A Procedure for Calculating the Best Exponents for Signal Representation

by

Yih-Guang John Jan

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF Master of Science

Thesis Director's Signature

Houston, Texas

May, 1970
ABSTRACT
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YIH-GUANG JOHN JAN

The approximation of a given signal over \((0, \infty)\) by a linear combination of a given number \(n\) of exponentials in such a sense that the integrated squared error is minimized over both the coefficients of the linear combination and the exponents used is discussed. The necessary conditions for the minimizations lead to nonlinear equations. Analog and digital computer implementations have been developed that are capable of adjusting the exponents of the basis functions and the coefficients of the linear combinations to minimize the integrated squared error between a signal and its representation.

The influence of each parameter of the orthonormal filters on the performance criterion is determined using an iterative convergent process. These parameter changes are introduced into the analog and digital systems to form a closed loop system.
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ACKNOWLEDGEMENTS

The author would like to express his gratitude to Dr. T. W. Parks - Research Director, for his many suggestions during the past two years.

Special thanks are due to Drs. R. J. P. de Figueiredo and C. S. Burrus for many constructive criticisms.

The research was supported by National Aeronautics and Space Administration under International Fellowships program and the Ministry of Education of the Republic of China.
Any signal can be considered as a signal vector in an infinite-dimensional signal space. The least-squares approximation of a signal is just the projection of the signal vector in certain finite dimensional vector space by the basis functions which span the vector space.

The most important thing is to choose the basis functions in such a way that uses the least number of basis functions to represent the signal. One of the most powerful sets of basis functions is that constructed from exponentials. A simple and elegant method for the orthogonalization of exponential functions has been described by Kautz (1) using the pole and zero cancellation concepts associated with the frequency domain representation. Orthogonal filters can easily be implemented with analog components, using Kautz's orthogonalization procedures.

Using the analog computation, it is possible to implement the iterative operations that automatically use the results of one analog computation to derive the operations and parameter values of filters to be used in the subsequent computations.

Owing to the development of the high-speed digital components, it is possible to perform the signal representation with a digital computer. The
discrete form (i.e., the sampled data) of orthonormal exponentials had been derived. (9)

In dealing with the sampled data, expressed mathematically by their $z$-transforms, it is to be noted that the $z$-transforms of the continuous orthonormal exponentials are not orthogonal to each other in the $z$-domain. To avoid the difficulty, the discrete orthonormal exponentials are chosen in such a way that the poles in the $z$-domain corresponding to those in the frequency domain, the zeros are different, but they are chosen in such a manner that when the sampling interval approaches zero, the discrete orthonormal exponentials will approach the continuous orthonormal exponentials.

Using McDonough and Huggins' criterion (3), many signal representation systems will be performed with analog and digital computers. The experimental results will be compared with previous analytic calculations and compared with previous experimental results (6).
Chapter 1

Signal Representation

There are many different meanings of signal. We will consider the representation of a signal as a real, continuous time function can be denoted as \( f(t) \) for \( t_1 \leq t \leq t_2 \) where \( 0 \leq t_1, t_2 < \infty \).

We want to approximate the signal \( f(t) \) by a collection (or class) of basis functions. The choice of the basis functions may be based that the maximum approximation error to be minimized, or the \( p \)th norm of the approximation error to be minimized. For practical considerations, only a finite number of basis functions will be used, and in order to obtain an explicit expression for the coefficients of representation we will use the integrated squared error criterion.

Let \( \{ \Xi_l(t) \} \) \( l=1,2,3,...,n \) be the set of basis functions which are used to represent the signal \( f(t) \). Then the approximation function can be expressed in the following manner:

\[
 f^*(t) = \sum_{l=1}^{n} \alpha_l \Xi_l(t) \quad t_1 \leq t \leq t_2 \tag{1-1}
 \]

where \( \alpha_l \)'s are the coefficients of the linear combination for the representation of \( f(t) \).

Although the basis functions \( \Xi_l(t) \) in equation are not necessarily orthogonal or normalized, however, the evaluation of the coefficients is much simplified if the basis functions are orthonormalized.
The orthonormality is defined by:
\[
\int_{t_1}^{t_2} \Phi_i(t) \Phi_j(t) \, dt = \delta_{ij}
\]
where \( \delta_{ij} \) is the kronecker delta,
\[
\delta_{ij} = 1 \quad \text{for} \ i = j
\]
\[
= 0 \quad \text{for} \ i \neq j
\]

The set of basis functions corresponding to the set of orthonormal functions \( \Phi_i(t) \)'s are called orthonormal functions. For given orthonormal functions, we want to find the coefficients \( a_i \)'s such that the integrated squared error:
\[
E = \int_{t_1}^{t_2} e(t)^2 \, dt
\]
\[
= \int_{t_1}^{t_2} (f(t) - \sum_{i=1}^{n} a_i \Phi_i(t))^2 \, dt
\]
\[
= \int_{t_1}^{t_2} f^2(t) \, dt - 2 \sum_{i=1}^{n} a_i \int_{t_1}^{t_2} f(t) \Phi_i(t) \, dt + \sum_{i=1}^{n} a_i^2 \int_{t_1}^{t_2} \Phi_i^2(t) \, dt
\]
is minimized.

Where \( e(t) \triangleq f(t) - \sum_{i=1}^{n} a_i \Phi_i(t) \)

Because of the orthonormality of \( \Phi_i(t) \)'s
\[
E = \int_{t_1}^{t_2} f^2(t) \, dt - 2 \sum_{i=1}^{n} a_i \int_{t_1}^{t_2} f(t) \Phi_i(t) \, dt + \sum_{i=1}^{n} a_i^2 \int_{t_1}^{t_2} \Phi_i^2(t) \, dt
\]

For minimizing \( E \), we take derivative of \( E \) with respect to \( a_i \) for \( i = 1, 2, \ldots, n \) and set them equal to zero.

i.e.
\[
\frac{\partial E}{\partial a_i} = -2 \int_{t_1}^{t_2} f(t) \Phi_i(t) \, dt + 2 a_i = 0
\]
\[
\therefore a_i = \int_{t_1}^{t_2} f(t) \Phi_i(t) \, dt, \quad i=1, 2, \ldots, n
\]
(1-4)

So the corresponding integrated squared error is:
\[
E = \int_{t_1}^{t_2} e^2(t) \, dt = \int_{t_1}^{t_2} f^2(t) \, dt - \sum_{i=1}^{n} a_i^2
\]
(1-5)

In the above, we have first assumed that \( f(t) \) is finite energy i.e.
\[
\int_{t_1}^{t_2} f^2(t) \, dt < \infty
\]
One of the most useful sets of basis functions is that constructed from exponentials. Generally the exponential functions in the form $e^{(-a+jb)t}$ are not orthogonal to one another. We can construct orthonormal functions from the given set by Gram-Schmidt orthonormalization method. The constructed orthonormal functions are just the linear combination of the original exponential functions.

Kautz (1) has described a simple and effective way to construct a set of orthonormal functions from a given set of exponentials $\{e^{(-S_it)}\} t \geq 0$, where the exponents $S_i$ may be real or complex and $\Re S_i > 0$.

In frequency domain, the representation of the orthonormal functions $\{E_i(s)\}$ $i=1, \ldots n$, if $\{S_i\}$ are all reals, are as follows:

$$E_1(s) = \sqrt{2S_1} \frac{1}{S+S_1}$$
$$E_2(s) = \sqrt{2S_2} \frac{(S-S_1)}{(S+S_1)(S+S_2)}$$
$$\vdots$$
$$E_n(s) = \sqrt{2S_n} \frac{(S-S_1)(S-S_2) \cdots (S-S_{n-1})}{(S+S_1)(S+S_2) \cdots (S+S_{n-1})(S+S_n)} \quad (1-6)$$

If $\{S_i\}$ are complex and occur in complex conjugate pairs, the corresponding $\{E_i(s)\}$ are:

$$E_{2k-1}(s) = \sqrt{2P_{2k}} \frac{S}{S^2+P_{2k}S+P_{2k}} \prod_{i=1}^{K-1} \frac{S-P_iS+q_i}{S^2+P_iS+q_i}$$
$$E_{2k}(s) = \sqrt{2P_{2k}} \frac{S}{S^2+P_{2k}S+P_{2k}} \prod_{i=1}^{K-1} \frac{S-P_iS+q_i}{S^2+P_iS+q_i} \quad (1-6')$$

with a final

$$E_n(s) = \sqrt{2S_n} \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{S^2-P_iS+q_i}{S^2+P_iS+q_i} \quad \text{if } n \text{ is odd}$$

$$E_{\left\lfloor \frac{n}{2} \right\rfloor}(s) = \sqrt{2S_{\frac{n}{2}}} \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{S^2-P_iS+q_i}{S^2+P_iS+q_i} \quad \text{if } n \text{ is even}$$

Where $P_i = S_{2i-1}^2 + S_{2i}$

$q_i = S_{2i-1}^2 S_{2i}$

are all reals.
The construction of the orthonormal exponentials described by Kautz is somewhat efficient, we can use its properties to formulate an orthonormal exponential filter by cascading a number of similar filter sections as shown in Figure (1-1):

If a signal \( V(t) \) is applied to the input of the filter, the response at the \( i \)th terminal will be:

\[
 a_i(t) = \int_0^\infty v(t-\tau) \overline{A_i}(\tau) \, d\tau 
\]  

(1-7)

which is the convolution integral.

From equation (1-4) we know that for a physical signal \( f(t) \) starts from \( t=0 \) and extends to future, if we want to represent it in the integrated squared error sense, the coefficient \( a_i \) is found from:

\[
 a_i = \int_0^\infty f(\tau) \overline{A_i}(\tau) \, d\tau 
\]  

(1-8)

There is a great similarity between equations (1-4) and (1-7), if we let \( V(t) = f(-t) \) and evaluate the convolution integral of the filter as specified by equation (1-7) at \( t=0 \) then

\[
 a_i(t) \bigg|_{t=0} = \int_0^\infty f(-t-\tau) \overline{A_i}(\tau) \, d\tau \bigg|_{t=0} 
\]

\[
 = \int_0^\infty f(\tau) \overline{A_i}(\tau) \, d\tau 
\]

which is exactly the equation (1-4). As a result, we can use the exponential filters to determine the coeffi-
cients of the representation of signal. By applying
the time-reversed signal \( f(-t) \) to the input, one can
measure the coefficients from the output terminals at
the instant corresponding to the start of the signal.

Young and Huggins (2) have shown that the error
energy associated with the approximation of the sig-
nal that occurred prior to the instant \( t \) is given by:

\[
E(t) = \int_0^\infty |b(t-\tau)|^2 d\tau (1 - q)
\]

where \( b(t) \), the "present instant error," is the out-
put of the time-invariant complementary filter whose
transfer function is:

\[
H(s) = \prod_{i=1}^{\frac{n}{2}} \frac{s^2 - p_i s + q_i}{s^2 + p_i s + q_i} (1-10)
\]

and Young and Huggins have shown a lot of particular
properties of the complementary filter \( H(s) \).
Chapter 2

ANALOG COMPUTATION AND EXPERIMENTAL RESULTS

In the previous chapter, we have considered only the approximation of a given real time function \( f(t) \), for \( t \geq 0 \), with a given set of exponential functions, \( \{e^{\lambda(t)}\}, \Re \lambda > 0 \). In this chapter, we shall treat the approximation problem by a linear combination of a given number \( n \) of exponentials such that the least-square's errors over the \( n \) coefficients of the linear combination and the \( n \) exponents used is minimized.

From the work of McDonough and Huggins (3), the necessary conditions for strict simultaneous minimization over both sets of \( n \) coefficients and \( n \) exponents are

\[
\int_0^\infty \varepsilon(t) e^\lambda(-\lambda t) \, dt = 0 \quad (2-1)
\]

\[
\int_0^\infty \varepsilon(t) t e^\lambda(-\lambda t) \, dt = 0 \quad (2-2)
\]

where \( \varepsilon(t) \triangleq f(t) - f^*(t) \) is the error function. The equivalent optimality criterion is to choose the exponents of the exponential functions \( f(t) \) has no component on the functions:

\[
\Phi_{n+1}(s) = H(s) \Phi_i(s) \quad i = 1, 2, 3, \ldots, n \quad (2-3)
\]

where

\[
H(s) = \prod_{i=1}^{n} \frac{s^2 + \lambda_i s + \mu_i}{s^2 + \lambda_i s + \mu_i}
\]

the complementary filter as defined in equation (1-9), and the coefficients are then to be chosen as stated.
in the previous chapter to minimize the integrated squared error using \( \exp\{-St\} \) found with above criterion.

For \( f(t) \) has no components on \( \{ \tilde{\varphi}_{n+1}(s) \} \), \( i = 1, 2, \ldots n \), it is just means (from equation (2-3)):

\[
C_i = f(-t) \ast \tilde{\varphi}_{n+1}(t) \ \triangleq \ \int_0^\infty f(-t+c) \tilde{\varphi}_{n+1}(c) \, dc \Bigg| \text{sample at } t = 0
\]

vanishes for all \( i \).

\[
\int_0^\infty f(c) \tilde{\varphi}_{n+1}(c) \, dc
\]

Equation (2-4) can be interpreted in the following block diagram:

\[
\begin{array}{c}
\text{f(-t)} \\
\downarrow \\
H(s) \text{ } \text{at}_{t=0} \rightarrow \{c_i\} \ \text{To Vanish for } i=1,2,\ldots \ n
\end{array}
\]

or from equation (2-3) it is equivalent to:

\[
\begin{array}{c}
\text{f(-t)} \\
\downarrow \\
\tilde{\varphi}_{n+1}(s) \rightarrow \text{Sample} \text{ at } t = 0 \rightarrow \{c_i\} \ \text{To Vanish Simultaneously}
\end{array}
\]

or the "present-instant" error \( a(-t) \) is orthogonal to the subspace spanned by \( \{ \tilde{\varphi}_{n+1}(s) \} \).

The circuit diagram for the simultaneous measurement of the coefficients of representation and the integrated squared error by passing the time reversed signal through a bank of filters and sampling the correct outputs at the time corresponding to the start of the signal to be represented is shown in Figure (2-1) for two exponential functions.

![Circuit Diagram](Fig.(2-1))
We will use the iterative techniques to find an algorithm for the iterative optimization of the exponents \((p,q)\) that specify the basis. The input signal will be operated repeatedly until the filter parameters converge to a final set of values. Therefore using the knowledge of the adaptive control techniques (4) the representation of each different input signal can be adequately gotten if the input signal can be repeatedly operating.

The block diagram of Figure (2-2) is used to illustrate the adaptive principle and to demonstrate the parameter influence coefficients can be used to provide closed loop optimization of representation systems. (5)

Suppose for the simple case, we want to use two exponential functions to represent some input signals, then the coefficients \(a_1, a_2\) for the representation are just the outputs of two orthonormal filters by passing the time-reversed signal \(f(-t)\) and sampling at the time corresponding to the start of the signal to be represented, i.e., as in equation (1-4)

\[
\begin{align*}
a_1 &= \int_{0}^{\infty} f(\tau) \mathcal{I}_1(\tau) \, d\tau \\
a_2 &= \int_{0}^{\infty} f(\tau) \mathcal{I}_2(\tau) \, d\tau
\end{align*}
\]  

(1-4)

and the approximation signal is

\[f^*(t) = a_1 \mathcal{I}_1(t) + a_2 \mathcal{I}_2(t)\]
then the integrated squared error corresponding to \( f(t) \) and \( f^*(t) \) is:

\[
E = \int_{0}^{\infty} f(t) \, dt - (a_{1}^{2} + a_{2}^{2})
\]

The parameter effect, as discussed in Figure (2-2), is just the partial derivatives of \( E \) with respect to the filter parameters \( (p, q) \), i.e.,

\[
\frac{\partial E}{\partial p} = -2a_{1} \frac{\partial a_{1}}{\partial p} - 2a_{2} \frac{\partial a_{2}}{\partial p}
\]

(2-5)

\[
\frac{\partial E}{\partial q} = -2a_{1} \frac{\partial a_{1}}{\partial q} - 2a_{2} \frac{\partial a_{2}}{\partial q}
\]

Since \( f(t) \) and \( \{ \mathbb{F}_{i}(t) \} \) are continuous (or at least piece-wise continuous), we can take derivatives of \( a_{1} \) and \( a_{2} \) with respect to \( (p, q) \) inside the integral sign (i.e., interchange the integral and derivative roles), then we get:

\[
\frac{\partial a_{1}}{\partial p} = \int_{0}^{\infty} f(t) \frac{\partial \mathbb{F}_{1}(t)}{\partial p} \, dt
\]

(2-6)

\[
\frac{\partial a_{1}}{\partial q} = \int_{0}^{\infty} f(t) \frac{\partial \mathbb{F}_{1}(t)}{\partial q} \, dt
\]

(2-7)

\[
\frac{\partial a_{2}}{\partial p} = \int_{0}^{\infty} f(t) \frac{\partial \mathbb{F}_{2}(t)}{\partial p} \, dt
\]

(2-8)

\[
\frac{\partial a_{2}}{\partial q} = \int_{0}^{\infty} f(t) \frac{\partial \mathbb{F}_{2}(t)}{\partial q} \, dt
\]

(2-9)

Substitute equations (2-6)---(2-9) into equations (2-5) it becomes:

\[
\frac{\partial E}{\partial p} = -2a_{1} \int_{0}^{\infty} f(t) \frac{\partial \mathbb{F}_{1}(t)}{\partial p} \, dt - 2a_{2} \int_{0}^{\infty} f(t) \frac{\partial \mathbb{F}_{2}(t)}{\partial p} \, dt
\]

(2-10)

\[
\frac{\partial E}{\partial q} = -2a_{1} \int_{0}^{\infty} f(t) \frac{\partial \mathbb{F}_{1}(t)}{\partial q} \, dt - 2a_{2} \int_{0}^{\infty} f(t) \frac{\partial \mathbb{F}_{2}(t)}{\partial q} \, dt
\]

(2-11)

When the complementary error signal \( a(-t) \) is orthogonal to the subspace spanned by \( \exp(-s_{i}t) \), the integrated squared error \( E \) is stationary with respect to the variations in the exponents \( (p, q) \) (6), which implies: \( \frac{\partial E}{\partial p} = \frac{\partial E}{\partial q} = 0 \).
We can recognize the integrals in equations (2-10), (2-11) which are just the sampling outputs of the filters \( \frac{\partial E(t)}{\partial p}, \frac{\partial E(t)}{\partial q} \) at the start of the input signal.

Let's define:

\[
\begin{align*}
B_1 &= \int_0^\infty f(t) \frac{\partial E(t)}{\partial p} \, dt \\
G_1 &= \int_0^\infty f(t) \frac{\partial E(t)}{\partial q} \, dt \\
B_2 &= \int_0^\infty f(t) \frac{\partial E(t)}{\partial p} \, dt \\
G_2 &= \int_0^\infty f(t) \frac{\partial E(t)}{\partial q} \, dt
\end{align*}
\]

Then the values of parameter changes are:

\[
\begin{align*}
\Delta p &= a_1 \beta_1 + a_2 \beta_2 \\
\Delta q &= a_1 \gamma_1 + a_2 \gamma_2
\end{align*}
\] (2-6)

The procedure for searching the parameters is to repeatedly process the signal; so that the influence of the parameters on \( E \) as shown in equation (2-6) for one computer run can be used to calculate the values of \((p, q)\) for the next run.

The parameter values during the \((n+1)\)th iterations are:

\[
\begin{align*}
(p)_{n+1} &= (p)_n - U_p (\Delta p)_n \\
(q)_{n+1} &= (q)_n - S_q (\Delta q)_n
\end{align*}
\]

The feedback coefficients \( U_p \) and \( S_q \), were chosen that provided convergence can be gotten from any starting point in the parameter space, variable coefficients can also be used to get rapid convergence for more complex problems. The iteration process is stopped when the sampling outputs \( a_1 \) and \( a_2 \) are simultaneously reduced to an acceptable level.

The analog circuits that correspond to the block diagram (2-2) is shown in Figure (2-3).
The symbols and notations for the diagram (2-3) are as follows:

(1) \[
\begin{array}{c}
\times \\
x
\end{array}
\]

multiplication with inputs \(x\) and \(y\)

(2) \[
\begin{array}{c}
\text{inverter}
\end{array}
\]

(3) \[
\begin{array}{c}
\text{integrator}
\end{array}
\]

(4) \[
\begin{array}{c}
\text{square}
\end{array}
\]

(5) \[
\begin{array}{c}
\text{pot (manual or servo-set pot)}
\end{array}
\]

(6) \[
\begin{array}{c}
\text{track and hold}
\end{array}
\]

The principle and operation of each analog element can be referred to any analog computation books. (7,8)
Experimental Results

1. **Square pulse**

We want to use the algorithm such as shown in Figure (2-3) for the iterative optimization of the parameters \((p, q)\) of two exponentials to represent the square pulse in minimizing the integrated squared error \(E\).

Square pulse has been treated by many authors due to the fact that it is easy to generate and it is also one of the few signals we can solve the non-linear equations as equations (2-1) and (2-2). The calculated parameter values \((6)\) for the representation of square pulse are: \(p = 3.02\) \(q = 5.62\).

With the iterative convergent process as in Figure (2-3), the feedback coefficients \(U_p, S_q\) are adjusted so that the process could be convergent for any set of values of \(p\) and \(q\) where \(0 < p < 1\), \(0 < q < 1\) (or equivalently \(0 < p < 10^V\), \(0 < q < 10^V\), the reference voltage of EAI 680 Analog computer is \(10^V\)). The convergence is shown in Figure (2-4) where both the temporal convergence of each parameter is investigated. The initial point was shown as \(p_0 = 1\) (equivalent \(10^V\)), \(q_0 = 1\) so that the trajectory could be easily followed and we can compare the results with the calculated values and O'Neill's result \((5)\).

Although we have used the constant feedback coefficients, it is often necessary to employ variable gains to obtain rapid convergence of the process.

In Figure (2-4) we showed the convergence phenomena of the parameters \(p\) and \(q\) for the representation of square pulse.
Initial values: \( p_0 = 10^v \)
\( q_0 = 10^v \)

Convergent parameters: \( p = 3.100^v \)
\( q = 5.700^v \)

Errors (with respect to calculated values):

\[
p: \frac{\Delta p}{p} = \frac{3.1 - 3.02}{3.02} = 2.65\%
\]
\[
qu: \frac{\Delta q}{q} = \frac{5.7 - 5.62}{5.62} = 1.41\%
\]

Oneill's error:

\[
p: \frac{\Delta p}{p} = \frac{3.2 - 3.02}{3.02} = 5.96\%
\]
\[
qu: \frac{\Delta q}{q} = \frac{6.0 - 5.62}{5.62} = 6.76\%
\]
Initial Value
P₀: 10 V
Convergent Value
P: 3.1 V

Initial Value
Q₀: 10 V
Convergent Value
Q: 5.7 V

Fig. (2-4)
2. Second-order network

Suppose we have a RLC network as shown in the following diagram:

![RLC Network Diagram]

Where \( x_1 \), \( x_2 \) are state variables representing the voltage across the capacitance and the current flowing through the inductor \( L \) respectively, \( I \) is the external current source. We can write the state equations as follows:

\[
\begin{align*}
L \frac{d}{dt} x_1 + x_2 &= I \\
- x_1 + L \frac{d}{dt} x_2 + R x_2 &= 0
\end{align*}
\]

The characteristic equation corresponding the state variables \( x_1 \) and \( x_2 \) is:

\[
\begin{vmatrix}
\lambda & \frac{1}{C} \\
-\frac{1}{L} & \lambda + \frac{R}{L}
\end{vmatrix} = \lambda^2 + \frac{R}{L} \lambda + \frac{1}{LC} = 0
\]

Let the two roots of equation (2-7) be denoted by \( S_1 \) and \( S_2 \), then from the properties of equations we can get the parameter values \( p \) and \( q \) as:

\[
\begin{align*}
p &= S_1 + S_2 = -\frac{R}{L} \\
q &= S_1 S_2 = \frac{1}{LC}
\end{align*}
\]

i.e., the parameter values \( p, q \) are determined only from the products of \( R, \frac{1}{L} \) and \( \frac{1}{L} \), \( \frac{1}{C} \) respectively.

The way to simulate equation (2-6) is shown in Figure (2-5) and let the network be forced only by the initial values \( x_1(0) \) and \( x_2(0) \), i.e., the external current source \( I \) has been removed.
we can get a corresponding waveform \( x_2(t) \). Arbitrarily we set \( R = 2.000 \), \( \frac{1}{L} = 0.2050 \), \( \frac{1}{C} = 4.7000 \), i.e., the corresponding values of \( p \) and \( q \) are \( 4.1^V \) and \( 9.63^V \) respectively. Using Figure (2-3), we want to find two best exponentials to approximate the waveform \( x_2(t) \) such that the integrated squared error is minimized over both the coefficients of the linear combinations and the two parameters used. To get \( x_2(-t) \), we have reversed the original signal \( x_2(t) \) with tape recorder, i.e., recorded the signal in the forward direction and played it in the reversed direction. The convergent trajectory of each parameter is investigated and is shown in Figure (2-6). The initial point was chosen as \( p_0 = 3.0^V \), \( q_0 = 6.7^V \), and the trajectory can be easily followed.

Initial values:

\[
\begin{align*}
p_0 &= 3.0^V \\
q_0 &= 6.7^V
\end{align*}
\]

Convergent parameters:

\[
\begin{align*}
p &= 3.96^V \\
q &= 10.4^V
\end{align*}
\]
Errors (with respect to $p=4.1^V$, $q=9.63^V$):
\[ \frac{\Delta p}{p} = \frac{4.1 - 3.96}{4.1} = 3.4\% \]
\[ \frac{\Delta q}{q} = \frac{9.4 - 9.63}{9.63} = 3\% \]

In Figure (2–7), we also showed the convergent phenomena for another initial point $p_0=2.55^V$, $q_0=5.50^V$:

Initial values: $p_0=2.55^V$
\[ q_0=5.50^V \]

Convergent values: $p = 4.00^V$
\[ q = 9.40^V \]

Errors:
\[ \frac{\Delta p}{p} = \frac{4.1 - 4}{4.1} = 2.44\% \]
\[ \frac{\Delta q}{q} = \frac{9.63 - 9.4}{9.63} = 2.4\% \]
Initial value:
$P_0: 3.0\, V$
Convergent value:
$P: 3.96\, V$

Initial value:
$Q_0: 6.7\, V$
Convergent value:
$Q: 10.4\, V$

Fig. (2-6).
Initial value
$P_0 = 2.55\, V$

Convergence value,
$P = 4\, V$

Fig. (2-7)
3. Analysis of measurement errors

Three kinds of errors have been found in our experiment which are:

(a) Random noise: the noise comes from the wire probes connections etc. produced insensitivity in the converging process in the parameters.

(b) Nonlinear analog elements: such as the multiplication element that the error we can not compensate or predict.

(c) Fast computer operations: normally the computer was run in normal-second mode, while we used the fast-millisecond mode so the integrator outputs would introduce a big noise if there were some noise or disturbance at its input.

There is still another measurement error found in the operation of second-order network approximation.

There is a certain distortions occurred in the recorder reversing process. That is why the error found in the latter experiment is larger than the square-pulse operation.
Consider a sequence of numbers \( x(0), x(T), x(2T), \ldots, x(nT), \ldots \) which were derived by sampling a continuous waveform \( x(t) \). The \( z \)-transform of the sequence is defined to be:

\[
x(z) = \sum_{n=0}^{\infty} x(nT) z^{-n}
\]

\( z \) is a complex variable and \( x(z) \) is a function of a complex variable. The question of convergence of this series and many proper properties of \( z \)-transforms have been intensively treated by Gold and Rader.\(^{(10)}\)

Now suppose we can express the function \( x(t) \) as a linear combination of exponential functions as 

\[
x(t) = \sum_{k} c_k e^{s_k t}
\]

then the expression in the frequency domain is 

\[
X(s) = \sum_{k} \frac{c_k}{s - s_k}
\]

or equivalently its \( z \)-transformation is 

\[
x(z) = \sum_{k} \frac{c_k}{1 - e^{s_k T} z^{-1}}
\]

From Cauchy's integral theorem, since \( c_k \) is the residue of \( x(s) \) with a pole at \( s=s_k \), then there exists a transformation pair between \( x(s) \) and \( x(z) \)\(^{(11)}\):

\[
X(z) = \oint_{C_1} \frac{X(s)}{1 - e^{s T} z^{-1}} \frac{d s}{2\pi j}
\]

(3-1)

where the contour \( C_1 \) encloses the singularities of \( x(s) \) only.

\[
X(s) = \oint_{C_2} \frac{X(z)}{s - \ln z} \frac{d z}{2\pi j z^{T}}
\]

(3-2)

and \( C_2 \) encloses the poles of \( x(z) \) only.

With the transformation pair (3-1) and (3-2), then the \( z \)-transformations corresponding to the first pair of
orthonormal exponential function

$$\Phi_i(s) = \sqrt{2p} \frac{s}{s^2 + ps + q},$$

are (9, 10):

$$\Phi_1(z) = \sqrt{2p} \frac{Z^2 - z e^{-\alpha T} [\alpha^2 \sin \beta T + \cos \beta T]}{Z^2 - 2Z e^{-\alpha T} \cos \beta T + e^{-2\alpha T}}$$

$$\Phi_2(z) = \sqrt{2p} \frac{Z e^{-\alpha T} \left( \frac{1}{\beta} \sin \beta T \right)}{Z^2 - 2Z e^{-\alpha T} \cos \beta T + e^{-2\alpha T}}$$

where

$$\alpha = \frac{p}{T} \quad \beta = \left( \frac{p}{T} \right)^2 - q.$$

Equations (3-3) and (3-4) are the sampled-data equivalent of the orthonormal exponentials. However with the calculation of the residues of the integrand of equation (3-2) at the poles of $$x(z)$$ (9), and from Shannon's sampling theorem (12), it can be shown that equations (3-3) and (3-4) are not normalized and even they are not orthogonal.

To construct a set of discrete orthonormal exponentials, we choose at the outset that, the poles in the z-domain are corresponding to the frequency domain poles, the zeros are different; however, they are chosen in such a way that as the sampling interval $$T$$ approaches 0, the chosen discrete basis functions will approach the frequency domain orthonormal exponentials.

The discrete orthonormal exponentials according to the above considerations are: (9)

$$\Phi_{ak_1}(z) = b_{ak_1} \frac{Z^2 - Z}{(Z - Z_k)(Z - Z_k^*)} \prod_{i=1}^{k-1} \frac{Z_i Z_i^*}{(Z - Z_i)(Z - Z_i^*)}$$

$$\Phi_{ak}(z) = b_{ak} \frac{Z^2 + Z}{(Z - Z_k)(Z - Z_k^*)} \prod_{i=1}^{k-1} \frac{Z_i Z_i^*}{(Z - Z_i)(Z - Z_i^*)}$$
where $z_k^*$ is the complex conjugate of $z_k$, and the orthonormalization constants $b_{2k-1}$ and $b_{2k}$ are:

$$b_{2k-1} = \left\{ \frac{1}{2} \left( |z_k|^2 \right) \left[ 1 + (z_k + z_k^*) + |z_k|^2 \right] \right\}^{1/2}$$

$$b_{2k} = \left\{ \frac{1}{2} \left( |z_k|^2 \right) \left[ 1 - (z_k + z_k^*) + |z_k|^2 \right] \right\}^{1/2}$$

The pole $z_k$ in the $z$-domain is related to the frequency domain pole $s_k$ with:

$$Z_k = e^{-s_k T} = e^{(-\alpha_k + j\beta_k)T}$$

and

$$Z_k^* = e^{-s_k^* T} = e^{(-\alpha_k - j\beta_k)T}$$

Corresponding to the s-domain complementary filter $H(s)$, as equation (1-9), the discrete complementary filter is:

$$H(z) = \sum_{i=1}^{\eta/2} \frac{Z_i Z_i^* (z - \frac{1}{Z_i})(z - \frac{1}{Z_i^*})}{(z - Z_i)(z - Z_i^*)}$$

(3-6)

As in the analog computation, applying McDonough and Huggins' criterion and adaptive control principle, we will use digital computer with discrete orthonormal filters to find the best exponents for the signal representation in the sense that the sum of the squared error is minimized over the coefficients of representation and the discrete exponents used.

Our operation is just searching for the values of the filter parameters such that the discrete complementary signal $a(-t)$ is orthogonal to the subspace spanned by the discrete orthonormal exponentials $(Z_k)$.

The first pair of the pulse response of the discrete orthonormal filters are:

$$\Phi_1(z) = \frac{b_1(z^2 - z)}{z^2 + \rho z + Q}$$

$$\Phi_2(z) = \frac{b_2(z^2 + z)}{z^2 + \rho z + Q} = \frac{b_2(z^2 + z)}{b_1(z^2 - z)} \Phi_1(z)$$

(3-7)
where \( P = -(Z_1 + Z_1^*) \) and \( q = Z_1 Z_1^* \)

\[
b_1 = \left\{ \frac{1}{2} (1 - |Z|^2) \left[ |1 + (Z_1 + Z_1^*) + |Z_1^2| \right] \right\}^{\frac{1}{2}}
\]

\[
b_2 = \left\{ \frac{1}{2} (1 - |Z|^2) \left[ |1 - (Z_1 + Z_1^*) + |Z_1^2| \right] \right\}^{\frac{1}{2}}
\]

and the discrete complementary filter corresponding to \( \mathbb{E}_1(z) \) and \( \mathbb{E}_2(z) \) is:

\[
H(z) = \frac{QZ^2 + Pz + 1}{Z^2 + Pz + Q}
\]

The simulation diagram for \( \mathbb{E}_1(z), \mathbb{E}_2(z), H(z) \) in the digital computation at \( \text{I} \)th sampling instant is shown in Figure (3-1) when the reversed \( \text{I} \)th sampled input is \( \hat{x}(I) \).

Let

\[
A(z) = \frac{b_1(Z^2 - Z)}{Z^2 + Pz + Q} \times(z)
\]

\[
C(z) = \frac{b_2(Z^2 + Z)}{Z^2 + Pz + Q} \times(z) = \frac{b_2(Z^2 + Z)}{b_1(Z^2 - Z)} A(z)
\]

then the coefficients \( c_1, c_2 \) of the linear combination for the signal representation are just the sampled values corresponding to the start of the approximated signal by passing the time reversed signal \( x(z) \) through the bank of orthonormal filters \( \{ \mathbb{E}(z) \} \).

![Simulation Diagram](image-url)
The parameter effect, i.e., the partial derivatives of \( \mathbb{P}(z) \), \( \mathbb{Q}(z) \) with respect to the filter parameters, can easily be derived from equation (3-7). i.e.:

\[
\begin{align*}
\frac{\partial \mathbb{P}(z)}{\partial p} x(z) & \overset{\Delta}{=} W(z) = \frac{-b_1(z^2-z)}{(z^2+pz+q)^2} x(z) \\
\frac{\partial \mathbb{Q}(z)}{\partial p} x(z) & \overset{\Delta}{=} F(z) = \frac{-b_2(z^2-z)}{(z^2+pz+q)^2} x(z) \\
\frac{\partial \mathbb{P}(z)}{\partial q} x(z) & \overset{\Delta}{=} Y(z) = \frac{-b_1(z^2-z)}{(z^2+pz+q)^2} x(z) \\
\frac{\partial \mathbb{Q}(z)}{\partial q} x(z) & \overset{\Delta}{=} Z(z) = \frac{-b_2(z^2-z)}{(z^2+pz+q)^2} x(z)
\end{align*}
\]

The over-all digital simulation circuit for \( \mathbb{P}(z) \), \( \mathbb{Q}(z) \), \( W(z) \), etc. is shown in Figure (3-2).

Let the values of parameter changes be denoted as \((\Delta p)\), \((\Delta q)\), then the parameter values during the \((n+1)\)th iteration are:

\[
\begin{align*}
P_{n+1} &= P_n - u_p (\Delta p)_n \\
Q_{n+1} &= Q_n - u_q (\Delta q)_n
\end{align*}
\]

Like the analog computation, the feedback coefficients \( u_p \) and \( u_q \) are chosen to provide the convergence for each parameter.

The flow chart for the set-up of an orthonormal set to find the filter parameters is shown in Figure (3-3).
\[
\frac{(Z+Z)(Z^{2}+PZ+Q)}{(Z^{2}+PZ+Q)^{2}}
\]

\[
X(z) \rightarrow \frac{Z^{2}-Z}{Z^{2}+PZ+Q} X(z)
\]

\[
\frac{Z^{2}+Z}{Z^{2}+PZ+Q} X(z)
\]

\[
\frac{Z^{2}-Z}{Z^{2}+PZ+Q} X(z)
\]

\[
H(z) (-1)
\]

\[
\frac{(Z^{2}-Z)(Z^{2}+PZ+Q)}{(Z^{2}+PZ+Q)^{2}}
\]

\[
\frac{\bar{y}_{1}(z) - b_{1}(Z^{2}-Z)Z}{(Z^{2}+PZ+Q)^{2}} X(z) = \frac{\partial \bar{y}_{1}(Z)}{\partial P} X(z) \triangleq W(z)
\]

\[
\frac{-b_{2}(Z^{2}+Z)Z}{(Z^{2}+PZ+Q)^{2}} X(z) = \frac{\partial \bar{y}_{2}(Z)}{\partial P} X(z) \triangleq F(z)
\]

\[
\frac{-b_{1}(Z^{2}-Z)}{(Z^{2}+PZ+Q)^{2}} X(z) = \frac{\partial \bar{y}_{1}(Z)}{\partial Q} X(z) \triangleq \gamma(z)
\]

\[
\frac{-b_{2}(Z^{2}+Z)}{(Z^{2}+PZ+Q)^{2}} X(z) = \frac{\partial \bar{y}_{2}(Z)}{\partial Q} X(z) \triangleq \zeta(z)
\]

Fig. (3-2)
Experimental Results

A. The square pulse fitted with two exponentials

In classical network synthesis problem, it was mainly to design a network when subject to a certain kind of input to produce a particular desirable output functions. McBride, Schaefgen, and Steiglitz (13) have proposed an iterative procedure to determine a rational transfer function of a given order whose output when subjected to a known input function best approximates, in the least-squares sense, to a given desired output function. They have used the iterative techniques to synthesize lumped delay lines of order from second through eighth for step response.

With initial orthonormal filter parameters $p_0 = 1.2$ and $Q_0 = 0.55$, and choosing proper feedback coefficients, we get the final values $p_n = 1.93697$, $Q_n = 0.94731$, and we can compare these results with the values by solving the necessary conditions for the optimum criterion as equations (2-1) and (2-2). We take these calculated values as our reference, the errors corresponding to the reference values are $\frac{\Delta p}{p} = 2\%$ and $\frac{\Delta Q}{Q} = 4\%$. 
B. The second-order network fitted with two exponentials

As in the analog computation, we want to use two exponentials to fit a known wave form (the same wave form as in analog case) which is generated from a RLC network with given element values. Since there are no recorder reversing process errors, we expect that the best exponents we find with a digital computer will be closer to the true values than analog results. The initial parameter values were chosen arbitrarily and the feedback coefficients were selected in such a way that to provide the iteration process converged in the parameter space.

With sampling rate \( T = 0.1 \) seconds, and the computing intervals over 20 seconds, we get 200 sample points. The reason for choosing 20 seconds is that it is about four time constants of the wave form and the results approximates that would be obtained over a semi-infinite interval since the value of the wave form is small above 20 seconds.

Using \( p_0 = 1.0; \) \( Q_0 = 0.3 \) as the initial parameter values, the process converged to values which may be taken as: \( p = 1.9216, \) \( Q = 0.9734 \). With these convergent values we found that the corresponding errors with respect to the calculated values \( (P_t = 1.9494, \) \( Q_t = 0.9587) \) are 1.5\% and 3\% respectively.
C. Some medical data fitted with three exponentials

In the past, the analysis of some medical wave form such as the electrocardiograph has proved to be very useful for the physician, who by recognizing the different wave forms to classify them into different categories. Due to the high-speed digital components, the realization of medical data with digital computer has helped the physicians in making more reliable analysis and reduced the time and work of examining data.

Another example is that a simple model on the analog of the cardiovascular system has been discussed for a long time by some medical groups. The cardiovascular system can be simulated as:

![Diagram](3-5)

Fig. (3-5)

The system is a distributed parameter system and the model is lumped. In the system, $C_1$ corresponds roughly to the blood capacity of the aorta; $L$ to the mass or inertia of the blood; $C_2$ and $R$, respectively, to the peripheral capacity and resistance of the capillaries and small arteries. The source is the heart. The overall problem is to observe the behavior of the heart through the unknown system by monitoring blood pressure in an artery and make some conclusions about the nature of the heart itself.
To gain some insight into how the parameter may influence response, we should first find an efficient representation for the system output wave forms.

Applying McDonough and Huggins' criterion and the iterative convergent values, we will use digital computation to find three best exponents to represent some response ensembles. Using the exponents found we can calculate the system element values. So we can examine how the parameters influence the response and other phenomena.

We have used some medical data taken from human subjects at the VA Hospital in Houston. The data consists of 36 samples and the sampling period is 0.02 seconds.

From equation (3-5), we can construct the discrete orthonormal filters for three exponential functions as follows:

\[
\begin{align*}
\mathcal{F}_1(z) &= b_1 \frac{z - z_1}{z^2 + p z + Q} \\
\mathcal{F}_2(z) &= b_2 \frac{z^2 + z}{z^2 + p z + Q} \\
\mathcal{F}_3(z) &= b_3 \frac{z}{z - z_3} \cdot \frac{Q z^2 + p z + 1}{z^2 + p z + Q}
\end{align*}
\]

where \( p = -(z_1 + z_2) \)

\( Q = z_1 z_2 \)

\( z_1 = e^{-s_1 T} \)

\( z_2 = e^{-s_2 T} \)

\( z_3 = e^{-s_3 T} \)

and \( b_1, b_2, b_3 \) are the corresponding orthonormalization constants.
The parameter effect, i.e., the partial derivatives of $\mathcal{F}_1(z)$, $\mathcal{F}_2(z)$, $\mathcal{F}_3(z)$ with respect to the filter parameters $p, q, z_j$ can also be calculated from equation (3-8).

With the 36 sampled data, and reversing the sequence orders, we finally use the result values as our filter input. With arbitrary initial parameter values (we set $p_0=1.0$, $q_0=0.2$, $z_j^0=-0.1$) and appropriate feedback gains we have the following convergent values: $p=-1.43030$, $q=0.55668$, $z_j=0.78334$ and the corresponding poles in the z-domain are:

\[
\begin{align*}
  z_1 &= 0.71515 + j0.4242 \\
  z_2 &= 0.71515 - j0.4242 \\
  z_j &= 0.78384
\end{align*}
\]

The most important thing we should note is that in the McDonough and Huggins' criterion, the optimum condition for least squares errors based on the fact that the function to be approximated is over the semi-infinite interval. Or we can say that over some proper time $T_L$, the energy of the signal over $(T_L, \infty)$ would be small compared with the whole signal energy.

From the above convergent values over the 36 sampled values we found that the energy over the final sample to $\infty$ is comparatively large compared with the signal energy. The way to correct this shortage we used the approximation equation found from the 36 sample values to find other samples over the 36th sample point. We chose total 500 sample values which corresponds to four time constants of the approximation equation. With the same procedure as above and the same initial filter parameter values
(\(P_0 = 1.0, \ \dot{\omega}_0 = 0.2, \ \omega_0 = -0.1\)), we finally get the following convergent parameter values:
\[
\begin{align*}
P &= -1.764502 \\
\xi &= 0.869245 \\
\omega &= 0.998475
\end{align*}
\]
which corresponds the poles in the z-domain are:
\[
\begin{align*}
z_1 &= 0.8820 + j0.3021 \\
z_2 &= 0.8820 - j0.3021 \\
z_3 &= 0.998475
\end{align*}
\]
The approximated sampled values can be found from:
\[
f_a(k) = \sum_{i=1}^{3} a_i z_i(k)
\]
where \(a_i\) is the output of the filters when we applied the reversed signal of those original sampled data to the filters and \(\{z_i(k)\}\) is the response of the filters when the number sequence \((1, 0, 0, \ldots, 0)\) is applied.

The result of the calculated sampled values and comparison between the calculated and original sampled values is shown in figure (3-6).

The rms error of the measured data \(f_a(k)\) to the original sampled data is defined as:
\[
\|e\| = \left[ \frac{1}{k} \sum_{i=0}^{k-1} (f_a(i) - f_i)^2 \right]^{\frac{1}{2}}
\]
and for \(k=36\) we found \(\|e\| = 0.837959313\) which can be compared with the results of Newton Raphson done by Dr. T. B. Watt, Jr. (with \(\|e\| = 0.60\)) and Burrus, Parks, and Watt (with \(\|e\| = 2.18\) (14).
CONCLUSION AND DISCUSSION

We have outlined the general principle of signal representation. The procedure for finding the best exponents for the signal representation in the sense that the minimization of the integrated squared error over the coefficients of linear combinations and a given number of exponents using the analog and digital computation has been investigated. They are based on the determination on the influence of each parameter on the performance criterion, and applying the adaptive control principle to implement the operations. The analog and digital computer experimental results revealed that the digital computation has less error than the former due to the noise and the distortion introduced to the analog components and the signal reversing. The hybrid simulation utilizing both analog and digital computation, using the benefits of wide bandwidths on analog components and high-speed digital components, to speech representation such as format tracking is a great interesting topic to investigate.
REFERENCES


