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CHEBYSHEV APPROXIMATION FOR NON-RECURSIVE DIGITAL FILTERS

by

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ABSTRACT

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An efficient procedure for the design of finite length impulse response filters with linear phase is presented. The algorithm obtains the optimum Chebyshev approximation on separate intervals corresponding to pass and/or stop bands, and is capable of designing very long filters. This approach allows the exact specification of arbitrary band-edge frequencies as opposed to previous algorithms which could not directly control pass and stop band locations and could only obtain \( \frac{N-1}{2} \) different band-edge locations for a length \( N \) low-pass filter, for fixed \( \delta_1 \) and \( \delta_2 \).

As an aid in practical application of the algorithm, several graphs are included to show relations among the parameters of filter length, transition width, band-edge frequencies, passband ripple, and stopband attenuation.
ACKNOWLEDGEMENTS

I would like to thank my thesis advisor, Dr. T. W. Parks, who provided the invaluable ideas necessary to start this research, and my wife, Carolyn, who has tolerated my long hours of work in completing this research.
I. INTRODUCTION

The filtering of time-signals by RLC networks is a familiar problem in Electrical Engineering. A similar operation can be performed by a digital computer if the signal is sampled prior to filtering and reconstructed afterwards. It is well known that the mathematical form of an analog filter is the ratio of two polynomials in the Laplace frequency variable $s$. Likewise, a digital filter can be expressed as the ratio of two polynomials in the $z$-transform variable $z$.

However, a key feature of digital filters is that realization of a polynomial in $z$ (called a non-recursive filter) is possible, in contrast to analog filters which always have a denominator polynomial. This realization is made practical by the Fast Fourier Transform (FFT) algorithm which enables high order polynomials to be implemented efficiently.

Non-recursive filters have some special properties which make them important for digital signal processing. First, since a non-recursive filter has no poles, it is not necessary to worry about the stability of the filter. In the case of recursive filters (filters with poles)
stability is a major concern. Secondly, a non-recursive filter is less sensitive to realizations in finite length arithmetic than a recursive filter. These two properties are important because the recursive filter is a much more compact design in terms of the number of filter coefficients needed. Typically, a recursive filter might have ten coefficients whereas a non-recursive filter might have one or two hundred. Thus, there is a tradeoff between the complexity of the filter and the problems of stability and coefficient quantization. Thirdly, exactly linear phase can be obtained with a non-recursive filter. This is important in applications where the delay distortion must be kept small. Finally, the design of non-recursive filters is easier because the problem is linear, whereas recursive filter design is a non-linear problem.

The techniques for designing low pass non-recursive filters are classified as least squares designs and Chebychev (or min-max) designs. In the former, the basic approach is to truncate the Fourier series of the rectangular low pass shape

\[ R(F) = \begin{cases} 
1 & 0 \leq F \leq F_c \\
0 & F_c \leq F \leq 0.5 
\end{cases} \]

and then use the Fourier series coefficients as the impulse
response coefficients. In the latter, the approach is to interpolate to a function which has the equiripple property of Chebychev approximation, namely, the error reaches its maximum value \( n \) times where \( n \) is the number of approximating functions. The recent review articles [1], [2] survey the status of available design techniques.

The least squares designs are handicapped by the Gibbs phenomenon which results from trying to approximate the discontinuity in \( R(F) \) at \( F = F_c \). The windowing technique [3] improves this situation by multiplying the frequency response by a carefully chosen window function to smooth out the overshoot. Thus the impulse response is convolved with the impulse response of the window function. However, the "optimum" window functions are very complicated.

The Chebychev norm is a more natural norm for filter design since the object is to minimize the peak value of the error. The frequency sampling method [4] was the first procedure to use this norm. By allowing only one, two, or three of the parameters in the design to vary the maximum stopband error is minimized subject to the constraint that the maximum passband deviation be less than some fixed value.

The idea of Hermann [5], which was made computationally feasible by Hofstetter [6], is to fix the passband
deviation $\delta_1$ and the stopband deviation $\delta_2$ and interpolate the approximating function through the points $1 \pm \delta_1$ in the passband and $\pm \delta_2$ in the stopband. The resulting filter "looks" like an optimal Chebychev design since it has the equiripple property. In view of the results which will be presented here, this filter is a special case of the optimum filters and will be referred to as an extraripple filter.

Since the goal is to design an optimum min-max filter, it is important to classify optimum filters and then view the previous results from this perspective. A new procedure results which includes the extraripple filters as a special case.
II. CHEBYCHEV APPROXIMATION FOR NON-RECURSIVE FILTERS

A. Problem Formulation

The detailed description of the new procedure described in this paper is in terms of low pass filters. Modifications for the general bandpass case are included. Linear phase, digital filters of length $2n + 1$ have a transfer function

$$G(Z) = \sum_{k=0}^{2n} h_k Z^{-k}$$  \hspace{1cm} (1)

with $h_k = h_{2n-k}$. The frequency response $\hat{G}(F)$ is obtained with

$$Z = \exp(j\omega T) = \exp(j2\pi F)$$  \hspace{1cm} (2)

where $F$ is the normalized frequency variable, $F = \omega T/2\pi$.

$$\hat{G}(F) = \exp(-j2\pi nF) \sum_{k=0}^{n} d_k \cos2k\pi F$$

$$= \exp(-j2\pi nF) H(F)$$  \hspace{1cm} (3)

where $d_{n-k} = 2h_k$, $k=0,\ldots,n-1$ and $d_0 = h_n$.  

5
The frequency domain design problem is to find the \( \{h_k\}, \ k=0, \ldots, 2n, \) or equivalently the \( d_k, \ k=0, \ldots, n, \) so that \( H(F) \) has the desired characteristic shown in Figure 1. The parameters in the standard tolerance scheme of Figure 1 are defined as follows:

\[
\begin{align*}
\delta_1 &= \text{allowed passband deviation.} \\
\delta_2 &= \text{allowed stopband deviation.} \\
F_P &= \text{desired passband edge.} \\
F_S &= \text{desired stopband edge.}
\end{align*}
\]

There are several ways to meet the prescribed tolerances. The most straightforward approach is to approximate the rectangle function

\[
R(F) = \begin{cases} 
1 & 0 < F < \frac{F_P + F_S}{2} \\
0 & \frac{F_P + F_S}{2} \leq F \leq 0.5
\end{cases}
\]

The discontinuity in \( R(F) \) causes difficulty in both the least square and Chebyshev approximation and has led to procedures where the region between \( F_P \) and \( F_S \) is filled in with a continuous function such as a straight line with slope of \(-\frac{(F_S - F_P)^{-1}}{F_P}\) or a section of a cosine. Another approach is to use a weighted least square approximation.
with zero weighting in the transition region between $F_p$ and $F_s$. The theory of weighted Chebyshev approximation on an interval will not allow such a zero weighting function. An indirect approach which ignores the error in a transition region makes use of the equiripple property of Chebyshev approximation by specifying the number of pass and stop band ripples $n_p$ and $n_s$ \cite{5}, \cite{6}.

This method requires the specification of $n_p$, $n_s$, $\delta_1$ and $\delta_2$. A limitation of this approach is that the relationship between the desired band-edge frequencies $F_p$ and $F_s$ and an a priori choice of $n_p$ and $n_s$ is unknown. A trial and error procedure is necessary to meet the $F_p$ and $F_s$ specifications if indeed they can be met by any set of $n_p$ and $n_s$.

A new approach described in this paper which meets the tolerance scheme of Figure 1 and allows specification of band-edge frequencies formulates the problem as one of Chebyshev approximation on two disjoint intervals (three for the bandpass case). For a filter of length $2n + 1$, the $n$ possible extraripple filters arise as special cases of this approach when appropriate values of $F_p$ and $F_s$ are specified. Rather than minimize the error in the stop band only as in the frequency-sampling method \cite{4}, this approach does optimization on several bands. An efficient algorithm will be developed which uses interpolation as in \cite{6} and
achieves comparable speed.

The filter design problem is stated as a weighted Chebyshev approximation problem by defining the passband $B_p$ and stopband $B_s$ as

\[ B_p = \{ F: 0 \leq F \leq F_p \} \quad (4) \]

\[ B_s = \{ F: F_s \leq F \leq 0.5 \} \quad . \quad (5) \]

The desired form for a low-pass $H(F)$ in (3) is then

\[ D(F) = \begin{cases} 
1 & F \in B_p \\
0 & F \in B_s .
\end{cases} \quad (6) \]

The weight function

\[ W(F) = \begin{cases} 
\frac{1}{K} & F \in B_p \\
1 & F \in B_s .
\end{cases} \quad (7) \]

allows the designer freedom to specify the relative magnitude of the error in the two bands.

With

\[ \Omega = B_p \cup B_s , \quad (8) \]
the union of pass and stop bands, the problem of designing a lowpass linear phase filter of length $2n+1$ becomes the problem of finding the $\{d_k\}$ $k = 0, \ldots, n$ in

$$H(F) = \sum_{k=0}^{n} d_k \cos2k\pi F$$

which minimize

$$\max_{F \in \Omega} W(F) |D(F) - H(F)|$$

The next section develops an efficient algorithm to solve the above problem and the general bandpass problem where there are three disjoint intervals.

B. Design Procedure

The theory of Chebyshev approximation on compact sets has established that the problem as formulated in (10) has a unique solution. More important, however, is the fact that necessary and sufficient conditions which characterize the best approximation are given by the following alternation theorem which uses the fact that the set of functions $\{\cos2\pi kF\}$ $k = 0, \ldots, n$ satisfies the Haar condition [7].
Theorem: Let $\Omega$ be any closed subset of $[0, 1/2]$. In order that

$$H(F) = \sum_{k=0}^{n} d_k \cos 2\pi k F$$

be the unique best approximation on $\Omega$ to $D(F)$ it is necessary and sufficient that the error function $E(F) = W(F)[D(F) - H(F)]$ exhibit on $\Omega$ at least $n + 2$ "alternations"

Thus: $E(F_i) = -E(F_{i-1}) = \pm ||E||$ with $F_0 < F_1 < \ldots < F$ and $F_i \in \Omega$. Here $||E|| = \max_{F \in \Omega} |E(F)|$.

The extraripple filters in [5], [6] are optimal in terms of this theory and are special cases obtained with the algorithm to be described. To see this, consider an example filter of length 11 with $n_p = 3$ and $n_s = 3$.

(Figure 2). For simplicity, let $\delta_1 = \delta_2$. A length 11 filter is an approximation with the set of 6 functions $\{1, \cos 2\pi F, \ldots, \cos 10\pi F\}$ ($n=5$). If we consider that the approximation is done on $[0, F_p] \cup [F_s, 0.5]$, then the alternation theorem says that there must be at least 7 alternations on this set. A careful count, including the points 0, $F_p$, $F_s$, and 0.5, of the alternations shows that there are 8 alternations in the sense of the alternation theorem. Therefore, the extraripple filters are optimum filters on the set $[0, F_p] \cup [F_s, 0.5]$. Note, however, that this set on which the approximation is best is not known beforehand,
but rather is a result of the design procedure. The fact that there are \( n + 3 \) alternations for the extraripple filters is true in general (this motivates the name) and tells us intuitively that the extraripple designs are a very special case of Chebyshev approximation. In fact, there are only \( n \) different extraripple filters for fixed \( \delta_1 \) and \( \delta_2 \) which means that for an arbitrary location of \( F_p \) and \( F_s \) and choice of weight function the optimum design will usually have only \( n + 2 \) alternations.

The extraripple design algorithm proposed by Hofstetter is capable of designing high order filters very efficiently; in fact the computation time is roughly proportional to \( n^2 \). Hofstetter mentions that the algorithm is similar to the classical Remes exchange algorithm. As will be shown, the computational aspects of the Hofstetter algorithm can be merged with the ascent property of the Remes exchange method [8] to give the following new algorithm which converges quadratically to the best weighted Chebyshev approximation on the set \( \Omega \).

The algorithm as indicated in Figure 3, begins with an initial guess of \( n + 2 \) points and exchanges points until it obtains the \( n + 2 \) points of alternation (extremal frequencies) of the alternation theorem. This is accomplished in the following way.

At the \( i \)-th step of the algorithm there are \( n + 2 \)
points on which the error function is forced to have magnitude $|\rho|$ with alternating signs.

For a given set of $n+2$ frequencies $\{F_k\}$ $k=0,\ldots,n+1$, this requires the solution of the $n+2$ equations

$$W(F_k) (H(F_k) - D(F_k)) = (-1)^k \rho \quad k=0,\ldots,n+1 \quad (11)$$

where $H(F)$, $D(F)$ and $W(F)$ are defined in (6), (7) and (9).

Equation (11) may be written matrix form as follows:

$$\begin{bmatrix}
1 \cos 2\pi F_0 & \cos 4\pi F_0 & \ldots & \cos 2\pi n F_0 & \frac{1}{W(F_0)}
1 \cos 2\pi F_1 & \frac{-1}{W(F_1)}
\vdots & \ddots & \ddots & \ddots & \frac{1}{W(F_2)}
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots
1 \cos 2\pi F_{n+1} & \frac{(-1)^{n+1}}{W(F_{n+1})}
\end{bmatrix}
\begin{bmatrix}
d_0 \\
d_1 \\
\vdots \\
\vdots \\
\vdots \\
d_n \\
\end{bmatrix}
= 
\begin{bmatrix}
D(F_0) \\
D(F_1) \\
\vdots \\
\vdots \\
\vdots \\
\bullet \\
\end{bmatrix}
$$

(12)

The system (12) is always non-singular [7] and thus there is a unique solution for $\rho$ and the $d$'s. However, the direct solution of (12) is time consuming. In fact as
pointed out by Temes [9], it is the most difficult part of the algorithm. It is more efficient to first calculate $\rho$ analytically as

$$
\rho = \frac{a_0 D(F_0) + a_1 D(F_1) + \ldots + a_{n+1} D(F_{n+1})}{a_0 \frac{1}{W(F_0)} - a_1 \frac{1}{W(F_1)} + a_2 \frac{1}{W(F_2)} \ldots (-1)^{n+1} a_{n+1} \frac{1}{W(F_{n+1})}}
$$

where

$$
a_k = \frac{n+1}{\prod_{j=0}^{n} \frac{1}{(x_k - x_j)}} \quad j \neq k
$$

and

$$
x_k = \cos 2\pi F_k.
$$

Next, Lagrange's interpolation formula in the barycentric form is used to interpolate $H(F)$ on the $n+1$ points $F_0 \ldots F_n$ to the values

$$
c_k = D(F_k) - (-1)^k \frac{\rho}{W(F_k)} \quad k = 0, \ldots, n
$$

$$
H(F) = \sum_{k=0}^{n} \frac{b_k}{(x-x_k)} c_k
$$

where

$$
\sum_{k=0}^{n} \frac{b_k}{(x-x_k)}
$$
Note $H(F)$ will also interpolate to $D(F_{n+1}) - (-1)^{n+1}\rho D(F_{n+1})^{-}$ since it satisfies (11). Then the error function $E(F) = W(F)[D(F) - H(F)]$ is evaluated on $\Omega$. However, it is only necessary to evaluate $E(F)$ at a finite number of points. A grid of $20n$ equally spaced points has proved to be sufficiently dense in $\Omega$. If the error function is such that $|E(F)| < |\rho|$ for $F \in \Omega$ then we have found the best approximation. If $|E(F)| > |\rho|$ for some $F \in \Omega$ then a new set of $n + 2$ frequencies must be chosen as candidates for the extremal points. The philosophy of the Remes method [8] is to choose these new frequencies such that $|\rho|$ will be increased at the next iteration. If the new points are chosen to be the peaks of the error curve (i.e., points where $|E(F)| > |\rho|$ and $E(F)$ is a local extremum) then $|\rho|$ is forced to increase and ultimately converge to its upper bound which corresponds to the solution of the problem.

The local extrema are determined by finding those points where $H(F_{k+1}) - H(F_k)$ changes sign. The endpoints $F_p$ and $F_s$ are always local extrema. Among these extrema we want to choose the $n + 2$ points where $|E(F)|$ is the greatest with a search. This search is very easy to implement because of the following properties of the error

$$b_k = \prod_{j=0, j \neq k}^{n-1} \frac{1}{x_k - x_j}.$$
function.

**Theorem:** For a filter of length $2n + 1$ (the approximation is being done with $n + 1$ functions) the $i$-th error curve must exhibit $n + 2$ or $n + 3$ peaks thus: $\text{sgn } E(F_k) = -\text{sgn } E(F_{k+1})$ for $k = 0, 1, \ldots, n+1$ with $|E(F_k)| \geq |\rho|$; the $F_k$ are the local maxima and minima of the error curve; two of the critical peaks are located at $F_p$ and $F_s$; at least one of the endpoints is a peak; and both endpoints are peaks if there are $n + 3$ peaks.

Figure 4 shows the two cases which can occur for a length 11 filter.

**Proof:** The properties of $E(F)$ depend on the fact that a function of the form

$$P(F) = \sum_{k=1}^{n} a_k \sin 2\pi kF$$

has zeros at $F = 0$ and $F = 1/2$ and at most $n - 1$ zeros in the open interval $(0, 1/2)$. For a given set of points the optimal $\rho$ has been calculated and

$$H(F) = \sum_{k=0}^{n} g_k \cos 2\pi kF$$

has been fit through $n + 1$ points. It is well known that on the set $\Omega$ which is the union of disjoint closed intervals that the relative min and max occur at points where
\[
\frac{dE}{dF} = \frac{dH}{dF} = 0 \text{ or at the endpoints.}
\]

\[
\frac{dH}{dF} = \sum_{k=1}^{n} a_k \sin 2\pi k F
\]

has at most \( n + 1 \) zeros in the set \( \Omega \) and the two boundary points \( F_p \) and \( F_s \) are the only other candidates for extremal points. Hence, there are at most \( n + 3 \) peaks of the error curve. Also, the only way that there can be \( n + 3 \) peaks is for both endpoints, 0 and 1/2, to be peaks. Since the error curve interpolates \( n + 2 \) points in \( \Omega \) where \( |E(F)| = |\rho| \) with alternating sign, there must be at least \( n + 2 \) local extrema where \( |E(F)| \geq |\rho| \) with alternating sign. If there are \( n + 3 \) peaks, compare the magnitude of the error at 0 and 1/2 and delete the point with smaller error to obtain the new set of \( n + 2 \) extremal points.

This method can be used to design bandpass filters by doing the approximation problem on three disjoint intervals. The search is complicated by the fact that the error curve can have \( n + 2, n + 3, n + 4 \) or \( n + 5 \) extremum points. In each case \( n + 2 \) points of alternation can be chosen by the search procedure.

C. Results

The design algorithm is implemented by specifying
the filter order \( n \), the weighting factor \( K \), and \( F_p \) and \( F_g \) which fix the transition width; then the deviations \( \delta_2 \) and also \( \delta_1 = K\delta_2 \) are minimized. Figure 5 shows experimental results of the relationships between these parameters.

Figure 5a shows \( \delta_2 \) versus \( F_p \) for a length 29 filter. The local minima at .13765 and .17265 correspond to Hofstetter designs. (See Table 1.) Between the maxima at .1195 and the minima at .13765 there is a difference of 3.32db, indicating what is to be gained by an optimal location of the transition region.

Figure 5b shows the control over \( \delta_2 \) exerted by the weighting factor. The a priori choice of \( K \) is a part of the design procedure. However, if the specifications require a certain stopband attenuation \( \delta_2 \) while not exceeding a given passband ripple \( \delta_1 \), then the natural choice for \( K \) is the ratio \( \frac{\delta_1}{\delta_2} \). It is also possible to fix \( F_p \) and \( F_g \) and obtain \( \frac{n_{\text{f}}}{2} \) extraripple filters by varying \( K \).

Figure 5c indicates how much is to be gained by increasing the length of the filter, \( n_f \), and Figure 5d shows that reducing the transition width results in larger deviations in the 2 bands.

One principal merit of the finite-length linear phase filter designs is the speed with which they can be obtained. Figure 6 shows experimental data from a
single-precision program run on the Burroughs B-5500 computer. In all cases the number of iterations never exceeded 10 and was usually about 6 or 7. (A length 95 filter was designed in 200 seconds with 10 iterations.) Most of the time is spent in the evaluation of the interpolation formula on 20n points. This operation is proportional to \( n^2 \) and any efforts to improve the speed would have to be concentrated in this part of the algorithm. The barycentric interpolation formula is used for this reason since there are \( n \) fewer operations than with the standard Lagrange interpolation formula.

The length of filters which can be designed is limited by the precision of the machine in the calculation of the \( a_k \) coefficients. For the large \( n \) many of the differences \( (x_i - x_k) \) are very small and thus

\[
a_k = \prod_{i \neq k} \frac{1}{(x_i - x_k)}
\]

is very large and overflow results. Double precision can be used to design extremely long filters but the time will be increased by about a factor of four.
III. SUMMARY

A technique for the weighted Chebyshev design of non-recursive linear phase digital filters has been presented. Central to this approach is the approximation of the ideal shape on two disjoint intervals. This allows the exact specification of the band edge frequencies, whereas previous procedures obtained the band edge indirectly. Furthermore, the extraripple filters were shown to be special cases of the new design procedure. Finally, an efficient algorithm has been developed to implement the method.

A limitation inherent in this procedure is that it is not possible to independently specify all of the desired parameters. Several graphs have been included based on computational experience showing the relations between parameters. These empirical relationships are an aid to design and show the best that can be attained.
REFERENCES


TABLE I. Comparison of Weighting Coefficients Obtained from Hofstetter's Algorithm and New Algorithm for a Length 29 Filter

<table>
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<tr>
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<th>New Design</th>
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Fig. 1. Desired low-pass filter characteristic.

Fig. 2. Extraripple filter showing application of alternation theorem.
CHECK WHETHER THE EXTREMAL POINTS CHANGED

INITIAL GUESS OF n + 2 EXTREMAL FREQUENCIES

CALCULATE THE OPTIMUM ρ ON EXTREMAL SET

INTERPOLATE THRU n + 1 POINTS TO OBTAIN H(F)

CALCULATE ERROR E(F) AND FIND LOCAL MAXIMA WHERE |E(F)| ≥ ρ

n+2 OR n+3 MAXIMA

END POINTS MUST BE MAXIMA. DISCARD SMALLER OF THESE

n+2

CHECK WHETHER THE EXTREMAL POINTS CHANGED

UNCHANGED

BEST APPROXIMATION

FIG. 3. BLOCK DIAGRAM OF THE DESIGN ALGORITHM
Fig. 4. Two error-curve possibilities for length 11 filter.
Fig. 5. (a) Influence of passband edge $F_p$ on stopband deviation $\delta_s$ for fixed transition width $TW = 0.09$ and weighting factor $K = 10$ with length 29 filter. (b) Control of stopband deviation $\delta_s$ exerted by weighting factor $K$ for filter lengths $n_f = 11, 21, 31,$ and $41$, with $F_p = 0.2$ and $F_s = 0.3$. (c) Dependence of stopband deviation $\delta_s$ on filter length for passband edge $F_p = 0.2$, stopband edge $F_s = 0.24$, and weighting factor $K = 50$. (d) Effect of transition width $TW = F_s - F_p$ on stopband deviation $\delta_s$ for passband edge $F_p = 0.2$, weighting factor $K = 50$, and filter length $n_f = 21$. 
Fig. 6. Computation time as a function of filter length.