THE RICE INSTITUTE

AN INVESTIGATION OF A CRITERION FOR THE OPTIMUM TRANSIENT RESPONSE OF SERVOMECHANISMS

by

Betsalel Tasini

A THESIS SUBMITTED TO THE FACULTY IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE

Houston, Texas May, 1957
ACKNOWLEDGEMENTS

The author would like to express his thanks to Dr. Paul E. Pfeiffer who suggested the problem for investigation and whose patience, perseverance and helpful criticisms made the investigation possible. Dr. Pfeiffer's assistance was particularly invaluable in the experimental phase of this investigation.

The author also would like to thank Mr. William Peters for his help in constructing auxiliary apparatus and in data taking during the experimental stage.

The author is deeply indebted to his wife who patiently helped in arranging, proof reading and typing of this manuscript.
A new criterion for the optimum transient response of linear systems (in particular regulating systems) is discussed and investigated. The criterion is evaluated analytically for systems of the second and third order.

Auxiliary apparatuses which were needed for experimental testing of the criterion are discussed and evaluated. The investigation concludes of the unsuitability of the criterion and its poor merits.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>THE SECOND ORDER SYSTEM</td>
<td>5</td>
</tr>
<tr>
<td>THE THIRD ORDER SYSTEM</td>
<td>10</td>
</tr>
<tr>
<td>DESCRIPTION OF APPARATUS</td>
<td>20</td>
</tr>
<tr>
<td>a) The Analog Computer</td>
<td>20</td>
</tr>
<tr>
<td>b) The Multipliers</td>
<td>23</td>
</tr>
<tr>
<td>1) The Servo Multiplier</td>
<td>23</td>
</tr>
<tr>
<td>2) The Coincidence Multiplier</td>
<td>26</td>
</tr>
<tr>
<td>3) The Squaring Circuit</td>
<td>35</td>
</tr>
<tr>
<td>4) The Triangular Carrier Multiplier</td>
<td>37</td>
</tr>
<tr>
<td>EXPERIMENTAL PROCEDURE</td>
<td>57</td>
</tr>
<tr>
<td>RESULTS AND DISCUSSION</td>
<td>58</td>
</tr>
<tr>
<td>CONCLUSIONS</td>
<td>60</td>
</tr>
<tr>
<td>APPENDICES</td>
<td>62</td>
</tr>
<tr>
<td>APPENDIX I</td>
<td>62</td>
</tr>
<tr>
<td>APPENDIX II</td>
<td>65</td>
</tr>
<tr>
<td>APPENDIX III</td>
<td>68</td>
</tr>
<tr>
<td>APPENDIX IV</td>
<td>76</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>79</td>
</tr>
</tbody>
</table>
INTRODUCTION

The title of this investigation suggests rather a limited scope of application of a criterion for the optimum transient response. However, it should be realized that although this investigation will deal with servomechanisms and regulating systems the result may be applied to any transmission system characterized by an analogous differential equation (or transfer function).

In the design of servomechanism systems, after all specifications as to the performance and stability of a system had been met, there always remains one important question to be answered, namely, "Has an optimum system been attained?" The term "optimum system" should be regarded with reserve, since a system can yield optimum performance only with respect to some parameters and under certain conditions. In this investigation a criterion for the optimum transient response of servomechanism systems will be sought. Mainly systems of the follower (or duplicator) type will be dealt with. The requirements of such systems are quite stringent and very broadly speaking the systems should have a fast response (small rise time) and ideally no overshoots (or at the most a very small percentage overshoot). However, these two very general requirements are incompatible. An attempt to increase the speed of response will be accompanied by an increase in the percentage overshoot and vice versa.
The subject has been dealt with quite exhaustively in the technical literature from various points of view and several criteria have been developed. However, in this project it is proposed to investigate a criterion of the form:

\[ I_e = \int_0^\infty (1 - e^{-\lambda t}) e^2(t) \, dt \]

Where:
- \( e \) = the base of the Natural logarithm
- \( \lambda \) = a parameter
- \( e(t) \) = the error signal

The criterion will be referred to as the I.W.S.E. criterion (Integral of the Weighted Squared Error).

The above definite integral will be a function of the parameter \( \lambda \) and the system parameters. It is proposed to take the minimum value of this integral for a given \( \lambda \) as the criterion for optimum transient response of given system.

No known criterion can be so absolute as to apply to all systems and under any circumstances, but it can provide a yardstick of regulating systems belonging to the same class. Therefore it is proposed to investigate the class of systems belonging to the so-called followers (or duplicators) class, which have at least a zero steady-state dis-

1) Superscripts refer to reference number in the bibliography.
placement error when subjected to a step function input. This, it is felt, will cover a considerable range in practical types of system.

The advantages of this type of criterion over others already available are:

(1) The existence of an exact analytical method for the evaluation of the criterion integral.

(2) The inclusion of a parameter $\lambda$, renders a certain flexibility in selecting eventually the best criterion. Also for certain values of the parameter comparison is possible with other available criteria. For example, if $\lambda \to \infty$ the criterion is reduced to the well known Integral of the Square of the Error criterion (I.S.E.). On the other hand for very small values of $\lambda$ the criterion approaches to the criterion

$$I_e = 2 \int_0^\infty e^2(t) dt$$

(3) The criterion "penalizes" systems for large errors more than for small ones by the presence of the term $e^2(t)$ in the integral. The presence of the weighting function $(1- e^{-2\lambda t})" makes" the criterion cognizant of the fact that the initial errors are inevitable. On the other hand later errors are more pronounced and are weighted with an increasing weight function as time elapses. Expressed
differently, the criterion "attaches" great importance to later errors and none to initial ones.

Ideally, a criterion should have three basic attributes:

(1) Reliability
(2) Ready applicability
(3) Selectivity (i.e. the ability to select only one set of system parameters distinctively).

Of the above mentioned three, (1) and (3) are actually the core of the proposed investigation and as for (2) it can be said that it is as easy to apply experimentally as any other criterion.

Although an analytical treatment will be given to some systems, such a treatment becomes quite lengthy as the order of the system is increased and therefore most of the investigation will be carried out using analog computer techniques.
THE SECOND-ORDER SYSTEM

Consider the following linear positioning system, which schematically may appear as in the diagram below.

\[ E(s) \text{ is the Laplace Transform of the error signal } = R(s) - C(s). \]

This system is defined by the differential equation

\[ J \frac{d^2c(t)}{dt^2} + B \frac{dc(t)}{dt} + Kc(t) = Kr(t) \]

Using the normalizing procedure (see appendix I) the following transfer function of the system is obtained.

\[ \frac{c(p)}{r(p)} = \frac{1}{p^2 + ap + 1} \]

From stability consideration the parameter "a" is always positive.

This system is chosen first, since it is the simplest and many systems can be approximated by it. The second order system also has a zero steady state displacement error to a unit step input. This can be proven by the final value
Theorem as follows

\[ c(p) = \frac{1}{p(p^2 + ap + 1)} \]

\[ e(p) = r(p) - c(p) = \frac{1}{p} - \frac{1}{p(p^2 + ap + 1)} = \]

\[ = \frac{p^2 + ap}{p(p^2 + ap + 1)} \]

\[ \lim_{t \to \infty} e(t) = \lim_{\rho \to 0} pe(p) \]

Thus

\[ \lim_{t \to \infty} e(t) = 0 \]

Evaluating the criterion integral by the method of Appendix III the following expression is obtained

\[ I_\varepsilon = \frac{\varepsilon^3(1 + a^2) - a\varepsilon^2(1 + a^2) - \varepsilon/2(2 + a^2 + a^3) - a/2}{a(2\varepsilon + a)(\varepsilon^2 + a\varepsilon + 1)} \]

It can be seen that even for a simple system of the second order the expression obtained is rather cumbersome.

The integral was evaluated with the aid of a digital computer and the results tabulated in Appendix IV for values
of "a" from 0.1 through 4.0 and values of \( \lambda \) from 0.5 through 10^6 in steps as indicated.

For values of the parameter \( \lambda = 0.5 \), the minimum of the criterion integral has been found to occur at \( a = 1.1 \). Minima at \( a = 1.1 \) were also located at values of \( \lambda = 3.0, 3.5, 4.0, 4.5, 5.0 \). For all values of \( \lambda \geq 10.0 \) the minima of the criterion integral were found to be at \( a = 1.0 \).

For values of \( 1.0 \leq \lambda < 3.0 \), the minima were found to be at \( a = 1.2 \). In the range of \( \lambda \)'s for which investigation was carried out no minimum of the minima could be obtained. However, the indication is in the direction of smaller \( \lambda \).

The value \( a = 1.1 \) can be selected as an average optimum parameter for the system considered. This value of the parameter \( a \) corresponds to a relative damping coefficient of 0.55, which is not an obviously undesirable value.

It seems, that at least for this transfer function the selection of the system parameter "a" was insensitive to variations of \( \lambda \).

The criterion integral can be split into two integrals

\[
I_e = \int_0^\infty e^2(t) dt - \int_0^\infty e^{-2\lambda t^2} e^2(t) dt = I_1 - I_2
\]

where

\[
I_1 = \int_0^\infty e^2(t) dt
\]

\[
I_2 = \int_0^\infty e^{-2\lambda t^2} e^2(t) dt
\]

x) Detailed computations are available in the copy of the Electrical Department only.
and

\[ I_2 = \int_0^\infty e^{-2 \alpha t} e^{2(t)} dt \]

In terms of system parameters, \( I_1 \) was evaluated and found to be

\[ I_1 = \frac{1 + a^2}{2a} \]

and

\[ I_2 = \frac{\left( \alpha^2 + a + 1 \right) + a^2 + a + 1}{2(2\alpha + a)(\alpha^2 + a\lambda + 1)} = \]

\[ = \frac{1}{(2\alpha + a)} + \frac{a^2 - \alpha^2}{(\alpha^2 + a\lambda + 1)} \]

Differentiating \( I_1 \) with respect to \( a \), the following is obtained

\[ \frac{dI_1}{da} = \frac{4a^2 - (1 + a^2)2}{4a^2} = \]

\[ = \frac{2a^2 - 2}{4a^2} = \frac{a^2 - 1}{2a^2} \]

For a maximum or a minimum

\[ a = \pm 1 \]
The negative root is inadmissible from stability consideration.

It can be seen that considering only $I_1$ the result obtained for the minimum of the integral for values of $\xi \geq 10.0$ is the same as if the total integral would have been considered. Furthermore $I_1$ is the integral obtained when the I.S.E. (Integral of the Squared Error) criterion is evaluated.
THE THIRD-ORDER SYSTEM

By the same procedure as for the second-order system
the expression for the criterion integral was obtained for
the closed loop transfer function of the form:

\[
\frac{C(p)}{R(p)} = \frac{1}{p^3 + ap^2 + bp + 1}
\]

The result was found to be:

\[
I_c = \frac{N}{D}
\]

where

\[
N = 8\lambda^5 + 16a\lambda^4 + 10(b + a^2)\lambda^3 +
+ (5 + 11ab + 2a^3)\lambda^2 + (4a + 2b^2 + 3a^2b)\lambda
+ (a^2 - b + ab^2)
\]

and \(D = 16\lambda^6 + 32a\lambda^5 + 20(b + a^2)\lambda^4 + (22ab + 4a^3 +
+ 14)\lambda^3 + 2(3a^2b + 17a + 2b^2)\lambda^2
\]

\[
(2b + 4a^2 + 2ab^2)\lambda + (2ab - 1)
\]

(From Hurwitz-Routh stability Criterion \(ab - 1 > 0\))

For values of

"b" = 1.25, 1.50, 1.75, 2.00,
2.145, 2.150, 2.155

and for values of
\[ \alpha = 0.1, 1.0, 10.0, 100.0, 1000.0, 10000.0, \]

the criterion integral was evaluated for values of \( \alpha \) = 0.1 through 2.5 in steps of 0.1. The results are tabulated in Appendix IV, with a summarizing table of the location of the minima on pages 12-17.
THE MINIMA OF THE CRITERION INTEGRAL FOR A THIRD-ORDER SYSTEM

$\chi = 0.1$

For $\chi = 0.1$:

- $b = 1.25$
  - Min. at $a = 2.0$, $I_e = 0.70955$
- $b = 1.5$
  - Min. at $a = 1.9$, $I_e = 0.518196$
- $b = 1.75$
  - Min. at $a = 1.8$, $I_e = 0.43462$
- $b = 2.0$
  - Min. at $a = 1.8$, $I_e = 0.4031323$
- $b = 2.145$
  - Min. at $a = 1.8$, $I_e = 0.398869$
- $b = 2.15$
  - Min. at $a = 1.8$, $I_e = 0.398858$
- $b = 2.155$
  - Min. at $a = 1.8$, $I_e = 0.398856$
\[ \alpha = 1.0 \]

<table>
<thead>
<tr>
<th>( b )</th>
<th>( \text{Min. at } a = )</th>
<th>( I_e = )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.25</td>
<td>1.75</td>
<td>1.4552854</td>
</tr>
<tr>
<td>1.5</td>
<td>1.3</td>
<td>1.18802030</td>
</tr>
<tr>
<td>1.75</td>
<td>1.2</td>
<td>1.07824340</td>
</tr>
<tr>
<td>2.0</td>
<td>1.0</td>
<td>1.04978350</td>
</tr>
<tr>
<td>2.145</td>
<td>0.9</td>
<td>1.05802340</td>
</tr>
<tr>
<td>2.15</td>
<td>0.9</td>
<td>1.05841070</td>
</tr>
<tr>
<td>2.155</td>
<td>0.9</td>
<td>1.05881860</td>
</tr>
</tbody>
</table>
\[ \chi = 10.0 \]

<table>
<thead>
<tr>
<th>( b )</th>
<th>Minimum at ( a )</th>
<th>( I_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.25</td>
<td>1.6</td>
<td>1.8550115</td>
</tr>
<tr>
<td>1.50</td>
<td>1.3</td>
<td>1.5894852</td>
</tr>
<tr>
<td>2.00</td>
<td>1.0</td>
<td>1.4500118</td>
</tr>
<tr>
<td>2.145</td>
<td>0.9</td>
<td>1.4577619</td>
</tr>
<tr>
<td>2.15</td>
<td>0.9</td>
<td>1.4581669</td>
</tr>
<tr>
<td>2.155</td>
<td>0.9</td>
<td>1.4585924</td>
</tr>
</tbody>
</table>
\[ L = 100.0 \]

\[
\begin{align*}
\text{b}=1.25 & \quad \text{Min., at } a = 1.6 & I_e = 1.90000 \\
\text{b}=1.50 & \quad \text{Min., at } a = 1.35 & I_e = 1.6345 \\
\text{b}=2.00 & \quad \text{Min., at } a = 1.0 & I_e = 1.4950 \\
\text{b}=2.145 & \quad \text{Min., at } a = 0.9 & I_e = 1.5027 \\
\text{b}=2.15 & \quad \text{Min., at } a = 0.9 & I_e = 1.5031 \\
\text{b}=2.155 & \quad \text{Min., at } a = 0.9 & I_e = 1.5035
\end{align*}
\]
\( \lambda = 1000.0 \)

\[
\begin{align*}
\text{b} = 1.25 & \quad \text{Min. at } a = 1.6 \quad I_e = 1.9045 \\
\text{b} = 1.50 & \quad \text{Min. at } a = 1.3 \quad I_e = 1.6389 \\
\text{b} = 2.00 & \quad \text{Min. at } a = 1.0 \quad I_e = 1.4995 \\
\text{b} = 2.145 & \quad \text{Min. at } a = 0.9 \quad I_e = 1.50725 \\
\text{b} = 2.15 & \quad \text{Min. at } a = 0.9 \quad I_e = 1.50765 \\
\text{b} = 2.155 & \quad \text{Min. at } a = 0.9 \quad I_e = 1.50808
\end{align*}
\]
\[ \lambda = 10000.0 \]

| \( b = 1.25 \) | \( \text{Min. at } a = 1.6 \) | \( I_0 = 1.904095 \) |
| \( b = 1.50 \) | \( \text{Min. at } a = 1.3 \) | \( I_0 = 1.6394 \) |
| \( b = 1.75 \) | \( \text{Min. at } a = 1.1 \) | \( I_0 = 1.52900 \) |
| \( b = 2.00 \) | \( \text{Min. at } a = 1.0 \) | \( I_0 = 1.49995 \) |
| \( b = 2.145 \) | \( \text{Min. at } a = 0.9 \) | \( I_0 = 1.5077 \) |
| \( b = 2.15 \) | \( \text{Min. at } a = 0.9 \) | \( I_0 = 1.5081 \) |
| \( b = 2.155 \) | \( \text{Min. at } a = 0.9 \) | \( I_0 = 1.5085 \) |
For the third-order system the minimum of the minima for the selection of the system parameters were chosen. For $\lambda = 0.1$ it can be seen from the summary table that at least in the range investigated no least value of the minima could be located, but at least one can say that values for $a = 1.8$ $b = 2.155$ indicated.

For all other values of $\lambda$ investigated the values $b = 2.0$ $a = 1.0$ were found to be the minimum value sought. Graham and Lathrop in their investigation of the I.T.A.E. criterion obtained the values $a = 1.75$ $b = 2.15$ for the third-order system. The transient response obtained with these parameters is an excellent compromise between rise time and overshoots. Bearing this in mind and taking $ab - 1 > 0$ as a qualitative yardstick it can intuitively be suspected that the I.W.S.E. criterion selects a rather more oscillatory system than that selected by the I.T.A.E. However, this will have to be investigated more fully.

Adopting the same procedure as for the second-order system the total integral $I_c$ is split into its two components $I_1$, which is independent of $\lambda$, and $I_2$, which is dependent upon $\lambda$. Attention will be paid now only to $I_2$ which is given by:
\[ I_1 = \frac{a^2 + b(ab-1)}{2(ab-1)} \]

\[ \frac{\partial I_1}{\partial a} = \frac{a(ab-2)}{2(ab-1)^2} \]

For a minimum \( a(ab - 2) = 0; \quad a \neq 0 \)

so that \( (ab - 2) = 0; \) or \( ab = 2 \)

\[ \frac{\partial I_1}{\partial b} = \frac{2a^2b - 2ab - 2a^3 + 2}{4(ab - 1)^2} \]

For minimum \( 2a^2b^2 - 4ab - 2a^3 + 2 = 0 \)

Since \( ab = 2 \) from above

\( a^3 = 1; \quad a = 1 \)

Again it can be seen that the results obtained for the third-order system for all values of \( \lambda > 1 \) are those which would have been obtained by the I.S.E. criterion.
DESCRIPTION OF APPARATUS

a). The Analog Computer

As was stated in the introduction to this investigation, most of the analysis will be performed using an analog computer.

Fig. 1 (page 22) depicts schematically the general arrangement employed.

The analog computer, which was constructed at the Electrical Department, The Rice Institute, Houston, Texas, employs 30 Operational Amplifiers, 10 of which are Chopper-Stabilized Integrators, 10 Summer Amplifiers, and 10 amplifiers installed in a special purpose chassis for function generators and simulation of non-linear devices. Fifty precision potentiometers are available out of which only 5 are dial calibrated to 0.1\% linearity. Ten potentiometers are used for setting the initial condition on the integrators. The others can be used for coefficient setting to a very high degree of accuracy by means of comparison with a single high precision "Dekapot" voltage divider under load conditions. A very flexible and convenient interconnecting scheme is provided, allowing ample permutation in communication between components. By means of a switch every integrator can be converted into a Chopper-Stabilized Summing Amplifier.

The Operational Amplifiers used are the K2X units
manufactured by George A. Phillbrick Inc., Boston, Mass. The integrators are stabilized with the K2P Chopper Stabilizing Amplifier units.

The power supplies for the analog computer are regulated to within 0.1%.

An overload protection circuit is incorporated which will hold computation at the last value attained. The voltage can be relaxed when the overload condition is "recognized" and computation will proceed from the point at which it was interrupted. The condition of overload is transmitted to the operator by means of a light indicator on the central control panel.

The analog computer was designed, engineered and its construction supervised by Dr. Paul E. Pfeiffer of the Electrical Engineering Department, The Rice Institute, Houston, Texas.

In the analysis only the normal response to the input excitation will be studied. By normal response it is meant that the systems under investigation are at rest initially, so that no initial condition settings will be required anywhere along the flow of information. Evidently this facilitates the investigation and reduces the time for computer set up. Furthermore, since only a minimum value of the integral criterion is being sought, the knowledge of the absolute value of the minima is of no interest.
b). The Multipliers

The experimental part of this investigation is rather routine in nature, consisting mainly of analog computer set-ups. Consequently, the author deemed it of prime importance to construct either an original or a tested multiplier unit.

Two multipliers were needed, one to perform $e^2(t)$ and another one to mechanize the product $(1- e^{-2 \alpha t}) e^2(t)$.

Various schemes for multipliers exist, but their complexities do not always result in a high degree of accuracy. The object in this investigation was to develop a simple, inexpensive multiplier unit having a tolerable accuracy. Four schemes were investigated and will be described subsequently.

(1) Servo-multiplier

Figure 2 (page 25) shows the schematic diagram of a servo-multiplier by means of which it was hoped to mechanize the function $(1- e^{-2 \alpha t}) e^2(t)$.

The servo-modulator (MA102) and the A.C. servo-amplifier (SA104H) are modular units manufactured by Servomechanisms Inc. (for specification see Appendix IV). The type MA102 is an electro-mechanical modulator designed essentially for servo systems wherein a D.C. error signal is to be converted into a 400 c.p.s. carrier voltage. The
modulator is equipped for two inputs. The output of the modulator is a 400 c.p.s. A.C. signal whose magnitude is proportional to the difference between the two inputs and whose phase indicates the polarity of the difference.

Type SA104H is a miniaturized, hermetically sealed plug-in, electronic amplifier designed to control a 400 c.p.s. 2-phase motor. The motor used was a Kearfott 2-phase servo motor-generator type R801-1.

By means of varying the effective damping of the servo loop, which can be achieved by increasing the excitation on the tachometer generator and also by increasing the rate signal feedback, different exponents of the function \((1 - e^{-2\lambda t})\) can be obtained. However, investigation of this multiplier proved that the range of variation of \(\lambda\) is small and unsuitable for this investigation.
Fig. 2.
(2) The Coincidence Multiplier

The Coincidence Multiplier makes use of a probability law (see Ref. 5, p. 676) and may be used to compute the product of any number of inputs. It is a well established experimental fact that if several events occur with random distribution in time, the probability of simultaneous occurrence of all of them is the product of their separate probabilities. This result may be extended to periodic wave forms whose periods have no common divisor. In particular, if rectangular waveforms of independent frequencies are superimposed on a gating circuit ("and" circuit), the time during which all waveforms are positive will be proportional to the product of the duty ratios of the waveforms. The duty ratio being defined as the ratio of the "mark" time to the period of a rectangular waveform. It is now necessary only to provide a system to linearly modulate the duty ratio of a rectangular wave in accordance with a given signal. Certain properties of the Operational Amplifier were utilized in an attempt to construct the Coincidence Multiplier.

The Operational Amplifier, by itself (i.e. without feedback) serves as a sensitive voltage comparator. When the input voltage, $e_1$, (see Fig. 3(a), 3(b), p. 31) changes but a few millivolts up or down from a fixed or variable reference voltage, $e_{\text{ref}}$, the output $e_1$ will swing from $-150$ to $+120$ VDC (for the K2K Operational Amplifier Unit).
Consider now the circuit configuration shown in Figure 3(c). The functional equation describing the action of the operational amplifier can be written as:

\[ e = f(e_{\text{ref}} - e_1) \]

The function \( f(e_{\text{ref}} - e_1) \) can be considered as an operator such that when

\[ e_{\text{ref}} - e_1 \leq 0; \quad e < 0 \quad (= \text{full negative saturation}) \]

and when

\[ e_{\text{ref}} - e_1 \geq 0; \quad e > 0 \quad (= \text{full positive saturation}) \]

The equality sign in the two inequalities above corresponds to the state of transition between full negative saturation and full positive saturation, but this change-over takes place in a hysteresis-like manner as follows:

Consider the first case for transition,

\[ e < 0 \]

\[ e_{\text{ref}} = e_1 \]

but

\[ e_{\text{ref}} = -e_1 \quad [\text{see Fig. 3(c)}] \]
Expressed in words, the last equation states that stable transition from full negative saturation to full positive saturation will take place when $e_1$ (which can be called the input) has reached a negative voltage equal in magnitude to $be$. For the condition of transition from full positive saturation to full negative saturation the following is obtained:

$$e > 0;$$
$$e_{ref} = + be$$

Thus, the transition from full positive saturation to full negative saturation will take place when $e_1$ attains the value $+be$. From the above it can be seen that a two stable state circuit has been obtained.

Figure 4(a) (page 32) shows schematically the characteristic of the circuit in Figure 3(c). If the saturation limits in both positive and negative directions were the same, a symmetrical characteristic about the $e_1 = 0$ axis would have been obtained. These, however, are not equal (for the K2X Unit) resulting in the characteristic shown.

Oscilloscope pictures were taken with a triangular wave form (100 c.p.s. freq.) input signal and with the sweep activated by the same signal. The various traces obtained by means of increasing the fraction $b$ of the potentiometer $R$ thus increasing the magnitude of $e_{ref}$. 
Broadening of the characteristic corresponds to increasing b.

All the foregoing preliminaries can now be assembled leading to a circuit which was used to obtain the duty-ratio modulation.

Figure 5(a) (page 33) depicts the circuit used to pulse-time modulate an astable multivibrator. For the purpose of explaining the free-running multivibrator action of the circuit ignore the presence of the signal $e_n$ and $R_1$ which are only relevant to the application of the modulating signal.

Assume that the circuit resides in one of its stable states, say the positive saturation level (i.e. $e > 0$). For this stable state to be maintained, $e_0$ will have to stay negative and well out of the $\Delta$ region [c.f. Fig. 4(a)] which is the transition region, however the condenser is charged and the point $e_0$ rises in potential positively until it reaches a value which will cause it to "flop" to the other state where now the process of condenser charging will go in the opposite direction. Thus, oscillations once started will continue. The period of the resulting rectangular waveform depends jointly on the time constant $RC$ and the threshold $\Delta$. The duty-ratio is determined by the ratio in which $\Delta$ is subdivided. In Figure 5(b) are shown the various waveforms obtained with the components indicated.
To produce pulse-time modulation, $o_m$ the modulating signal is applied as shown. $o_m$ biases the threshold level off zero resulting in pulse-time modulation as shown in Figure 6(a) (page 34).

Figure 6(b) depicts the final circuit arrangement by means of which the mechanizing of the product of two machine variables $x$ and $y$ was attempted. $x$ and $y$ modulate rectangular waveform generators of the type discussed above. The resulting output is mixed and fed into a summing amplifier. This amplifier is used as a gating amplifier delivering an output pulse to the RC integrating network, whenever both inputs are positive. The time average output voltage represents the desired product.

No attempt was made to investigate whether the duty ratio modulation was linear, because of the difficulty in measuring time magnitudes to a reasonable degree of accuracy. The circuit was constructed and tested for the final outcome. It was found that the product did not follow a linear relationship. Difficulties were also encountered in obtaining zero condition when either of the variables was set equal to zero and the other varied.
Positive Saturation

120 VDC

Negative Saturation

150 VDC

\[ e_{\text{ref}} - e \]

\[ e \]

\[ e_{\text{ref}} \]

\[ e_i \]

Fig. 3
(a)

$C = 0.01 \mu F$

$R = 100 K$

$R_1 = 10 K$

$R_2 = R_3 = 100 K$

(b)

Output Voltage $e$

Fig. 5.
Output Voltage

(a)

Fig. 6
(3) The Squaring Circuit

A squaring circuit (subsequently described) was used to form the function \( e^2(t) \). The plate currents of certain triodes (such as the 12AU7) vary closely with a constant power of the plate voltage for a fixed grid voltage over a restricted range. This non-linear resistor may be used in the input path of an operational amplifier to provide an output which is the square of the input voltage. The signal has to be rectified before being fed to the squaring circuit. This is achieved by means of an absolute value circuit. Figure 7(a) (page 37) shows the arrangement used.

The layout for the absolute value circuit is shown in Figure 7(b). The reference is grounded and a conventional inverting amplifier follows whenever the input \( e_1 \) is negative. But positive values of \( e_1 \) cause the reference to be connected directly to the input, and converts the circuit to a voltage follower. It was found that using a grid bias of -1.0 volt on the 12AU7 a tolerable square law was obtained over a range of the input voltage of 0-4 volts. The combination of absolute value and squaring circuits was followed by an amplifier having a gain of 10. The calibration of the squaring circuits is tabulated on page 36.
## THE CALIBRATION OF THE SQUARING CIRCUIT

<table>
<thead>
<tr>
<th>$e_{in}$ (volts)</th>
<th>$e_0$ (volts)</th>
<th>$e_{in}$ (volts)</th>
<th>$e_0$ (volts)</th>
</tr>
</thead>
<tbody>
<tr>
<td>.2</td>
<td>.05</td>
<td>- .2</td>
<td>.03</td>
</tr>
<tr>
<td>.3</td>
<td>.11</td>
<td>- .3</td>
<td>.09</td>
</tr>
<tr>
<td>.4</td>
<td>.19</td>
<td>- .4</td>
<td>.18</td>
</tr>
<tr>
<td>.5</td>
<td>.28</td>
<td>- .5</td>
<td>.29</td>
</tr>
<tr>
<td>.6</td>
<td>.42</td>
<td>- .6</td>
<td>.4</td>
</tr>
<tr>
<td>.7</td>
<td>.53</td>
<td>- .7</td>
<td>.5</td>
</tr>
<tr>
<td>.8</td>
<td>.66</td>
<td>- .8</td>
<td>.64</td>
</tr>
<tr>
<td>.9</td>
<td>.81</td>
<td>- .9</td>
<td>.81</td>
</tr>
<tr>
<td>1.0</td>
<td>.97</td>
<td>-1.0</td>
<td>.97</td>
</tr>
<tr>
<td>1.1</td>
<td>1.12</td>
<td>-1.1</td>
<td>1.18</td>
</tr>
<tr>
<td>1.2</td>
<td>1.3</td>
<td>-1.2</td>
<td>1.31</td>
</tr>
<tr>
<td>1.3</td>
<td>1.43</td>
<td>-1.3</td>
<td>1.5</td>
</tr>
<tr>
<td>1.4</td>
<td>1.65</td>
<td>-1.5</td>
<td>1.95</td>
</tr>
<tr>
<td>1.5</td>
<td>1.95</td>
<td>-2.0</td>
<td>3.35</td>
</tr>
<tr>
<td>1.6</td>
<td>2.25</td>
<td>-3.0</td>
<td>9</td>
</tr>
<tr>
<td>1.8</td>
<td>2.8</td>
<td>-4.0</td>
<td>17</td>
</tr>
<tr>
<td>2.0</td>
<td>3.85</td>
<td>-5.0</td>
<td>32.5</td>
</tr>
<tr>
<td>2.5</td>
<td>6.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>16.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.5</td>
<td>25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.0</td>
<td>31</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
CALIBRATION OF THE SQUARING CIRCUIT
(a) **Squaring Circuit**

(b) **Absolute Value Circuit**

**Fig. 7**
(4) The Triangular Carrier Waveform Multiplier

This multiplier was conceived by Dr. Paul E. Pfeiffer and constructed at the Electrical Department, The Rice Institute, Houston, Texas. The unit accepts two machine variables x and y and generates the product to a very reasonable degree of accuracy. A full discussion on the performance will be given.

The multiplier is a modified form of the "quarter-squarer" multiplier, which depends in its operation on mechanizing the relationship:

\[-xy = \frac{[1 + (x - y)]^2 - [1 - (x + y)]^2 - 4x}{4}\]

In the triangular carrier multiplier the squaring is achieved by taking advantage of the fact that the area of two similar triangles is proportional to the square of one dimension. Fig. 8(a)(p. 45) shows the open circuit voltage waveform (i.e. in the absence of the diode) which prevails at the point A Fig. 8(b) when a triangular waveform is "resistance mixed" with a waveform whose frequency is very much lower than that of the triangular carrier, so that can be considered to be constant over at least one period of the carrier \( \Delta \). The current waveform resulting by the action of the diode in Fig. 8(b) is represented by the positive half of the waveform in Fig. 8(a) to a suitable scale.
This scale is determined by the forward resistance of the diode. The area of the shaded portion of the triangular waveform Fig. 8(a) is:

\[
\text{Area} = \frac{1}{2}(a + \delta) \cdot \frac{T}{2} \left(\frac{a + \delta}{2}\right) = \frac{T}{4} a (1 + \frac{\delta}{a})^2
\]

where

\[a = \text{half the peak-to-peak amplitude of the triangular carrier}\]

\[T = \text{the period of the triangular waveform.}\]

The average ordinate is:

\[
\text{Average Ordinate} = \frac{\text{Area}}{T} = \frac{a}{4} (1 + \frac{\delta}{a})^2
\]

Note, that the average ordinate depends on \(a\) and \(\delta\) and is independent of the frequency of the triangular carrier to the extent mentioned above and the frequency limitations of the entire circuit. The operation is also independent of the shape of the triangular waveform so long as it is a true triangular wave shape and does not contain any d.c. component at its source. In other words the wave shape should have symmetrical excursions about the zero-voltage reference.

The average of the current \(I_1\) indicated in Fig. 8(b) is given by:
where diode forward resistance assumed to be negligible compared to $R$. The diode is assumed to be an ideal diode, having infinite backward resistance and a constant finite forward resistance. By the same process of reasoning, the average of the current $I_2$ shown in Fig. 8(c) is given by:

$$\text{Av. } I_2 = \frac{e}{4k} (1 - \frac{\delta_2}{a})^2$$

Assuming identical diodes. Now,

$$\text{Av.} (I_1 + I_2) = \frac{a}{4k} \left[ (1 + \frac{\delta_1}{a})^2 - (1 - \frac{\delta_2}{a})^2 \right] =$$

$$= \frac{a}{4k} \left( 1 + \frac{2\delta_1}{a} + \frac{\delta_1^2}{a^2} - 1 + \frac{2\delta_2}{a} - \frac{\delta_2^2}{a^2} \right) =$$

$$= \frac{1}{2k} (\delta_1 + \delta_2) + \frac{1}{4ak} (\delta_1^2 - \delta_2^2)$$

Let, $\delta_1 = x - y$

$\delta_2 = x + y$
Then,

\[
\frac{1}{2R} (\delta_1 + \delta_2) = \frac{x}{R}
\]

and

\[
\frac{1}{4aR} (\delta_1^2 - \delta_2^2) = -\frac{xy}{aR}
\]

Fig. 8(d) illustrates the way in which the circuits of Figs. 8(b) and 8(c) are combined together with the signal \(-x\) to deliver the current \(I = -\frac{xy}{aR}\) to the summing point \(C\). This point is maintained at virtual ground potential as a result of the action of the operational amplifier. The output of the operational amplifier represents the proportional product desired.

A glance at the equation representing the averages of \(I_1\) and \(I_2\) will reveal that these would not hold for:

\[
|x - y| > a
\]

\[
|x + y| > a
\]

These conditions were found to be true in practice. Thus the dynamic range of the inputs is dependent on the peak amplitude of the triangular carrier. The operational amplifier as shown acts as a current measuring and averaging device.
Fig. 9 (p. 46) is the actual circuit diagram of the multiplier unit. The multiplier was built as a compact plug-in unit using the special chassis of the Analog Computer. The operational amplifiers are the K2X units. All resistors are 1% high stability resistors and were matched as shown. The amplifiers are balanced individually by grounding their inputs and connecting their outputs to the tie-points (TP) by means of switches $S_{2a}$, $S_{2b}$, $S_{3a}$, $S_{3b}$ and $S_{4a}$, $S_{4b}$. After balancing has been completed the switches are returned to the "compute" position. The alignment of the unit as a whole is performed by the rotary switch ($S_{1a}$, $S_{1b}$) which follows the sequence below:

Position 1: Adjust $\Delta$ potentiometer for null
Position 2: Adjust $x$ potentiometer for null
Position 3: Adjust $x'$ potentiometer for null
Position 4: Check $x'$ potentiometer for null (alternating between positions 3 and 4)
Position 5: Adjust $y$ potentiometer for null
Position 6: Check $y$ potentiometer for null (alternating between positions 5 and 6)
Position 7: Compute

The null is measured at the output of amplifier G. The alignment procedure was developed as a consequence of the balancing conditions required of the unit under the various
circuit configurations.

The calibration of the unit was carried out in great detail for the first quadrant. The rest of the quadrants were found operative but only a single range in each quadrant was investigated. The results are tabulated on pp. 48-56 and calibration curves produced on p. 47.

The linearity of the multiplier was found to be much better than 25 over the ranges of variables scanned. This figure was computed as a percentage of any given output product. Relative to the maximum output (which was found to be 15 volts in the range investigated) linearity was kept within 1/3 of one percent. It was observed that gross errors were committed whenever the magnitude of the sum of the two variables exceeded 60 volts. This exceeded the expected theoretical limit for this multiplier.

In the range of correct operation the deviations from linearity never exceeded 20mV. This behaviour was attributed to the lack of null on the y amplifier alignment positions (c.f. p. 42 and Fig. 9) there was always a minimum residual output of 20mV. Any mismatch in the characteristics of the diodes (especially near the origin) can cause this residual output. This can be easily shown by considering the circuit configuration when the amplifier is being adjusted.

The "zero tracking" of the multiplier was also affected by the residual output discussed above. When x was set
equal to zero and \( y \) varied from 0 to 45 volts the output observed stayed essentially constant at 20mV. When \( y \) was set equal to zero and \( x \) varied from 0 to 45 volts the output rose to the maximum value of 25mV.

Although drift in the operational amplifiers used is acceptably low errors might have been caused, particularly at low values of the variables.

Note that the multiplier unit operated without using the ideal diodes postulated theoretically. It is believed that the operation of the multiplier can be improved by matching the diodes and chopper-stabilizing at least the final amplifier.
Fig. 8
All resistors are in megohms unless otherwise specified.

CR, CR₂  ½ 6AL5

CR₁, CR₂  0.1
THE TRIANGULAR CARRIER MULTIPLIER CALIBRATION

<table>
<thead>
<tr>
<th>x = 1.0</th>
<th></th>
<th>x = 2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>xy</td>
<td>x</td>
</tr>
<tr>
<td>0.20</td>
<td>0.005</td>
<td>0.5</td>
</tr>
<tr>
<td>0.25</td>
<td>0.0055</td>
<td>1.0</td>
</tr>
<tr>
<td>0.30</td>
<td>0.0075</td>
<td>1.5</td>
</tr>
<tr>
<td>0.50</td>
<td>0.0075</td>
<td>2.0</td>
</tr>
<tr>
<td>1.00</td>
<td>0.0150</td>
<td>2.5</td>
</tr>
<tr>
<td>2.00</td>
<td>0.0320</td>
<td>3.0</td>
</tr>
<tr>
<td>4.00</td>
<td>0.0625</td>
<td>3.5</td>
</tr>
<tr>
<td>6.00</td>
<td>0.0950</td>
<td>4.0</td>
</tr>
<tr>
<td>8.00</td>
<td>0.1250</td>
<td>4.5</td>
</tr>
<tr>
<td>10.00</td>
<td>0.1500</td>
<td>5.0</td>
</tr>
<tr>
<td>15.00</td>
<td>0.2420</td>
<td>6.0</td>
</tr>
<tr>
<td>16.00</td>
<td>0.2450</td>
<td>7.0</td>
</tr>
<tr>
<td>20.00</td>
<td>0.3250</td>
<td>8.0</td>
</tr>
<tr>
<td>30.00</td>
<td>0.5000</td>
<td>9.0</td>
</tr>
<tr>
<td>32.00</td>
<td>0.5300</td>
<td>10.0</td>
</tr>
<tr>
<td>40.00</td>
<td>0.6900</td>
<td>11.0</td>
</tr>
<tr>
<td>45.00</td>
<td>0.7900</td>
<td>12.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>14.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>16.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>18.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>20.0</td>
</tr>
</tbody>
</table>

Note: x, y and xy are in volts

Frequency of triangular carrier 1200 c.p.s.
<table>
<thead>
<tr>
<th>( x = 2.0 )</th>
<th>( x = 4.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y )</td>
</tr>
<tr>
<td>22.0</td>
<td>0.760</td>
</tr>
<tr>
<td>24.0</td>
<td>0.835</td>
</tr>
<tr>
<td>26.0</td>
<td>0.900</td>
</tr>
<tr>
<td>28.0</td>
<td>0.975</td>
</tr>
<tr>
<td>32.0</td>
<td>1.110</td>
</tr>
<tr>
<td>36.0</td>
<td>1.275</td>
</tr>
<tr>
<td>40.0</td>
<td>1.415</td>
</tr>
<tr>
<td>45.0</td>
<td>1.650</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td>XY</td>
</tr>
<tr>
<td>----</td>
<td>-----</td>
</tr>
<tr>
<td>10.0</td>
<td>1.12</td>
</tr>
<tr>
<td>17.0</td>
<td>1.19</td>
</tr>
<tr>
<td>18.0</td>
<td>1.26</td>
</tr>
<tr>
<td>19.0</td>
<td>1.32</td>
</tr>
<tr>
<td>20.0</td>
<td>1.40</td>
</tr>
<tr>
<td>22.0</td>
<td>1.56</td>
</tr>
<tr>
<td>24.0</td>
<td>1.70</td>
</tr>
<tr>
<td>26.0</td>
<td>1.83</td>
</tr>
<tr>
<td>28.0</td>
<td>1.98</td>
</tr>
<tr>
<td>30.0</td>
<td>2.11</td>
</tr>
<tr>
<td>32.0</td>
<td>2.27</td>
</tr>
<tr>
<td>34.0</td>
<td>2.40</td>
</tr>
<tr>
<td>36.0</td>
<td>2.56</td>
</tr>
<tr>
<td>38.0</td>
<td>2.70</td>
</tr>
<tr>
<td>40.0</td>
<td>2.85</td>
</tr>
<tr>
<td>42.0</td>
<td>3.00</td>
</tr>
<tr>
<td>44.0</td>
<td>3.14</td>
</tr>
<tr>
<td>45.0</td>
<td>3.20</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$x = 8.0$</td>
<td>$y$</td>
</tr>
<tr>
<td>----------</td>
<td>-----</td>
</tr>
<tr>
<td>0.20</td>
<td>0.0250</td>
</tr>
<tr>
<td>0.25</td>
<td>0.0350</td>
</tr>
<tr>
<td>0.30</td>
<td>0.0425</td>
</tr>
<tr>
<td>0.40</td>
<td>0.0500</td>
</tr>
<tr>
<td>0.50</td>
<td>0.0700</td>
</tr>
<tr>
<td>0.60</td>
<td>0.0850</td>
</tr>
<tr>
<td>0.70</td>
<td>0.1000</td>
</tr>
<tr>
<td>0.80</td>
<td>0.1200</td>
</tr>
<tr>
<td>0.90</td>
<td>0.1350</td>
</tr>
<tr>
<td>1.00</td>
<td>0.1450</td>
</tr>
<tr>
<td>1.50</td>
<td>0.2180</td>
</tr>
<tr>
<td>2.00</td>
<td>0.2900</td>
</tr>
<tr>
<td>2.50</td>
<td>0.3600</td>
</tr>
<tr>
<td>3.00</td>
<td>0.4300</td>
</tr>
<tr>
<td>3.50</td>
<td>0.5000</td>
</tr>
<tr>
<td>4.00</td>
<td>0.5600</td>
</tr>
<tr>
<td>4.50</td>
<td>0.6300</td>
</tr>
<tr>
<td>5.00</td>
<td>0.7200</td>
</tr>
<tr>
<td>6.00</td>
<td>0.8400</td>
</tr>
<tr>
<td>7.00</td>
<td>0.9750</td>
</tr>
<tr>
<td>8.00</td>
<td>1.1200</td>
</tr>
<tr>
<td>9.00</td>
<td>1.2500</td>
</tr>
</tbody>
</table>

$\text{a}^8$
<table>
<thead>
<tr>
<th>$x = 10.0$</th>
<th>$x = 15.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$xy$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.08</td>
</tr>
<tr>
<td>1.0</td>
<td>0.18</td>
</tr>
<tr>
<td>2.0</td>
<td>0.36</td>
</tr>
<tr>
<td>4.0</td>
<td>0.72</td>
</tr>
<tr>
<td>8.0</td>
<td>1.42</td>
</tr>
<tr>
<td>16.0</td>
<td>2.85</td>
</tr>
<tr>
<td>20.0</td>
<td>3.60</td>
</tr>
<tr>
<td>25.0</td>
<td>4.50</td>
</tr>
<tr>
<td>30.0</td>
<td>5.40</td>
</tr>
<tr>
<td>40.0</td>
<td>7.20</td>
</tr>
<tr>
<td>45.0</td>
<td>8.00</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$y$</td>
<td>$xy$</td>
</tr>
<tr>
<td>------</td>
<td>-------</td>
</tr>
<tr>
<td>0.1</td>
<td>0.040</td>
</tr>
<tr>
<td>0.2</td>
<td>0.070</td>
</tr>
<tr>
<td>0.4</td>
<td>0.140</td>
</tr>
<tr>
<td>0.5</td>
<td>0.180</td>
</tr>
<tr>
<td>1.0</td>
<td>0.360</td>
</tr>
<tr>
<td>2.0</td>
<td>0.730</td>
</tr>
<tr>
<td>4.0</td>
<td>1.460</td>
</tr>
<tr>
<td>8.0</td>
<td>2.900</td>
</tr>
<tr>
<td>10.0</td>
<td>3.650</td>
</tr>
<tr>
<td>16.0</td>
<td>5.950</td>
</tr>
<tr>
<td>20.0</td>
<td>7.400</td>
</tr>
<tr>
<td>25.0</td>
<td>9.200</td>
</tr>
<tr>
<td>32.0</td>
<td>11.60</td>
</tr>
<tr>
<td>40.0</td>
<td>14.100</td>
</tr>
<tr>
<td>45.0</td>
<td>15.250</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$x = 30.0$</td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>$x$</td>
<td>$xy$</td>
</tr>
<tr>
<td>0.1</td>
<td>0.040</td>
</tr>
<tr>
<td>0.2</td>
<td>0.085</td>
</tr>
<tr>
<td>0.3</td>
<td>0.150</td>
</tr>
<tr>
<td>0.4</td>
<td>0.210</td>
</tr>
<tr>
<td>0.5</td>
<td>0.300</td>
</tr>
<tr>
<td>1.0</td>
<td>0.580</td>
</tr>
<tr>
<td>2.0</td>
<td>1.110</td>
</tr>
<tr>
<td>4.0</td>
<td>2.220</td>
</tr>
<tr>
<td>8.0</td>
<td>4.350</td>
</tr>
<tr>
<td>16.0</td>
<td>8.600</td>
</tr>
<tr>
<td>20.0</td>
<td>10.900</td>
</tr>
<tr>
<td>25.0</td>
<td>13.500</td>
</tr>
<tr>
<td>30.0</td>
<td>16.000</td>
</tr>
<tr>
<td>40.0</td>
<td>20.000</td>
</tr>
<tr>
<td>45.0</td>
<td>21.750</td>
</tr>
</tbody>
</table>
\[
\begin{array}{ll|ll|ll}
\hline
x = +20.0 & x = -20.0 \\
\hline
x & xy & x & xy \\
\hline
-0.1 & -0.050 & -0.1 & +0.040 \\
-0.2 & -0.080 & -0.2 & +0.070 \\
-0.4 & -0.165 & -0.4 & +0.145 \\
-0.8 & -0.300 & -0.8 & +0.285 \\
-1.6 & -0.600 & -1.6 & +0.580 \\
-3.2 & -1.180 & -3.2 & +1.170 \\
-4.0 & -1.480 & -6.4 & +2.300 \\
-8.0 & -2.900 & -8.0 & +2.850 \\
-16.0 & -5.800 & -16.0 & +5.700 \\
-32.0 & -11.600 & -20.0 & +7.200 \\
-40.0 & -14.200 & -32.0 & +11.300 \\
\hline
\end{array}
\]
\( x = -20.0 \)

<table>
<thead>
<tr>
<th>( Y )</th>
<th>( XY )</th>
</tr>
</thead>
<tbody>
<tr>
<td>+0.1</td>
<td>-0.040</td>
</tr>
<tr>
<td>+0.2</td>
<td>-0.080</td>
</tr>
<tr>
<td>+0.4</td>
<td>-0.170</td>
</tr>
<tr>
<td>+0.8</td>
<td>-0.335</td>
</tr>
<tr>
<td>+1.6</td>
<td>-0.620</td>
</tr>
<tr>
<td>+4.0</td>
<td>-1.490</td>
</tr>
<tr>
<td>+8.0</td>
<td>-2.900</td>
</tr>
<tr>
<td>+16.0</td>
<td>-5.800</td>
</tr>
<tr>
<td>+32.0</td>
<td>-11.500</td>
</tr>
<tr>
<td>+40.0</td>
<td>-14.000</td>
</tr>
<tr>
<td>+45.0</td>
<td>-15.500</td>
</tr>
</tbody>
</table>
EXPERIMENTAL PROCEDURE

The Integral Criterion was tested experimentally for systems of the second, third and fourth order. The transfer function representing the systems under investigation was set on the analog computer. The response and error signals were extracted; the former was fed to the recorder and the latter to the squaring circuit (see Fig. 1, p. 22). The function generator for generating \(1 - e^{-2\lambda t}\) was an integrator, the time constant of which was varied to represent the various values of the parameter \(\lambda\). The output of the integrator representing \(I_0\) (the integral criterion) was transmitted to another channel of the recorder.

The coefficient potentiometers representing the system parameters were set at convenient values. The system was excited by a unit step input and records were obtained of the response and the integral criterion. This was repeated for various values of the system parameters and a fixed \(\lambda\), until a minimum of the criterion could be located. The above procedure was repeated for various values of \(\lambda\).
RESULTS AND DISCUSSION

It was found from the experimental results that although the location of the minima followed those established theoretically (for the second and third order systems) the results cannot be quoted to any reasonable degree of accuracy. The selectivity was found to be very poor. The differences between the value of the criterion at the minimum and any other point near it was so small that it could have been within the margin of experimental error and the computer accuracy. This behaviour became more pronounced as the order system was increased. For the fourth order system no minima could be established.

The lack of selectivity of the integral criterion with respect to variation of the system parameters was already evident in the digital machine computation of the integral criterion for the second and third order systems. There it was observed that for the second order system a change of approximately 10% in the value of the parameter \( a \) produced a change of less than 1.0% in the value of the integral criterion near the minimum point. This was true for any given value of \( \lambda \).

For the third order system for values of \( \lambda < 1.0 \) and for a given value of the parameter \( b \) a variation of about 10% in the value of the parameter \( a \) resulted in about 1.0% change in the value of the integral criterion near the minima. The selectivity became worse as \( \lambda \) was increased,
a change of about 0.5% occurred in the value of the integral criterion near the minima, for an approximate 10% change of the parameter a.

The lack of sensitivity of the criterion to variation of the parameter $\lambda$ can be explained by the form of that part of the integral criterion ($I_2$) which is dependent on $\lambda$. Referring to the forms obtained for the second and third order systems it can be seen that $I_2$ is an algebraic function whose denominator and numerator are polynomials in $\lambda$. The degree of the numerator is lower by one than that of the denominator. Thus $I_2$ varies as $1/\lambda$. For small values of $\lambda$ there can be some variation in the criterion due to $I_2$; this depends on the relative magnitude of the constants in the $\lambda$-polynomials, which are the system parameters. However, as $\lambda$ is increased the contribution becomes negligible and vanishes in the limit. This explanation of the behaviour of $I_2$ can be extended to higher order systems having transfer function similar to those investigated here.
At the outset of this investigation certain attributes of a criterion for the optimum transient response of linear systems were enumerated and speculations were made as to the advantages of the criterion under investigation. Unfortunately none of the above aspects was realized.

The existence of an exact analytical method for the evaluation of the criterion was found to be rather lengthy and cumbersome although not impossible with the aid of computing machinery. If the criterion were found to contribute something of value the effort would have been well justified to carry computations to higher order systems.

The inclusion of the parameter $\lambda$ did not add to the scope of the criterion but rather divided the space of the minima into two distinctive sets corresponding to the small values of $\lambda$ and large values of $\lambda$ only. These are trivial cases in so far as this investigation was concerned since they yield the results obtained by the I.T.A.E. and I.S.E. respectively. When a choice of parameters could be made (only analytically) on the basis of the minimum of the minima the criterion selected a system whose transient behaviour was quite oscillatory. Although this in itself is not entirely undesirable the response is not satisfactory for systems of the duplicator type, with which this investigation was primarily concerned. The system selected by the criterion was the one that would have been selected...
by the I,S,E, criterion.

The criterion was shown to have very poor selectivity, deteriorating with increase of the order of the system. To generalize this statement beyond the third order system would be rather difficult. However, experimentally this was found to be so within the accuracy of the apparatus used.

It was found that because of the poor selectivity, especially near the minima, the experimental apparatus used for evaluating of the criterion was quite useless since the experimental error very much exceeded the percentage changes in criterion values.

Although one single class of systems was investigated, this was such a basic and simple class that in view of the results any further investigation was deemed of no particular value.
APPENDIX I

Normalization of Transfer Functions

In general, linear transfer systems of the duplicator class can be described by an explicit differential equation of the form:

\[
\begin{align*}
\frac{d^n}{dt^n} (Q_{n dt^n} + Q_{n-1 dt^{n-1}} + \cdots + Q_{2 dt^2} + Q_{1 dt} + Q_0) c(t) &= \\
\frac{d^m}{dt^m} (P_{m dt^m} + P_{m-1 dt^{m-1}} + \cdots + P_{2 dt^2} + P_{1 dt} + P_0) r(t)
\end{align*}
\]

In which the \( P_i \) and \( Q_i \)'s are constants \((i = 1, 2, 3, \ldots \ldots, n)\) and \( m \leq n-1 \). \( c(t) \) is the controlled variable (or output) and \( r(t) \) is the exciting function (or the input).

The corresponding transfer function is,

\[
\frac{C(s)}{R(s)} = \frac{P_m s^m + P_{m-1} s^{m-1} + \cdots + P_2 s^2 + P_1 s + P_0}{Q_n s^n + Q_{n-1} s^{n-1} + \cdots + Q_2 s^2 + Q_1 s + Q_0}
\]

(1)

Where \( s \) is the Laplace Transform Complex variable.

To convert the above equation to a more convenient form, normalize as follows:

(a) Define a new constant \( v_0 \) such that,

\[
v_0^n = \frac{Q_0}{Q_n}
\]

(2)
(b) Define new coefficients for the denominator in eqn. (1) by

\[ q_i = \frac{Q_i}{v_0^{n-1} Q_n} \quad (i = 1, 2, 3, \ldots, n) \quad (3) \]

Define new coefficients for the numerator in equation (1) by,

\[ p_i = \frac{P_i}{v_0^{n-1} Q_n} \quad (i = 0, 1, 2, \ldots, m) \quad (4) \]

(c) Divide the numerator and the denominator of equation (1) by \( Q_n \) and apply the transformations defined by eqn. (2), (3) and (4).

The transfer function then becomes,

\[ C(s) = \frac{p_m v_0^n + \cdots + p_2 v_0^{n-2} + p_1 v_0^{n-1} + p_0 v_0^n}{s^n q_{n-1} v_0^{n-1} + \cdots + q_2 v_0^{n-2} + q_1 v_0^{n-1} + v_0^n} \]

(d) Divide each term of the numerator and the denominator by \( v_0^n \) and introduce a new complex variable \( p = \frac{s}{v_0} \).

Then the transfer function becomes the normalized form desired:
\[
\frac{C(p)}{R(p)} = \frac{r_m p^m + r_{m-1} p^{m-1} + \cdots + r_2 p^2 + r_1 p + r_0}{p^n + q_{n-1} p^{n-1} + \cdots + q_2 p^2 + q_1 p + 1}
\]

The introduction of the new complex variable is equivalent to substituting a new independent variable \( T \) in the time domain such that \( T = v_0 t \) which converts the time variable into a dimensionless entity.

Note that the transfer function has been reduced to a form in which the first and last coefficients in the denominator had been made equal to unity.
In this investigation an integral of the following form is to be evaluated:

\[ I_e = \int_0^\infty (1-e^{-\lambda t}) e^2(t) dt = \mathcal{L}[e^2(t)] \bigg|_{s=0} - \mathcal{L}[e^2(t)] \bigg|_{s=2} \]

\( \lambda \) is a positive parameter.

The theorem of complex convolution (see Ref. 4, p. 275) yields the result:

\[ I_e = \frac{1}{2\pi j} \int_{C-j\infty}^{C+j\infty} E(s)E(-s)ds - \frac{1}{2\pi j} \int_{C_1-j\infty}^{C_1+j\infty} E(s)E(2\lambda-s)ds \]

If \( E(s) \) is analytic \( \Re(s) > \sigma_0 \) The first integral holds for \( C > \sigma_0 \); \( \sigma_0 < 0 \) The second integral holds for \( 2\lambda > \sigma_0 \); \( C_1 > \sigma_0 \) where \( \sigma_0 \) is the abscissa of absolute convergence of \( e(t) \). The second integral holds for,

\[ 2\lambda > \sigma_0 + C_1 \]
In particular \( c_1 \) may take any value in the interval:
\[
\frac{c_0}{2} < c_1 < 2 \lambda + |c_0|
\]

Let the first integral be denoted by \( I_1 \) and the second by \( I_2 \) then
\[
I_2 = \int_{c_1 - j\infty}^{c_1 + j\infty} E(s)E(2\lambda - s)ds
\]

Let
\[
s = z + \lambda \quad ds = dZ
\]
when
\[
s = c_1 + j\omega \quad z = c' + j\omega
\]
or
\[
z = (c_1 - \lambda) + j\omega
\]
where
\[
c' = c_1 - \lambda
\]
then
\[
2\pi j I_2 = \int_{c' - j\infty}^{c' + j\infty} E(\lambda + z)E(\lambda - z)dz
\]
where

\[
\frac{c_0}{2} - \lambda < c' < \lambda + |c_0|
\]
Suppose \( E(s) \) is analytic in \( R(s) > -\varepsilon \)
then \( E(\lambda - s) \) is analytic in \( R(s) > \lambda - \varepsilon \)
and \( E(\lambda - s) \) is analytic in \( R(s) \leq \lambda + \varepsilon \)

Hence in the evaluation of \( I_2 \) we may put \( c^* = \lambda \)
The criterion integral reduces now to its final form:

\[
I_c = \frac{1}{2\pi j} \int_{\gamma} E(s)E(-s)ds - \frac{1}{2\pi j} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} E_a(s)E_a(-s)ds =
\]

\[
= I_1 - I_2
\]

where

\[
E_a(s) = E(\lambda + s)
\]

The main task now is to evaluate integrals of the form:

\[
I = \frac{1}{2\pi j} \int_{\gamma} E(s)E(-s)ds
\]

and to proceed and find their minima. The method of evaluation is given in Appendix III.
APPENDIX III

As was seen in Appendix II the main task is to evaluate integrals of the form:

\[ I = \frac{1}{2\pi j} \int_{C-j\infty}^{C+j\infty} E(-s) E(s) ds \]

In the discussion to follow \( E(s) \) is always an algebraic function of \( s \).

To proceed, consider

\[ I = \int_{C-j\infty}^{C+j\infty} \frac{s^m}{p_n(s)} ds \]

where \( p_n(s) \) is a polynomial in \( s \) of degree \( n \) with \( R(s) = 0 \)

\[ p_n(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 \]

Two cases will have to be considered.

CASE I: \( m < n-1 \)

This case will be the usual one being one of the requirements of a stable system which will be dealt with exclusively. \( m = n-1 \) is of course also permissible and will
be investigated separately,

\[
I = \int_{C+J \infty}^{C-J \infty} \frac{s^m}{P_n(s)} \, ds = \lim_{R \to \infty} \int_{C} s^m \frac{1}{P_n(s)} \, ds = 0
\]

Where \( C \) is the equivalent Bromwich contour and \( R \) is the radius of the semi-circle in the right half of the complex plane. The integral vanishes by Cauchy's first integral theorem since the integrand is analytic in the right-half of the complex plane.

**CASE II: \( m = n-1 \)**

In this case the integrand has a multiple zero at \( s = 0 \).

This can be shown as follows:

\[
F(s) = \frac{s^{n-1}}{P_n(s)} = \frac{c_1}{s} + \frac{c_2}{s^2} + \cdots
\]

The only contribution to the integral is that of the small semi-circle enclosing the origin on the left. The value of \( I \) is given by:

\[
I = c_1 j \pi
\]

But

\[
c_1 = \lim_{s \to \infty} s F(s) = \frac{1}{a_n}
\]
The following results were obtained:

\[ I = \frac{1}{2\pi j} \int_{C-J\infty} \frac{s^m}{P_n(s)} \, ds = \begin{cases} 0; & m < n-1 \\ \frac{1}{2a_n}; & m = n-1 \end{cases} \]

Now consider the following integral

\[ I' = \frac{1}{2\pi j} \int_{C-J\infty} \frac{s^m}{P_n(s)} \, ds \]

where \( P_n(s) = P_n(-s) \) and is analytic in \( \Re(s) < 0 \)

\[ P_n(s) = a^n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 \]

where

\[ a_n' = (-1)^n a_n \]

By the same process of reasoning as for case I the following is obtained:

\[ I' = \frac{1}{2\pi j} \int_{C-J\infty} \frac{s^m}{P_n(s)} \, ds = \begin{cases} 0; & m < n-1 \\ -\frac{1}{2a_n'} = \frac{(-1)^n}{2a_n}; & m = n-1 \end{cases} \]
Let \( E(s) = \)

\[
\frac{b_ms^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \ldots + b_0}{a_ns^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_0} = \frac{E(s)}{A(s)} \quad ; \quad m < n
\]

Consider

\[
F(s) = \frac{B(s)}{A(s)} - \frac{A(-s)}{A(s)} = \frac{C_1(s)}{A(s)} + \frac{C_2(-s)}{A(s)}
\]

Now \( C_1(s) \) and \( C_2(s) \) are of degree \( r = n-1 \) by the Partial Fraction Expansion theorem. Also \( C_1(s) \) and \( C_2(s) \) cannot have factors in common with either \( A(s) \) or \( A(-s) \).

Now \( F(s) = F(-s) \) so that

\[
\frac{C_1(s) - C_2(s)}{A(s)} + \frac{C_2(-s) - C_1(-s)}{A(-s)} = 0
\]

At zeros of \( A(s) \), \( A(-s) \neq 0 \), \( C_1(s) \neq 0 \), \( C_2(s) \neq 0 \) hence \( C_1(s) = C_2(s) \) at the \( n \) zeros of \( A(s) \) which implies that they are identical. Thus the following expression may be written

\[
F(s) = \frac{C(s)}{A(s)} + \frac{C(-s)}{A(-s)}
\]
\[ I = \frac{1}{2\pi j} \int_{C-j\infty}^{C+j\infty} \frac{G(s)}{A(s)} + \frac{G(-s)}{A(-s)} \, ds = \]

\[ = \frac{c_{n-1}}{2a_n} - \frac{(-1)^n c_{n-1}}{2(-1)^n a_n} = \frac{c_{n-1}}{a_n} \]

To find \( c_{n-1} \) use the relation

\[ B(s)B(-s) = A(-s)C(s) \quad A(s)C(-s) \]

and equate the coefficients of the highest power on both sides. Note that no factoring of \( A(s) \) is required.

The integrals have been evaluated by R. W. Bond in an M.I.T. Memorandum and the results shown in Table I.
TABLE I

**B(s)-to-I Transformation**

\[ \begin{align*}
\frac{B(s)}{A(s)} & = I = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} E(s)E(-s)ds \\
A(s) & = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0 \\
B(s) & = b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + \ldots + b_0
\end{align*} \]

**n = 1**

\[ I = \frac{b_0^2}{2a_1 a_0} \]

**n = 2**

\[ I = \frac{a_0 b_1^2 + a_2 b_0^2}{2a_2 a_1 a_0} \]

**n = 3**

\[ I = \frac{a_1 a_0 b_2^2 + a_3 a_0 b_1^2 + a_3 a_2 b_0^2 - 2a_3 a_0 b_2 b_0}{2a_3 a_0 (a_2 b_1 - a_3 b_0)} \]
\[ n = 4 \]

\[ I = \frac{N}{D} \]

where

\[ N = (a_2a_1 - a_3a_0)a_0b_3^2 + a_4a_1a_0(b_2^2 - 2b_3b_1) + 
+ a_4a_3a_0(b_1^2 - 2b_2b_0) + (a_3a_2 - a_4a_1)a_4b_0^2 \]

and

\[ D = 2a_4a_0(a_3a_2a_1 - a_4a_1^2 - a_3^2a_0) \]

\[ n = 5 \]

\[ I = \frac{N}{D} \]

where

\[ N = (a_2a_1a_0 - a_4a_1^2 + a_3a_2a_1 - a_3^2a_0)a_0b_4^2 + 
+ (a_2a_1 - a_3a_0)a_5a_0b_3^2 + 2(a_3a_0 - a_2a_1)a_5a_0b_4b_2 + 
+ 2(a_4a_1 - a_5a_0)a_5a_0b_3b_1 + (a_4a_1 - a_5a_0)a_5a_0b_2^2 + 
+ 2(a_5a_0 - a_4a_1)a_5a_0b_3b_1 + (a_4a_3 - a_5a_2)a_5a_0b_1^2 + 
+ 2(a_5a_2 - a_4a_3)a_5a_0b_2b_0 + (a_5a_4a_0 - a_5a_2^2 - a_4a_1 + a_4a_3a_2)a_5b_0^2 \]
and

\[ D = 2a_5a_0(2a_5a_4a_1a_0 + a_5a_3a_2a_0 - a_5^2a_1 - \\
- a_5a_0^2 + a_4a_3a_2a_1 - a_4^2a_1^2 - a_4a_3^2a_0) \]
APPENDIX IV

ELECTRICAL SPECIFICATIONS FOR THE SERVO-MODULATOR
AND SERVO-AMPLIFIER

(1) Servo-Modulator Type SA104H

Power Requirements
(i) 150-300 Volt D.C. at 2mA. (Max.)
(ii) 6.3 Volts at 10% R.M.S.,
     400 c.p.s. at 0.2 amp.

Frequency Tolerance
400 c.p.s. ±5%

Tubes
1 - 12AX7

Input Impedance
Over 0.5 Meg. per input

Output Impedance
Approx. 30 Kilohms

Transfer Gain
3.0 Volts R.M.S. output per 1 Volt D.C. input difference (high value)
1.2 Volts R.M.S. O/P per volt D.C. input difference.
Residual Output at Null 25mV R.M.S.

Maximum Output Voltage 50 Volts

(2) **Servo-Amplifier Type SA104H**

**Input Impedance** 200 Kilohms approx.

**Gain**  The amplifier will have a voltage gain of 375 \( \pm 15\% \) for a 0.2 volt input at 400 c.p.s.

**Saturation** 0.35 Volt approx.

**Phase Shift** 120° 20 deg. lagging at 400 c.p.s.

**Damping** Derivative control (error-rate damping) is provided by a parallel T network. Optional velocity damping furnished by tachometer generators, can be applied to the "auxiliary input." (Low impedance output of amplifier
enhances velocity damping characteristic of motor).

<table>
<thead>
<tr>
<th>Carrier Frequency</th>
<th>The amplifier is designed for carrier signals of 380-420 c.p.s.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>9 watts at 115 volts, 400 c.p.s. (Max.)</td>
</tr>
</tbody>
</table>
BIBLIOGRAPHY


