THE RICE INSTITUTE

SOME THEOREMS ON FEEDBACK

by

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CONTENTS

I. Introduction

Feedback theory is the study of interactions in a system, and it is convenient to characterize the system in terms of flow graphs and the corresponding equations. The basic problem in feedback theory is one of determining the effect of certain branch transmission factors on various system transmission factors. It may be desired to determine the influence on system stability of a transmission factor or the sensitivity of the overall system response to changes in the value of certain branch transmission factors.

A detailed analysis of stability and sensitivity with respect to a single transmission element in a system is discussed by Pfeiffer[1] in a completely rigorous manner. A logical extension to the case of several transmission factors was presented by Bode[2] and is also discussed by Pfeiffer[1] and Truxal[3]. This extension is basically one of considering the system in several stages. The system is first modified by removing all the branches whose effects are to be determined. The effect of the first branch on the modified system is determined. With this result, the effect of the second branch on the modified system with the first branch replaced is found. This technique is continued for all the branches under consideration. Analytically, such a procedure is feasible, but experimentally such a procedure may be untenable. That is because the system with several branches replaced may be unstable and the addition of the remaining branches may be necessary to insure overall system stability. Another disadvantage to the above method is that it does not allow for the determination of the sensitivity with respect to a system parameter which may appear in several branches of the flow graph used to represent the system.

The study of several questions leads to the method of analysis developed in this paper. The questions of prime interest are as follows:

1. Is it possible to find an order of replacing the branches such that the system will be stable at each stage in the process if the original system and the final system are both stable?

2. Is it always possible to make measurements on a stable system to determine if this system, with several branches added, will be stable?

3. How can the sensitivity with respect to a system parameter which appears in several branches be found?

Question 1 is answered by an example in section III of this paper.
Questions 2 and 3 are discussed at length in section II.
The treatment that follows is algebraic in nature for completeness and rigor, but it appeals to signal flow graphs for visualization and physical interpretations.

II. Mathematical Formulation and Development of the Theorems

The basic mathematical model used to study a discrete parameter, time-invariant, linear system is the set of equations

$$\sum_{k=1}^{n} a_{ik}h_k = \sum_{j=1}^{m} b_{ij}f_j, \qquad 1 \leqq i \leqq n. \tag{1}$$

The coefficients a_{ik} and b_{ij} may be linear integro-differential operators, phasor operators, or transformed operators. The h_k and f_j may be, correspondingly, functions of time, complex amplitudes, or Laplace transforms. For the study at hand, it will be assumed that none of the a_{ii} is identically zero.

Eqs. (1) may be rewritten by solving the ith equation for h_i as follows:

$$h_i = \sum_{k=1}^{n} (\delta_{ik} - a_{ik})h_k + \sum_{j=1}^{m} b_{ij}f_j, \qquad 1 \leqq i \leqq n, \tag{2}$$

or

$$h_i = \sum_{k=1}^{n} t_{ki}h_k + d_i \tag{3}$$

where

$$t_{ki} = \delta_{ik} - a_{ik} \quad \text{and} \quad d_i = \sum_{j=1}^{m} b_{ij}f_j. \tag{4}$$

δ_{ik} is the Kronecker delta defined as

$$\delta_{ik} = \delta_{ki} = \begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases}. \tag{5}$$

A typical flow graph may be drawn to express Eqs. (3) as in Fig. 1. The driving element represents the direct effect of the driving variables on the h_i response in terms of a composite driving variable d_i. The response element represents the effect of the response elements themselves along with d_i on the h_i response.

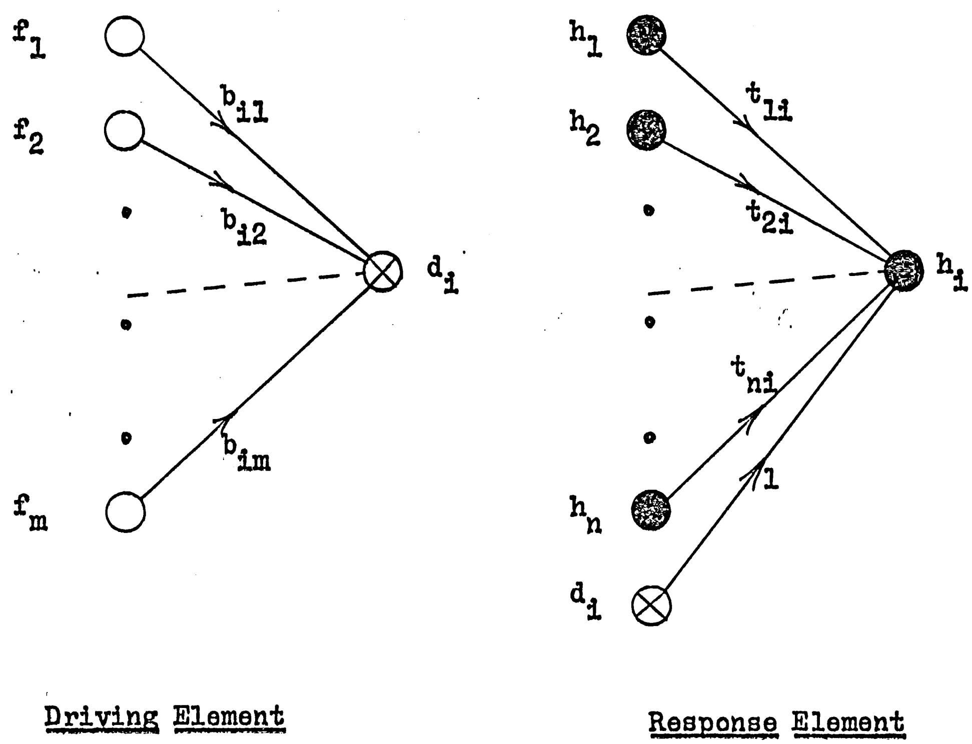

Figure 1. Flow graph elements representing the ith equation in the set Eqs. (3).

It will be convenient to define two cases of special interest with regard to the effect of driving variables on system response.

Case I: For this case, it is assumed that the f_j driving variable is connected directly to the node representing the h_j response. All other driving variables are identically zero. Mathematically this may be stated

$$d_i = f_j \delta_{ij} , \qquad 1 \leqq i \leqq n . \tag{6}$$

In terms of the flow graph, the only h_i having inputs other than the system responses is the h_j node which in addition has the input f_j. For this case, Eqs. (1) may be solved by Cramer's rule to give

$$h_k = f_j \frac{\Delta_{jk}}{\Delta} \tag{7}$$

where Δ is the system determinant for the set of equations described by Eqs. (1) and Δ_{jk} is the cofactor of the jth row and kth column of this determinant. For this case, a system transmission factor T_{jk} will be defined as

$$T_{jk} = \frac{h_k}{f_j} = \frac{\Delta_{jk}}{\Delta} . \tag{8}$$

Case II: For this case, it is assumed that all driving variables except f_j are identically zero. Mathematically this is

$$f_i = \delta_{ij} f_j \quad \text{and hence} \quad d_i = b_{ij} f_j . \tag{9}$$

Again, the response h_k may be solved for by Cramer's rule to be

$$h_k = f_j \frac{\sum_{i=1}^{n} b_{ij} \Delta_{ik}}{\Delta} . \tag{10}$$

For this case, a system transmission factor G_{jk} will be defined as

$$G_{jk} = \frac{h_k}{f_j} = \frac{1}{\Delta} \sum_{i=1}^{n} b_{ij} \Delta_{ik} = \sum_{i=1}^{n} b_{ij} T_{ik} \tag{11}$$

with application of Eqs. (8) and (10).

The problem in feedback theory is to determine the effect of several branches on the overall system response. Assume that the effect of transmission elements from node r_μ to node s_μ where $\mu = 1, 2, \ldots, p$ is of interest. Also assume that

$$t_{r_\mu s_\mu} = t_\mu + t^o_{r_\mu s_\mu} \tag{12}$$

where t_μ is one of the transmission elements whose effect on the overall system response is to be determined. The system with the t_μ's in place will be referred to as the <u>original</u> <u>system</u> and the system with the t_μ's identically zero will be termed the <u>modified</u> <u>system</u>. The original system is characterized by a_{uv}, t_{uv}, Δ, Δ_{uv}, T_{uv}, and G_{uv} and the similar properties for the modified system are written a^o_{uv}, t^o_{uv}, Δ^o_{uv}, T^o_{uv}, and G^o_{uv}.

In terms of a flow graph, a portion of the original system could be shown as in Fig. 2. A portion of the modified system could be depicted as shown in Fig. 3. The flow graph of Fig. 3, which is just a portion of the complete system flow graph, will be quite useful in visualizing the physical significances of the properties to be developed. If the various e_{r_μ} are identically zero, the actual modified system is shown. If $e_{r_\mu} = h_{r_\mu}$ for all μ, then the original system is being shown.

In terms of the notation mentioned above and the relation expressed in Eq. (4)

$$a^o_{ik} = a_{ik} = \delta_{ik} - t_{ki} \tag{13}$$

for both $i \neq s_\mu$ and $k \neq r_\mu$. Also

$$a_{s_\mu r_\mu} = \delta_{s_\mu r_\mu} - t^o_{r_\mu s_\mu} - t_\mu = a^o_{s_\mu r_\mu} - t_\mu \tag{14}$$

for $\mu = 1,2,\ldots,p$.

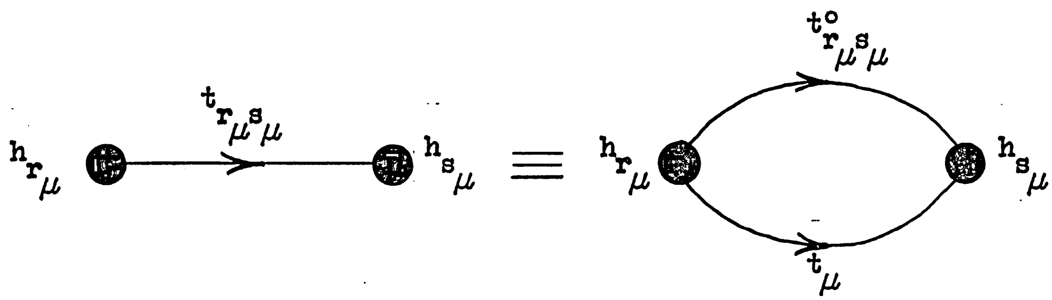

Figure 2. A portion of the original system flow graph.

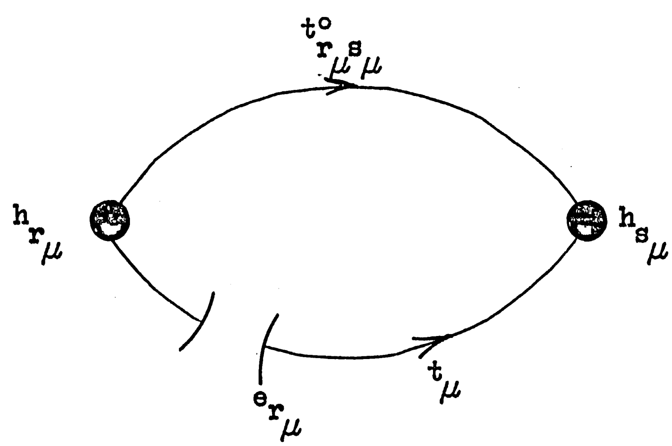

Figure 3. A portion of the modified system flow graph.

The actual problem of interest can now be formulated. It is desired to determine the system transmission factor G_{jk} in terms of the t_{μ}'s and characteristics of the modified system. With this in mind, Eqs. (3) may be written as

$$h_i = \sum_{k=1}^{n} t_{ki} h_k + b_{ij} f_j \,, \qquad i \neq s_{\mu} \,, \tag{15}$$

$$h_{s_{\mu}} = \sum_{\substack{k=1 \\ k \neq r_{\mu}}}^{n} t_{ks_{\mu}} h_k + t^{o}_{r_{\mu} s_{\mu}} h_{r_{\mu}} + t_{\mu} e_{r_{\mu}} + b_{s_{\mu} j} f_j \,. \tag{16}$$

Assume that given the pair $(r_{\mu'} \,, s_{\mu'})$ that there exists no $(r_{\mu} \,, s_{\mu}) = (r_{\mu'} \,, s_{\mu'})$ except for $\mu = \mu'$. This insures p distinct equations of the form given by Eq. (16).

Eq. (15) can then be written as

$$\sum_{k=1}^{n} (\delta_{ik} - t_{ki}) h_k = \sum_{k=1}^{n} a^{o}_{ik} h_k = b_{ij} f_j, \qquad i \neq s_{\mu} \,. \tag{17}$$

With Eq. (14), Eq. (16) may be written as

$$\sum_{\substack{k=1 \\ k \neq r_{\mu}}}^{n} (\delta_{s_{\mu} k} - t_{ks_{\mu}}) h_k + (\delta_{s_{\mu} r_{\mu}} - t^{o}_{r_{\mu} s_{\mu}}) h_{r_{\mu}} = \sum_{k=1}^{n} a^{o}_{s_{\mu} k} h_k = b_{s_{\mu} j} f_j + t_{\mu} e_{r_{\mu}} \,. \tag{18}$$

It should be noted again that $\mu = 1,2,\ldots,p$.

Eqs. (17) and (18) may be solved by Cramer's rule for the h_i response to give

$$h_i = \frac{f_j \sum_{k=1}^{n} b_{kj} \Delta^{o}_{ki} + \sum_{\mu=1}^{p} e_{r_{\mu}} t_{\mu} \Delta^{o}_{s_{\mu} i}}{\Delta^{o}} \,. \tag{19}$$

With application of Eqs.(8) and (11), Eq. (19) may be simplified to

$$h_i = f_j G^{o}_{ji} + \sum_{\mu=1}^{p} e_{r_{\mu}} (t_{\mu} T^{o}_{s_{\mu} i}) \,. \tag{20}$$

When the t_{μ}-branches are replaced, Eq. (20) becomes

$$h_i = f_j G^o_{ji} + \sum_{\mu=1}^{p} h_{r_\mu} (t_\mu T^o_{s_\mu i}) \; . \tag{21}$$

Eqs. (21) may be visualized in terms of a signal flow graph depicting f_j, h_k, and the various h_{r_μ}. To simplify drawing the graph, the graph for the case p = 2 is shown in Fig. 4. The nodes marked ⊗ do not, in general, represent actual system responses. This flow graph may be termed a <u>basic flow graph</u>, for it depicts the effect of the t_μ-transmission factors on the system transmission factor G_{jk}.

A problem commonly considered in a study of this type is that of determining system stability. Usually the various transmission factors for the modified system are themselves stable transfer functions. The problem is then one of determining if the original system will be stable when the t_μ-branches are replaced. From examination of Eqs. (21), it can be noted that the h_k response can be unstable, i.e. grow exponentially with time, only if the various h_{r_μ} themselves become unstable. If the T's do not have zeros in the right half plane, then a test for stability of the h_{r_μ}'s is a test for overall system stability.

Before proceeding with the question of stability, it will be useful to define two tests that may be performed on the modified system.

<u>Test I</u>: For this test, $f_j = 0$ and $e_{r_\mu} = 0$ for all μ except for $\mu = y$. With the system in this condition, Eq. (20) shows that the response at the h_{r_x} node is

$$h_{r_x} = e_{r_y} (t_y T^o_{s_y r_x}) \; . \tag{22}$$

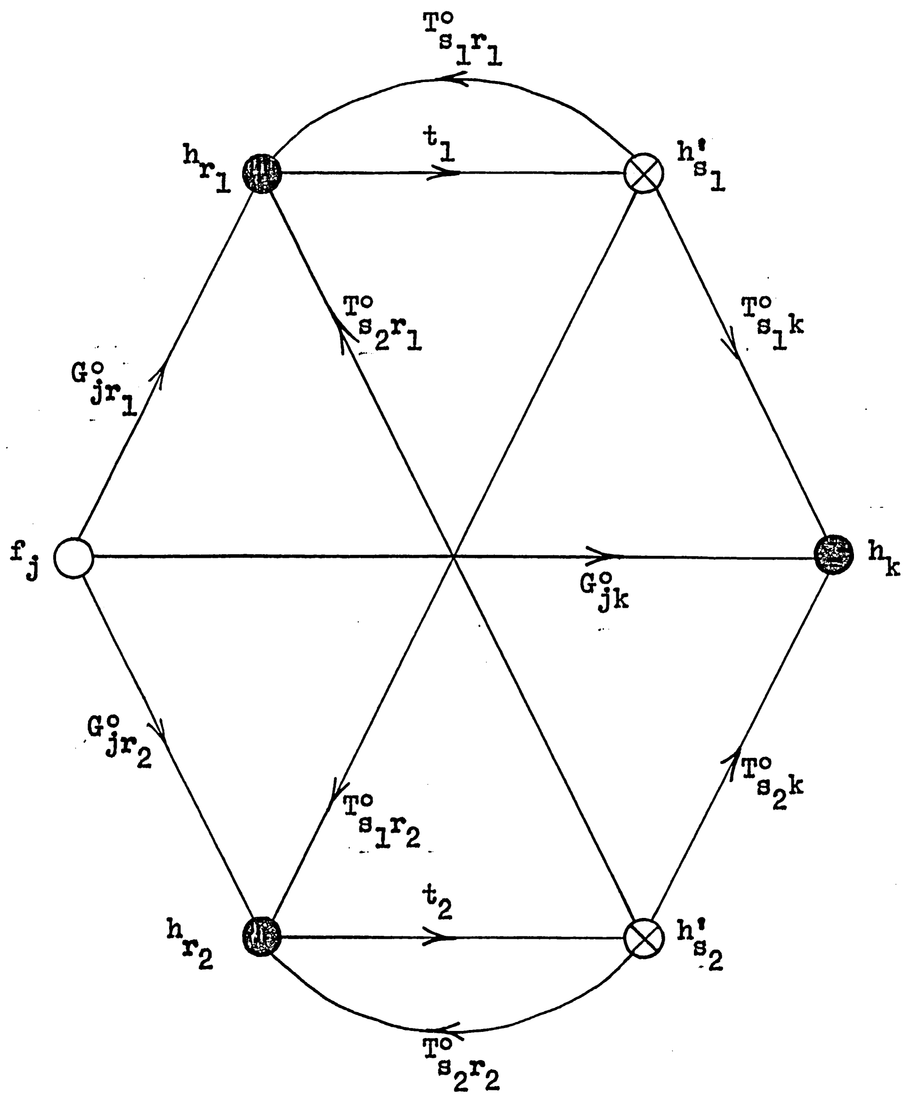

Figure 4. <u>Basic flow graph</u> for the case p = 2.

R_{xy} is defined as the ratio of h_{r_x} to e_{r_y} which from Eq. (22) is

$$R_{xy} = t_y T^o_{s_y r_x} . \tag{23}$$

R_{xy} is called the <u>return ratio from y to x</u>. In many cases this ratio may be determined experimentally in the laboratory and its physical significance is quite important.

<u>Test II</u>: For this test, $e_{r_\mu} = 0$ for all μ except for $\mu = y$. The input f_j is adjusted to an amount sufficient to reduce h_k to zero. Eq. (20) shows the amount of f_j sufficient to reduce h_k to zero to be

$$f_j = - e_{r_y} \frac{t_y T^o_{s_y k}}{G^o_{jk}} . \tag{24}$$

Application of Eqs. (20), (23), and (24) yields the response at h_{r_x} to be

$$h_{r_x} = e_{r_y} R_{xy} - e_{r_y} t_y \frac{T^o_{s_y k} G^o_{j r_x}}{G^o_{jk}} . \tag{25}$$

R^N_{xy} is defined as the ratio of h_{r_x} to e_{r_y} which from Eq. (25) is

$$R^N_{xy} = R_{xy} - t_y \frac{T^o_{s_y k} G^o_{j r_x}}{G^o_{jk}} . \tag{26}$$

R^N_{xy} is called the <u>null return ratio from y to x</u>.

While the algebraic evaluations of the return ratio and the null return ratio have been made from the equations, the visualization of their significance may be seen from the basic flow graph. It can be observed that the same evaluation is obtained when one thinks of opening all the t_μ-branches on the basic flow graph and then performing the tests outlined above.

By application of Eq. (23), the responses at the h_{r_μ}'s can be solved from Eq. (21). The system to be solved can be expressed in the following matrix form:

$$[F][H] = f_j[G] \tag{27}$$

where

$$[F]= \begin{bmatrix} 1 - R_{11} & - R_{12} & \cdots & - R_{1p} \\ - R_{21} & 1 - R_{22} & \cdots & - R_{2p} \\ & \vdots & & \\ - R_{p1} & - R_{p2} & \cdots & 1 - R_{pp} \end{bmatrix}, \tag{28}$$

$$[H]= \begin{bmatrix} h_{r_1} \\ \vdots \\ h_{r_p} \end{bmatrix}, \tag{29}$$

and

$$[G]= \begin{bmatrix} G^o_{jr_1} \\ \vdots \\ G^o_{jr_p} \end{bmatrix}. \tag{30}$$

[F] can be rewritten from Eq. (28) to be

$$[F] = [I] - [R] \tag{31}$$

where

$$[I] = \text{the unit matrix} \tag{32}$$

and

$$[R]= \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1p} \\ & \vdots & & \\ R_{p1} & R_{p2} & \cdots & R_{pp} \end{bmatrix}. \tag{33}$$

From Eq. (27) the response h_{r_μ} can be found by Cramer's rule to be

$$h_{r_\mu} = \frac{f_j \sum_{i=1}^{p} G^o_{jr_i} F_{i\mu}}{F} \tag{34}$$

where F is the determinant of [F] and $F_{i\mu}$ is the cofactor of the element in the ith row and μth column of this determinant. It should be pointed out that a test for system instability can be made in the usual case by testing F for zeros in the right half plane. This, of course, is assuming the various R_{xy} to be functions of the Laplace transformed variable s.

Substitution of Eq. (34) in Eq. (21), where $h_i = h_k$, will allow for the determination of the ratio of h_k to f_j. This ratio, by definition, is G_{jk}. When the substitution is made and the resulting equation is simplified, G_{jk} is found to be

$$G_{jk} = G^o_{jk} + \frac{\sum\limits_{\mu,i=1}^{p} t_\mu T^o_{s_\mu k} G^o_{jr_i} F_{i\mu}}{F} . \tag{35}$$

It will now be convenient to consider the ratio

$$\frac{G_{jk}}{G^o_{jk}} = \frac{F + \sum\limits_{\mu,i=1}^{p} F_{i\mu} t_\mu \frac{T^o_{s_\mu k} G^o_{jr_i}}{G^o_{jk}}}{F} . \tag{36}$$

As can be observed from Eq. (26),

$$R_{i\mu} - R^N_{i\mu} = t_\mu \frac{T^o_{s_\mu k} G^o_{jr_i}}{G^o_{jk}} . \tag{37}$$

Substitution of Eq. (37) in Eq. (36) yields

$$\frac{G_{jk}}{G^o_{jk}} = \frac{F + \sum\limits_{\mu,i=1}^{p} F_{i\mu}(R_{i\mu} - R^N_{i\mu})}{F} . \tag{38}$$

It will now be convenient to introduce another matrix $[F^N]$ which is defined as follows:

$$[F^N] = [I] - [R^N] \tag{39}$$

where

$$[R^N] = \begin{bmatrix} R^N_{11} & R^N_{12} & \cdots & R^N_{1p} \\ & \vdots & & \\ R^N_{p1} & R^N_{p2} & \cdots & R^N_{pp} \end{bmatrix} . \tag{40}$$

Let F^N be the determinant of $[F^N]$ and let

$$N = F + \sum_{\mu, i=1}^{p} F_{i\mu}(R_{i\mu} - R_{i\mu}^{N}) \quad . \tag{41}$$

Then if it can be established that $F^N = N$, the formula

$$\frac{G_{jk}}{G_{jk}^{o}} = \frac{F^N}{F} \tag{42}$$

will be established.

With this thought in mind, a more compact notation will be introduced. Let U be a determinant having elements u_{ij} where $u_{ij} = \delta_{ij} - R_{ij}$. Let V be a determinant having elements v_{ij} where $v_{ij} = \delta_{ij} - R_{ij}^{N}$. If X is a determinant having elements x_{ij} where $x_{ij} = v_{ij} - u_{ij}$, then use of the definitions in this paragraph shows that $x_{ij} = R_{ij} - R_{ij}^{N}$. Eq. (41) may then be rewritten as

$$N = U + \sum_{i,j=1}^{p} U_{ij}x_{ij} \tag{43}$$

where U_{ij} is the cofactor of the element in the ith row and jth column of the U determinant.

In terms of the notation above, one can show that Eq. (26) may be written as

$$\frac{x_{ab}}{x_{ad}} = \frac{x_{cb}}{x_{cd}} \quad . \tag{44}$$

Eq. (44) may be written in the form

$$\begin{vmatrix} x_{ab} & x_{ad} \\ x_{cb} & x_{cd} \end{vmatrix} = 0 \tag{45}$$

where Eq. (45) holds for any a, b, c, and d. This equation is equivalent to stating that every 2 x 2 minor of X is zero, which is equivalent to the statement that X is of rank 1. Therefore, any determinant containing more than one row (or column) of x_{jk}'s is identically zero. In terms of this new notation, if it can be established that V = N, then Eq. (42) will have been verified.

Let

$$\mu_j = \begin{bmatrix} u_{1j} \\ \vdots \\ u_{pj} \end{bmatrix} \quad \text{and} \quad \omega_j = \begin{bmatrix} x_{1j} \\ \vdots \\ x_{pj} \end{bmatrix} \tag{46}$$

and hence, $V = \left| \mu_1 + \omega_1 \quad \mu_2 + \omega_2 \quad \cdots \quad \mu_p + \omega_p \right|$ (47)

with application of the relations introduced in the paragraph preceding Eq. (43).

Now consider $0 \leqq k \leqq P = 2^p - 1$ and write out the following array for k considering it to be in binary form:

$$\text{P + 1 rows} \left[\begin{array}{c|ccccc} & \multicolumn{5}{c}{\text{p columns}} \\ & j=1 & j=2 & \cdots & j=p-1 & j=p \\ k=0 & 0 & 0 & \cdots & 0 & 0 \\ k=1 & 0 & 0 & \cdots & 0 & 1 \\ k=2 & 0 & 0 & \cdots & 1 & 0 \\ k=3 & 0 & 0 & \cdots & 1 & 1 \\ & & & \vdots & & \\ k=P & 1 & 1 & \cdots & 1 & 1 \end{array} \right] .$$

Let $C_k = \left| c_1^k \quad c_2^k \quad \cdots \quad c_p^k \right|$ where $c_j^k = \begin{cases} \mu_j, & \text{if kth row and jth column of array has a 0} \\ \omega_j, & \text{if kth row and jth column of array has a 1} \end{cases}$.

Then

$$V = \sum_{k=0}^{P} C_k \, . \tag{48}$$

Now Eq. (45) shows that all C_k's will be zero except for k = 0, $2^0, 2^1, 2^2, \ldots, 2^{p-1}$. Eq. (48) then becomes

$$V = \left| \mu_1 \, \mu_2 \cdots \mu_p \right| + \left| \mu_1 \;\; \mu_2 \cdots \omega_p \right| + \left| \mu_1 \;\; \mu_2 \cdots \omega_{p-1} \, \mu_p \right| + \cdots + \left| \omega_1 \;\; \mu_2 \cdots \mu_p \right| . \tag{49}$$

But Eq. (49) is equivalent to the following equation:

$$V = U + \sum_{i=1}^{p} U_{ip}x_{ip} + \dots + \sum_{i=1}^{p} U_{i1}x_{i1} = U + \sum_{i,j=1}^{p} U_{ij}x_{ij}. \qquad (50)$$

Therefore, since Eq. (43) is identical to Eq. (50), V = N and Eq. (42) is correct. Since j and k are arbitrary but fixed, it is useful to write Eq. (42) as

$$\frac{G}{G^o} = \frac{F^N}{F} . \qquad (51)$$

It should be mentioned that throughout the above derivations, it has been assumed that G^o is not identically zero. If G^o is identically zero, this case may be treated by considering G^o to be ϵ and then taking the limit as ϵ approaches zero. This may be done whenever the system transmission factors are continuous functions of the system parameters. In writing Eq. (27), no mention was made of the fact that there might not be p distinct h_{r_μ}. If there are not p distinct h_{r_μ}, then the [R] matrix would be singular but the [F] matrix would not be singular. Again, no difficulty is encountered.

It is quite useful in linear system analysis to determine the effect of a percentage change in one of the transmission factors t_j on the percentage change in G. In other words, it is desirable to speak of a sensitivity with respect to the parameter t_j having the property that

$$\frac{\delta G}{G} \doteq S_j \frac{\delta t_j}{t_j} . \qquad (52)$$

To give a precise definition, S_j will be defined to be the <u>sensitivity of G with respect to the parameter t_j</u> and the mathematical definition will be taken as

$$S_j = \frac{t_j}{G} \frac{\partial G}{\partial t_j} . \qquad (53)$$

Substitution of Eq. (51) in Eq. (53) and performing the indicated differentiation, with simplification, yields

$$S_j = \frac{F\, t_j \dfrac{\partial F^N}{\partial t_j} - F^N t_j \dfrac{\partial F}{\partial t_j}}{F\, F^N} \; . \tag{54}$$

Performing the indicated differentiation and using properties of determinants, one sees easily that the following hold:

$$t_j \frac{\partial F^N}{\partial t_j} = F^N - F^N_{jj} \; , \tag{55}$$

and

$$t_j \frac{\partial F}{\partial t_j} = F - F_{jj} \tag{56}$$

where F^N_{ij} is the cofactor of the element in the ith row and jth column of the F^N determinant. Eqs. (55) and (56) may be substituted in Eq. (54) and the resulting equation simplified to give

$$S_j = \frac{F_{jj}}{F} - \frac{F^N_{jj}}{F^N} \; . \tag{57}$$

For the case in which w of the t_j are functions of a parameter α, it is convenient to define the total sensitivity S with respect to α as

$$S = \frac{\alpha}{G} \frac{dG}{d\alpha} \; . \tag{58}$$

Now assume that $t_1 = f_1(\alpha)$, $t_2 = f_2(\alpha)$, . . . , $t_w = f_w(\alpha)$.

Then $$S = \sum_{j=1}^{w} \frac{\alpha}{G} \frac{\partial G}{\partial t_j} \frac{dt_j}{d\alpha} = \sum_{j=1}^{w} \left(\frac{t_j}{G} \frac{\partial G}{\partial t_j} \right) \left(\frac{\alpha}{t_j} \frac{dt_j}{d\alpha} \right) \; . \tag{59}$$

Now if $$s_i = \frac{\alpha}{t_j} \frac{dt_j}{d\alpha} \; ,$$

then with Eq. (53)

$$S = \sum_{j=1}^{w} S_j s_j \ . \qquad (60)$$

Now consider a very special case. Assume that all the t_μ-transmission factors leave the same node h_{r_a}. Then from Eq. (21)

$$h_{r_a} = f_j G^o_{jr_a} + h_{r_a} \sum_{\mu=1}^{p} (t_\mu T^o_{s_\mu r_a}) \ . \qquad (61)$$

With application of Eq. (23), Eq. (60) may be rewritten and solved for h_{r_a} to yield

$$h_{r_a} = \frac{f_j G^o_{jr_a}}{1 - R^c} \qquad (62)$$

where R^c is defined by the relation

$$R^c = \sum_{\mu=1}^{p} R_{a\mu} \ . \qquad (63)$$

From Eqs. (21), (61), and (26), the ratio of G_{jk} to G^o_{jk} is found as

$$\frac{G_{jk}}{G^o_{jk}} = \frac{1 - R^{c^N}}{1 - R^c} \qquad (64)$$

where R^{c^N} is defined by the relation

$$R^{c^N} = \sum_{\mu=1}^{p} R^N_{a\mu} \ . \qquad (65)$$

It should be observed that both R^c and R^{c^N} have physical significance. They can be thought of as a composite return or null return ratio when the t_μ-transmissions are tied together and a common test input is applied. The response is again observed at the h_{r_a} node. An example that shows application of the above ideas is given in section III of this paper.

III. Examples Indicating Applications of the Theorems

Example 1: The example shown in Fig. 5 is a simple feedback system. $G(s)$ is the forward transmission factor and $-K_1$ and $-K_2$ are transmission factors for feedback branches. This example has been chosen for two reasons:

(1) It illustrates a system which is stable with both feedback branches removed and with both feedback branches replaced, yet is unstable when either of the individual feedback branches alone is replaced. The possibility of such a situation lead to the formulation of this thesis problem.

(2) It is an application of the method of analysis suggested by Eq. (63).

The transfer function that is plotted in the usual polar form in the function plane is similar to that given by Truxal[3], p. 596. This plot is shown in Fig. 6 where

$$G(s) = \frac{427(s+5)(s+50)}{s(s+1)(s+4)(s+200)(s+400)} .$$

The following data have been tabulated from Truxal:

$$a = 1.78 \times 10^{-5} ,$$

$$b = 6.3 \times 10^{-4} ,$$

and

$$c = 1.0 \times 10^{-1} .$$

The individual return ratios are $-K_1G(s)$ and $-K_2G(s)$. For $K_1 = K_2 = K = 10^3$,

$$Kc = 100 > 1$$

and

$$Kb = 0.63 < 1 .$$

Therefore the critical point, (1,0), is encircled and, from the Nyquist stability criterion, the system is unstable with either K_1 or K_2 replaced.

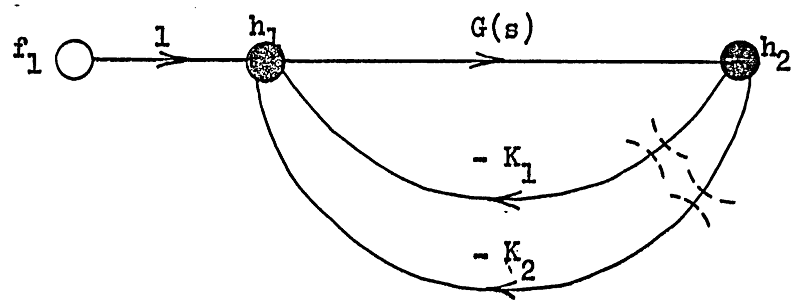

Figure 5. A simple two branch feedback system.

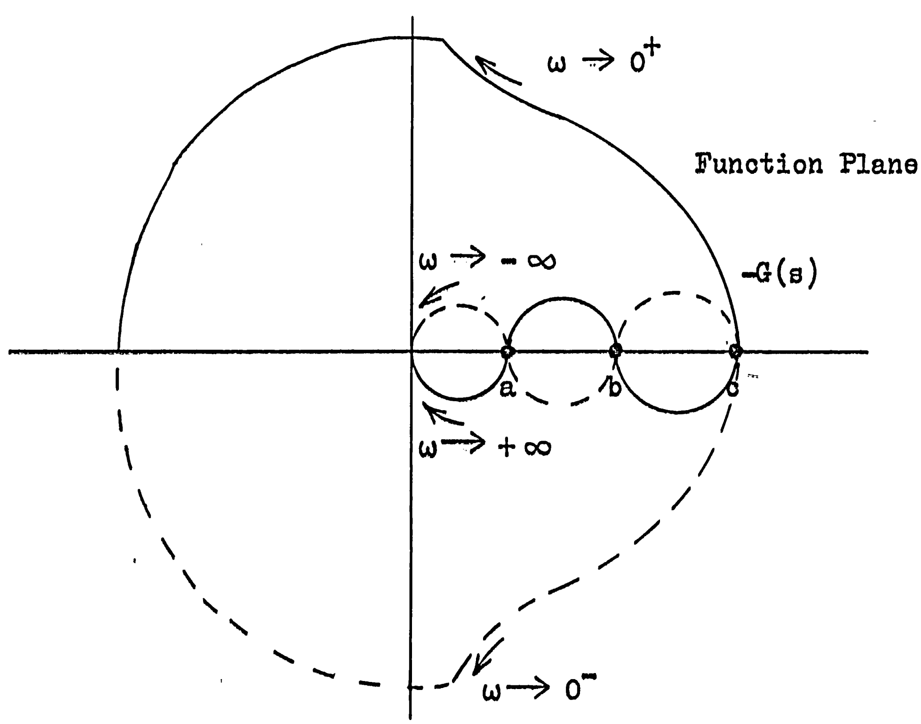

Figure 6. "Nyquist plot" of return ratios.

If both K_1 and K_2 are to be replaced simultaneously, then the composite return ratio is $-(K_1 + K_2)G(s)$. Hence

$$(K_1 + K_2)b = 1.26 > 1$$

and

$$(K_1 + K_2)a = 0.0356 < 1.$$

Therefore the system with both branches replaced is stable.

<u>Example 2</u>: Consider the vacuum tube amplifier shown in Fig. 7. It is desired to determine the output impedance of the amplifier using the method of analysis discussed in the preceding pages. This may be done by finding the transmission from i_o to e_o as indicated in terms of the flow graph of Fig. 8. For the analysis, it is convenient to consider the t_μ-transmission factors as indicated in the flow graphs of Fig. 9. The analysis may now be carried out in the following steps:

1° Compute the output impedance with e_{T_1} and $e_{T_2} = 0$, i.e. grids grounded. Let r_7 = the parallel combination of r_3 and r_{p_1},

$$r_8 = r_4 + r_5 + r_7 ,$$

and r_9 = the parallel combination of r_8, r_{p_2}, and r_6.

Then G^o = the open branch output impedance

= the parallel combination of r_9 and $(r_1 + r_2)$.

For r_{p_2}, r_5, and $r_1 \gg r_6$, $G^o \doteq r_6$.

2° With e_{T_2} grounded and $i_o = 0$, compute R_{j1} for $j = 1, 2$. To simplify this calculation, consider $e_{T_1} = 1$.

Then

$$e_{p_1} = -\mu_1 \frac{r_{10}}{r_{p_1} + r_{10}}$$

where r_{10} = the parallel combination of r_3 and $[r_4 + r_5 +$ (the parallel combination of r_{p_2}, r_6, and $r_1 + r_2$)].

For r_{p_1} and $r_5 \gg r_3$,

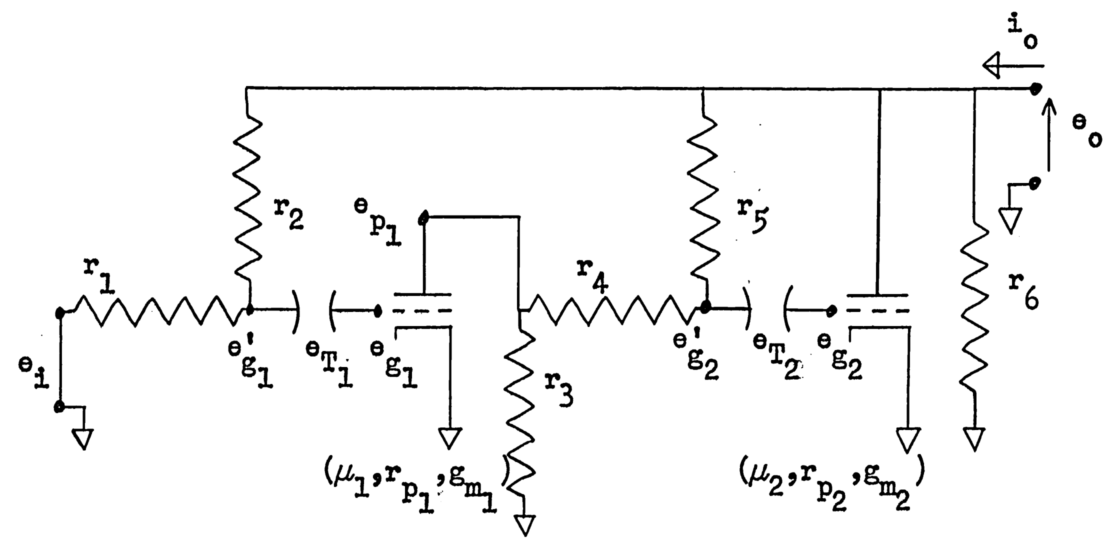

Figure 7. A vacuum tube amplifier with two active elements.

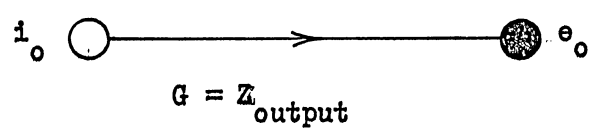

Figure 8. A flow graph showing that the output impedance is the relation between i_o and e_o.

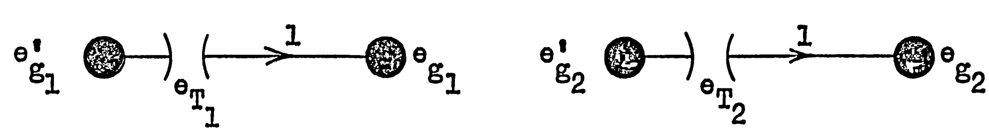

Figure 9. A flow graph representation of the return ratio tests.

$$e_{p_1} \doteq -g_{m_1} r_3 .$$

Now $$e'_{g_2} = e_{p_1} \frac{r_{11}}{r_4 + r_{11}}$$

where $r_{11} = r_5 +$ the parallel combination of r_{p_2}, r_6, and $r_1 + r_2$.

With the above assumptions,

$$R_{21} \doteq -g_{m_1} \frac{r_3 r_5}{r_4 + r_5} .$$

Now $$e_o = e_{p_1} \frac{r_{12}}{r_4 + r_5 + r_{12}}$$

where $r_{12} =$ the parallel combination of r_{p_2}, r_6, and $r_1 + r_2$.

With the above assumptions,

$$e_o \doteq e_{p_1} \frac{r_6}{r_4 + r_5}$$

and $$R_{11} = e_o \frac{r_1}{r_1 + r_2} \doteq -g_{m_1} \frac{r_1 r_3 r_6}{(r_1 + r_2)(r_4 + r_5)} .$$

3° With e_{T_2} grounded and $i_o = 0$, compute R_{j2} for $j = 1, 2$. To simplify this calculation, consider $e_{T_2} = 1$.

Then $$e_o = -\mu_2 \frac{r_{13}}{r_{p_2} + r_{13}}$$

where $r_{13} =$ the parallel combination of r_6, r_8, and $r_1 + r_2$.

With the assumptions listed above under 1°,

$$e_o \doteq -g_{m_2} r_6 .$$

Then $$R_{12} = e_o \frac{r_1}{r_1 + r_2} \doteq -g_{m_2} \frac{r_1 r_6}{r_1 + r_2}$$

and $$R_{22} = e_o \frac{r_4 + r_7}{r_4 + r_5 + r_7} \doteq -g_{m_2} \frac{r_4 r_6}{r_4 + r_5}$$

with the added assumption that $r_4 \gg r_3$.

4° Calculate F.

$$F = \begin{vmatrix} 1 - R_{11} & - R_{12} \\ - R_{21} & 1 - R_{22} \end{vmatrix} \doteq \frac{g_{m_1} g_{m_2} r_1 r_3 r_6 [r_4 r_6 - r_5(r_4 + r_5)]}{(r_1 + r_2)(r_4 + r_5)^2}$$

for $|R_{11}|$ and $|R_{22}| \gg 1$.

Since by an assumption under 1°, $r_5 \gg r_6$, therefore

$$F \doteq - \frac{g_{m_1} g_{m_2} r_1 r_3 r_5 r_6}{(r_1+r_2)(r_4+r_5)} \ .$$

5° Calculate the R^N_{jk} for j, k = 1, 2. For any of these calculations, an e_T is applied and an i_o of sufficient strength to reduce e_o to zero is applied. Hence R^N_{11}, R^N_{12}, and R^N_{22} are zero. Therefore $F^N = 1$.

6° Calculate G.

$$G = G^o \frac{F^N}{F} \doteq - \frac{(r_1+r_2)(r_4+r_5)}{g_{m_1} g_{m_2} r_1 r_3 \ r_5} \ .$$

The sensitivity of the output impedance with respect to the μ's of the tubes could be determined quite easily since breaking the branches as indicated is equivalent to breaking the μ_1 or μ_2 transmission in the tube. This is because neither tube has a cathode resistor.

While a number of approximations have been made to make the calculations less tedious, if the experiments were performed in the laboratory, no approximations would be necessary to find the various R's.

IV. Conclusions

The method of analysis presented in this paper clearly depicts the possibility of determining system stability from measurements that can be made experimentally on a stable system. This, however, assumes that breaking a physical branch in the system can be related to breaking a transmission on a flow graph. The question of determining whether the replacing of several branches of a system will yield a stable system may be answered by considering appropriate minors of F. This may be stated in other terms as follows: Suppose $q \geq p$ branches are removed; consider the determinant F for the p branches to be replaced; it is possible to determine if the system will be stable when those p branches are replaced. Example 1 indicates what is meant by the above statement. For that example, $q = 2$ and with the two branches removed the system is stable. For $p = 1$, the system is unstable when either branch was to be closed individually. For $p = 2$, i.e. both branches were to be replaced simultaneously, the result is a stable system. If p branches can be replaced to yield a stable system, a new problem results with fewer t_μ branches to consider.

The sensitivities with respect to the various transmission factors in the system can be determined experimentally from strictly open-branch measurements. The sensitivity with respect to a variable which is a parameter for several t_μ may be determined open-branch. An example considered in this paper indicates how a system may be decomposed into groups of simple elements for the sake of easier analysis and/or measurement.

The problem considered in this paper is one of analysis. It remains to be seen if the method suggested here gives more insight into the synthesis problem. The ideas discussed need to be correlated more closely with some of the terminology in the current literature, if the results are to be most useful to the practicing engineer. A method of synthesis of multibranch systems using the root-locus technique was presented by Glomb[4]. It is felt that some of his ideas could be generalized using the method suggested by this paper to point ultimately in the direction of synthesis. One obvious generalization is suggested by the special case considered at the end of section II of this paper.

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VI. References

Direct references:

1. Pfeiffer, P. E., LINEAR SYSTEMS ANALYSIS, McGraw-Hill Book Company, Inc., New York, 1961, Chapter 12.

2. Bode, H. W., NETWORK ANALYSIS AND FEEDBACK AMPLIFIER DESIGN, D. Van Nostrand Company, Inc., New York, 1945.

3. Truxal, J. H., AUTOMATIC FEEDBACK CONTROL SYSTEM SYNTHESIS, McGraw-Hill Book Company, Inc., New York, 1955.

4. Glomb, J. D., "On the Synthesis of Multiloop Systems", Proceedings of the IRE, Vol. 48, No. 2, February, 1960, pp. 245-246.

Background references:

5. Mason, S. J., "Feedback Theory--Some Properties of Signal Flow Graphs," Proceedings of the IRE, Vol. 41, No. 9, September, 1953, pp. 1144-1156.

6. Mason, S. J., "Feedback Theory--Further Properties of Signal Flow Graphs," Proceedings of the IRE, Vol. 44, No. 7, July, 1956, pp. 920-933.