Dynamic Response of Single-Degree-of-Freedom Bilinear Systems

by

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ABSTRACT

An analytical investigation is made of the dynamic response of bilinear single-degree-of-freedom systems excited by transient lateral motions of the base. Both elastic and hysteretic systems are studied. The input motions considered include a half-cycle velocity pulse, a half-cycle displacement pulse, a full-cycle displacement pulse, and an earthquake motion. The response quantity studied is the maximum deformation of the spring. Making use of the piece-wise linearity of the resistance-deformation relationship of the spring, the differential equation of motion is solved exactly in a step-wise manner.

The specific problems studied are:

1. The response of elastic systems with Coulomb damping and the relationship of this response to that of systems with viscous damping.

2. The response of bilinear hysteretic systems with unequal yield levels in the two directions of deformation.

3. The response of bilinear elastic systems and its relationship to that of bilinear hysteretic systems.

The results are summarized in the form of response spectra.
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FIGURES
I. INTRODUCTION

1.1 Object and Scope

This investigation deals with the response of bilinear single-degree-of-freedom systems subjected to ground motions of various forms. The objectives of the study are:

1. To develop information regarding the response of elastic systems with Coulomb friction.

2. To study the response of bilinear hysteretic systems with unequal yield levels in the two directions of deformation, and

3. To compare the response of bilinear elastic systems with that of bilinear hysteretic systems.

Although the response of viscously damped systems has been studied extensively, there has been comparatively little attention given to the response of systems with Coulomb friction. Several studies on the latter topic are reported in references (1-4). Jacobsen studied the steady-state response of systems to a sinusoidal excitation using an approximate method and Den Hartog presented the exact solution to the steady-state vibrations of systems damped by a combination of dry and viscous friction. Mindlin and Thomson investigated the transient response due to a sudden velocity change and a triangular velocity pulse with a peak at the origin, respectively. In Chapter III the effect of Coulomb friction on the response of elastic systems subjected to more complex types of excitation is assessed and compared to that of viscous damping. The applicability of these results to a certain class of inelastic systems is also shown.

Previous investigations (5-19) on the response of inelastic systems to ground
motions have been restricted to systems with equal yield levels in the two directions of deformation. Systems with unequal yield levels may arise when the restoring properties of the spring in the two directions are different, or when the spring, even though it has the same restoring properties, is deformed prior to the action of the ground excitation. This deformation may be due to the weight of the mass or to prestressing. Since the position of static equilibrium under such conditions does not correspond to the undeformed position of the spring, the system behaves dynamically as one with unequal restoring properties. In Chapter IV, response of elastoplastic systems with unequal yield levels in the two directions of deformation under various ground excitations is considered. Primary attention is given to systems that yield only in one direction and the results compared with those obtained for linear systems and elastoplastic systems with equal yield levels in the two directions of deformation.

Nonlinear systems having the same force-deformation curves for initial loading may have different behavior under dynamic excitation, depending on the shape of the unloading force-deformation curves. In Chapter V the response of bilinear systems with equal yield levels in the two directions of deformations is considered. Both elastic and hysteretic behavior are investigated and maximum deformations are obtained in each case. Attention is focused on systems with a horizontal second slope, and some consideration is given to systems with non-zero second slope.

The ground motions considered in this investigation are presented in Chapter II. They include a half-cycle velocity pulse, a half-cycle displacement pulse, a full-cycle displacement pulse, and one earthquake motion.
characteristics of the systems, the method of solution, and the capabilities of the computer program used to obtain the numerical data required in this study are also discussed in Chapter II.

1.2 Notation

The symbols used are defined when first introduced in the text, and the most important ones are summarized in the following:

<table>
<thead>
<tr>
<th>Symbol</th>
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<tr>
<td>$A$</td>
<td>pseudo-acceleration, defined by $p^2 U$ for a linear system and by $p_1^2 U$ for a bilinear system</td>
</tr>
<tr>
<td>$c$</td>
<td>$Q_y/Q_0$ = yield factor. Used also to denote the coefficient of viscous damping</td>
</tr>
<tr>
<td>$c'$</td>
<td>$Q'_y/Q_0$ = yield factor</td>
</tr>
<tr>
<td>$f$</td>
<td>undamped natural frequency of a linear system, in cycles per unit of time; also the undamped natural frequency corresponding to the initial elastic region of a bilinear system</td>
</tr>
<tr>
<td>$F$</td>
<td>constant force produced by Coulomb damping</td>
</tr>
<tr>
<td>$k$</td>
<td>spring stiffness for a linear system or current spring stiffness for a bilinear system</td>
</tr>
<tr>
<td>$k_1$</td>
<td>first spring stiffness of a bilinear system</td>
</tr>
<tr>
<td>$k_2$</td>
<td>second spring stiffness of a bilinear system</td>
</tr>
<tr>
<td>$m$</td>
<td>mass</td>
</tr>
<tr>
<td>$p$</td>
<td>$\sqrt{k/m} = \text{undamped circular natural frequency}$</td>
</tr>
<tr>
<td>$p_1$</td>
<td>$\sqrt{k_1/m} = \text{undamped circular natural frequency corresponding to the initial elastic region of a bilinear spring}$</td>
</tr>
<tr>
<td>$p_2$</td>
<td>$\sqrt{k_2/m} = \text{undamped circular natural frequency corresponding to the}$</td>
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second portion of a bilinear spring

\[ Q = \text{spring force} \]

\[ Q_{m} = \text{absolute maximum value of } Q \text{ for an } "\text{elastoplastic-elastic}\” \text{ system without regards to sign} \]

\[ Q_{m, e} = \text{absolute maximum spring force in the elastic direction of deformation for an } "\text{elastoplastic-elastic}\” \text{ system} \]

\[ Q_{o} = \text{absolute maximum value of } Q \text{ for the linear system associated to a bilinear system} \]

\[ Q_{y} = \text{smaller yield force for a bilinear spring yield force of systems with equal yield forces in the two directions of deformation} \]

\[ Q'_{y} = \text{higher yield force for a bilinear spring} \]

\[ t = \text{time} \]

\[ t_{d} = \text{total duration of a pulse} \]

\[ t_{1} = \text{duration of a half-cycle velocity pulse} \]

\[ u = x - y = \text{displacement of the mass relative to the ground, spring deformation} \]

\[ u_{o} = \text{absolute maximum value of } u \text{ for the linear system associated to a bilinear system} \]

\[ u_{y} = \text{smaller yield point deformation. Yield point deformation of systems with equal yield levels of deformation in the two directions} \]

\[ u'_{y} = \text{higher yield point deformation} \]

\[ U = \text{absolute maximum value of } u \text{ without regards to sign} \]

\[ V = \text{pseudo-velocity, defined by } pU \text{ for linear system and by } p_{1}^{2}U \text{ for a bilinear system} \]

\[ x = \text{absolute displacement of mass} \]
\[ y = \text{displacement of ground} \]
\[ y_o = \text{absolute maximum value of} \ y \text{ without regards to sign} \]
\[ \beta = \text{viscous damping factor, defined by} \ c/2mp \text{ for a linear system and by} \]
\[ c/2mp_1 \text{ for a bilinear system} \]
\[ \gamma = F/m\ddot{y}_o = \text{friction factor} \]
\[ \mu = U/u_y = \text{ductility ratio} \]

1.3 Acknowledgements

The author wishes to express his sincere appreciation to Professor A. S. Veletsos for his guidance and many helpful suggestions in the preparation of this dissertation. Special appreciation is also extended to Professor E. C. Holt, Jr. for his earlier direction of the author's work.
II. METHOD OF ANALYSIS

2.1 System Considered

The single-degree-of-freedom system considered in this study is illustrated in Fig. 2.1a. It consists of a rigid mass, m, a weightless spring exerting a force, Q, a dashpot which exerts a force proportional to the velocity of the mass relative to the ground, and a frictional device exerting a constant force, F, opposite to the direction of motion. The absolute displacement of the mass is denoted by \( x \), the displacement of the ground by \( y \), and the relative displacement of the mass with respect to the ground, the spring deformation, is denoted by \( u \), i.e.,

\[ u = x - y \]  

(2.1)

Positive directions of \( x \) and \( y \) are shown in the figure. A dot superscript on any of the above quantities represents differentiation with respect to time.

The resistance-deformation relationship for the spring is considered to be of the bilinear type with arbitrary "yield levels" in the two directions of deformation as indicated in Fig. 2.1b. The stiffness of the initial portion of the bilinear spring is denoted by \( k_1 \) and the stiffness of the second portion by \( k_2 \). Maximum and minimum absolute yield levels of \( Q \) are denoted by \( Q_y \) and \( Q'_y \), respectively, and the corresponding deformations by \( u_y \) and \( u'_y \).

The resistance-deformation relationship may be elastic or hysteretic as explained below and illustrated in Fig. 2.2.

a.) Elastic behavior - The force in the spring is a single valued function of the relative displacement, i.e., loading and unloading take place along the same path.

b.) Hysteretic behavior - The force in the spring is not only a function of the
relative displacement, but also of the time history of motion. Unloading is assumed to take place along a line parallel to the initial portion of the diagram.

2.2 Ground Motions Considered

In all cases, the variation of the ground acceleration is considered to be piece-wise linear. Displacement, velocity and acceleration diagrams for the ground excitations considered are shown in Figs. 2.3 and 2.4. The subscript "o" denotes the maximum value of the function to which it is attached, and the symbols $t_d$ and $t_1$ denote the total duration of the pulse and the duration of a velocity pulse, respectively.

The three simple motions shown in Fig. 2.3 are referred to as "the half-cycle velocity input," "the half-cycle displacement input" and "the full-cycle displacement input," respectively. The input shown in Fig. 2.4 represents the first portion of the North-South component of the earthquake motion recorded at El Centro, California on May 18, 1940. It will be referred to as the "El Centro Earthquake Input." The time histories of this motion were obtained from a straight-line approximation of the recorded accelerogram by applying a parabolic correction to its base line according to the criterion proposed by Berg and Housner\(^{20}\). Although this correction took into account 29.16 sec. of the recorded motion, only the first 6.29 sec. of the resulting record are used in this study. This initial time interval includes the most intense motion of the earthquake, or more specifically, the maximum absolute values of the ground acceleration, velocity and displacement. The exact cut-off point was chosen so that there would be no final ground velocity and the residual ground displacement would have a
2.3 Solution of Differential Equation of Motion

The equation of motion for the system considered can be written in the form:

\[ m\ddot{x} + c\dot{u} + Q + (\text{sgn } \dot{u})F = 0 \quad (2.2) \]

where \( c \) is the coefficient of viscous damping, \( (\text{sgn } \dot{u}) \) assigns to the force \( F \) the sign of the relative velocity \( \dot{u} \).

The system is assumed to be initially at rest. Because of the nonlinear character of the resisting force \( Q \), a complete solution of Eq. (2.2) cannot be obtained. However, an exact stepwise integration is possible due to the piecewise linear character of the resistance-deformation relationship. A description of the method follows.

Let \( Q_i, u_i, \) and \( \dot{u}_i \) be the values of the force in the spring, the relative displacement, and the relative velocity of the mass, respectively, at time \( t = t_i \), and let it be desired to evaluate the corresponding quantities after an increment of time \( \tau \). The force \( Q \) and the relative displacement, \( u \), may be expressed as

\[ Q(t) = Q_i + k\overline{u}(\tau) \quad (2.3) \]

and

\[ u(t) = u_i + \overline{u}(\tau) \quad (2.4a) \]

where \( \overline{u}(\tau) \) is the change in deformation and \( k \) is the current stiffness of the spring (\( k_1 \) or \( k_2 \)). The relative velocity \( \dot{u}(t) \) is obtained by differentiation of Eq. (2.4) as

\[ \dot{u}(t) = \overline{\dot{u}}(\tau) \quad (2.4b) \]

where a dot superscript over a barred quantity denotes differentiation with respect to \( \tau \). Making use of Eq. (2.3) and noting that \( x = u + y \), Eq. (2.2)
becomes

\[ m(\ddot{u} + \dot{y}) + cu + Q_i + k\ddot{u} + (\text{sgn} u) F = 0 \]  

(2.5)

The initial conditions of this equation are obtained from Eqs. (2.4) as

\[ \ddot{u}(0) = 0 \quad \text{and} \quad \dot{u}(0) = \dot{u}_i \]  

(2.6)

Letting

\[ p = \sqrt{k/m}, \quad p_1 = \sqrt{k_1/m}, \quad p_2 = \sqrt{k_2/m} \quad \text{and} \quad \beta = \frac{c}{2mp_1} \]  

(2.7)

we obtain

\[ \dddot{u} + 2\beta p_1 \ddot{u} + p^2 u = -\ddot{y}(t) - \frac{Q_i}{m} - (\text{sgn} u) \frac{F}{m} \]  

(2.8)

Note that \( p \) is equal to \( p_1 \) or \( p_2 \) depending on whether \( k = k_1 \) or \( k_2 \).

Since the ground acceleration-time diagram is considered to be piece-wise linear, \( \ddot{y}(t) \) may be expressed as

\[ \ddot{y}(t) = \ddot{y}_1 + b_j (t - t_i) = \ddot{y}_1 + b_j \]  

(2.9)

where \( \ddot{y}_1 \) is the ground acceleration at time \( t = 0 \) (\( t = t_i \)) and \( b_j \) is defined as

\[ b_j = \frac{\ddot{y}_{j+1} - \ddot{y}_j}{(\Delta t)_j} \]

\[ b_j \] = rate of change of the acceleration \( \ddot{y} \) in the \( j^{th} \) interval of the ground acceleration-time diagram

\[ \ddot{y}_{j+1} \] = value of \( \ddot{y} \) at the end of the \( j^{th} \) interval

\[ \ddot{y}_j \] = value of \( \ddot{y} \) at the beginning of the \( j^{th} \) interval

\[ (\Delta t)_j \] = time elapsed between \( \ddot{y}_j \) and \( \ddot{y}_{j+1} \)

Note that \( b_j \) changes every time a peak in the input function is crossed. Using Eqs. (2.7) and (2.9), Eq. (2.8) becomes,

\[ \dddot{u} + 2\beta p_1 \ddot{u} + p^2 u = -b_j \ddot{y}_j - \dddot{y}_1 - \frac{Q_i}{m} - (\text{sgn} \dot{u}) \frac{F}{m} \]  

(2.10)
Note that in Eq. (2.10), $p_1$, $\beta$ and $F$ are constant for a given problem, whereas $\dot{y}_1$, $Q_j$, $b_j$ and $p$ are known for a given $t_i$.

A solution for Eq. (2.10) with initial conditions given by Eqs. (2.6) may now be obtained. Different expressions will result depending on whether the instantaneous value of $p = \sqrt{k/m}$ is equal to or different from zero. Expressions for the relative displacement, relative velocity and relative acceleration of the mass are given for the various cases in the Appendix. For the present discussion, it is sufficient to consider the solution of Eq. (2.10) when $p \neq 0$ and $\beta = 0$

$$\ddot{u} = -\frac{R_1}{p^2} \left(1 - \cos \frac{\pi t}{p}\right) + \left(\frac{1}{p} + \frac{1}{3}\right) \sin \frac{\pi t}{p} - \frac{b_j}{2}$$

(2.11)

where

$$R_1 = \dot{y}_1 + \frac{Q_j}{m} + \frac{(\text{sgn} \dot{u})}{m}$$

The velocity $\dot{u}$ and acceleration $\ddot{u}$ are given by

$$\dot{u} = \left(\frac{1}{p} + \frac{1}{2}\right) \cos \frac{\pi t}{p} - \frac{R_1}{p} \sin \frac{\pi t}{p} - \frac{b_j}{2}$$

(2.12)

and

$$\ddot{u} = -R_1 \cos \frac{\pi t}{p} - \left(\frac{1}{p} + \text{p} \dot{u}\right) \sin \frac{\pi t}{p}$$

(2.13)

The evaluation of the absolute maximum deformation experienced by systems subjected to ground motion is of primary interest in this investigation. To this end, it is necessary to obtain the value of all the relative minimum and maximum deformations. These will occur at times when $\dot{u} = 0$. It may be noted that the determination of the times when $\dot{u} = 0$ is essential for the stepwise integration of Eq. (2.2) since the direction of the friction force, $F$, must be changed when the velocity $\dot{u}$ changes sign. Also, for hysteretic systems, unloading takes place
along a line of stiffness \( k_1 \), hence, if the current stiffness is \( k_2 \), a break will occur when \( \ddot{u} = 0 \). Since an explicit solution for these times of occurrence cannot be obtained in a convenient manner from Eq. (2.12), a searching procedure is adopted. Noting that a zero of the velocity function is always contained between two adjacent zeros of the acceleration function, a time \( \dot{t} \), measured from \( t_1 \) to the first zero of the \( \ddot{u} - t \) diagram is chosen to ensure that at the most one zero of the velocity function occurs during this interval. From Eq. (2.13) we obtain

\[
\dot{t} = \frac{1}{p} \tan^{-1} \left( -\frac{pR_i}{p^2 \ddot{u}_1 + b} \right) \quad \text{with} \quad 0 < p \dot{t} \leq \pi
\]

Expressions for \( \dot{t} \) for the other combinations of \( p \) and \( \beta \) are given in the Appendix.

If \( \ddot{u} \) has changed sign between \( t_1 \) and \( t_1 + \dot{t} \), a back iteration is performed and the "exact" time for which \( \ddot{u} = 0 \) is thus determined. A back iteration is also necessary to determine the time when the deformation \( u \) crosses a kink of the resistance-deformation diagram.

The sequence of the computations is summarized by means of the Block Diagram shown on the next page.

2.4 Capabilities of Computer Program

a.) System - The system considered is the general bilinear system described in section 2.1. All the parameters in the system must be equal or greater than zero with the exception of \( k_1 \) that must be positive.

b.) Input Motion - The acceleration-time history (accelerogram) must be specified. Any piece-wise linear variation may be considered.

c.) Response Quantity - The relative displacement is used as a response quantity. The maximum absolute deformation and its time of occurrence are
recorded. In addition, u may be printed out at constant increments of time and/or at the points where it attains stationary values. The integration may be interrupted at the end of the accelerogram or continued until the first extremum value of u is attained during free vibration.

2.5 Block Diagram

```
Read system parameters and ground motion

\[ t_1 = 0 \]

Compute \( t^- \)

Is \( t = t_1 + t^- \) greater than \( t_j \) (\( t_j \) time when the next peak in \( \dot{y} \) occurs)?

Yes

\[ t^- = t_i - t_1 \]

Evaluate \( \dot{u} \) at \( t = t_i + t^- \)

Has \( \dot{u} \) changed sign?

Yes

Iterate back to find new \( t^- \) for which \( \dot{u} = 0 \)

Compute \( u \) at \( t = t_i + t^- \)

Has \( u \) crossed a kink in the resistance deformation diagram?

Yes

Iterate back to find new \( t^- \) for which \( u = u_{\text{kink}} \)

Record \( u \), if desired compute \( u \) between \( t_i \) and \( t_i + t^- \) at constant increments of time, and if necessary, set the proper value of \( p \) for the next increment and/or new kinks

\[ t_i = t_i + t^- \]

Has the whole time-acceleration diagram (plus free vibration period) been analyzed?
```

No

III. RESPONSE OF LINEAR SYSTEMS WITH COULOMB DAMPING

3.1 Linear Systems

Linear systems are first discussed in detail and the applicability of the results of bilinear hysteretic systems is given consideration in section 3.2.

The absolute maximum value of the spring deformation, without regard to sign, is denoted by $U$, and the pseudo-velocity and pseudo-acceleration are denoted by $V$ and $A$, respectively. The latter quantities are alternative measures of $U$, defined as

$$V = pU$$
$$A = p^2 U$$

where $p$ is the undamped circular natural frequency of the system.

Results are summarized in the form of response spectra. A response spectrum is a plot of the maximum value of a response quantity as a function of the natural frequency of the system. The spectrum for the maximum deformation $U$ will be referred to briefly as the "deformation spectrum."

3.1.1 Presentation and Discussion of Results

Figures 3.1 - 3.3 give the deformation spectra for the half-cycle velocity, half-cycle displacement and full-cycle displacement inputs shown in Fig. 2.3. The spectra are plotted on a log-log scale. The ordinate represents the pseudo-velocity normalized with respect to the maximum ground velocity $y_o$, and the abscissa represents the dimensionless product of the natural frequency $f$ of the system in cycles per unit time and the duration of the dominant half-cycle portion of a velocity pulse, $t_i$.

The maximum ground deformation $U$, normalized with respect to the maximum ground displacement, $y_o$, may be read on the diagonal scale at the left, and the pseudo-acceleration $A$, normalized with respect to the maximum ground accelera-
tion $y_o$, may be read on the diagonal scale at the right. Each curve is for a fixed value of the dimensionless friction parameter $\gamma$, defined as

$$\gamma = \frac{F}{m \ddot{y}_o}$$

(3.1)

Values of $\gamma$ may range between 0 and 1. If $\gamma = 0$, the system has no friction, and if $\gamma = 1$, the friction is equal to the maximum force applied to the system, $m \ddot{y}_o$; thus, no deformations will occur. Friction is often measured by the coefficient of friction $\mu_f$, defined as

$$\mu_f = \frac{F}{mg}$$

(3.2)

where $g$ is the gravitational acceleration. In general, $\mu_f$ may have different values for static and kinetic friction. In this study it is assumed to be constant. This coefficient is related to $\gamma$ by the equation

$$\mu_f = \gamma \frac{\dot{y}_o}{g}$$

(3.3)

General: The overall effect of Coulomb damping is to reduce the maximum deformations experienced by the system and to smooth out the humps and undulations of the spectra.

The response of a system with Coulomb damping is equivalent to that obtained for a frictionless system with same natural frequency subjected to the modified input motion

$$\ddot{y}_e(t) = \dot{y}(t) \pm \frac{F}{m}$$

(3.4)

where the + sign is applicable while the relative velocity $\dot{u}$, is positive. In the regions where

$$F > \left| m \ddot{y}(t) + Q(t) \right|$$
no motion results unless the system is already in motion; thus, \( \dot{y}_e(t) = 0 \)
in such regions. The quantity \( \dot{y}_e(t) \) will be referred to as the "effective ground acceleration" and the associated velocity and displacement will be identified by the symbols \( \dot{y}_e(t) \) and \( y_e(t) \), respectively.

**Low Frequency Systems** In general, the sign of \( F \) is unknown and cannot be determined before the response has been evaluated. For the limiting case of low frequency systems, the mass of the system may be considered to remain constant, hence

\[
\ddot{u}(t) = -\dot{y}_e(t) \tag{3.5}
\]
\[
\dot{u}(t) = -\dot{y}_e(t) \tag{3.6}
\]
and

\[ u(t) = -y_e(t) \tag{3.7} \]

The direction of \( F \) will therefore reverse when \( \dot{u} = \dot{y}_e = 0 \). Now, the time when \( \dot{y}_e = 0 \) may be determined by integration of the effective ground acceleration, which will also give the time of reversal of \( F \). Fig. 3.4 shows the effective ground acceleration, velocity and displacement diagrams corresponding to a low frequency system with a friction factor \( \gamma = 0.3 \), subjected to a half-cycle velocity input. The shaded area in Fig. 3.4a depicts the effective ground acceleration, obtained by superposition of the ground acceleration \( \ddot{y}(t) \), represented by a solid line, and the acceleration \( F/m \) due to Coulomb damping given by the dashed line. The time of reversal of the friction force \( F \) is determined as explained above. The effective ground velocity and displacement diagrams shown in Figs. 3.4b and 3.4c, respectively, are obtained by successive integrations of the effective ground acceleration. From Eq. (3.6) it is observed that
the diagrams shown in Fig. 3.4, except for a change of sign, also represent
the time histories of relative acceleration, velocity and displacement of the
system considered.

Fig. 3.5 gives the effective ground displacement, or equivalently, the
relative displacement of low frequency systems (ft_1 > 0) with Coulomb damping
for the input motions shown in Fig. 2.3. Changes in shape occur as the friction
factor \( \gamma \) increases, in particular, for \( \gamma = 0.5 \) the full-cycle displacement pulse
transforms into a half-cycle displacement pulse. Notice also that the peak
value of the effective ground displacement, \( (y_e)^o \), does not necessarily decrease
as the value of \( \gamma \) increases, and that for the full-cycle displacement input, for
\( \gamma = 0.1 \) and 0.3, \( (y_e)^o \) is greater than the maximum ground displacement, \( y_o \).

The peak value \( (y_e)^o / y_o = U/y_o \) for each of the three pulses is given in
Fig. 3.6 as a function of the friction factor \( \gamma \). These deformations represent
the low frequency limiting values of the spectra shown in Figs. 3.1 - 3.3. Two
different half-cycle velocity pulses are considered in Fig. 3.6a, one with a
continuous and the other with a step input acceleration, denoted as pulses A and
B, respectively. Analytical solutions for these pulses may be readily determined.
For pulse A we obtain

\[
\frac{U}{y_o} = 1 - 4\gamma - \frac{2}{3}\gamma \sqrt{2\gamma} \quad \text{for} \quad \gamma \leq 3 - 2\sqrt{2} \quad 0.17 \quad (3.8)
\]

and

\[
\frac{U}{y_o} = \frac{3 + 2\sqrt{2}}{6} (1 - \gamma)^3 \quad \text{for} \quad \gamma \geq 3 - 2\sqrt{2} \quad (3.9)
\]

Corresponding to pulse B, the maximum deformation will be given by

\[
\frac{U}{y_o} = \frac{1 - \gamma}{1 + \gamma} \quad \text{for all values of} \ \gamma. \quad (3.10)
\]
Notice that different values of $U/y_0$ are obtained for each input for a given value of the friction parameter, $\gamma$. As opposed to undamped systems, Fig. 3.6a shows that the maximum deformation of low frequency systems with Coulomb damping depends, not only on the characteristics of the displacement trace of the input motion but on the detailed shape of the acceleration trace of the input motion as well.

**High Frequency Systems.** For very high values of the frequency parameter (high frequency systems or input motions of long duration), the pseudo-acceleration $A$ of linear systems with Coulomb damping, if the input acceleration does not have any discontinuities, is equal to the maximum effective ground acceleration, $(\ddot{y}_e)_0$, that is, the maximum deformation equals the value that would be produced by a static force of magnitude $m(\ddot{y}_e)$ on a fixed-base system. It may be noted that the effective ground acceleration is a continuous function for very large values of the frequency in spite of the discontinuous character of the friction force; therefore

$$A = (\ddot{y}_e)_0 = \ddot{y}_o - \gamma \ddot{y}_o \quad (3.11)$$

or

$$\frac{A}{\ddot{y}_o} = 1 - \gamma \quad (3.12)$$

Thus, as the parameter $f_t$ becomes large, the spectra shown in Figs. 3.1 - 3.3 converge to $(1 - \gamma)$ on the diagonal scale $A/\ddot{y}_o$.

**Applicability of Limiting Values to Intermediate Frequency Regions.** It is of interest to investigate the extent to which the response spectra for the systems considered may be predicted from the effective motions corresponding to the limiting cases referred to above. This is shown in Fig. 3.7 for the special
case of systems with \( \gamma = 0.4 \) subjected to the half-cycle velocity input. The diagrams on the left and right indicate the effective ground motions at the respective limits, and the dashed and dotted curves represent the corresponding spectra. The solid curve represents the spectrum for the actual input. It may be observed that the spectrum corresponding to the left-hand input is in reasonable agreement with the exact spectrum for a fairly wide range of the low frequency region and the same is also true for the corresponding spectra in the high frequency region.

3.1.2 Comparison of the Effect of Coulomb and Viscous Damping on the Response of Linear Systems. The deformation spectra for viscously damped linear systems subjected to the pulses depicted in Fig. 2.3 are given in Figs. 3.8 - 3.10. On comparing these spectra with the corresponding spectra for systems with Coulomb damping shown in Figs. 3.1 - 3.3, it may be observed that the relative effects of viscous and Coulomb damping in reducing the magnitude of the maximum deformation are quite different in the various regions of the spectrum. In particular, in the low and high frequency ranges of the spectrum for which the effect of viscous damping may be considered to be negligible, the effect of Coulomb damping is extremely important. These results show clearly that even though it is possible to express the effect of Coulomb damping in terms of a fixed amount of "equivalent viscous damping" for a given frequency in the medium frequency range, this equivalent damping cannot possibly hold but for a very limited range of frequencies.

3.1.3 Deformation Spectra of Systems Subjected to Earthquake Motion. The deformation spectra of linear systems with Coulomb damping subjected to
the El Centro Earthquake input are given in Figs. 3.11 and 3.12, respectively.

The most striking feature of the earthquake spectra presented is their similarity to the spectra for simple pulses, in particular, the half-cycle displacement pulse. This similarity arises from the fact that the dominant pulse in the displacement trace of the input motion is almost the same as a half-cycle displacement pulse.

The one major difference between the results presented in this section and those given earlier concerns the reduction obtained for small values of damping, both Coulomb and viscous. Whereas the reduction achieved for simple pulses is very small, for the earthquake motion and particularly for systems with Coulomb damping, this reduction is quite significant, especially in the low frequency region of the spectrum. This difference arises from the fact that the response of systems with Coulomb damping depends to a large extent on the detailed shape of the acceleration trace of the input motion which is radically different for the El Centro Earthquake input from that of the half-cycle displacement pulse.

3.2 Applicability of Results to Bilinear Hysteretic Systems

Consider the resistance-deformation relationship of the bilinear hysteretic system shown in Fig. 3.13a. In this discussion, the yield levels in the two directions of deformation are considered to be equal, and unloading from a point of maximum deformation is assumed to take place along a line parallel to the initial portion of the curve. The yield point deformation is denoted by $u_y$ and the yield point resistance by $Q_y$. If the slope of the first portion of the diagram, $k_1$, becomes very large, a system with the resistance diagram shown
in Fig. 3.13b is obtained. This diagram also represents the resistance-deformation relationship of a linear system with Coulomb damping. Consequently, the results presented for systems with Coulomb damping are directly applicable to the limiting cases of bilinear hysteretic systems, the yield point resistance \( Q_y \) corresponding to the Coulomb force \( F \) and the slope of the second portion of the resistance diagram of the bilinear spring, \( k_2 \), corresponding to the stiffness of the linear system, i.e., \( F = Q_y \) and \( f = \frac{1}{2\pi} \sqrt{k_2/m} \), respectively.

The results presented in section 3.1 are exact only for the cases of bilinear hysteretic systems with a vertical first slope, but it would seem reasonable to expect that they remain valid for systems with large values of the first slope.

It is necessary to establish a relationship between the friction factor \( \gamma \), used to measure the friction force \( F \), and the yield point resistance, \( Q_y \), of bilinear systems. The force \( F \) is given by

\[
F = \gamma m \ddot{y}_0
\]  

(3.13)

It is convenient to express the yield point resistance, \( Q_y \), as a fraction of the absolute value of the maximum force, \( Q_o \), of a linear system having the same stiffness as the initial stiffness of the bilinear system; the yield factor, \( c \), is defined as

\[
c = \frac{Q_y}{Q_o}
\]  

(3.14)

For high frequency linear systems we have

\[
Q_o = m \ddot{y}_o
\]  

(3.15)

Substituting this into Eq. (3.14) we obtain

\[
Q_y = cm \ddot{y}_o
\]  

(3.16)
Setting \( Q_y = F \) gives

\[ \gamma = c \]  

(3.17)

that is, for the purposes of this discussion, the parameters \( \gamma \) and \( c \) may be used interchangeably. The spectra shown in Figs. 3.1, 3.2, 3.3 and 3.11 may now be looked upon as the deformation spectra for high frequency bilinear hysteretic systems, where \( f \) is the frequency corresponding to the second slope of the resistance diagram and \( \gamma \) represents the yield factor.

The maximum deformation \( U/y_0 \) as a function of the yield factor, \( c = \gamma \), is given in Fig. 3.6 for rigid plastic systems (\( f \to 0 \)). It is most interesting to observe that \( U/y_0 \) does not necessarily increase as the yield factor is reduced. Moreover, for the full-cycle displacement pulse, \( U/y_0 \) attains a value of 1.8 when \( c = \gamma = 0.15 \), whereas when the yield level goes down to zero, the maximum deformation becomes equal to the maximum ground displacement.

For a given value of the yield factor, \( c \), one might expect that a small increase from zero in the slope of the second portion of the resistance diagram would produce a significant reduction in the maximum deformation. Contrary to this expectation, the spectra given in Figs. 3.1, 3.2, 3.3 and 3.11 show that the maximum deformation \( U/y_0 \) remains almost constant if the second slope is increased considerably. An increase in the value of \( U/y_0 \) may occur if this slope is increased further, which means that the maximum deformation of a high frequency bilinear system is not necessarily bounded by the corresponding values obtained for linear and elastoplastic systems. (It may be noted that a high frequency elastoplastic system experiences a maximum deformation greater than that of the corresponding linear system.) After a peak in the pseudo-velo-
city, $V/\dot{y}_o$, is attained, the maximum deformation, $U/y_o$, decreases rapidly as the second slope increases.
IV. RESPONSE OF ELASTOPLASTIC SYSTEMS

4.1 General

This chapter is devoted to a discussion of the response of elastoplastic systems having unequal yield levels in the two directions of deformation. Primary attention is given to systems for which the two yield levels are such that yielding occurs only in one direction of deformation, the response in the other direction being linearly elastic. Such systems will be designated as "elastoplastic-elastic" systems. Their resistance-deformation diagram is depicted in Fig. 4.1a. Two different cases must be distinguished:

a.) The initial deformation is along the linear or elastic branch of the resistance diagram, and

b.) The initial deformation is along the bilinear or elastoplastic branch.

Unloading from a region of inelastic deformation is assumed to take place along a line parallel to the initial linear path.

The yield level of the system will be expressed in terms of the dimensionless yield factor \( c \), defined as

\[
c = \frac{Q_y}{Q_o} = \frac{u_y}{u_o}
\]  

(4.1)

In these expressions \( Q_y \) and \( u_y \) represent the yield force and the yield deformation, as indicated in Fig. 4.1a, and \( Q_o \) and \( u_o \) represent the maximum force and maximum deformation experienced by a linear system having the same stiffness as the initial stiffness of the inelastic system. For a linear system, \( Q_y \) and \( u_y \) may be considered to be equal to \( Q_o \) and \( u_o \), respectively. Accordingly, the yield factor is equal to unity in this case.
The symbols referred to above will also be used to specify the yield resistance of elastoplastic systems with equal yield levels in the two directions of deformation, and such systems will be referred to briefly as "elastoplastic." For systems with unequal yield levels, the quantities \( Q_y \) and \( u_y \) will refer to the smaller yield force and the smaller yield deformation, respectively, and primed symbols will be used to define the quantities corresponding to the higher yield level.

The absolute maximum deformation of a system, without regards to sign, will be denoted by \( U \).

4.2 Elastoplastic-Elastic Systems

In the following sections, the maximum deformation of elastoplasticelastic systems is evaluated for a range of structural parameters and input motions, and the results are compared with the corresponding deformation of both elastoplastic and linear systems.

4.2.1 Half-Cycle Velocity Input Figure 4.2 presents spectra for undamped systems subjected to the half-cycle velocity input shown in Fig. 2.3a. The yield factor of the inelastic systems is considered to be \( c = 0.2 \). The quantity \( f \) on the abscissa represents the natural frequency of the system, in cycles per unit of time, corresponding to the initial elastic portion of its resistance diagram. The important features of these curves in the various frequency regions of the spectrum are as follows.

Low Frequency Region For both elastic and elastoplastic systems, it is known that at the low-frequency limit of the spectrum the maximum deformation \( U \) is equal to the maximum ground displacement \( y_0 \). It is important to
note that this trend is also true for elastoplastic-elastic systems if the initial
deformation is along the inelastic or bilinear branch of the resistance diagram,
but it is not true if the initial deformation is along the linear branch. In the
latter case, the maximum deformation may be appreciably greater than the
maximum ground displacement.

To explain this behavior, we investigate the effect of an instantaneous dis-
placement change of the base. The direction of the input motion is considered
to be such that the initial deformation is along the linear branch of the resistance
diagram, as shown by the segment o-a in Fig. 4.1b. The value of this initial
deformation will be \(-y_o\). The strain energy stored in the system at that instant is

\[
E_1 = \frac{1}{2} k_1 y_o^2 \tag{4.2}
\]

The system then deforms along the path a-o-b-c, and eventually oscillates along
the path c-d-e with \(cd = de\). The maximum deformation \(U\) corresponds to
point c, and this value may be determined by equating \(E_1\) to the area of the
trapezoid obcf. Noting that

\[
E_2 = \frac{1}{2} k_1 u_y^2 + k_1 u_y (U - u_y) \tag{4.3}
\]

and recalling that

\[
u_y = cu_o = cy_o \tag{4.4}
\]

we obtain

\[
\frac{U}{y_o} = \frac{1 + c^2}{2c} \tag{4.5}
\]

It can be seen that for values of \(c < 1\), \(U/y_o > 1\).

**Medium Frequency Region** The most important characteristic of
the spectra for the elastoplastic-elastic systems presented in Fig. 4.2 is that,
irrespective of the direction of initial deformation, the maximum deformation of these systems is always greater than or equal to that of the associated linear systems. In contrast, for systems with equal yield levels in the two directions of deformation the maximum deformation in the medium frequency region may be smaller than for the linear systems. The response of elastoplastic systems in this region has been studied in some detail in Ref. 19.

**High Frequency Region** At the high frequency limit, the maximum deformation of elastoplastic systems may be determined directly from the data presented in Chapter III for elastic systems of extremely low frequency. As already indicated, it is only necessary to interpret the frictional damping factor of the elastic system, $\gamma$, as the coefficient $c$ in the expression for the yield of the elastoplastic system

$$Q_y = c Q_o = c m\ddot{y}_o$$

(4.6)

In particular, the relationship between $U/y_o$ and $c$ for the half-cycle velocity input is the same as that presented in Fig. 3.6a.

For elastoplastic-elastic systems, the deformation history of a system at the high frequency limit may be evaluated by double integration of the "effective input acceleration" in a manner entirely analogous to that used for low-frequency elastic systems with Coulomb damping. It is only necessary to consider that in this case the structure can deform in one direction only. The details of computation are illustrated in part (a) of Fig. 4.3 for the half-cycle velocity input and in part (b) for the half-cycle displacement input. The initial deformation of the system is considered to be along the inelastic branch of the resistance diagram. The solid line in each of the upper diagrams denotes the input acceleration,
and the dashed line the acceleration corresponding to the resisting force Q.

The effective acceleration is represented by the ordinates of the shaded portions of the diagrams. The resisting acceleration reverses direction at the instant that the velocity of the system, represented by the area of the effective acceleration diagram, is equal to zero. In the regions where the resisting acceleration is numerically greater than the input acceleration, no motion results unless the system is already in motion. The deformation histories are given by the lower diagrams. From the computation of these diagrams it should be apparent that the maximum deformation of inelastic systems in the high-frequency limit is a function of the detailed history of the input acceleration. This result should be contrasted with the fact that for the associated linear systems the maximum deformation depends only on the peak value of acceleration if the acceleration diagram is a continuous function.

Referring now back to Fig. 4.2, it should be noted that the limiting deformations referred to above approximate with reasonable accuracy the response for a fairly wide range of the frequency parameter in the right-hand region of the spectrum. These data and those presented in the following sections indicate that these limiting values may be used for the entire range of the spectrum for which the pseudo-acceleration A of the associated linear system may be considered to be equal to the maximum ground acceleration.

4.2.2 Half-Cycle Displacement Input The spectra for this input are given in Fig. 4.4. As before, four different cases are considered, and the yield factor of the inelastic systems is taken as \( c = 0.2 \).

With one major difference, the overall relationship between the various
curves in this figure is similar to that for the half-cycle velocity input considered in Fig. 4.2. In particular, for systems yielding only in one direction the maximum deformation is always greater than for the associated linear systems, whereas for elastoplastic systems, it is smaller in the medium frequency region of the spectrum. The difference occurs in the low frequency region. For the half-cycle velocity input, the value \( U \) of elastoplastic-elastic systems is greater than \( y_0 \) when the initial deformation is along the elastic branch of the resistance diagram. For the half-cycle displacement input, on the other hand, \( U \) is greater than \( y_0 \) when the initial deformation is along the inelastic branch of the diagram.

This behavior may be understood by investigating the response of an extremely flexible system (\( f \to 0 \)), with reference to the resistance diagram shown in Fig. 4.1c. During the period of application of the pulse, the system deforms along the path \( o-a-b-c \), and experiences a maximum deformation \( y_0 \). Since the spring force at the end of the pulse is different from zero, the system will continue to deform after termination of the pulse. It will either oscillate along the path \( c-d-c' \) with \( c'd = cd \), or deform beyond point \( b \) to a point such as \( e \). The absolute maximum deformation \( U \) corresponding to point \( e \) may be determined by equating the strain energy of the system at the end of the pulse, \( E_1 \), to the strain energy \( E_2 \) associated with the deformation \( U \). These quantities, represented by the areas of the triangle \( odc \) and the trapezoid \( dbef \), respectively, are given by the expressions

\[
E_1 = \frac{1}{2} k_1 (y_0 - u_y)^2 \tag{4.7}
\]

and

\[
E_2 = \frac{1}{2} k_y y^2 + k_1 u_y (U - y_0) \tag{4.8}
\]
Setting $E_1 = E_2$ and making use of Eq. (4.4), one obtains

$$\frac{U}{y_0} = \frac{1}{2c}$$

This equation should be used only for values of $c < 0.5$. For $c \geq 0.5$, the maximum deformation occurs during the pulse and $U/y_0 = 1$.

As a further illustration of the behavior of these systems, in Fig. 4.5 are shown histories of the resistance-deformation paths followed by systems with $f_{t_1} = 0.3$. The numerals in these diagrams indicate the order of the various relative maxima, and the numbers in parentheses show the associated times, expressed as fractions of the total duration of the pulse, $t_d$. The top diagrams refer to elastic and elastoplastic systems, and the lower ones to elastoplastic-elastic systems. The yield factor of the inelastic systems is $c = 0.2$.

4.2.3 El Centro Earthquake Input In their low-frequency and medium-frequency regions, these spectra are similar to those presented in Fig. 4.4 for the half-cycle displacement input. This similarity should not be surprising, since the response of the systems in these regions is governed by the gross characteristics of the displacement and velocity trace, and the dominant pulses in these traces are similar to those for the half-cycle displacement input. In the high-frequency region, on the other hand, the response of inelastic systems depends on the detailed history of the ground acceleration. Since the acceleration diagrams in the two cases are different, the results are not directly comparable in this region.

4.2.4 Additional Spectra The spectra for inelastic systems presented so far in this chapter were limited to a value of $c = 0.2$. In Figs. 4.7 through 4.12
are presented spectra for other values of $c$. The first three are for elastoplastic-elastic systems and the last three for elastoplastic systems. Inputs considered are those shown in Figs. 2.3a, 2.3b and 2.4. The general trends of these spectra are similar to those discussed in the preceding sections.

4.2.5 Spectra for Maximum Spring Force The maximum spring deformation has been the response quantity depicted in the previous figures. It is also of interest to evaluate the maximum spring force, $Q^m$. For a linear system, $Q^m = Q_o = ku_o$, and for an elastoplastic system with equal yield levels, it is equal to the yield force $Q_y$. For an elastoplastic-elastic system, $Q^m$ cannot be determined from the data that have been presented, and additional information is required. Let $Q_{m,e}$ denote the maximum force experienced by the system in the direction of linear behavior. The absolute maximum force $Q_m$ would then be the larger of the values $Q_{m,e}$ and $Q_y$. In Figs. 4.13 and 4.14 the values of $Q_{m,e}$, normalized with respect to $Q_o$, are plotted for the half-cycle velocity and the half-cycle displacement inputs considering each of the two possible directions of initial deformation and fixed values of the yield factor $c$.

Consider now an elastoplastic system with unequal yield levels, and let $c$ denote the yield factor associated with the yield resistance in one direction of deformation. If the yield force of the system in the other direction is less than the value of $Q_{m,e}$ determined from Figs. 4.13 or 4.14 corresponding to the particular input under consideration, then the system will yield in one direction only, and the resulting maximum deformation may be determined from the spectra for elastoplastic-elastic systems that have been presented.
4.3 **Systems Yielding in Both Directions**

In order to provide an indication of the response of such systems, some data are presented in Fig. 4.19 for the case of systems subjected to a half-cycle velocity input. The yield factor corresponding to the lower yield level is taken as \( c = 0.3 \), and a range of values of \( c' = \frac{Q_y}{Q_o} \) is considered for three values of the frequency parameter. Two possibilities are investigated for each combination of the parameters.

a.) The initial deformation in the direction of the higher yield level, and

b.) Initial deformation in the direction of the lower yield level.

The central and limiting values in each of these diagrams were reported previously. The central value, for which \( c = c' \), corresponds to an elastoplastic system with equal yield levels, and the limiting values correspond to elastoplastic-elastic systems. It is of interest to note that for some ranges of the parameters the ordinates of these curves are smaller than those corresponding to systems with equal yield levels in the two directions of deformation.
5.1 **General**

This chapter is concerned with the response of bilinear elastic and bilinear hysteretic systems with equal "yield" levels in the two directions of deformation. Primary attention is given to systems with a horizontal second slope, and to systems with a stiffness ratio \( k_2/k_1 = 0.1 \). Where a distinction must be made between the maximum deformations of a bilinear elastic system and of the corresponding hysteretic system, these deformations will be designated by \( U_e \) and \( U_h \), respectively.

5.2 **Presentation and Discussion of Results**

5.2.1 **Bilinear Systems with a Horizontal Second Slope.** Figure 5.1 presents spectra for undamped systems subjected to the half-cycle velocity input. The yield factor and the stiffness ratio of the bilinear systems are considered to be \( c = 0.2 \) and \( k_2/k_1 = 0 \), respectively. Corresponding spectra for the half-cycle displacement input and the El Centro earthquake records are given in Figs. 5.2 and 5.3. In the latter case, the systems are assumed to have two percent critical viscous damping. This amount of damping has been introduced to reduce the irregularities that spectra for undamped systems would exhibit for this input motion.

It can be seen that the maximum deformation experienced by a bilinear elastic system with a horizontal second slope is, in general, greater than that of the corresponding linear system for all values of the frequency parameter, \( f_1 \). In contrast, for a bilinear hysteretic system, i.e., an elastoplastic system, the maximum deformation is smaller than that for the corresponding linear
system in the medium frequency region of the spectrum. In the low frequency region, the maximum deformation is equal to the maximum ground displacement for all the systems under consideration. At the high frequency limit, the maximum deformation of bilinear systems with zero second slope may be determined by successive integration of the effective ground acceleration, $\ddot{y}_e(t)$, defined by

$$\ddot{y}_e(t) = \ddot{y}(t) \pm \frac{Q_y}{m}$$

(5.1)

where the $+$ sign holds for positive values of $\dot{u}$ or $u$ depending on whether the system has hysteric or elastic behavior, respectively. Notice that in the high frequency region, the maximum deformation of a bilinear elastic system is greater than that of the corresponding hysteric system.

Spectra for the ratio $U_e/U_h$ of the maximum deformations for bilinear elastic and hysteretic systems with a horizontal second slope are given in Figs. 5.4 - 5.6 for each of the pulses. In addition to results for the yield factor $c = 0.5$, data are presented for $c = 0.1$, 0.3 and 0.7. If $c = 1$ no "yielding" occurs and both the bilinear elastic and hysteretic systems behave linearly, thus the ratio $U_e/U_h$ is equal to one for all values of the frequency parameter. If $c = 0$ there is effectively no spring between the mass and the base, therefore $U_e = U_h = y_o$ and $U_e/U_h = 1$. Notice that for other values of $c$, the ratio $U_e/U_h$ is in general, greater than one attaining the largest values in the high frequency region of the spectrum. Indicated in the diagrams are the points to the right of which the ductility factor, $\mu$, corresponding to the bilinear elastic system, becomes greater than 10. The ductility factor is defined as
\[ \mu = \frac{U}{u_y} \]  

(5.2)

which is a measure of the maximum deformation of a system in terms of the "yield" point deformation.

Figures 5.7 - 5.9 show the effect of progressively reducing the yield level of bilinear systems. The ductility factor, \( \mu \), is plotted on a log-log scale as a function of the "yield" factor, \( c \), for fixed values of the frequency parameter and for the three pulses considered. The diagonal lines represent the equation

\[ \mu = \frac{1}{c} \]  

(5.3)

or alternatively

\[ U = u_o \]  

(5.4)

as may be obtained by substituting Eqs. (4.1) and (5.2) into Eq. (5.3). Points below the diagonal lines correspond to bilinear systems for which the maximum deformation is smaller than that of the associated linear systems, that is

\[ U < u_o \]  

for points below the line \( \mu = 1/c \)

Similarly

\[ U > u_o \]  

for points above the line \( \mu = 1/c \)

Effectively, Figs. 5.7 - 5.9 represent cross-sections of the plots shown in Figs. 5.1 - 5.3 for selected values of the frequency parameter, the maximum deformation being expressed by means of the ductility factor, \( \mu \). It is noted that for small values of the frequency parameter, the curves corresponding to the bilinear and linear systems are almost the same irrespective of the yield level involved. The curves \( \mu - c \) are very sensitive to variations in the value of \( f_{i1} \), and as \( f_{i1} \) becomes large, these curves approach a horizontal line, i.e.,
slight reductions of the yield factor result into very large deformations. These figures emphasize the fact that, in general, the maximum deformation of bilinear elastic systems with a horizontal second slope is greater than that of the corresponding hysteretic systems.

5.2.2 Bilinear Systems with a Non-zero Second Slope. Figure 5.10 presents deformation spectra for undamped systems subjected to the half-cycle displacement input. For the bilinear systems, two values of the stiffness ratio, \( k_2/k_1 = 0 \) and 0.1, and one value of the yield factor, \( c = 0.3 \) are considered.

In the low and medium frequency regions of the spectrum, the maximum deformation of systems with a stiffness ratio \( k_2/k_1 = 0.1 \) is very similar to that obtained for \( k_2/k_1 = 0 \), but as the frequency parameter becomes large, a drastic reduction occurs in the maximum deformation of the bilinear systems with \( k_2/k_1 = 0.1 \). It is also observed that, in contrast to systems with \( k_2/k_1 = 0 \), the maximum deformation of a high frequency bilinear elastic system with \( k_2/k_1 = 0.1 \) is very close to that of the corresponding hysteretic system. At the limit, the value of \( U/y_0 \) for a bilinear hysteretic system, which may be interpreted as an elastic system with Coulomb damping, is given in Fig. 3.2. In this figure \( f \) represents the frequency corresponding to the slope of the second portion of the bilinear system.

Spectra for the ratio \( U_e/U_h \) of the maximum deformations for the elastic and hysteretic bilinear systems with a stiffness ratio \( k_2/k_1 \) of 0.1 are given in Fig. 5.11 for the half-cycle displacement input. Results are given for values of the yield factor, \( c = 0.1, 0.3, 0.5 \) and 0.7. In the low and medium frequency regions of the spectrum, the ratio \( U_e/U_h \) for systems with a stiffness ratio \( k_2/k_1 = 0.1 \) is, in general, similar to that obtained for systems with a hori-
zontal second slope, i.e., close to one in the low frequency region and somewhat larger in the medium frequency region of the spectrum. It is in the high frequency region where the ratio $U_e/U_h$ is greatly reduced due to the increase of the stiffness ratio $k_2/k_1$. Moreover, whereas for systems with a horizontal second slope the ratio $U_e/U_h$ is in general, greater or equal than one, for systems with a stiffness ratio $k_2/k_1$ of 0.1 this ratio may become smaller than one.

Further study of the effect of the stiffness ratio $k_2/k_1$ on the ratio $U_e/U_h$ for systems in the high frequency region of the spectrum should prove valuable.
REFERENCES


13. Iwan, W. D., "The Dynamic Response of the One Degree of Freedom Bilinear Hysteretic System," Third World Conference on Earthquake Engineering,
New Zealand, (January 1965)


APPENDIX

EXPRESSIONS FOR RESPONSE OF SYSTEM CONSIDERED

The solution of Eq. (2.10) with initial conditions given by Eqs. (2.6) is given by the following expressions depending on the instantaneous values of $p$ and $\beta$.

1. For $p \neq 0$

   a) If $p \neq \beta p_1$

\[
\begin{align*}
\bar{u} &= \left[ \frac{\beta_1}{2} \frac{2\beta b_j}{p^3} \right] (1 - e^{-\beta p_1 \bar{t}} \cos p_d \bar{t}) + \left[ \frac{\beta p_1 b_j}{p} - \frac{b_j}{p} \right] \\
&\quad \left( 1 - 2\beta^2 \right) + \bar{u}_1 \left[ \frac{-\beta p_1 \bar{t}}{p} \right] \frac{\sin p_d \bar{t}}{p_d} - \frac{b_j}{p^2} \bar{t}
\end{align*}
\]

(A.1)

where

\[
p_d = \sqrt{p^2 - (\beta p_1)^2}
\]

The symbols $\sin$ and $\cos$ are taken as the trigonometric or hyperbolic sine or cosine depending on whether $p$ is greater or smaller than $\beta p_1$, respectively.

\[
\begin{align*}
\dot{u} &= e^{-\beta p_1 \bar{t}} \left( \frac{\beta_1}{p} + \bar{u}_1 \right) e^{-\beta p_1 \bar{t}} \cos p_d \bar{t} + \left( -R_1 + \frac{\beta p_1 b_j}{p} - \beta p_1 \bar{u}_1 \right) \\
&\quad \left( 1 - 2\beta^2 \right) + \bar{u}_1 \left[ \frac{-\beta p_1 \bar{t}}{p} \right] \frac{\sin p_d \bar{t}}{p_d} - \frac{b_j}{p^2} \bar{t}
\end{align*}
\]

(A.2)

\[
\begin{align*}
\ddot{u} &= e^{-\beta p_1 \bar{t}} \left[ -(R_1 + 2\beta p_1 \bar{u}_1) \cos p_d \bar{t} + \left( R_1 \beta p - b_j - (1 - 2\beta^2) \right) \right] \\
&\quad \bar{u}_1 \left[ \frac{-\beta p_1 \bar{t}}{p} \right] \frac{\sin p_d \bar{t}}{p_d} - \frac{b_j}{p^2} \bar{t}
\end{align*}
\]

(A.3)

b) If $p = \beta p_1$

\[
\begin{align*}
\bar{u} &= \left[ \frac{\beta_1}{2} \frac{2\beta b_j}{p^3} \right] (e^{-\beta p_1 \bar{t}} - 1) + \left[ \frac{\beta p_1 b_j}{p} + \frac{b_j}{p} \right] (1 - 2\beta^2) + \bar{u}_1
\end{align*}
\]
\[ e^{-\beta p_1 t} \frac{-b_j}{p^2} - \frac{b_j}{p} \]  
(A.4)

\[ \dot{u} = e^{-\beta p_1 t} \frac{-b_j}{p} \left\{ \frac{1}{p^2} + \dot{u}_1 + \left( -R_1 + \frac{\beta p_1 b_j}{p} - \beta p_1 \dot{u}_1 \right) \tau \right\} - \frac{b_j}{p} \]  
(A.5)

\[ \ddot{u} = e^{-\beta p_1 t} \left\{ \left[ -R_1 - 2\beta p_1 \dot{u}_1 + \left( R_1 - b_j - (1 - 2\beta^2) p^2 \ddot{u}_1 \right) \tau \right] \right\} \]  
(A.6)

2. For \( p = 0 \)

a) If \( \beta \neq 0 \)

\[ u = \left\{ -\frac{\dot{u}_1}{2\beta p_1} - \frac{R_1}{(2\beta p_1)^2} + \frac{b_j}{(2\beta p_1)^3} \right\} \left( e^{-2\beta p_1 \tau} - 1 \right) - \frac{b_j}{4\beta p_1} \tau^2 + \]  
\[ \frac{-b_j}{(2\beta p_1)^2} \left( \frac{R_1}{2\beta p_1} \right) \tau \]  
(A.7)

\[ \dot{u} = (\dot{u}_1 + \frac{R_1}{2\beta p_1} - \frac{b_j}{(2\beta p_1)^2}) e^{-2\beta p_1 \tau} - \frac{b_j}{2\beta p_1} \tau - \frac{R_1}{2\beta p_1} + \frac{b_j}{(2\beta p_1)^2} \]  
(A.8)

\[ \ddot{u} = (-2\beta p_1 \dot{u}_1 - R_1 + \frac{b_j}{2\beta p_1}) e^{-2\beta p_1 \tau} - \frac{b_j}{2\beta p_1} \]  
(A.9)

b) If \( \beta = 0 \)

\[ \ddot{u} = \dot{u}_1 \tau = -\frac{R_1 \tau^2}{2} - \frac{b_j \tau^3}{6} \]  
(A.10)

\[ \ddot{u} = \dot{u}_1 - R_1 \tau - \frac{b_j \tau^2}{2} \]  
(A.11)

\[ \dddot{u} = -R_1 - b_j \tau \]  
(A.12)

Expression for the time increment \( \tau \).

1. For \( p \neq 0 \)

a) If \( p > \beta p_1 \)
\[ t = \frac{1}{\rho_d} \tan^{-1} \left( \frac{-p_d (R_1 + 2\beta \rho_1 \hat{u}_j)}{-\beta \rho_1 (R_1 + 2\beta \rho_1 \hat{u}_j) + b_j + p^2 \hat{u}_j} \right) \]  

(A. 13)

b) If \( p < \beta \rho_1 \)

\[ t = \frac{1}{2p_d} \left( \frac{(\beta \rho_1 + p_d) (-R_1 - 2\beta \rho_1 \hat{u}_j) + b_j + p^2 \hat{u}_j}{(\beta \rho_1 - p_d) (-R_1 - 2\beta \rho_1 \hat{u}_j) + b_j + p^2 \hat{u}_j} \right) \]  

(A. 14)

c) If \( p = \beta \rho_1 \)

\[ t = \frac{-R_1 - 2\beta \rho_1 \hat{u}_j}{\beta \rho_1 (-R_1 - 2\beta \rho_1 \hat{u}_j) + b_j + p^2 \hat{u}_j} \]  

(A. 15)

2. For \( p = 0 \)

a) If \( \beta \neq 0 \)

\[ t = \frac{1}{2\beta \rho_1} \left( \frac{2\beta \rho_1 (-R_1 - 2\beta \rho_1 \hat{u}_j) + b_j}{b_j} \right) \]  

(A. 16)

b) If \( \beta = 0 \)

\[ t = -\frac{R}{b_j} \]  

(A. 17)
FIG. 2.1 SYSTEM CONSIDERED.
FIG. 2.2 RESISTANCE-DEFORMATION RELATIONSHIP.
FIG. 2. SIMPLIFIED INPUT MOTIONS CONSIDERED.
FIG. 2.4 FIRST PORTION OF N-S COMPONENT OF EL CENTRO, CALIFORNIA EARTHQUAKE RECORD OF MAY 18, 1940
FIG. 3.1  DEFORMATION SPECTRA FOR ELASTIC SYSTEMS WITH COULOMB DAMPING.
Input motion shown in Fig. 2.3a
FIG. 3.2 DEFORMATION SPECTRA FOR ELASTIC SYSTEMS WITH COULOMB DAMPING.
Input motion shown in Fig. 2.3b
FIG. 3.3 DEFORMATION SPECTRA FOR ELASTIC SYSTEMS WITH COULOMB DAMPING.
Input motion shown in Fig. 2.3c
FIG. 3.4 EFFECTIVE GROUND MOTION FOR LOW FREQUENCY ELASTIC SYSTEMS WITH COULOMB DAMPING.
FIG. 3.5 EFFECTIVE GROUND DISPLACEMENT OF LOW FREQUENCY SYSTEMS WITH COULOMB DAMPING.
FIG. 3.6 MAXIMUM DEFORMATION OF LOW FREQUENCY SYSTEMS WITH COULOMB DAMPING.
FIG. 3.7 DEFORMATION SPECTRA FOR ELASTIC SYSTEMS WITH COULOMB DAMPING. Input motion shown in Fig. 2.3a
FIG. 3.8 DEFORMATION SPECTRA FOR ELASTIC SYSTEMS WITH VISCOUS DAMPING.

Input motion shown in Fig. 2.3a.
FIG. 3.9 DEFORMATION SPECTRA FOR ELASTIC SYSTEMS WITH VISCOUS DAMPING.
Input motion shown in Fig. 2.3b
FIG. 3.10 DEFORMATION SPECTRA FOR ELASTIC SYSTEMS WITH VISCOUS DAMPING.
Input motion shown in Fig. 2.3c
FREQUENCY, $f$, cps

DEFORMATION SPECTRA FOR ELASTIC SYSTEMS WITH COULOMB DAMPING.

FIG. 3.11  DEFORMATION SPECTRA FOR ELASTIC SYSTEMS WITH COULOMB DAMPING.
Input motion shown in Fig. 2.4
FIG. 3.12 DEFORMATION SPECTRA FOR ELASTIC SYSTEMS WITH VISCOUS DAMPING. 
Input motion shown in Fig. 2.4.
FIG. 3.13 RESISTANCE-DEFORMATION RELATIONSHIP OF BILINEAR Hysteric SYSTEMS.
FIG. 4.1 RESISTANCE-DEFORMATION RELATIONSHIP OF AN ELASTOPLASTIC-ELASTIC SYSTEM.
FIG. 4.2 DEFORMATION SPECTRA FOR SYSTEMS WITHOUT DAMPING. Input motion shown in Fig. 2. 3a
FIG. 4.3 EFFECTIVE GROUND MOTION OF LOW FREQUENCY ELASTOPLASTIC-ELASTIC SYSTEMS.

a) Half-Cycle Velocity

b) Half-Cycle Displacement
FIG. 4.4 DEFORMATION SPECTRA FOR SYSTEMS WITHOUT DAMPING. Input motion shown in Fig. 2.3b

- Initial Deformations Along
- a) Elastic Branch
- b) Elastoplastic Branch

Half-Cycle Displacement Input

4

U /
2

0.2

0.5

0.02

0.05

0.1

0.2

0.5

2

5

10
Fig. 4.5 Paths on Resistance-Deformation Diagram for Systems Without Damping. Half-Cycle displacement input shown in Fig. 2.3b.
FIG. 4.6 DEFORMATION SPECTRA FOR SYSTEMS WITH TWO PERCENT VISCOUS DAMPING. Input motion shown in Fig. 2.4.
FIG. 4.7 DEFORMATION SPECTRA FOR SYSTEMS WITHOUT DAMPING. Input motion shown in Fig. 2.3a
FIG. 4.8 DEFORMATION SPECTRA FOR SYSTEMS WITHOUT DAMPING. Input motion shown in Fig. 2. 3b
FIG. 4.9 DEFORMATION SPECTRA FOR SYSTEMS WITH TWO PERCENT VISCOUS DAMPING.
Input motion shown in Fig. 2.4
FIG. 4.10 DEFORMATION SPECTRA FOR SYSTEMS WITHOUT DAMPING. Input motion shown in Fig. 2.3a
FIG. 4.11 DEFORMATION SPECTRA FOR SYSTEMS WITHOUT DAMPING. Input motion shown in Fig. 2.3b
FIG. 4.12 DEFORMATION SPECTRA FOR SYSTEMS WITH TWO PERCENT VISCOUS DAMPING.
Input motion shown in Fig. 2.4
FIG. 4.13 SPECTRA FOR FORCE IN DIRECTION OF LINEAR BEHAVIOR OF ELASTOPLASTIC-ELASTIC SYSTEMS. Input motion shown in Fig. 2.3a
FIG. 4.14 SPECTRA FOR FORCE IN DIRECTION OF LINEAR BEHAVIOR OF ELASTOPLASTIC-ELASTIC SYSTEMS. Input motion shown in Fig. 2.3b
FIG. 4.15 MAXIMUM DEFORMATION OF SYSTEMS YIELDING IN THE TWO DIRECTIONS OF DEFORMATION. Half-cycle displacement input shown in Fig. 2.3b

Yield Factor $c = 0.3$

Initial deformation in the direction of:
(a) Lower yield level
(b) Higher yield level

Yield Factor $c'$
FIG. 5.1 DEFORMATION SPECTRA FOR BILINEAR SYSTEMS WITHOUT DAMPING.
Input motion shown in Fig. 2.3a
FIG. 5.2 DEFORMATION SPECTRA FOR BILINEAR SYSTEMS WITHOUT DAMPING.

Input motion shown in Fig. 2.3b.
FIG. 5.3 DEFORMATION SPECTRA FOR BILINEAR SYSTEMS WITH TWO PERCENT VISCOUS DAMPING. Input motion shown in Fig. 2.4
FIG. 5.4 SPECTRA FOR RATIO OF MAXIMUM DEFORMATIONS OF BILINEAR ELASTIC AND HYSTERETIC SYSTEMS. Input motion shown in Fig. 2.3a.
FIG. 5.5 SPECTRA FOR RATIO OF MAXIMUM DEFORMATIONS OF BILINEAR ELASTIC AND HYSTERETIC SYSTEMS. Input motion shown in Fig. 2.3b.
FIG. 5.6 SPECTRA FOR RATIO OF MAXIMUM DEFORMATIONS OF BILINEAR ELASTIC AND HYSERETIC SYSTEMS. Input motion shown in Fig. 2.4
FIG. 5.7 RELATION BETWEEN DUCTILITY FACTOR AND YIELD FACTOR.
Input motion shown in Fig. 2.3a
FIG. 5.8 RELATION BETWEEN DUCTILITY FACTOR AND YIELD FACTOR.
Input motion shown in Fig. 2.3b
FIG. 5.9 RELATION BETWEEN DUCTILITY FACTOR AND YIELD FACTOR.
Input motion shown in Fig. 2.4
FIG. 5.10 DEFORMATION SPECTRA FOR BILINEAR SYSTEMS WITHOUT DAMPING.

Input motion shown in Fig. 2.3b.
FIG. 5.11 SPECTRA FOR RATIO OF MAXIMUM DEFORMATIONS OF BILINEAR ELASTIC AND HYSTERETIC SYSTEMS. Input motion shown in Fig. 2.3b