AN EXPERIMENTAL AND THEORETICAL STUDY
OF THE INSTANTANEOUS FREQUENCY OF
SOME AUDIO FREQUENCY WAVES

by

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Introduction

The resultant of two simple harmonic vibrations of different frequencies is a single vibration the amplitude of which varies with the difference frequency and the phase of which varies also at the difference frequency. The variation in amplitude is the well known phenomenon of beats, and the difference frequency is the beat frequency. Through the use of Fourier analysis, a complex wave may ordinarily be represented as a sum of sinusoidal components\(^1\), and these simple harmonic vibrations may likewise be represented as a single "sinusoidal" vibration the amplitude and phase of which vary. The time derivative of the phase is defined as the instantaneous frequency function, which can be visualized as the instantaneous angular velocity of the vector representing the resultant of the Fourier components.

The immediate objective of this investigation was to design and build a device which would produce the instantaneous frequency function as a voltage proportional to instantaneous frequency for audio frequency input signals and to make a theoretical and experimental study of the instantaneous frequency of a few audio frequency waves. Two units were constructed, using slightly different approaches.

\(^1\)Superscripts refer to numbers in the bibliography; see Appendix I.
Instantaneous frequency theory

The well-known steady-state theory of alternating currents is simplified by the fact that the e.m.f. and the currents in all the branches of a network in which the e.m.f. is impressed involve the time $t$ only through the common factor $e^{j\omega t}$ where $j = \sqrt{-1}$ and $\omega$ is the constant angular frequency. The network is completely specified by its complex admittance $Y(j\omega)$. If the e.m.f. is $E_0 e^{j\omega t}$, the steady-state current is $I_{ss} = E_0 Y(j\omega) e^{j\omega t}$. According to Carson and Fry, if the frequency is variable, the impressed e.m.f. may be written as

$$E \exp \left[ j \int_0^t \Omega(t) \, dt \right],$$

where $\Omega(t)$ is the instantaneous frequency. This can also be written as

$$E e^{j\Phi(t)}$$

if $\frac{d}{dt} \Omega(t)$ is the instantaneous frequency $\Omega(t)$ in radians per second and the notation $\Phi(t)$ means a function of time. Carson and Fry further state that "a pure frequency-modulated wave may be defined as a high-frequency wave of constant amplitude, the 'instantaneous' frequency of which is varied in accordance with a low-frequency signal wave." Thus

$$y = \exp \left[ j \omega_0 t + \int_0^t \Omega(t) dt \right]$$

$$= \exp \left[ j \omega_0 t + \Phi(t) \right], \quad \text{if} \quad \frac{d}{dt} \Phi(t) = \Omega(t),$$

is a pure frequency modulated wave. Here $\omega_0$ is the constant carrier frequency and $\Omega(t)$ is the low-frequency modulating frequency. The instantaneous frequency is then defined as $\omega_0 + \Omega(t)$. In corresponding terms, the pure amplitude-modulated wave is of the form

$$y = \Omega(t) e^{j\omega t},$$

and the frequency, $\omega$, is constant.
If a voltage is of the form
\[ V(t) = a(t) \cos (\omega_0 t + \phi(t)) \]
\[ = a(t) \cos \omega_0 t - b(t) \sin \omega_0 t \quad c(t) = \sqrt{a^2 + b^2} \]
\[ = \frac{1}{2} [a(t)+ib(t)] e^{i\omega_0 t} + \frac{1}{2} [a(t)-ib(t)] e^{-i\omega_0 t}, \]
the function \( c(t) = \sqrt{a(t)^2 + b(t)^2} \) is called the amplitude function or envelope of the voltage with respect to the radian frequency \( \omega_0 \), and \( \frac{d}{dt} \phi(t) \) is the instantaneous deviation of the radian frequency from the reference value \( \omega_0 \). If such a modulated voltage wave is applied to an ideal linear detector, the output across the load circuit is the envelope \( c(t) \). More generally, if a voltage is of the form \( a(t) \sin b(t) \), then \( a(t) \) is the envelope and the time derivative of \( b(t) \) is the instantaneous frequency.

In order to demonstrate the use of the instantaneous frequency concept, consider an elementary discriminator consisting simply of a pure inductance through which passes an a-c current,
\[ i = I \sin (\omega_0 t + \phi(t)). \]
Here \( I \) is a constant, equal to the maximum value of the current, and \( \phi(t) \) is some function of time. The voltage across the inductance is then
\[ e_L = L \frac{di}{dt} \]
\[ = LI \cos [\omega_0 t + \phi(t)] \frac{d}{dt} [\omega_0 t + \phi(t)] \]
\[ = [\omega_0 + \frac{d}{dt} \phi(t)] LI \cos [\omega_0 t + \phi(t)]. \]

In this case an applied current of constant amplitude and varying frequency yields a voltage response also of varying frequency but in addition of varying amplitude. The amplitude variations may be separated by the conventional process of detection; i.e., by rectifying the voltage response and by means of appropriate filters, recovering
the low-frequency variations. Here, of course, it is assumed that the variations to be separated are of considerably lower frequency than that of the carrier. The output voltage of this elementary discriminator is, then, a voltage proportional to a constant plus \( \frac{d}{dt} \theta(t) \), if the carrier amplitude and the inductance are constant. The output of the discriminator including the rectifier and filter is the instantaneous frequency function. If \( \omega_0 \) is a high-frequency carrier, then the frequency modulation applied to the carrier is \( \frac{d}{dt} \theta(t) \), and this is the dynamic part of the output of the discriminator. The concept of instantaneous frequency is used in frequency modulation theory, wherein the equation for the sinusoidally modulated wave is usually written:

\[
y = A \sin \left( 2\pi F t + \frac{\Delta F}{\mu} \sin 2\mu t \right).
\]

The argument of the sine is the phase angle, and the instantaneous frequency is the time derivative of the phase angle, divided by \( 2\pi \):

\[
\frac{d}{dt} \left( 2\pi F t + \frac{\Delta F}{\mu} \sin 2\mu t \right) = F + \frac{\Delta F}{\mu} \cos 2\mu t \quad \text{cycles per second},
\]

showing that the carrier frequency \( F \) is varied by the modulating frequency \( \mu \). \( \Delta F \) is a constant.

In order to apply the concept of instantaneous frequency, a complex wave must be written in the form \( a(t) \sin b(t) \). It can be seen that the definition of instantaneous frequency is in a sense based upon sinusoidal motion as fundamental in frequency. Without further specification, however, some ambiguity would be introduced. For example, the sum of two sine waves \( y = \cos \omega_1 t + \cos \omega_2 t \) might be written in a number of different forms.
\[ y = [2 \cos \frac{\omega_1 + \omega_2}{2} t] \cos \left[ \frac{\omega_1 - \omega_2}{2} t \right] \]

or \[ y = \left[ 1 + \cos \frac{\omega_1 t}{\cos \omega_2 t} \right] \cos \left[ \omega_2 t \right] , \text{ etc.}, \]

but in order to make our definition completely unambiguous and corresponding to physical facts, it is required that the Fourier components add vectorially. The voltage \( A \sin \omega t \) can be represented as a vector of length \( A \) rotating in the \( x,y \) plane with a constant angular velocity \( \omega \), in which case the voltage is the projection of the vector on the \( y \)-axis. The angle with the \( x \) axis is \( \omega t \) radians.

If two vectors represent two voltages of different frequency, they rotate at different angular velocities, and the angular velocity of the resultant \textit{varies} over each cycle, as does the length of the resultant vector. This vector is the desired \( a(t) \cos b(t) \). Such vector representation is considered more fully in a recent paper by Cocci and Sartori.

The amplitude function or envelope and the instantaneous frequency function for the sum of two sinusoidal waves may be derived in a manner representing vector addition: if

\[ u_1(t) = A_1 \cos \omega_1 t = \Re A_1 e^{i \omega_1 t} \]

and \[ u_2(t) = A_2 \cos (\omega_1 + 2\mu)t = \Re A_2 e^{i(\omega_1 + 2\mu)t} \]

wherein the angular frequencies are \( \omega_1 = 2\pi f_1 \) and \( \omega_2 = 2\pi f_2 \) and \( \mu = f_2 - f_1 \) (i.e., the difference frequency in cps.), then

\[ u_1 + u_2 = \Re \left[ A_1 e^{i \omega_1 t} + A_2 e^{i(\omega_1 + 2\mu)t} \right] \]

\[ = \Re \left[ e^{i \omega_1 t} (A_1 + A_2 e^{2i\mu t}) \right] \]

but \( A_1 + A_2 e^{2i\mu t} = (A_1 + A_2 \cos 2\mu t) + j(A_2 \sin 2\mu t) \)
therefore \( u_1 + u_2 = \text{Re} \left[ e^{j\omega t} \sqrt{(A_1 + A_2 \cos 2\omega t)^2 + (A_2 \sin 2\omega t)^2} \right] \)

\[
= \sqrt{(A_1 + A_2 \cos 2\omega t)^2 + (A_2 \sin 2\omega t)^2} \cos \left( \omega_1 t + \tan^{-1} \frac{A_2 \sin 2\omega t}{A_1 + A_2 \cos 2\omega t} \right)
\]

The instantaneous phase is the argument of the cosine and the instantaneous frequency is the time derivative of the phase. The instantaneous amplitude function is the multiplier of the cosine function. Writing \( x = A_2 / A_1 \) and taking the derivative of the phase,

\[
\omega = \omega_1 + 2\mu \left[ \frac{\mu^2 + x \cos 2\mu t}{1 + \mu^2 + 2\mu x \cos 2\mu t} \right], \quad \text{or}
\]

\[
f = f_1 + \mu \left[ \frac{\mu^2 + x \cos 2\mu t}{1 + \mu^2 + 2\mu x \cos 2\mu t} \right] \quad \text{in cycles per second.}
\]

As shown by Stumpere\(^5\), a further analysis of the formula for the instantaneous frequency of two sine waves shows that the function

\[
f(x, 2\mu t) = \frac{x^2 + x \cos 2\mu t}{1 + x^2 + 2x \cos 2\mu t}
\]

may be written as a Fourier series,

\[= x \cos 2\mu t - x^2 \cos 4\mu t + x^3 \cos 6\mu t - \ldots,\]

and that \( f(x, 0) = x/1+x, \ f(x, \pi/2) = x^2/1+x^2, \) and if \( x \neq 1 \) \( f(x, \pi) = -x/1-x. \) Also

\[
\lim_{x \to 1} f(x, 2\mu t) = 1/2 \quad \text{if } 2\mu t \neq \pi
\]

and

\[
\lim_{x \to 1} f(x, \pi) = -\infty
\]
As an example, consider the case of two sidebands produced in sinusoidal amplitude modulation: since the amplitudes are the same, \( f(x, 2\pi t) = 1/2 \) except at the instant when \( 2\pi t = \pi \), thus the instantaneous frequency is

\[
f = f_1 + \frac{1}{2} \mu, \quad \text{except at } 2\pi t = \pi,
\]

which is a frequency halfway between the two sidebands and equal to the carrier frequency, and constant except for a unit impulse at \( 2\pi t = \pi \), showing that for all practical purposes there is no frequency variations accompanying the amplitude modulation. In general the instantaneous frequency of the sum of two sine waves of equal amplitude is a constant frequency equal to the arithmetic average of the two frequencies.

The instantaneous frequency function for the sum of two sine waves is plotted in figure 2 on the next page. The instantaneous frequency is

\[
f_1 + \mu f(x, 2\pi t),
\]

and the second term in the above expression is plotted against \( 2\pi t \), for one cycle of \( f(x, 2\pi t) \), in units of \( \mu \). Zero on the vertical axis thus represents the frequency \( f_1 \), and \( 2\mu \), for example, represents a frequency of \( f_1 + 2\mu \). As the ratio of the amplitudes approaches unity, the instantaneous frequency swing approaches infinity, but the width of the peak approaches zero. With reference to the \( \frac{1}{2} \mu \) baseline, the area of the curves is constant as \( x \) is varied. In a practical circuit the accuracy of reproduction would be limited by the frequency response of the circuit.
INSTANTANEOUS FREQUENCY
FOR THE SUM OF TWO SINE WAVES
\[ f = \frac{1}{2} (f_1 + f_2) \]

At different times for one cycle of beat freq.

\[ f_1 = \frac{\cos \omega t + \cos \omega_1 t}{1 + \omega^2 + 2 \omega \omega_1 \cos \Omega t} \]

Plotted in units of \( \mu \),
\( \mu \leq 0 \) is one frequency and
\( \mu > 0 \) is the other frequency.
HETERODYNE ENVELOPE
FOR THE SUM OF TWO SINE WAVES
\[ e^{i \omega_1 t} + e^{i \omega_2 t} \]
\[ v = e^{i \omega_1 t} + e^{i \omega_2 t} \]
\[ y = \text{difference freq.} \]

FIGURE 2.

envelope \( e_{\text{envelope}} = e^{\sqrt{1 + x^2 + 2x \cos \omega_1 t}} \)
plotted in units of \( \omega_1 \) for one complete frequency cycle.

For \( x < 1 \) envelope \( e_{\text{envelope}} = e^{i + x \cos \omega_1 t} \)
for \( x = 1 \) envelope \( = 2e, |\cos \omega_1 t| \)
The frequency swing from $2\mu t = 0$ to the peak value at $2\mu t = \pi$ is $(x/1+x) + (x/1-x) = [2x/1-x^2]$. A few illustrative values are tabulated:

<table>
<thead>
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<th>$x$</th>
<th>swing units of $\mu$</th>
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<tbody>
<tr>
<td>0.1</td>
<td>0.22</td>
</tr>
<tr>
<td>0.2</td>
<td>0.416</td>
</tr>
<tr>
<td>0.4</td>
<td>0.954</td>
</tr>
<tr>
<td>0.6</td>
<td>1.875</td>
</tr>
<tr>
<td>0.7</td>
<td>2.744</td>
</tr>
<tr>
<td>0.8</td>
<td>4.440</td>
</tr>
<tr>
<td>0.9</td>
<td>9.48</td>
</tr>
<tr>
<td>0.99</td>
<td>99.1</td>
</tr>
</tbody>
</table>

For an amplitude ratio of 0.9 the maximum instantaneous frequency swing is 9.48 times the frequency difference of the two sine waves.

A mechanical analogy of the adding of two sine waves is sketched in Figure 1 on the next page. $a$ and $b$ represent the vectors which generate the two sine waves; they are represented by crank arms, $a$ pivoted on a fixed mount at point 1 and $b$ pivoted to $a$ at point 2. $a$ and $b$ are made to rotate at the proper angular velocities $\omega_a$ and $\omega_b$. Then the resultant vector is crank $g$ connecting the fixed pivot point 1 and the point 3. Crank arm $g$ must be telescopic so as to allow for variation in length, and the angular velocity of $g$ as determined by the two other cranks varies considerably in each revolution. The length of $g$ as a function of time is the amplitude function of the sum of two sine waves, and the angular velocity as a function of time is the instantaneous frequency. From this analogy it can be seen that the instantaneous frequency function is continuously defined; it does not depend upon the completion of one cycle of the frequency. If the crank $g$ in the analogy were connected to a siren, the pitch of its sound would vary according to the instantaneous frequency.
FIG. 1 - MECHANICAL ANALOGY

\[ E = a \sin \omega_a t + b \sin \omega_b t = c(t) \sin \omega_c(t) t \]

Inst. freq. = \( \frac{d}{dt} \left[ \omega_c(t) t \right] = \text{angular velocity} \)

\[ a = \text{constant} \]
\[ b = \text{constant} \]
\[ c = c(t) \]
\[ \omega_a = \text{constant} \]
\[ \omega_b = \text{constant} \]
\[ \omega_c = \omega_c(t) \]

max. value of \( C(t) \) = \( a + b \)

min. value of \( C(t) \) = \( a - b \)
In general a complex wave may be written as a sum of cosinusoidal components, either through the use of Fourier series or the Fourier Integral\(^\text{10}\). Then if the general wave is
\[[y = \sum a_n \cos (2\pi F t + \delta_n(t)), \quad n = 0, 1, 2, \ldots, \text{and } \delta_0 = 0;\]
this may be written\(^\text{11}\)
\[[y = R \cos (2\pi F t + \Delta), \quad \text{wherein } R = \sqrt{\left(\sum a_n \cos \delta_n\right)^2 + \left(\sum a_n \sin \delta_n\right)^2};\]
and
\[[\tan \Delta = \frac{\sum a_n \sin \delta_n}{\sum a_n \cos \delta_n}.\]

Then \(R\) is the instantaneous amplitude function and the time derivative of \([2\pi F t + \Delta]\) is the instantaneous frequency function.

\[[2\pi F t + \Delta] = 2\pi F t + \tan^{-1} \frac{\sum a_n \sin \delta_n}{\sum a_n \cos \delta_n}.\]

and taking the derivative,
\[[\omega = 2\pi F + \frac{1}{1 + \left(\sum a_n \sin \delta_n\right)^2} \frac{d}{dt} \left\{\sum a_n \sin \delta_n\right\} = 2\pi F + \frac{\sum a_n \cos \delta_n \frac{d\delta_n}{dt} + \sum a_n \sin \delta_n \frac{d\delta_n}{dt}}{\left(\sum a_n \cos \delta_n\right)^2 + \left(\sum a_n \sin \delta_n\right)^2}.\]

This is the general equation for the instantaneous frequency of the general wave in Fourier form \(y = \sum a_n \cos (2\pi F t + \delta_n(t))\). For harmonically related components, \(\delta_0 = 0, \delta_1 = 2\pi F t + k_1, \delta_2 = 2\pi F t + k_2\), etc., where the \(k\)'s are phase constants. If the sum of two sine waves is considered using these general equations, the results will be exactly the same as in the previous special analysis (Page 487).
Practical considerations

The graph of the instantaneous frequency function for the sum of two sine waves (Figure 2) shows that as the ratio of the amplitudes, \( x \), goes from values less than unity to values greater than unity, the polarity of the peak is reversed. Replacing the ratio of the amplitudes by its reciprocal changes only the polarity of modulation and not the shape of the curve. The envelope, or amplitude function, on the other hand, is always positive, however it goes through zero when the ratio of the amplitudes is unity, as shown in Figure 3.

Referring again to the analysis of the simple discriminator, it was necessary to the operation of the discriminator that the current input to the discriminator be of constant amplitude. In the practical consideration, a limiter is suggested. An ideal limiter clips the input wave severely as it swings positive or negative, so that its output is a square wave, and has a very low internal generator impedance. The square wave thus formed is applied to a tuned circuit tuned to the carrier frequency, so that the small frequency variations due to modulation are easily transmitted unattenuated but harmonics of the carrier are suppressed and the square wave is again made very nearly sinusoidal. The original frequency variations may then be separated by the discriminator.

In order to produce the envelope and the instantaneous frequency functions for general audio-frequency waves, the signal may be amplitude modulated upon a high-frequency carrier and the carrier and one set of sidebands suppressed; then the envelope of the high-frequency wave is the same as the envelope of the low-frequency modulating signal and the frequency variations of the high-frequency
wave are also the same as the frequency variations of the low-frequency modulating wave. This may be shown mathematically by considering the equation for the amplitude modulated wave:

\[ y = \left[ 1 + s(t) \right] \cos \omega_c t \]

wherein the low frequency modulating wave is the general \( s(t) \) and the carrier frequency is \( \omega_c \). If the carrier is suppressed, the equation becomes

\[ y = s(t) \cos \omega_c t. \]

Expressing the modulating function \( s(t) \) as

\[ s(t) = S(t) \cos b(t) , \]

which has been shown on page 13 to involve no loss in generality, the amplitude modulated wave with carrier suppressed becomes

\[ y = S(t) \cos b(t) \cos \omega_c t \]

\[ = \frac{1}{2} S(t) \left[ \cos (\omega_c t + b(t)) + \cos (\omega_c t - b(t)) \right] \]

and suppressing the difference frequencies, i.e., the lower sidebands, there remains

\[ \frac{1}{2} S(t) \cos [\omega_c t + b(t) ]. \]

This is the upper sideband spectrum and contains radian frequencies of \( \frac{d}{dt} [\omega_c t + b(t) ] \). The phase angle is that of the modulating function plus a linear function, so the derivative of the phase angle is the derivative of the phase angle of the modulating function plus a constant. Therefore the same variations in frequency appear in the single sideband suppressed carrier signal as in the modulating signal. The envelope of the single sideband suppressed carrier signal is a constant times the envelope of the audio frequency signal.
Applying this wave to an ordinary amplitude modulation detector of suitable design the output is proportional to $S(t)$, the original envelope or instantaneous amplitude function of the modulating signal. Applying this wave to a limiter and discriminator, the output is proportional to the time derivative of $b(t)$, the instantaneous frequency of the modulating signal.
Experimental Approach

The objective of the laboratory work was to build a device which would derive from any audio frequency input wave its instantaneous frequency, and would indicate this as an output voltage proportional to frequency. For repetitive waves the frequency function could then be observed on an oscilloscope and compared directly to the input wave. The instantaneous frequency corresponding to each point on the input wave would be indicated.

The experimental method of obtaining the frequency function was relatively simple: a single sideband suppressed carrier signal was obtained at 455 kc using conventional methods and this was applied to a conventional limiter and discriminator; as shown on page 15, the frequency variations in the 455 kc single sideband signal are the same as the frequency variations in the modulating signal, and the output of the discriminator is a voltage proportional to the instantaneous frequency of the audio frequency input wave.

The Circuit. (M1)

The first unit constructed (M1) is described by the block diagram of Figure 4 and the Circuit diagram of Figure 5. The input signal is amplified in the audio amplifier and modulated upon the 10 kc signal from the 10 kc oscillator in a balanced modulator. In the balanced modulator, the 10 kc carrier frequency is balanced out and only the sum and difference frequencies (the sidebands) appear in the output. The frequency of the output voltage of the first balanced modulator is thus 10 kc plus and minus the modulating frequency. This output is then applied to the National F-22 single sideband filter, which has a very sharp cutoff below 10 kc and a relatively flat passband above 10 kc.
FIGURE 4.
BLOCK DIAGRAM OF M-1 ECF UNIT.
FIGURE 5.
WIRING DIAGRAM: M-1 CCF UNIT
DESIGNED AND BUILT BY
LOUIS A. STEVENSON, JR.
MAY 1, 1950 - THE RICE INSTITUTE
FIGURE NO. 1. CHARACTERISTIC CURVE OF F-22.

NOTE
CURVE BASED ON AUDIO AMPLIFIER OUTPUT OF 250 TO 3000 CYCLES

F-22 SIDE BAND FILTER

FIGURE NO. 5b. SCHEMATIC DIAGRAM OF F-22 AND ADJACENT STAGES.
Figure 6.
Test setup for observing instantaneous frequency.

Figure 7.
Connections for observing INST. Amplitude function.
to 13 kc, as shown in Figure 5b. The output of the sideband filter contains only the upper sidebands, or the sum frequencies, but not the carrier and not the lower sidebands or difference frequencies. This signal is then applied to a second balanced modulator and modulated upon a 445 kc carrier. The carrier is again balanced out, and the output contains sum and difference frequencies of 445 kc plus and minus the frequencies between 10 and 13 kc from the sideband filter. The upper and lower sidebands are here separated by a minimum of 20 kilocycles, and therefore are easily separated in the selectivity of the following 456 kc amplifier stages. Only the upper sidebands are amplified. These are then applied to a limiter stage to remove amplitude variations and then to a conventional narrow-band FM discriminator designed to work at 456 kc. The output is a low-level rectified 456 kc voltage, essentially d-c, filtered for audio-frequency variation, which is proportional to the frequency of the input wave.

The circuit for the production of the single sideband signal at 456 kilocycles is essentially one that appeared in QST. The limiter and discriminator for narrow-band FM (NBFM), is described in recent issues of the APRL Handbook. The specifications of the National NBFM discriminator transformer indicate a linear range of approximately 12 kilocycles, centered about 456 kilocycles.

The envelope function may be obtained in practice in either of two ways: for viewing the envelope of the sum of two sine waves on the oscilloscope, the input wave itself may be connected to the oscilloscope and the scope sweep synchronized with the difference frequency. Then, if the two sinusoidal components are not both
multiples of the difference frequency, the envelope will be displayed. This was the method used in Figures 14 through 19. A simple method of synchronizing the oscilloscope at the difference frequency is to synchronize with the instantaneous frequency pulse. Of course, if the frequency components of the input wave are multiples or sub-multiples of the sweep frequency, only the input wave will appear. The more general method of observing the amplitude or envelope function is through the use of a simple AM detector as shown in Figure 7, connected to the secondary of the second IF output transformer. This additional load considerably disturbs the operation of the limiter and discriminator, so it was not made permanently on the experimental unit; however the photographs of envelope functions of Figures 31, 34, and 35 were taken using this method.

**Limiter:**

After tuning the unit (see Appendix II for tuning procedure) the data of Figures 8 and 9 was taken. In order to test the operation of the limiter, the output voltage was measured as the input voltage was varied, at constant frequency. Since the output of a discriminator is proportional to the input voltage as well as to the input frequency, the output voltage should increase until the limiter begins to operate. The limiter curve shown in Figure 8 shows that the variation in output was only 7 percent for variation of input from 4 to 40 millivolts RMS with the gain control at maximum. Input voltages over 40 millivolts RMS may block the audio amplifier stages, as was indicated by observation of waveforms at the plate of the second audio amplifier stage. This would also be indicated as additional frequency components in the output if the harmonics
thus generated fell within the frequency range of the unit. The input voltage for all tests was accordingly adjusted for a maximum of approximately 40 millivolts RMS in order to obtain the maximum limiting. In later tests, (Figures 32 and 33,) the sudden rise of a square wave input voltage did not cause any exactly corresponding rapid rise in the output voltage, indicating that the limiting is probably even better than indicated by the curve of static response of Figure 8.

**Discriminator: M1**

Proper operation of the limiter permits proper operation of the discriminator. Figure 9 shows the output voltage of the unit for an input voltage of a sine wave of approximately 30 millivolts RMS as the frequency of the input voltage is varied. In the tuning of the unit, the 10 kc oscillator may be shifted slightly in frequency, making it possible to place the carrier frequency as close as is desired to the 3 kc passband of the sideband filter. In order to obtain best rejection of the undesired sidebands and the remaining carrier which could not be perfectly balanced out, the 10 kc carrier was placed about 100 cycles below the sharp cutoff of the filter; then the filter determines the frequency range of the unit as approximately 100 to 3000 cycles. As shown in Figure 9, the output voltage above and below this range will drop sharply to voltages below the output for 100 cycles. If the output is used to control other equipment it would probably be desirable to include a clipper to prevent the output voltage from going below the voltage representing zero frequency, since the output goes below this value and is very
**Figure B**

**Constant-Frequency Characteristic, ecf, Unit, M.I**

(MAX. GAIN SETTING)

Within 7% over 0.004 volts input.

Audio system blocks at 50 millivolts.

Output constant for inputs greater than 0.05 volts.
No distortion in output at 1 volt input.

R.M.S. input voltage 2200 cps sine wave.
erratic for input frequencies outside the operating range of the unit or for no input at all. For a pure sine wave input voltage, the output is a constant voltage, as expected. For an input wave containing harmonics, the output voltage varies as discussed in the following section. For an input of swept-frequency audio frequency sine waves from the Clough-Brengle sweep oscillator, the output is the exponential curve corresponding to the logarithmic sweep of frequency, up to the cutoff of the unit.

Photographs of waves from the oscilloscope

The simplest, and the fundamental, complex wave considered is that of the sum of two sine waves. The mathematical discussion indicated what the instantaneous frequency function should look like. (See Figure 2). For small values of amplitude ratio, the instantaneous frequency function resembles a sinusoidal wave, while for values of the amplitude ratio near unity, the frequency function becomes a narrow peak, and as the ratio goes through unity the polarity of the modulation changes. The maximum frequency swing which can be indicated is limited by the frequency range of the discriminator; whereas the system before the limiter need only exhibit sufficient bandwidth to pass the approximately 3 kc range of frequency components, the limiter and discriminator must pass the instantaneous frequency swings, which approach infinity when the amplitude ratio approaches unity. If the discriminator has the specified 12 kc linear range and if the reference frequency is at the center of the frequency range, then the maximum frequency swing is only 6 kc. In order to allow for instantaneous frequency swings in both directions, the discriminator was
adjusted so that the sidebands fall approximately in the middle of the linear range; thus no more than 6 kc swing can be indicated.

If the difference frequency is 1000 cycles, this corresponds to an amplitude ratio of approximately 0.88. Amplitude ratios nearer unity would not be indicated. The shape of the instantaneous frequency curve for the sum of two sine waves depends only upon the ratio of the amplitudes and not on the frequencies or the phase relationships. The sharpness of the pulse obtained when the amplitudes are approximately the same will depend upon the frequencies but can never represent a swing of over 6 kc. These predictions were verified; the best photograph of the instantaneous frequency function for the sum of two sine waves is shown in Figure 20, and Figures 16 through 21 show other similar photographs. The repetition rate of the instantaneous frequency cycle was always the difference frequency; the polarity was observed to change as the ratio of the amplitudes went through unity; and the maximum peak occurred just as the polarity changed. The peak was observed to narrow and become higher as the amplitudes were made more nearly the same.

The envelope function for the sum of two sine waves is shown in Figures 14 through 19. The theory was verified in that the envelope function goes through zero only when the amplitudes of the two sine wave components are the same, and in that the envelope is a rectified cosine wave at half the difference frequency when the amplitudes are the same. For amplitudes very different, the envelope approaches a cosine wave at the difference frequency but of small amplitude, and the envelope is constant for only one sine wave input.
Large variations in the amplitude function seemed to accompany large variations in the frequency function, and vice versa. The peak in the frequency function corresponded in time to the minimum in the envelope function. For an input of the sum of two sine waves, the envelope function could be distinguished from the frequency function by two very evident characteristics: (1) the frequency function changes polarity when the amplitude ratio goes through unity and the envelope function does not, and (2) the frequency function becomes a narrow pulse while the envelope function becomes a rectified cosine wave as the ratio of the amplitudes approaches unity. Both methods discussed on Page 22 for obtaining the envelope function gave the same results for the sum of two sine waves. For more complicated waves, the second method was necessary, using an AM detector as shown in Figure 7, and the envelope waveforms of Figures 31, 34, and 35 were obtained using this method.

The envelopes for sawtooth waves of different frequencies are shown in Figure 31, and for square waves of different frequencies in Figures 34 and 35. The instantaneous frequency functions for the sawtooth are shown in Figures 22 to 30 and for the square wave in Figures 32 and 33. Since sawtooth waves contain all harmonics of the fundamental frequency, they are of particular interest here.

The sawtooth wave is of the form

\[ e(x) = \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \ldots \]

Placing the fundamental frequency of the sawtooth sufficiently close to the high frequency cutoff of the unit, the unit may be made to indicate the instantaneous frequency of the sum of the first 1, 2, 3, 4, etc., sine wave components, up to the limit imposed by the band-
width of the unit. Figure 22 shows the characteristic curve obtained by viewing the instantaneous frequency of a sawtooth of sufficiently high frequency that only two sinusoidal components are indicated. As can be seen from the series expansion of the sawtooth, the ratio of the amplitudes of the first two components is 2 or \( \frac{1}{2} \), and reference to the curves of Figure 2 will show that the instantaneous frequency indication is of the correct shape. Figure 30 shows the instantaneous frequency of the lowest frequency sawtooth, approximately 100 cycles per second. Within the bandwidth 100 to 3000 cycles there would be 30 harmonics of 100 cycles, which should allow a very good representation of a sawtooth. The ratio of amplitudes of harmonics above the 9th would be between 0.9 and 1, so large variation in the instantaneous frequency curve would be expected, but the height of the peak in the curve is limited by the capabilities of the unit. However Figure 30 is probably a fairly good representation of the instantaneous frequency of a sawtooth wave. Figures 22 to 30 are photographs of instantaneous frequency function for a sawtooth wave using the sawtooth to sweep the oscilloscope, so that the phase or time relationship between the trace and one linear sawtooth is very precise. Several facts may be noted in passing: that the instantaneous frequency function is not discontinuous during the flyback time of the trace, and that the polarity of the instantaneous frequency peak for very low frequencies is different than that indicated for higher frequencies.

The pictures of the envelope function of the sawtooth shown in Figure 31 were taken using the sawtooth wave for sweep, as before,
so again the time relationship is precise. As noted before, the variations in envelope correspond to variations in frequency; the maxima or minima usually occur at the same time. With no variation in envelope there is no variation in frequency, and vice versa.

Envelope and frequency functions for the square wave are shown in the photographs of Figures 32 to 35. The close similarity between the curves of instantaneous frequency of a square wave and the sawtooth wave is to be noted; the first curve of Figure 33 may in particular be directly compared to the curve of Figure 24. It will be recalled that the shape of the instantaneous frequency curve for the sum of two sine waves depends only upon the ratio of the amplitudes and not on the frequencies. For more than two sine components, the shape also depends upon the phase relationships; however in both the sawtooth and square waves all components go through zero at the start of the wave, so the phase relationships should be the same. The square wave may be written as

$$\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \ldots$$

The ratio of the amplitudes is slightly different from that of the sawtooth: 3 to 1 rather than 2 to 1 for the first two components, for example, but the ratios are sufficiently similar to give similar instantaneous frequency curves. The envelope functions of Figures 34 and 35 likewise show similarities with those of Figure 31.

A look at the general equations for the instantaneous amplitude or envelope function and the instantaneous frequency function will show that for more than a very few sinusoidal components, calculation.
of these functions would be very laborious if not prohibitive, but using the instantaneous frequency unit these functions may be viewed immediately for a large variety of input waves.

The second unit

The first instantaneous frequency unit (M1) constructed had several serious limitations: its input frequency range was too restricted (180 to 3000 cycles), its output voltage was very low, and its output fidelity was too restricted by the linear range of the discriminator and the required filter following the discriminator. To solve these problems, the following changes were proposed: to obtain the single sideband suppressed carrier signal using the new "phasing" system, which allows better audio frequency fidelity and better carrier rejection, and to operate the discriminator at 10 megacycles so that the necessary filtering would affect the output less and a discriminator transformer of much wider linear range could be obtained commercially. Discriminator transformers for 10.7 megacycles are available with a linear range of 150 kc. If more amplification were included in the unit and limiting were done at a higher level, the output voltage would be higher.

The circuit of the M2 unit is shown in block form in Figure 10 and the circuit schematic is shown in Figure 11. The circuit was adapted from one that appeared in The Radio Handbook, together with the discriminator circuit recommended for use with the National discriminator transformer. The audio input at a level of 2 \( \frac{1}{2} \) volts is applied to the Millen 75011 audio frequency phase shift unit, the circuit of which is shown in Figure 11a. This unit provides two
output voltages of the same magnitude but differing in phase by almost exactly 90° throughout the frequency range of 70 to 5400 cycles. Each output is balanced with respect to ground to allow push-pull operation in each output. These voltages are modulated in two balanced modulators upon 10 megacycle carriers shifted in phase so that they, too, differ in phase by 90°. The 10 megacycle carrier originates in a simple crystal-controlled tri-tet oscillator, and is shifted in phase by a simple R-C network in each of two branches. The carrier is balanced out in each of the two balanced modulators, and the outputs of the two balanced modulators are combined in such a manner that one set of the sidebands cancel out. For a further discussion of this method of producing single-sideband signals, see reference (16) in the bibliography. The single sideband suppressed carrier signal at approximately 10 megacycles is then limited in a conventional limiter stage and applied to a conventional FM discriminator. The output of the discriminator should be the instantaneous frequency function.

M2 Results

The frequency and limiter characteristics for the M2 unit are shown in Figures 12 and 13. Photographs of the output waveforms for various input waveforms are shown in Figure 40, and in Figure 41 is shown the output waveforms for the two instantaneous frequency units for the same inputs, using the same sweep for both traces so that the outputs correspond in phase. Several peculiarities were noted: (1) the instantaneous frequency function for the sum of two sine waves did not change polarity as the ratio of the
Figure 10.
Block diagram M-Z E-CF unit.
amplitudes was varied through unity, and (2) the output waveform resembled the envelope function more than the frequency function, and (3) the output waveform for square and sawtooth waveform inputs showed a discontinuity corresponding to the discontinuity of the input waveform and did not resemble output from the other unit. From the frequency characteristic curve it can be seen that the output does vary with frequency, as can also be seen from the sweep-frequency photograph of Figure 40, and from the limiter curve it can be seen that some limiting exists. However the signal level measured at the limiter was too low for the limiter to operate properly, and the indication seems to be that in operation the limiting is not sufficient, so that the output of the discriminator contains components proportional to the original wave and the amplitude function in addition to the component proportional to the frequency function.

The first recommendation for the improvement of the M2 unit is therefore to include two 10 megacycle intermediate-frequency type amplifier stages before the limiter, to make certain that limiting is very severe, and perhaps even to add another stage of limiting. Some developmental work remains to be done on this unit, however it is believed that the basic design is capable of producing very good results. Tuning procedure is given in Appendix III.

Summary of results

The M1 instantaneous frequency unit was capable of producing an essentially d-c output voltage proportional to the frequency of
the input signal for constant-frequency sine waves within its range of frequencies, and in the case of an input signal of the sum of two sine waves the unit indicated an instantaneous frequency function that corresponded to the predictions of the theory, with certain limitations due to the limited frequency response of the component circuits. On the basis of this and the theoretical analysis of the circuit, it was concluded that the unit produces an output voltage proportional to the instantaneous frequency for any audio frequency signal, within its limitations. These limitations seemed to be primarily that the input frequency range was 100 to 3000 cycles, and that the limit of frequency swing of the instantaneous frequency that could be indicated was about 6 kilocycles. Inputs having components outside of these limits, however, cause no ill effects but simply are not indicated in the output.

Conclusion

The instantaneous frequency of audio frequency waves, defined as the angular velocity of the vector representing the vector sum of the Fourier components of the waves, and the envelope function, defined as the length of that vector as a function of time, may be displayed on an oscilloscope through the use of equipment of the type described in this paper. The experimental results indicated that a number of improvements could be made in the equipment; if these were made the operating range could probably be extended considerably. Such a device could have possible uses in the analysis of audio frequency waves and related work.
FIGURE 14. In each picture: sum of two sine waves, above, and envelope, below. At left: amplitudes approximately the same; at right, different. Sweeps differ for wave and envelope so phase correspondence is not good.

FIGURE 15. Waves of greatly different frequency with envelopes below. Envelope cycle is at difference frequency.
FIGURE 16 (left) and 17 (right). In each case the top wave is the envelope, the second wave down is the instantaneous frequency, the third is the input wave, the sum of two sine waves, and at bottom the instantaneous frequency again. Phase correspondence is fairly good here; peaks in the frequency function correspond to minima in the envelope, and the frequency function was repeated in the picture with the original wave without disturbing the phase correspondence. To get the envelope picture, one frequency component was changed slightly in frequency so as to unsynchronize it, but the envelope is correct nevertheless.
FIGURES 18 and 19. Heading from top to bottom: envelope, instantaneous frequency, input waveform, and instantaneous frequency, for the sum of two sine waves. Instantaneous frequency peaks should correspond to minima in envelope; phase correspondence is fairly good.
FIGURE 20. Best photograph obtained of the instantaneous frequency function for the sum of two sine waves of almost the same amplitude. This is to be compared with the curves of Figure 2. Sweep frequency is the difference of the two input component frequencies.

FIGURE 21. The sum of two sine waves and the corresponding instantaneous frequency function. In this case the envelope of the wave is apparent from the photograph of the wave itself.
FIGURE 22. Instantaneous frequency of a sawtooth wave for one cycle of the sawtooth. This is at a sufficiently high frequency that only two sinusoidal components are indicated.

FIGURE 23. Instantaneous frequency of a sawtooth wave for one cycle of the sawtooth. This is at a slightly lower frequency than the above. Probably three components are indicated.

FIGURE 24. Instantaneous frequency of a sawtooth wave for one cycle of the sawtooth. Lower frequency than above; probably four components.
FIGURES 25, 26 and 27. Instantaneous frequency of a sawtooth for one cycle of the sawtooth. These photographs show progressively lower sawtooth frequencies, so that progressively more frequency components are indicated.
FIGURE 28. Inst. frequency of a sawtooth for one cycle of the sawtooth. These pictures show frequency functions for progressively lower frequency sawtooth waves.

FIGURE 29. Same as above but lower frequency sawtooth.

FIGURE 30. Frequency function for lowest frequency sawtooth. Note the change in polarity. Zero reference line was moved so as not to interfere. Frequency is about 100 cps.
Figure 31. The envelope function of a sawtooth wave. High frequency sawtooth at top, low frequency at bottom. Trace at right is for one sawtooth, using the sawtooth for sweep, so the time relationship is exact.
FIGURE 32. Instantaneous frequency function of a square wave. High frequency square wave at left has two components in the passband of the equipment, lower frequency wave at right probably has three components in the passband.

FIGURE 33. Instantaneous frequency of a square wave. Medium frequency square wave at left and low frequency square wave at right. Note the change in polarity; this is to be compared with figure 30. Wave at right is for square wave of approximately 100 cycles.
FIGURE 34. Envelope function for square wave. At left, high frequency square wave; at right, medium frequency square wave. Time correspondence is approximate.

FIGURE 35. Envelope function for square wave. At left, medium frequency square wave; at right, very low frequency square wave. These are to be compared with figure 31.
FIGURE 36. Instantaneous frequency and envelope for the sum of three sine waves. These functions vary considerably with different phase and amplitude relationships.

FIGURE 37. Instantaneous frequency of the sum of three sine waves and the sum itself. This is approximately the same wave as in Figure 36.
FIGURE 38. Direct comparison of frequency function (above) and envelope function (below) for the sum of two sine waves, as indicated by the MI unit. Connection of the AM detector disturbed the operation of the discriminator section, so these pictures are not the best.

FIGURE 39. Instantaneous frequency function and envelope function of a sawtooth wave showing exact time relationship. In the picture at left the envelope is below, at right the envelope is above. These pictures are not as good as the previous ones because the AM detector disturbed the operation of the discriminator.
FIGURE 40. Instantaneous frequency functions as indicated by the two units: M1 above and M2 below in each case. At top, the sawtooth, at left, the sum of two sine waves.
FIGURE 40. Instantaneous frequency functions as indicated by M2 unit. Above, sum of two sine waves. At left, audio sweep. Below, sawtooth and square waves. These are all incorrect.
APPENDIX I. BIBLIOGRAPHY


3. Ibid., p. 522.


16. Ibid., p. 82.
APPENDIX II. TUNING THE M1 UNIT

If the M1 unit is approximately in tune, some of these steps will be insignificant.

1. Check the 10 kc oscillator (6J5) for oscillation.

2. Check the 6SL7 audio stages for proper operation.

3. Connect a panoramic spectro scope to the output of the F-22 filter (with suitable beat-signal source for a 10 to 13 kc input); connect a 3 kc, 40 millivolt signal to the input of the unit, gain control at 3/4 maximum. Carrier and sideband will be visible on spectro scope. Adjust frequency of 10 kc oscillator to edge of selectivity curve of filter. Balance the 1st balanced modulator for minimum carrier using the 30 ohm pot.

4. Connect the panoramic spectro scope to the secondary of T3 (see Fig. 5). Check the 6V6 445 kc oscillator for oscillation. T7 may be tuned for maximum output from the oscillator. Balance the 2nd balanced modulator for minimum carrier using the 50 micromicrofarad trimmer and the 5000 ohm plateode pot.

5. Connect the panoramic spectro scope to one plate of the 6H6. Tune T3, T4, and T5 for maximum sideband signal.

6. Connect an oscilloscope to the output and a sweep frequency audio signal to the input, using the same sweep for the oscillator and the oscilloscope. Adjust T6 (only) for good curve.

APPENDIX III. TUNING THE M2 UNIT

The Millen unit does not require tuning.

1. Check the 6AG7 crystal oscillator for oscillation. C1 should be just slightly meshed. Tune T1 for maximum output.

2. Connect the panoramic spectro scope to one plate of the 6H6, with suitable beat signal for viewing a 10.7 mc. signal. Connect a 2½ volt 5 kc signal to the input to the M2 unit. Unbalance the cathode potentiometers in the balanced modulator. Carrier and sideband should be visible on the 'scope. Balance the cathode pots for zero carrier, balance the grid pots for zero indication of the other sideband. Tune the IF transformers for maximum desired sideband.

3. Connect sweep audio input and tune T2 for good output curve.
APPENDIX IV. ADDITIONAL PHOTOGRAPHS.