</th>
<th>Rotation ( w_y ) Multiplier ((qL^3/D))</th>
<th>Exact Solution</th>
<th>Percentage Error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( 4 \times 4 )</td>
<td>( 8 \times 8 )</td>
<td>( 4 \times 4 )</td>
</tr>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
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<td>2</td>
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<td>0</td>
</tr>
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<td>-.002121</td>
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<td>15</td>
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<td>-.002130</td>
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<td>---</td>
<td>-.002166</td>
<td>-.002200</td>
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<tr>
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<td>---</td>
<td>-.000787</td>
<td>-.000828</td>
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</table>
FIG. 8

COMPARISON OF RESULTS OF DEFLECTIONS AT THE CENTER OF A SQUARE CLAMPED
PLATE UNDER UNIFORM LOAD  -  v = .3
TABLE II

COMPARISON OF RESULTS OF MOMENTS $M_x$ FOR A SQUARE CLAMPED PLATE UNDER UNIFORM LOAD

8x8 MESH

<table>
<thead>
<tr>
<th>Point</th>
<th>$M_x$</th>
<th>Exact Solution (8)</th>
<th>Error %</th>
</tr>
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<tbody>
<tr>
<td>6</td>
<td>.022300</td>
<td>.022560</td>
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</tr>
<tr>
<td>7</td>
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<td>.021650</td>
<td>-6.20</td>
</tr>
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<td>10</td>
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<td>-.026760</td>
<td>-1.00</td>
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</tr>
<tr>
<td>17</td>
<td>.015630</td>
<td>.016650</td>
<td>-6.10</td>
</tr>
<tr>
<td>20</td>
<td>-.021020</td>
<td>-.021110</td>
<td>-.40</td>
</tr>
<tr>
<td>26</td>
<td>.006400</td>
<td>.006830</td>
<td>-6.30</td>
</tr>
<tr>
<td>29</td>
<td>-.011750</td>
<td>-.011530</td>
<td>1.90</td>
</tr>
</tbody>
</table>

$v = .3$ multiplier $qL^2$

<table>
<thead>
<tr>
<th>$v = .3$</th>
<th>Maximum Positive Moment</th>
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Multiplier $qL^2$
### TABLE III a

COMPARISON OF RESULTS OF DEFLECTIONS FOR A RECTANGULAR (Lx2L) CLAMPED PLATE UNDER UNIFORM LOAD

\( v = .3 \)

<table>
<thead>
<tr>
<th>Nodal Point</th>
<th>Deflection Multiplier ((qL^4/D))</th>
<th>Exact Solution ((8))</th>
<th>Percentage Error</th>
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TABLE III b

COMPARISON OF RESULTS OF ROTATIONS FOR A RECTANGULAR (Lx2L) CLAMPED PLATE UNDER UNIFORM LOAD

\( v = 0.3 \)

<table>
<thead>
<tr>
<th>Nodal Point</th>
<th>Rotation ( w_y )</th>
<th>Multiplier ( (qL^3/D) )</th>
<th>Exact Solution ((8))</th>
<th>Percentage Error</th>
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TABLE IV

COMPARISON OF RESULTS OF MOMENTS $M_x$ and $M_y$ FOR A RECTANGULAR (1x2L) CLAMPED PLATE UNDER UNIFORM LOAD 8x8 MESH

<table>
<thead>
<tr>
<th>Point</th>
<th>$M_x$</th>
<th>Exact Solution $^8$</th>
<th>Error%</th>
<th>$M_y$</th>
<th>Exact Solution $^8$</th>
<th>Error%</th>
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<td>.010600</td>
<td>- .60</td>
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<td>-.000300</td>
<td>-.000390</td>
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$v = .3$ Multiplier $ql^2$
# TABLE V

Comparison of results of deflections and rotations for a square simply supported plate under uniform load with $v = 0.3$

<table>
<thead>
<tr>
<th>Nodal Point</th>
<th>Deflection Multiplier ($qL^4/D$)</th>
<th>Exact Solution</th>
<th>Percentage Error</th>
</tr>
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<tbody>
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<table>
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<th>Nodal Point</th>
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<th>Exact Solution</th>
<th>Percentage Error</th>
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<td>.012000</td>
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TABLE VI

COMPARISON OF RESULTS OF MOMENTS $M_x$ FOR A SQUARE SIMPLY SUPPORTED PLATE UNDER UNIFORM LOAD

8x8 MESH

<table>
<thead>
<tr>
<th>Point</th>
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<th>Exact Solution (8)</th>
<th>Error %</th>
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<tbody>
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<td>8</td>
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$\nu = 0.3$  Multiplier $qL^2$
**TABLE VII**

**COMPARISON OF RESULTS OF DEFORMATIONS FOR A RECTANGULAR (L × 2L) SIMPLY SUPPORTED PLATE UNDER UNIFORM LOAD**

\( \nu = 0.3 \)

<table>
<thead>
<tr>
<th>Nodal Point</th>
<th>Deflection Multiplier ((gL^4/D))</th>
<th>Exact Solution</th>
<th>Percentage Error</th>
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<tbody>
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TABLE VIII

COMPARISON OF RESULTS OF ROTATIONS $w_y$ FOR A RECTANGULAR (Lx2L) SIMPLY SUPPORTED PLATE UNDER UNIFORM LOAD

$\nu = .3$

<table>
<thead>
<tr>
<th>Nodal Point</th>
<th>Rotation $w_y$ multiplier $4x4$, $qL^3/\pi8$</th>
<th>Exact Solution $(8)$</th>
<th>Percentage Error $4x4$, $8x8$</th>
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### Table IX

Comparison of Results of Moments $M_x$ and $M_y$ for a Rectangular (Lx2L) Simply Supported Plate Under Uniform Load

8x8 Mesh

<table>
<thead>
<tr>
<th>Point</th>
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<th>Exact Solution 8</th>
<th>Error %</th>
<th>$M_y$</th>
<th>Exact Solution 8</th>
<th>Error %</th>
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<td>0.027800</td>
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$\nu = 0.3$ Multiplier $\sqrt{2}$
TABLE X

COMPARISON OF RESULTS OF DEFLECTIONS AND MOMENTS \( M_x \)
AND \( M_y \) FOR A CANTILEVERED SQUARE
PLATE UNDER UNIFORM LOAD

<table>
<thead>
<tr>
<th>Nodal Point</th>
<th>Beam Theory (9)</th>
<th>Rayleigh-Ritz (9)</th>
<th>Point Matching Solution (9)</th>
<th>HERE 2x2</th>
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<td>.045500</td>
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\( v = .3 \) Multiplier \( qL^4/D \)

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<tr>
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<th>Point Matching Solution (9)</th>
<th>Here 3x3</th>
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<tr>
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<td>( M_y )</td>
<td>( M_x )</td>
<td>( M_y )</td>
</tr>
<tr>
<td>3</td>
<td>-.1500</td>
<td>.0075</td>
<td>-.1275</td>
</tr>
<tr>
<td>4</td>
<td>-.1350</td>
<td>-.0300</td>
<td>-.1200</td>
</tr>
<tr>
<td>5</td>
<td>-.1425</td>
<td>-.1425</td>
<td>-.3150</td>
</tr>
</tbody>
</table>

\( v = .3 \) Multiplier \( qL^2 \)
CHAPTER VI

DISCUSSION OF THE RESULTS AND CONCLUSIONS

The results obtained in all examples show reasonable agreement with the exact solutions. The clamped cases show better accuracy than the simply supported ones.

The maximum deflection for the square clamped plate is more accurate than the result given by Zienkiewicz and Cheung. The percentage errors are reduced by factors of 3.5, 2.5, and 1.5 respectively, for the 2x2, 4x4, and 6x6 meshes. For the 8x8 mesh the results are approximately the same for both formulations.

Although this does not happen with the simply supported cases, for the clamped plates the deflections at some points, for the 2x2 mesh, are larger than the exact solution and they are smaller for a larger number of divisions. However, if deflections and moments are calculated at several points of the plate the average of the results obtained converge monotonically, as shown in Tables I, II, III and IV.

The maximum positive moment for the clamped plate is more accurate than the solution given by Zienkiewicz and Cheung. For the 8x8 mesh, the percentage error is reduced by a factor of 1.22.

Excellent results were obtained for the cantilevered square plate where the results of deflections for the 2x2 mesh closely agree with the solutions given by the point-matching method, beam theory and the Rayleigh-Ritz method. The maximum error with respect to the point-matching solution is about .2%. The moments here obtained are closer to those given by the
point-matching solution than the ones obtained by Zienkiewicz and Cheung, by a factor varying from 1.3 to 4.2.

It may be concluded that the stiffness matrix presented here generally gives more accurate results than those obtained by Zienkiewicz and Cheung. Furthermore, the convergence provided by this new stiffness matrix is monotonic. The improvement in accuracy results from the fact that continuity of deformations and displacements along the element boundaries were satisfied when constructing the displacement function here presented.
LIST OF REFERENCES


APPENDIX A

SUGGESTION FOR THE DISPLACEMENT FUNCTION FOR AN ANNULAR SECTOR

FINITE ELEMENT

The same procedure used to construct the displacement function for a rectangular element, can be used to formulate a displacement function for an annular sector element (Fig. 10), for the analysis of the bending of plates bounded by coordinate lines in polar coordinates. Consequently, this displacement function also satisfies the compatibility requirements along the element boundaries.

The displacement function is given below:

\[
w = \frac{\Delta_1}{\varphi \theta^3} \left(2\varphi^3 + \theta^3 - 3\varphi \theta^2\right) \left[2(r-R)^3 + c^3 - 3c(r-R)^2\right]
- \frac{\varphi r_1}{\varphi \theta^3} \left(2\varphi^3 + \theta^3 - 3\varphi \theta^2\right) \left[2c(r-R)^2 - c^2(r-R) - (r-R)^3\right]
+ \frac{\varphi r_1}{\varphi \theta^3} \left(2\varphi^2 - \theta^2 - \varphi \theta^3\right) \left[2(r-R)^3 + c^3 - 3c(r-R)^2\right]
+ \frac{\Delta_2}{\varphi \theta^3} \left(3\varphi^3 - 2\theta^3\right) \left[2(r-R)^3 + c^3 - 3c(r-R)^2\right]
- \frac{\varphi r_2}{\varphi \theta^3} \left(3\varphi^2 - 2\theta^2\right) \left[2(r-R)^2 - c^2(r-R) - (r-R)^3\right]
- \frac{\varphi \theta}{\varphi \theta^3} \left(\theta^3 - \varphi \theta^2\right) \left[2(r-R)^3 + c^3 - 3c(r-R)^2\right]
+ \frac{\Delta_3}{\varphi \theta^3} \left(2\varphi^3 + \theta^3 - 3\varphi \theta^2\right) \left[3c(r-R)^2 - 2(r-R)^3\right]
+ \frac{\varphi r_3}{\varphi \theta^3} \left(2\varphi^3 + \theta^3 - 3\varphi \theta^2\right) \left[(r-R)^3 - c(r-R)^2\right] \text{(con't)}
\]
\[
\begin{align*}
&+ \frac{\varphi_3}{c^3 \theta^2} \left( 2 \theta \phi^2 - \theta^2 \phi - \phi^3 \right) \left[ 3c(r-R)^2 - 2(r-R)^3 \right] \\
&+ \frac{\Delta_4}{c^3 \theta^3} \left( 3 \theta \phi^2 - 2 \phi^3 \right) \left[ 3c(r-R)^2 - 2(r-R)^3 \right] \\
&+ \frac{\varphi_4}{c^2 \theta^3} \left( 3 \theta \phi^2 - 2 \phi^3 \right) \left[ (r-R)^3 - c(r-R)^2 \right] \\
&- \frac{\varphi_6}{c^3 \theta^2} \left( \phi^3 - \theta \phi^2 \right) \left[ 3c(r-R)^2 - 2(r-R)^3 \right].
\end{align*}
\]

Once the stiffness matrix for the plate element is derived from the displacement function, the plate of Fig. 9 can be solved following the same steps as for the rectangular plate.
FIG. 9

A PLATE BOUNDED BY COORDINATE LINES IN POLAR COORDINATES SUBDIVIDED INTO A NUMBER OF ANNULAR SECTOR ELEMENTS

FIG. 10

AN ANNULAR SECTOR ELEMENT
APPENDIX B
FLOW DIAGRAM AND COMPUTER PROGRAM

Read Elastic and Geometric Properties for each plate element

Set up K matrix for the entire plate

Read concentrated loads, if any, and set up load matrix

Read distributed loads, if any, and set up load matrix

Read boundary conditions and reduce stiffness matrix K

Solve equations for nodal deflections of the plate

Calculate nodal displacements for the elements

Calculate stresses

Print deflections and stresses

New Plate

New Plate

or

New Loading Case?

Neither

STOP
C*****JOSE CARNEIRO DE ANDRADE, AUGUST 1965
C*****DEFLECTIONS AND STRESSES IN A PLATE BY USING FIN. ELEMENTS
C*****RECTANGULAR PLATE WITH RECTANGULAR FINITE ELEMENTS
C*****QD MEANS DISTRIBUTED LOAD
C*****QC MEANS CONCENTRATED LOAD
C*****MOLD IS THE NUMBER OF PARTIALLY DISTRIBUTED LOAD
C*****MCLD IX THE NUMBER OF CONCENTRATED LOAD
C*****LIM GIVES MAX. NUMBER OF ITERATION IN GAUSS-SEIDEL PROCEDURE
C*****ITOT GIVES NUMBER OF ITERATION USED IN GAUSS-SEIDEL PROCEDURE
C*****TOL GIVES THE TOLERANCE IN THE GAUSS-SEIDEL ITERATION
C***» XQC, YQC ARE THE COORDINATES OF THE CONCENTRATED LOAD
C***» X2, X1, Y2, Y1 ARE THE COORDINATES OF THE PARTIALLY DISTRIBUTED LOAD
C***» IS THE FLEXURAL RIGIDITY OF EACH ELEMENT OF PLATE
C***» KE REPRESENTS THE STIFFNESS OF EACH ELEMENT
C***» KT REPRESENTS THE STIFFNESS MATRIX OF THE PLATE
C***» A REPRESENTS DIMENSION OF EACH STRIP OF PLATE IN Y DIRECTION
C***» B REPRESENTS DIMENSION OF EACH STRIP OF PLATE IN X DIRECTION
C***» IF NCTRL = 1 READ NEW PLATE CHARACTERISTICS, MESH, TOL, LIM OR BDRY.
C***» IF NCTRL = 2 READ NEW LOAD CASE
C***» MSET GIVES THE NUMBER OF THE PROBLEM SOLVED (PLATE NO.).
C***» MU IS THE POISSON,S RATIO
C***» F IS THE RELAXATION FACTOR FOR THE GAUSS-SEIDEL ITERATION
C***» MCX REPRESENTS THE MOMENT AT THE CENTER OF EACH ELEMENT ABOUT X-X
C***» MCY REPRESENTS THE MOMENT AT THE CENTER OF EACH ELEMENT ABOUT Y-Y
C***» MCXY REPRESENTS THE MOMENT AT THE CENTER OF EACH ELEMENT ABOUT XY
1 DIMENSION BDRY(4), ITEM(3), Y(1200), X(363)
2 DIMENSION IND(363,9), D(IOO), P(363), A(10), B(10)
3 DIMENSION KE(12,12), KT(363,27), MCX(100), MCY(100), MCXY(IOO)
4 DIMENSION DELTX(363), TITL(12)
5 REAL KE, KT, MCX, MCY, MU
6 INTEGERRBL, RBL, RBL, RBL, RBL, RBL, RBL, RBL, RBL, RBL
7 READ(5,57) MSET
11 READ (5,800) (TITL(I), I = 1,12)
16 800 FORMAT(12A6)
17 500 READ(5,57)M,N
22 LSET = 0
23 M=M-1
24 NA=N-1
25 MNA=MA*NA
26 READ(5,61) F
27 READ(5,61) (D(I), I = 1,MNA)
34 READ(5,61) (A(I), I = 1,NA)
41 READ(5,61) (B(I), I = 1,MA)
46 READ(5,577) MU,TOL,BDRY(1), I=1,4)
53 577 FORMAT(2F10.0,4I6)
54 READ(5,577) LIM
56 57 FORMAT(216)
57 61 FORMAT(8F10.0)
C*****SET UP INDEX MATRIX (IND)
60 IN=1
61 IL=2
62 DO 12 MM=1,M
63 IF(MM.EQ.M)IL = 2
66 JN=1
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DE ANDRADE

FINITE ELEMENTS

FORTRAN SOURCE LIST FERP

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67  JL=2
70  LC = 2
71  DO 11 NN=1,N
72  IF(NN.EQ.N) JL = 2
75  IF(NN.EQ.N) LC = 2
100  K=(MM-1)*N+NN
101  DO 10 I=1,1JL
102  DO 10 J=1,1JL
103  L=LC*(I-1)+J
104  10 IND(K,L)=K+(I-IN)*N+J-NJ
107  JN=2
110  LC=3
111  11 JL=3
113  IN=2
114  12 IL=3

C*****SET UP KE MATRIX FOR EACH ELEMENT
116  MNN = 3*MN
117  DO 3008 KM = 1,MN
118  DO 3008 KN = 1,27
119  3008 KT(KM,KN) = 0.
120  124  DO 16 NN = 1,MA
121  125  DO 16 MM=1,NN
122  126  IE = (NN-1)*(NN-1)+1
127  KE(1,1)=D(IE)/(A(MM)**4+3*B(NN)**4)*156.*(B(NN)**4*A(MM)**4)/35.+1
129  130  1 78.*A(MM)**4/35. + (12./50. + MU*6./5.)*A(MM)**2*B(NN)**2
131  131  1 KE(1,2)=D(IE)/(A(MM)**4+3*B(NN)**4)*156.*(B(NN)**4/A(MM)**4)/35.+1
133  133  1 KE(1,4)=D(IE)/(A(MM)**4+3*B(NN)**4)*156.*(B(NN)**4/A(MM)**4)/35.+1
136  136  1 KE(1,7)=D(IE)/(A(MM)**4+3*B(NN)**4)*156.*(B(NN)**4/A(MM)**4)/35.+1
139  139  1 132.*A(MM)**4/35. - (12./50. + MU*6./5.)*A(MM)**2*B(NN)**2
140  140  1 KE(1,10)=D(IE)/(A(MM)**4+3*B(NN)**4)*156.*(B(NN)**4/A(MM)**4)/35.+1
145  145  1 KE(2,2)=D(IE)/(A(MM)**4+3*B(NN)**4)*156.*(B(NN)**4/A(MM)**4)/35.+1
147  147  1 166.*A(MM)**4/210. - (12./100.+120.*MU/100.)*A(MM)**2*B(NN)**2
148  148  1 KE(2,4)=KE(1,5)
149  149  1 KE(2,5)=KE(1,6)
150  150  1 KE(2,6)=KE(1,7)
151  151  1 KE(2,3)=D(IE)/(A(MM)**4+3*B(NN)**4)
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1 78.*A(MM)**4/420.+(2./100.+MU/10.)*A(MM)**2*B(NN)**2)
150 KE(2,7)=-KE(1,8)
151 KE(2,8)=D(IE)/(A(MM)**3*B(NN))*(12.*B(NN)**2/140.+ 1
152 A(MM)**6/35.-12.*A(MM)**2*B(NN)**2/2/150.)
154 66.*A(MM)**4/210.+(2./100.+MU/10.)*A(MM)**2*B(NN)**2)
155 KE(2,10)=-KE(1,11)
156 KE(2,11)=D(IE)/(A(MM)**3*B(NN))*(12.*B(NN)**2/140.+ 1
157 18.*A(MM)**4/70.+12.*A(MM)**2*B(NN)**2/2/150.)
158 KE(2,12)=-D(IE)/(A(MM)**3*B(NN))*(39.*B(NN)**4/140.+ 1
159 118.*A(MM)**4/210.-2.*A(MM)**2*B(NN)**2/100.)
160 KE(3,3)=-D(IE)/(A(MM)**2*B(NN)**3)*(-52.*B(NN)**4/35.- 1
161 12.*A(MM)**4/105.-24.*A(MM)**2*B(NN)**2/2/75.1)
162 KE(3,4)=KE(1,6)
163 KE(3,5)=KE(2,6)
164 KE(3,6)=D(IE)/(A(MM)**3*B(NN)**3)*(-26.*B(NN)**4/35.- 1
166 KE(3,7)=KE(1,9)
167 KE(3,8)=KE(2,9)
168 KE(3,9)=-D(IE)/(A(MM)**3*B(NN)**3)*(-36.*B(NN)**4/70.+ 1
169 12.*A(MM)**4/105.+24.*A(MM)**2*B(NN)**2/2/75.1)
170 KE(3,10)=-KE(1,12)
171 KE(3,11)=KE(2,12)
172 KE(3,12)=-D(IE)/(A(MM)**3*B(NN)**3)*(-18.*B(NN)**4/70.+ 1
174 KE(4,4)=KE(1,1)
175 KE(4,5)=KE(1,2)
176 KE(4,6)=KE(1,3)
177 KE(4,7)=KE(1,10)
178 KE(4,8)=KE(1,11)
179 KE(4,9)=KE(1,12)
180 KE(4,10)=KE(1,7)
181 KE(4,11)=KE(1,8)
182 KE(4,12)=KE(1,9)
183 KE(5,5)=KE(2,1)
184 KE(5,6)=KE(2,3)
185 KE(5,7)=KE(2,11)
186 KE(5,8)=KE(2,12)
187 KE(5,9)=KE(2,1)
188 KE(5,10)=KE(4,11)
189 KE(5,11)=KE(2,8)
190 KE(5,12)=KE(2,9)
191 KE(6,6)=KE(3,3)
192 KE(6,7)=KE(3,10)
193 KE(6,8)=KE(2,12)
194 KE(6,9)=KE(3,12)
195 KE(6,10)=KE(3,7)
196 KE(6,11)=KE(2,9)
197 KE(6,12)=KE(3,9)
198 KE(7,7)=KE(1,1)
199 KE(7,8)=KE(1,1)
200 KE(7,9)=KE(1,3)
201 KE(7,10)=KE(1,4)
202 KE(7,11)=KE(1,5)
203 KE(7,12)=KE(1,6)
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226  KE(8,8)=KE(2,2)
227  KE(8,9)=KE(2,3)
230  KE(8,10)=KE(1,5)
231  KE(8,11)=KE(2,5)
232  KE(8,12)=KE(2,6)
233  KE(9,9)=KE(3,3)
234  KE(9,10)=KE(1,6)
235  KE(9,11)=KE(2,6)
236  KE(9,12)=KE(3,6)
237  KE(10,10)=KE(1,1)
238  KE(10,11)=KE(1,2)
239  KE(10,12)=KE(1,3)
240  KE(11,11)=KE(2,2)
241  KE(11,12)=KE(2,3)
243  KE(12,12)=KE(3,3)
245  DO 13 J=1,12
246  DO 13 I=J,12
247  C****STORE ELEMENT IN SYSTEM STIFFNESS MATRIX (KT)
252  C=(NN-1)*N+MM
253    K=C
254    DO 15 RBL=1,2
255    DO 14 HBL=1,2
256    J=G
257    DO 17 CBL=1,2
258    DO 18 I=1,9
261    IF(IND(I,I)-J18,19,18
262    18 CONTINUE
264    STOP
265    19 IRW=3*(K-1)
266    ICOL=3*(I-1)
267    DO 20 RST=1,3
270    DO 20 CST=1,6
271    ROW=6*(RBL-1)+3*(HBL-1)+RST
272    COL=6*(CBL-1)+CST
273    IRR=IRW+RST
274    ICC=ICOL+CST
275    20 KT(IRR,ICC)=KT(IRR,ICC)+KE(ROW,COL)
300    17 J=J+N
302    14 K=K+1
304    15 K=C+N
306    16 CONTINUE
311    MSET = MSET + 1
C****CALCULATE LOAD MATRIX
312    600 DO 38 I=1,1,30
313    38 P(I)=0.
315    WRITE (6,801) TITL(I), I = 1,12
322    WRITE(6,4008) BDRY(I), I=1,4
327    WRITE(6,4009) N,M,HU,TOL,LIM,F
330    WRITE(6,4001) (I, A(I), I=1,NA)
335    WRITE(6,4002) (I, B(I), I=1,MA)
342    WRITE(6,4005) (I, D(I), I=1,MNA)
347    READ(5,57)MCLD,MOLD
352    IF(MCLD)410,410,420
353    420 DO 400 I=1,MCLD
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354      READ(5,310)QC,XQC,YQC
355      WRITE(6,4006) QC,XQC,YQC
356      310 FORMAT(3F10.0)
357      DO 1000 MM = 1,NA
358      IF (XQC.LT.A(MM)) GO TO 1010
359      1000 QC = QC - A(MM)
360      1010 DO 1020 NN = 1,MA
361      IF (YQC.LT.B(NN)) GO TO 1030
362      1020 YQC = YQC - B(NN)
363      1030 JJ = (MM-1)*3+3*(NN-1)*N
364      JJ = 3*(MM-1)*3+3*NN*N+3
365      P(JJ+1) = P(JJ+1)+(1/QC/A(MM)**3*B(NN)**3)*(2.*XQC**3 +
366             1*A(MM)**3 - 3.*A(MM)**3*XQC**2))*(2.*YQC**3 + B(NN)**3 -
367             2.3*B(NN)**YQC**2)
368      P(JJ+2) = P(JJ+2)-(1/QC/A(MM)**3*B(NN)**3)*(2.*XQC**3 +
369             1*A(MM)**3 - 3.*A(MM)**3*XQC**2))*(2.*B(NN)**YQC**2 -
370             2*B(NN)**2*YQC - YQC**3)
371      P(JJ+3) = P(JJ+3)+(1/QC/A(MM)**3*B(NN)**3)*(2.*XQC**3 +
372             1*A(MM)**2*XQC - XQC**3)*2*(2.*YQC**3 + B(NN)**3 -
373             2.3*B(NN)**YQC**2)
374      P(JJ+4) = P(JJ+4)+1*(1/QC/A(MM)**3*B(NN)**3)*(3.*A(MM)**XQC**2 -
375             1.2*XQC**3)*(2.*YQC**3 + B(NN)**3 - 3.*B(NN)**YQC**2)
376      P(JJ+5) = P(JJ+5)+(1/QC/A(MM)**3*B(NN)**3)*(3.*A(MM)**XQC**2 -
377             1.2*XQC**3)*(2.*B(NN)**YQC**2 - B(NN)**2*YQC - YQC**3)
378      P(JJ+6) = P(JJ+6)+(1/QC/A(MM)**3*B(NN)**3)*(3.*A(MM)**XQC**2 -
379             1.2*XQC**3)*(2.*B(NN)**YQC**2 - B(NN)**2*YQC - YQC**3)
380      P(JJ+7) = P(JJ+7)+(1/QC/A(MM)**3*B(NN)**3)*(3.*A(MM)**XQC**2 -
381             1.2*XQC**3)*(2.*B(NN)**YQC**2 - B(NN)**2*YQC - YQC**3)
382      P(JJ+8) = P(JJ+8)+1*(1/QC/A(MM)**3*B(NN)**3)*(3.*A(MM)**XQC**2 -
383             1.2*XQC**3)*(2.*B(NN)**YQC**2 - B(NN)**2*YQC - YQC**3)
384      P(JJ+9) = P(JJ+9)+1*(1/QC/A(MM)**3*B(NN)**3)*(3.*A(MM)**XQC**2 -
385             1.2*XQC**3)*(2.*B(NN)**YQC**2 - B(NN)**2*YQC - YQC**3)
386      P(JJ+10) = P(JJ+10)+1*(1/QC/A(MM)**3*B(NN)**3)*(3.*A(MM)**XQC**2 -
387             1.2*XQC**3)*(2.*B(NN)**YQC**2 - B(NN)**2*YQC - YQC**3)
388      P(JJ+11) = P(JJ+11)+1*(1/QC/A(MM)**3*B(NN)**3)*(3.*A(MM)**XQC**2 -
389             1.2*XQC**3)*(2.*B(NN)**YQC**2 - B(NN)**2*YQC - YQC**3)
390      IF (X1.LT.A(MM)) GO TO 1060
391      1060 X1 = X1 - A(MM)
392      IF (Y1.LT.B(NN)) GO TO 1080
393      Y1 = Y1 - B(NN)
394      1080 J1 = 3*(MM-1)+3*(NN-1)*N
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441  J2*3*(MM-1)*3*(NN-1)*N+3
442  J3*3*(MM-1)*3*NN*N
443  J4*3*(MM-1)*3*NN*N+3
444  P(J1+1)=P(J1+1)+(QD/(A(MM)**3*B(NN)**3))*X2**4/2. -
445  1 X1**4/2. - A(MM)**3*X2**3 - X1**3 +
446  2 A(MM)**3*(X2 - X1)*(Y2**4/2. - Y1**4/2. -
447  3 B(NN)**3*(Y2**3 - Y1**3) + B(NN)**3*(Y2 - Y1))
448  P(J1+2)=P(J1+2)-(QD/(A(MM)**3*B(NN)**2))*X2**4/2. -
449  1 X1**4/2. - A(MM)**2*X2**3 - X1**3 +
450  2 A(MM)**2*(X2 - X1)*(Y2**3 - Y1**3) + B(NN)**3*(Y2 - Y1))
451  P(J1+3)=P(J1+3)+(QD/(A(MM)**3*(B(NN)**2)))*(X2**3/2.*Y2**4/2. -
452  1 X1**3/2. - (X2**3 - X1**3)*A(MM)**3/3.)*(Y2**4/2. -
453  2 Y1**4/2. - (Y2**3 - Y1**3) + B(NN)**3*Y2 -
454  3 B(NN)**3*Y1)
455  P(J1+4)=P(J1+4)-(QD/(A(MM)**3*B(NN)**3)))*(X2**3/2.*Y2**4/2. -
456  1 X1**3/2. - (X2**3 - X1**3)*A(MM)**3/3.)*(Y2**4/2. -
457  2 Y1**4/2. - (Y2**3 - Y1**3) + B(NN)**3*Y2 -
458  3 B(NN)**3*Y1)
459  P(J3+1)=P(J3+1)+(QD/(A(MM)**3*B(NN)**2)))*(X2**2/2. -
460  1 X1**2/2. - A(MM)**2*X2**2 - X1**2 +
461  2 A(MM)**2*X1**2/2. - A(MM)**2*X2**2 - X1**2 +
462  3 Y1**4/2. -
463  P(J3+2)=P(J3+2)-(QD/(A(MM)**3*B(NN)**2)))*(X2**2/2. -
464  1 X1**2/2. - A(MM)**2*X2**2 - X1**2 +
465  2 A(MM)**2*X1**2/2. - A(MM)**2*X2**2 - X1**2 +
466  3 Y1**4/2. -
467  P(J3+3)=P(J3+3)+(QD/(A(MM)**3*B(NN)**2)))*(X2**2/2. -
468  1 X1**2/2. - A(MM)**2*X2**2 - X1**2 +
469  2 A(MM)**2*X1**2/2. - A(MM)**2*X2**2 - X1**2 +
470  3 Y1**4/2. -
471  P(J4+1)=P(J4+1)-(QD/(A(MM)**3*B(NN)**3)))*(A(MM)**2*X2**3 -
472  1 A(MM)**2*X2**3 - X2**4/2. - X1**4/2. -
473  2 B(NN)**3*Y1**3 - Y1**4/2. -
474  P(J4+2)=P(J4+2)-(QD/(A(MM)**3*B(NN)**2)))*(A(MM)**3*X2**3 -
475  1 A(MM)**3*X2**3 - X2**4/2. - X1**4/2. -
476  2 B(NN)**3*Y1**3 - Y1**4/2. -
477  P(J4+3)=P(J4+3)-(QD/(A(MM)**3*B(NN)**3)))*(X2**4/2. -
478  1 X1**4/2. - A(MM)**2*X2**3 - X1**3) + B(NN)**3*Y2**3 -
479  2 B(NN)**3*Y1**3 - Y1**4/2. - Y1**4/2.)
480  CONTINUE
481  LSET = LSET + 1
482  WRITE(6,B111),MSET,LSET
483  C******BOUNDARY CONDITIONS
484  ITEM(1)=1
485  DO 54 I=1,4
486  IF(IBDRY(I)-4)411,54,41
487  41 GO TO (42,43,44,45,1)
DE ANDRADE  FINITE ELEMENTS

ISN   SOURCE STATEMENT

470  42 N1=1
471  N2=N
472  N3=1
473  GO TO 46
474  43 N1=N
475  N2=N*M
476  N3=M
477  GO TO 47
500  44 N1=(M-1)*N+1
501  N2=N*M
502  N3=1
503  GO TO 46
504  45 N1=1
505  N2=(M-1)*N+1
506  N3=M
507  47 ITEM(2)=2
510  ITEM(3)=3
511  GO TO 48
512  46 ITEM(2)=3
513  ITEM(3)=2
514  48 IF(BDRY(1)-2 )49,50,51
515  49 K1=3
516  K2=3
517  GO TO 52
520  50 K1=1
521  K2=2
522  GO TO 52
523  51 K1=1
524  K2=3
525  52 DD 53 II=N1,N2,N3
526  DD 53 J=K1,K2
527  KD(KROW)=3*(II-1)+ITEM(J)
530  P(KROW)=0.
531  DO 53 KTCOL=1,27
532  53 KI(KTCOL)=0.
536  54 CONTINUE

C****SET UP Y MATRIX SUCH THAT Y = A*X
541  DO 138 MM=1,MA
542  DO 138 NN=1,NA
543  CC=3*(MM-1)*N+3*NN-3
544  DD=CC+3*N
545  DD 138 J=1,6
546  KK=12*(NN-1)+12*(MM-1)*(N-1)+J
547  KC=CC+J
548  KD=DD+J
549  Y(KK)=X(KC)
551  Y(KK+6)=X(KD)
C****CALCULATE STRESSSES
556  DO 139 K = 1,MA
557  DO 139 L = 1,NA
560  J = (K-1)*(N-1) + L
561  I=12*(J-1)+1
562  MCX(J)=-(D(J)/2.)*((MU*(-Y(I+1)-Y(I+4)+Y(I+7)+Y(I+10)+Y(I+12)-Y(I+5)+Y(I+8)-Y(I+11))/A(L))
HE ANDRADE

FINITE ELEMENTS

SOURCE STATEMENT

FORTRAN SOURCE LIST FERP

ISN

553  MCY(J)=-D(J)/2.0*(Y(I+1)+Y(I)+Y(I+4)+Y(I+7)+Y(I+10))/B(K)+
     IMU*Y(I+2)-Y(I+5)+Y(I+8)-Y(I+11)/A(L))
564  MCXY(J)=D(J)*(1.-MU)*(Y(I+1)+Y(I+4)+Y(I+7)+Y(I+10))/3.*A(L)*B(K))+
801  *(Y(I)+Y(I+3)+Y(I+6)+Y(I+9))/9.*A(L)*B(K))+
139  2*(Y(I)+2.Y(I)+5)+Y(I+8)+Y(I+11))/3.*B(K))
565  CONTINUE
570  801 FORMAT (1HI,12A6)
571  522 FORMAT (1HO,9HPLATE NO. 12)
572  8111 FORMAT (1HO,9HPLATE NO.12,8X,13HLOAD CASE NO.12)
573  4008 FORMAT (1HO,8HBDRY(1)=15,5X,8HBDRY(2)=15,5X,8HBDRY(3)=15,5X,+
18HBDRY(4)=15)
574  4000 FORMAT (1HO,2HN=12,2X,2HM=12,2X,3HMU=F5.3,2X,4HTOL=F8.7,2X,4HLIM=+
113,2X,2HF=F5.3)
575  4001 FORMAT (1HO,2HA=13,2H)=F8.5,4X,2HA=13,2H)=F8.5,4X,2HA=13,2H)=F8.5)
576  4002 FORMAT (1HO,2HB=13,2H)=F8.5,4X,2HB=13,2H)=F8.5,4X,2HB=13,2H)=F8.5)
577  4005 FORMAT (1HO,2HD=13,2H)=F8.5,4X,2HD=13,2H)=F8.5,4X,2HD=13,2H)=F8.5)
578  14X,2HD=13,2H)=F8.5)
579  4006 FORMAT (1HO,3HCQ=1F10.5,6X,4HHCQ=F10.5,5X,4HHCQ=F10.5)
580  4007 FORMAT (1HO,3HCQ=1F10.5,6X,3HCQ=F10.5,6X,3HCQ=F10.5,6X,3HCQ=F10.5,+
16X,3HCQ=F10.5)
581  DO 6000 1=1,4
582  IF(BDRY(1)-2)6001,6002,6005
583  6001 WRITE(6,6003) I
584  6003 FORMAT (1HO,8HSIDE NO.11,1X,HIS GUIDED)
585  GO TO 6000
586  6002 WRITE(6,6004) I
587  6004 FORMAT (1HO,8HSIDE NO.11,1X,HIS HINGED)
588  GO TO 6000
589  6005 IF(BDRY(1)-3)6000,6006,6007
590  6006 WRITE(6,6008) I
591  6008 FORMAT (1HO,8HSIDE NO.11,1X,HIS CLAMPED)
592  GO TO 6000
593  6007 WRITE(6,6009) I
594  6009 FORMAT (1HO,8HSIDE NO.11,1X,THIS FREE)
595  6000 CONTINUE
596  622 WRITE (6,56)
597  56 FORMAT (1HI,1HDEFLECTIONS)
598  624 WRITE (6,55)
599  55 FORMAT (1HO,9HJOINT NO.,5X,1HW,17X,6HTHETA0,14X,6HTHETAY/)
600  MJ = M+N
601  DO 917 I = 1,MJ
602  II = 3*(I-1)+1
603  917 WRITE (6,918) I,XX(I),XXX(I),XX(I+2)
604  918 FORMAT (1H,13,F20.7,F18.7,F20.7)
605  920 FORMAT (1H,33HMOENTS AT CENTER OF EACH ELEMENT)
606  921 WRITE (6,921)
607  921 FORMAT (1HO,1HELEMENT NO.,3X,3HMCX,15X,3HMCY,17X,4HMCXY/)
640  DO 922 J = 1,MNA
641  922 WRITE(6,918) J,MCX(J),MCY(J),MCXY(J)
643  923 WRITE(6,8112)ITOT
644  8112 FORMAT (1HO,5HITOT=13)
645  READ(5,140)NCTRL
646  140 FORMAT (1I)
650  IF(NCTRL=2)500,600,700
DE ANDRADE  FINITE ELEMENTS
ISN    SOURCE STATEMENT
      FORTRAN SOURCE LIST FERP

  651    700 STOP
  652      END
SUBROUTINE GAUSS (MN,X,LIM,IND,KT,P,TOL,ITOT,F)

DIMENSION X(363),IND(363,9),P(363),DELTX(363),KT(363,27)

REAL KT
DO 21 I=1,MN
21 X(I)=0.

DO 24 ITOT=1,LIM
24 MARK=0
SUMR=0.
TEPO=MN
DO 23 I=1,MN
II=(I+2)/3
SUM=0.
DO 22 LL=1,9
JJ=IND(I,LL)
L=3*LL-3
J=3*JJ-3
DO 22 KL=1,3
A L = L +1
J=J+1
IF(I-J)26,25,26
25 MARK=1
A 1
26 X(I)=X(I)+DELTX(I)*F
IF(ABS(X(I)) .GT. XMAX) XMAX = ABS(X(I))
23 CONTINUE
END
STUDY OF A RECT. PLATE WITH RECT. ELEMENT- OCTOBER 65 / DE ANDRADE

BDRY(1)= 3  BDHY(2)= 1  BDRY(3)= 1  BDRY(4)= 3

N= 5  M= 5  MU=0.300  TOL=.0000010  LIM=300  F=1.600

A(1)= 0.12500  A(2)= 0.12500  A(3)= 0.12500
A(4)= 0.12500  A(5)=
B(1)= 0.25000  B(2)= 0.25000  B(3)= 0.25000
B(4)= 0.25000  B(5)=
D(1)= 1.00000  D(2)= 1.00000  D(3)= 1.00000  D(4)= 1.00000
D(5)= 1.00000  D(6)= 1.00000  D(7)= 1.00000  D(8)= 1.00000
D(9)= 1.00000  D(10)= 1.00000  D(11)= 1.00000  D(12)= 1.00000
D(13)= 1.00000  D(14)= 1.00000  D(15)= 1.00000  D(16)= 1.00000

QD= 1.00000  X2= 0.12500  X1= 0.  Y2= 0.25000  Y1= 0.
QD= 1.00000  X2= 0.25000  X1= 0.12500  Y2= 0.25000  Y1= 0.
QD= 1.00000  X2= 0.37500  X1= 0.25000  Y2= 0.25000  Y1= 0.
QD= 1.00000  X2= 0.50000  X1= 0.37500  Y2= 0.25000  Y1= 0.
QD= 1.00000  X2= 0.12500  X1= 0.  Y2= 0.50000  Y1= 0.25000
QD= 1.00000  X2= 0.25000  X1= 0.12500  Y2= 0.50000  Y1= 0.25000
QD= 1.00000  X2= 0.37500  X1= 0.25000  Y2= 0.50000  Y1= 0.25000
QD= 1.00000  X2= 0.50000  X1= 0.37500  Y2= 0.50000  Y1= 0.25000
QD= 1.00000  X2= 0.12500  X1= 0.  Y2= 0.75000  Y1= 0.50000
QD= 1.00000  X2= 0.25000  X1= 0.12500  Y2= 0.75000  Y1= 0.50000
QD= 1.00000  X2= 0.37500  X1= 0.25000  Y2= 0.75000  Y1= 0.50000
QD= 1.00000  X2= 0.50000  X1= 0.37500  Y2= 0.75000  Y1= 0.50000
QD= 1.00000  X2= 0.12500  X1= 0.  Y2= 1.00000  Y1= 0.75000
QD= 1.00000  X2= 0.25000  X1= 0.12500  Y2= 1.00000  Y1= 0.75000
QD= 1.00000  X2= 0.37500  X1= 0.25000  Y2= 1.00000  Y1= 0.75000
QD= 1.00000  X2= 0.50000  X1= 0.37500  Y2= 1.00000  Y1= 0.75000

PLATE NO.15  LUAD CASE NO. 1
SIDE NO.1 IS CLAMPED
SIDE NO.2 IS GUIDED
SIDE NO.3 IS GUIDED
SIDE NO.4 IS CLAMPED
<table>
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<th>ΘX</th>
<th>ΘY</th>
<th>ΘZ</th>
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<tr>
<td>2</td>
<td>0.</td>
<td>0.</td>
<td>0.</td>
</tr>
<tr>
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<td>0.</td>
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<td>6</td>
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</table>
### Moments at Center of Each Element

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<th>Element No.</th>
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<th>MCXY</th>
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ITOT=209