RICE UNIVERSITY

ON ASYMPTOTIC STABILITY PROPERTIES OF DIRECT CONTROL SYSTEMS
WITH A FEEDBACK CHARACTERISTIC LYING PARTLY OUTSIDE THE
HURWITZ SECTOR

by

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ABSTRACT
ON ASYMPTOTIC STABILITY PROPERTIES OF DIRECT CONTROL SYSTEMS
WITH A FEEDBACK CHARACTERISTIC LYING PARTLY OUTSIDE THE
POPOV SECTOR
Jacques A. Dutertre
In his original work, Popov developed a simple frequency
domain criterion by which global asymptotic stability of a
nonlinear control system is guaranteed for all nonlinearities
lying inside a certain sector called the Popov sector.
Following this result, particular attention has been
devoted in the scientific literature to the cases in which
the Popov sector differs from the Hurwitz sector, this with a
view to establishing additional conditions on the nonlinearity
that may permit the so-called Aizerman conjecture to be
verified.
In the present work, second order nonlinear control
problems, with finite Popov sector, are investigated with
respect to asymptotic stability of the origin for nonlinear
feedback characteristics lying partly outside the Popov-
Hurwitz sector, which seems to be a new viewpoint.
A new kind of stability, near asymptotic stability is
introduced and studied for second order control systems
with a piecewise linear feedback characteristic and is
extended to systems with a general feedback characteristic
of the type mentioned above.
A theorem is given which permits to predict near asymp-
totic stability for such second order systems.
All the studies carried out in this work are obtained through the use of a state space approach and their validity is illustrated both by analog and digital computer solutions.
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CHAPTER 1

INTRODUCTION TO THE RESEARCH TOPIC

1-1 Statement of the problem

The general direct control system that is to be considered throughout this work is represented in state variable formulation by the set of equations:

\[
\begin{align*}
\dot{x} &= Ax - b \phi(c) \\
\sigma &= c'x
\end{align*}
\]

(1-1)

where \( A \) is a stable \((n,n)\) constant matrix

the \( n \)-vector \( x \) the state vector

\( b \) and \( c \) are constant \( n \)-vectors

\( \sigma \) is the scalar output

\( \phi(c) \) is a Lipschitzian scalar function of the scalar variable \( \sigma \) satisfying the restriction

\( \phi \sigma > 0 \) for all \( \sigma \)

The system (1-1) is represented in block diagram form in Fig.1.

According to the Popov criterion, the above system is globally asymptotically stable for all nonlinearities \( \phi(c) \) inside the Popov sector, i.e. satisfying the inequality

\[ 0 < \phi(c)/\sigma < k \quad \text{for all } \sigma \]  

(1-2)

if there exists a real number \( \beta \) such that

\[ \text{Re} \left( 1+\beta \omega \right) G(\omega) + 1/k > 0 \]  

(1-3)

Several authors have weakened the Popov condition by allowing \( \phi(c) \) to leave the Popov sector but remain inside the Hurwitz sector. These conditions introduced more restrictions on the derivatives of \( \phi(c) \).
One of the main objectives of the present research was to study whether global asymptotic stability of the null solution could be achieved if $\phi(a)$ leaves the Hurwitz sector. For this purpose, we began considering second order systems, a detailed analysis of which is rendered amenable due to the fact that the state space reduces to a plane. For such systems, the Popov and Hurwitz sectors coincide. The interesting case is the one in which the Hurwitz sector is finite. This corresponds to non-minimum phase shift systems for which the transfer function is of the form:

$$G(s) = \frac{\delta s - \gamma}{s^2 + 2\alpha s + \omega^2}$$

The first part of this work will investigate the system (1-1) with discontinuous linear feedback characteristics of the type shown in Fig. 2, lying completely or partly outside the Popov-Hurwitz sector. It will be seen that the best type of stability one can achieve is the one in which a finite number of trajectories do not tend to the origin. We call such stability near global asymptotic stability.

Then, in the second part, asymptotic stability is studied for the system (1-1) with a piecewise linear feedback characteristic.

Sufficient conditions are given which permit, through the knowledge of the behavior of a few particular trajectories, to conclude about the near asymptotic stability of the system (1-1).

All this study relies heavily on phase plane methods and the general results established by Poincare, Andronov, Chaikin.
and other authors.

The last part of this work concerns two particular classes of third order control problems that may be represented in the block diagram form by Fig. 1 with the transfer functions:

\[ G(S) = \frac{s^2}{s^3 + as^2 + bs + c} \]

and

\[ G_1(S) = \frac{1}{s^2 + as^2 + bs + c} \]

It is shown that a simple state space study of these systems is not sufficient to conclude on global asymptotic stability, for it fails to disprove the possible occurrence of limit cycles for certain initial conditions.

However, analog and digital computer studies seem to indicate that the above systems are actually globally asymptotically stable and that further studies are needed in order to obtain a definite conclusion.

1-2 Relation to the results of other investigators

It seems that the study of nonlinear control systems with a feedback characteristic lying partly outside the Hurwitz sector has not been undertaken before, even in the second order case.

Of course this is not the case of the Aizerman conjecture. Among many works, for the first class of third order control systems here considered, it seems that the best results to date have been obtained by Brockett and Willems.

In their crucial paper, (see reference 4), they illus-
the limitations of the Popov theorem and, using Liapounov functions and the concept of positive real multipliers, they established a new frequency domain criterion which differs from Popov's in that it assumes stronger restrictions on the nonlinearity. In particular, for the class of third order control systems mentioned above, their criterion guarantee global asymptotic stability for all monotonically increasing nonlinearities contained in the first and third quadrants.
CHAPTER 2
NEAR GLOBAL ASYMPTOTIC STABILITY FOR A SECOND ORDER
NON LINEAR CONTROL SYSTEM

2-1 System under consideration

The state variable equations for the second order system of the type (1-1) to be considered in the present chapter are

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\omega_0^2 x_1 - 2a x_2 - \phi(\sigma) \\
\sigma &= -\gamma x_1 + \delta x_2
\end{align*} \] (2-1)

where \( \gamma \) and \( \delta \) are positive constants, \((x_1, x_2) \in \mathbb{R}^2\) is the state vector and the real variable \( \sigma \) the scalar output.

It is known that the Popov sector of the control system (2-1) may be obtained geometrically by construction of the so-called Popov locus of which the coordinates are given parametrically by:

\[ \begin{align*}
X &= \Re(G(j\omega)) \\
Y &= \omega \Im(G(j\omega))
\end{align*} \] (2-2)

where \( \Re \) and \( \Im \) respectively denote the real and imaginary part of the transfer function \( G(s) \).

The Popov sector \( k \) is determined by constructing the straight line \( X - \Re k + 1/k = 0 \) (\( \Re \) real) such that the Popov locus lies completely on the right hand side of this line.

In the present case, (2-2) is:

\[ \begin{align*}
X &= \frac{\gamma \omega_0^2 + \omega_1^2 \{ \gamma - \delta \}}{\left( \omega_0^2 - \omega_1^2 \right) + 4a^2 \omega^2} \\
Y &= \frac{\delta \omega_0 + 2a \omega_1}{\left( \omega_0^2 - \omega_1^2 \right) + 4a^2 \omega^2}
\end{align*} \] (2-3)
and it follows easily that the value of the Popov sector is given by:

\[ k = -1/X(0) = \frac{2}{\omega_0} \]

and is consequently the same as the Hurwitz sector.

2-2 Preliminary study of an elementary linear characteristic

As already stated in chapter 1, one of the aims of this work is to study the system (1-1) with a piecewise linear feedback characteristic lying partly outside the Popov sector. It is to be noted that such a characteristic may be considered as the juxtaposition of elementary discontinuous linear functions that will be called, from now on, elementary linear characteristics (see Fig. 2).

Such functions are defined by:

\[
\phi(\sigma) = \begin{cases} 
0 & \text{if } \sigma \notin (\sigma_1, \sigma_2) \\
\sigma + q & \text{if } \sigma \in (\sigma_1, \sigma_2)
\end{cases}
\]

2-3 Determination of the critical points

Preliminary definition

A critical point is said to be real if it lies in the region of space it influences. It is said to be virtual if it does not lie in the region of space it influences.

For the system considered the critical points are determined by writing \( \dot{x}_1 = \dot{x}_2 = 0 \) i.e.

\[
\begin{align*}
\dot{x}_2 &= 0 \\
\phi(\sigma) &= -\omega_0 x_1
\end{align*}
\]

which can be written as:

\[ \phi(\sigma) = k \sigma \]
and is equivalent to saying that these points correspond to the values of $a$ at which the nonlinearity leaves and reenters the Popov sector. In other words it can be said that the number of critical points of the system (2-1) is equal to the number of points at which the feedback characteristic intersects the Popov-Hurwitz sector.

First of all we will concentrate our attention on the system (2-1) with $0 < a/\omega < 1$ which means that we will first assume the origin to be a stable focus.

Substituting the expression for $\phi(a)$ into (2-1) yields:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{2}{\omega} x_1 - 2ax_2$$

and

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = (2a+p)\dot{x}_1 - q$$

if $\sigma(\sigma_1, \sigma_2)$

(2-7)

The characteristic equation of (2-7) is:

$$\lambda^2 + (2a+p)\lambda - \gamma(p-k) = 0$$

Consideration of the roots of this equation makes clear the need for separating the elementary linear functions into two classes depending upon the value of their slope $p$ for, if $p > k$, the roots of (2-9) are always real and of opposite sign and, consequently, the corresponding critical point is a saddle point; on the other hand, if $p < k$, they will have real parts of the same sign, i.e., the corresponding critical points are of the type node or focus.

In conclusion, in the preliminary study of the system (2-1) with an elementary linear feedback characteristic it
will be necessary to consider the four cases shown in Fig. 2.

Cases (a) and (b) correspond to a slope of the function larger than \( k \), but, for (a), there is one **real** critical point and, for (b), one **virtual** critical point. The same distinction holds for (c) and (d), but with a slope of the function less than \( k \).

2-4 **Study of the case in which \( p > k \)**

2-4-1 **Case (a)**

As already stated previously, the critical point is a saddle point for this case.

The phase plane behaviour is illustrated by Fig. 3 and reveals the important role played by the four separatrices of this saddle point \( S \). Denote by \( S_0 \) and \( S_0' \) the stable separatrices converging to \( S \) and by \( S_1 \) and \( S_1' \) the unstable separatrices emerging from \( S \) (see Fig. 3). From now on these symbols will always refer to the separatrices of a real saddle point.

It is clear that it is of interest to show that the separatrices \( S_1 \) and \( S_1' \) have a slope \( m \) less than the slope \( \sqrt{k} \) of the switching lines for, otherwise, no stability properties could be obtained.

We have:

\[
-(2a+p 6)/2 + \left[ (2a+p 6)^2/4 + \chi(p-k) \right]^{1/2} < \sqrt{k}
\]

or

\[
p-k > 1/\sqrt{k}(\sqrt{\chi}+2a+p 6)
\]

\[-k < \sqrt{\chi}(a^2 + 2a/\delta)
\]

which is always true since the right hand side of the inequality is always positive.

It is to be noted that, for the control system (2-1)
with a nonlinear characteristic leaving the Popov sector on some $\sigma$-interval, it will never be possible to obtain global asymptotic stability since the critical point corresponding to the value $q_0$ of $\sigma$ at which the nonlinearity leaves the sector (see section 2-3) is always a real saddle point; this is because the slope of the nonlinearity is then larger than $k$ for this value of $\sigma$.

Consequently, as shown in Fig. 3, there will always be at least two trajectories (the two stable separatrices of the saddle point) that will not tend to the origin as $t \to \infty$.

This prompts us to define a new kind of stability, the best one may hope to achieve in such a problem: near asymptotic stability.

**Definition 1**

A control system (1-1) is near asymptotically stable in a region $R$ of the state space if all but a finite number of trajectories originated in $R$ tend to the origin as $t \to \infty$.

**Definition 2**

The control system (1-1) is near globally asymptotically stable if it is near asymptotically stable in the whole state space.

**Definition 3**

The direct region of attraction of a stable critical point is the region of space from which all trajectories tend to this critical point as $t \to \infty$ without crossing any switching line.
Theorem 1

A necessary condition for near global asymptotic stability of the system (2-1) is that the unstable separatrix \( S_1 \) tend to the origin as \( t \rightarrow \infty \).

Proof

Assume that the unstable separatrix \( S_1 \) does not tend to the origin as \( t \rightarrow \infty \).

\( S_1 \) is the unstable separatrix of a saddle point and it is known from the works of Poincaré, Andronov and other authors that such a separatrix is orbitally stable, i.e. that there is an infinity of trajectories lying in an \( \epsilon \)-neighborhood of this separatrix that have the same set of limit points as this separatrix; in the present case, this implies that an infinity of trajectories do not reach the direct region of attraction of the origin and consequently the system (2-1) is not near asymptotically stable, which terminates the proof.

The above reasoning is geometrically illustrated by Figs. 4 and 5.

2-4-2 Case (b)

The corresponding phase plane configuration is shown on Fig. 6; the interesting feature observed in this case, i.e. the appearance of a "slipping segment", will be studied in detail since it plays an important part in the sequel of this work.

It is clear that if \( A \) and \( B \) respectively denote the points of tangency of trajectories belonging to the regions
I and II respectively to the switching line \(-x_1 + \Delta x_2 = c_1\), trajectories arriving onto the segment \(AB\) are not able to switch directly and, consequently, following Andronov and Chaikin, it is natural to assume that the operating point will move along part of the switching line.

The validity of the above assumption is proved as follows:

Instead of considering the feedback characteristic shown in Fig. 2, take \(\phi(\sigma)\) as the limit when \(\varepsilon\) tends to 0 of the characteristic with hysteresis shown in Fig. 2bis.

Then a chattering appears between the two switching lines \((\sigma_1, \sigma_1 + \varepsilon)\) as indicated in Fig. 2ter.

Since \(S\) is a saddle point, the slope of \(A_1 A_2\) in the neighborhood of \(A_1\) is positive. In addition, since the origin is a stable focus, the slope of \(A_0 A_1\) in the neighborhood of \(A_1\) is negative. Hence, if the width of the hysteresis segment is sufficiently small, \(A_2\) will intersect the lower switching line to the left of the point where this line is intersected by \(A_0 A_1\).

It follows then that the trajectories do indeed exhibit the slipping segment behavior as assumed by us.

We may now study the motion on this segment by transferring the origin to the point \(0_1\) and calling \(X_1\) and \(X_2\) the coordinates in the new frame of reference.

With \(x_{1c} = -a/\gamma(p-k)\) we have

\[
X_2 = x_2 \\
X_1 = x_1 - x_{1c}
\]
i.e. \(-\frac{dY}{dt} = \frac{\partial x}{\partial x_1} = 0\)

The interpretation appears immediately:

Trajectories arriving onto the segment \(0_1A\) will slip towards \(A\) and go along the trajectory tangent in \(A\) to the switching line and, in a similar fashion, the trajectories falling upon the segment \(0_1B\) will slip towards \(B\) and pursue their evolution along the trajectory tangent to the switching line in \(B\) (see Fig.6); in addition, there will be two trajectories that tend to \(0_1\). From now on the trajectories originated from the points \(A\) and \(B\) will be referred to as "slipping trajectories".

The importance of the two slipping trajectories originated from \(A\) and \(B\) is clearly shown by Fig.6 and it appears that they play the same role as the separatrices \(S_1\) and \(S_1^\prime\) in the case previously studied, which leads to a very similar theorem.

If \(SL_1\) denotes the slipping trajectory issued from \(A\),

**Theorem 2**

A necessary condition for the system (2.1) with an elementary linear function \((b)\) as feedback characteristic to be near globally asymptotically stable is that the slipping trajectory \(SL_1\) tend to the origin as \(t \to \infty\).

**Proof**

If the slipping trajectory \(SL_1\) does not tend to the origin as \(t \to \infty\), this means that an infinity of trajectories falling upon the slipping segment \(0_1A\) (see Fig.6) (which has
a positive length) will not converge to the origin, which contradicts the definition of near global asymptotic stability and terminates the proof.

2-5 Study of the case in which \( p < k \)

Differently from the previously studied case, the nature of the critical point involved in the present case will change depending upon the value of the slope of the elementary linear feedback characteristic.

Define

\[ \Lambda = \frac{2a + p}{2} \text{ and } \Omega = \gamma(p-k) \]

2-5-1 Case (c)

The corresponding critical point is real and it is immediately clear that if the critical point is a stable node or focus—which will be the case if \( \Lambda/\Omega > 0 \) i.e. \( p > -2a/\delta \)—then an infinite number of trajectories will tend to this stable critical point as \( t \to \infty \) and, therefore near global asymptotic stability can never be obtained (see Fig. 7).

On the other hand, if \( \Lambda/\Omega > 0 \), the critical point is an unstable node or focus and, consequently, near global asymptotic stability cannot be obtained as illustrated by Figs. 8 and 9.

2-5-2 Case (d)

The same classification of critical points holds but, this time, since these critical points are virtual, no particular problem arises.

The near global asymptotic stability situation is shown by Figs. 10 through 13.
It is clear that, for all the cases considered above, theorem 2 is still valid and decides whether the system may be near globally asymptotically stable.

2-6 Case of a piecewise linear feedback characteristic

Define by \( \Phi \) the following class of continuous piecewise linear feedback characteristics:

A function \( \phi(\sigma) \in \Phi \) if

1) the function \( \phi(\sigma) \) is continuous on \(( -\infty, +\infty )\)

2) countable set \( \xi \) of numbers \(-\infty < \xi_1, \xi_2, \ldots, \xi_n < +\infty\)

such that in each interval \((\xi_i, \xi_j)\), for all \( i \) and \( j \),

\[
\phi(\sigma) = p_i \sigma + q_i
\]

where \( p_i \) and \( q_i \) are finite constant real numbers on \((\xi_i, \xi_j)\)

3) the function \( \phi(\sigma) \) satisfies the following inequalities:

\[
0 < \phi(\sigma) < k \quad \text{if} \quad \sigma \in (\xi_0, \xi_1)
\]

\[
\phi(\sigma) < k \quad \text{if} \quad \sigma \in (\xi_0, \xi_1)
\]

where \( \xi_0 \) and \( \xi_1 \) are positive numbers.

Functions belonging to the class just defined are shown in Figs. 14 and 15.

From now on denote by \( S \) and \( U \) the critical points corresponding to the values \( \xi_0 \) and \( \xi_1 \) at which the feedback characteristic respectively leaves and reenters the Popov sector.

The study conducted in the previous section for an elementary linear characteristic immediately leads to a necessary condition to be imposed on the feedback linear
piecewise characteristic for near global asymptotic stability:

**Theorem 3**

A necessary condition for near global asymptotic stability of system (2-1) with a piecewise linear feedback characteristic belonging to the class $\tilde{\phi}$ is that reentry into the Popov sector takes place with a slope smaller than $-2\alpha/\delta$ i.e. $\frac{d\phi(q_i)}{dq}<2\alpha/\delta$.

**Proof**

If this was not true, the critical point $U$ would be stable and, in its neighborhood, an infinity of trajectories would converge to it, which violates the definition of near asymptotic stability.

From now on, we will assume that the necessary condition just stated is verified i.e. the critical point $U$ is an unstable node or focus.

The theorem 1 already given for an elementary linear characteristic is easily extended to the case of a piecewise linear feedback characteristic and the reasoning is very similar and based upon the properties of unstable separatrices of a saddle point.

Unstable separatrices of a saddle point, if they do not tend to a saddle point are orbitally stable paths, i.e., all trajectories originated in their immediate neighborhood will admit the same set of limit points. Hence, in the present case, if the unstable separatrix $S_1$ does not tend to the origin as $t \to \infty$, there can occur only two possibilities:
a) $S_1$ converges back to the saddle point $S$ and, consequently, an infinity of trajectories contained in the region of the space so delimited will never be able to reach the direct region of attraction of the origin.

b) $S_1$ does not converge back to the saddle point $S$ and it follows that $S_1$ is an orbitally stable path and that an infinity of trajectories will have the same set of limit points as this separatrix and will not tend to the origin as $t \to -\infty$.

So, in both cases, the definition of near global asymptotic stability is contradicted, which terminates the proof.

From the above necessary condition, one may be tempted to conclude that it is also sufficient but, in fact, one must be careful and investigate the limit cycles that may possibly occur for system (2-1).

As we have three critical points, one of them being a saddle point, the following possibilities may occur:

First of all, around the stable focus at the origin, no limit cycle whatsoever may occur since, if it existed, it would have to lie partly in the region of direct attraction of the origin, which is clearly impossible.

Around the unstable critical point $U$, the situation is different and one may encounter two types of limit cycles according to Poincaré:

a) As shown in Fig. 16, a stable limit cycle $C_0$ can exist and the unstable separatrix $S_1$ winds itself around it.

b) As in Fig. 17 bis, there may be a semi-stable limit cycle $C_1$ such that the unstable separatrix $S_1$ does tend to the origin as $t \to -\infty$ but the stable separatrix $S'$.
unwinds itself from $C_1$.

Finally, Poincare's theory predicts the possible existence of limit cycles surrounding the three critical points and as before they may be of two types:

a) As in Fig. 17, one can have a stable limit cycle $C_2$ which is approached by the unstable separatrix $S_1$.

b) There may be a limit cycle $C_3$ (semi-stable) and, if the unstable separatrix $S_1$ does tend to the origin, the stable separatrix $S_0$ unwinds from $C_3$.

The consideration of the above possibilities and the fact that the qualitative behavior of the trajectories of a control system is completely determined by the study of its limit cycles make obvious the following theorem:

**Theorem 4**

Sufficient conditions for near asymptotic stability of the system (2-1) with a piecewise linear feedback characteristic belonging to the class $\Phi$, in a closed region of the phase plane larger than the region of direct attraction of the origin are:

1) The unstable separatrix $S_1$ tends to the origin as $t \to \infty$

2) Some trajectory which originates in an $\varepsilon$-neighborhood of the unstable critical point $U$ (positive arbitrarily small) is either asymptotically stable to the origin or stable to the saddle point $S$.

The closed region of stability mentioned in the above theorem is obviously delimited by the limit cycle $C_3$. 
Remark 1

Theorem 4 is clearly extended to near global asymptotic stability by requiring, in addition to conditions (1) and (2) the following one:

3) Some trajectory originated from the point \( M \) of coordinates \((M, 0)\) where \( M \) is an arbitrarily big number tends to the origin as \( t \to \infty \).

2-7 Case of a nonlinear feedback characteristic

The nonlinearity considered from now on is assumed to have the following properties:

a) \( 0 < \phi'(\sigma)/\sigma < k \) if \( \phi' (\sigma, \sigma_r) \)

b) \( \phi'(\sigma)/\sigma > k \) if \( \sigma \in (\sigma_0, \sigma_r) \)

c) Reentry into the Popov sector takes place with a slope

\[
\frac{d\phi(\sigma)}{d\sigma} < -2\alpha/\delta
\]

where \( \sigma_0 \) and \( \sigma_r \) are positive values of \( \sigma \).

It follows that, as was the case with the piecewise linear characteristic studied before, there will be three critical points: the origin and two other points \( S \) and \( U \) respectively corresponding to the values \( \sigma_S \) and \( \sigma_R \) of \( \sigma \). \( S \) will be a saddle point (since \( \frac{d\phi(\sigma)}{d\sigma} > k \) and \( U \) an unstable node or focus (since \( \frac{d\phi(\sigma)}{d\sigma} > k \)), which means that the possible limit cycles already encountered in the previous section will have the same general location.

We already stated the fact that the qualitative behavior of a control system is completely determined by the knowledge of its critical points, separatrices and the general location of the limit cycles.
Hence, it can be concluded that all the results already established with piecewise linear feedback characteristics and in particular theorem 4 always hold.

Remark 2

All the above results and theorems have been obtained by considering feedback characteristics leaving the Popov sector only for positive values of $\sigma$; it is obvious that the results are the same if the values of $\sigma$ are negative.

In the case where the characteristic lies partly outside the Popov sector, both for positive and negative values of $\sigma$, the theorems already given are easily extended by imposing conditions on the two unstable separatrices (one for each saddle point) and on the two unstable critical points; this situation is illustrated by Fig. 18bis.

Remark 3

All the preceding work was conducted while assuming the origin to be a stable focus.

It is clear that the theorems and results so established remain completely valid when the origin is a stable node (i.e. $a/\omega > 1$)
CHAPTER 3
HIGH ORDER SYSTEMS

It would be of interest to gain some insight concerning high order control systems and study their stability behavior when the feedback characteristic $\phi(\sigma)$ leaves both the Popov and the Hurwitz sectors.

In particular, we tried to investigate this problem for a restricted class of third order control systems represented in state variable form by the equations:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -cx_1 - bx_2 - ax_3 - \phi(\sigma) \\
\sigma &= x_1 \\
a, b, c &> 0 \\
ab-c &> 0
\end{align*}
\]

The Popov and Hurwitz sectors coincide and have the value $k = ab-c$ and we considered as feedback characteristic leaving the Hurwitz sector the following simple ramp:

\[
\begin{align*}
\phi(\sigma) &= p(\sigma - \sigma_1) & \text{if} & & \sigma > \sigma_1 \\
\phi(\sigma) &= 0 & \text{if} & & \sigma \leq \sigma_1
\end{align*}
\]

where $p > ab-c$

It is then seen that such a system has only one real stable critical point located at the origin and one virtual critical point $0_1$ on the $x_1$ axis of abscissa:

\[
x_{1c} = \frac{p}{(c+p)\sigma_1} < \sigma_1
\]
By imposing additional restrictions on the coefficients $a, b, c$ namely:

$$(9c-ab)^2 - (6b-2a^2)(6ac-2b^2) > 0$$

and

$$27c-9ab + 2a^3 > 0$$

it is easily seen that $0_1$ is a virtual unstable saddle-focal point and, consequently, all trajectories originated in the region $x_1 > 0_1$ will intersect the switching plane $x_1 = 0_1$ in a finite time.

Unfortunately, we have not been able to guarantee that such a trajectory, after switching, does not intersect again the switching plane $x_1 = 0_1$ i.e. the possibility of a limit cycle occurring for certain initial conditions is not removed.

It is however of interest to note that analog and digital computer studies of this problem do not yield any limit cycle and all the solutions tend to the origin of the state space.

This problem deserves to be further investigated in order to bring a definite conclusion concerning global asymptotic stability.

Another third order system investigated by us with respect to global asymptotic stability was:

$$\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -c x_1 - b x_2 - a x_3 - \phi(\sigma) \\
\sigma &= x_3 \\
a, b, c &> 0 \\
ab-c &> 0
\end{align*}$$
The study of the Popov locus of this system (see Fig. 19) indicates that, for this system, if the Hurwitz sector is infinite, the Popov sector has a finite value and it is therefore of interest to study the validity of the Aizerman conjecture.

Taking piecewise linear continuous feedback characteristics which can be represented by the expressions:

\[ \Phi(\sigma) = p_1 \sigma + q_1 \]

on intervals \((\sigma_i, \sigma_j)\) of the variable \(\sigma, i=1, ... n, j=i+1\)

it is clear that there will be only one real stable critical point at the origin and all the other critical points, influencing the regions of the state space \(\sigma_i < x_j < \sigma_j, i=1, ... n\) will be virtual, i.e., on the \(x_j\) axis and have for abscissa:

\[ x_{i1} = -q/c \]

In each strip \(\sigma_i < x_j < \sigma_j\), the state equations, referred to the influencing critical point are:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -c x_1 - b x_2 - (a + p_1) x_3
\end{align*}
\]

and, consequently, if \(p_1 > -a + c/b\) for all \(i\), the corresponding critical points are stable.

In the state space, the reasoning already conducted for the first third order system considered may be applied and, if one was able to guarantee that trajectories cannot intersect the switching planes \(x_j = \sigma_i\) several times, one could readily conclude the validity of the Aizerman conjecture for the class of nonlinearities having their slopes
bounded from below by $-a+c/b$ for all $c$.

However we have not been able to disprove the possible existence of limit cycles although an analog computer study conducted for this system shows global asymptotic stability.

It seems that to be able, in the future, to draw any definite conclusions, one will have to use Liapounov methods.
CHAPTER 4
CONCLUSIONS

In the main part of this work, the behavior of second order nonlinear control systems has been investigated with respect to global asymptotic stability when the nonlinear feedback characteristic leaves the Popov-Hurwitz sector on a finite \( \alpha \)-interval.

It is shown that, with such a feedback characteristic, global asymptotic stability may never be obtained.

However, by imposing restrictions on the slope of the nonlinear feedback characteristic when it reenters the Popov-Hurwitz sector and by the stability behavior of a few particular trajectories, interesting conclusions may be obtained to the effect that all but a finite number of trajectories tend to the origin as \( t \to \infty \); this behavior was referred to by the term of near global asymptotic stability.

A worthwhile feature of this study was to establish that the trajectories that do not tend to the origin as \( t \to \infty \) are either stable separatrices of a saddle point or semi-stable limit cycles, i.e. trajectories that are not orbitally stable.

Consequently, in the real control system where small disturbances always exist, in effect, the stable separatrices and the semi-stable limit cycles mentioned above have no physical existence and it follows that, practically, the system will be globally asymptotically stable provided that condition (1) of theorem 4 holds, which can be checked easily.
by the computer program contained in the appendix.

The last part of this work, dealing with high order systems demonstrates the inadequacy of state space methods for the study of global asymptotic stability.

It seems that any additional work in this area will have to use Liapounov's methods in order to derive definitive conclusions.
Region of direct attraction of the origin.
Figure 7

Region of direct attraction of the origin.

Real stable focus.
Figure 9

Region of direct attraction of the origin.
virtual stable focus

region of direct attraction of the origin.
virtual unstable focus

Region of direct attraction of the origin.
APPENDIX

1 Computer program

The program we use is the classical application of a Runge-Kutta numerical method of integration applied to a system of $N$ simultaneous first order differential equations.

1-1 Nomenclature of the symbols used

$N = \text{dimension of the system}$

$\Delta T = \text{step of integration}$

$X_1, X_2, X_3, \ldots \ldots \text{etc} = \text{state variables}$

$\text{DEP}(1), \text{DEP}(2), \text{DEP}(3) \ldots \ldots \text{initial conditions}$

$T = \text{initial time}$

$\Sigma = \text{variable } \sigma \text{ in equations (1-1)}$

$\Phi = \text{feedback characteristic } \phi \sigma \text{ in equations (1-1)}$

The state variable equations are to be written with the following notations:

$P(I) = \frac{dX_I}{dt}$ \quad $I=1,2,\ldots, N.$

$DV(I) = X_I$

1-2 Display of the results

The state variables values are printed by columns.

In addition, the following rough plots of the solution may be available:

- $X_2 \text{ vs } X_1$
- $X_3 \text{ vs } X_2$
- $\vdots$
2 Examples studied

The following examples have been studied both with the digital computer and the analog computer.

The system under consideration is:

\[ \dot{x}_1 = x_2 \]
\[ x_2 = -x_1 - x_2 - \phi(\sigma) \]
\[ \sigma = -x_1 + x_2 \]

The Popov and Hurwitz sectors are confused and equal to 1.

The following different nonlinearities violating the Popov condition are simulated:

2-1

\[ \phi(\sigma) = 0 \quad \text{if} \quad \sigma \leq 1 \quad \text{or} \quad \sigma \geq 3 \]
\[ \phi(\sigma) = 3(\sigma - 1) \quad 1 \leq \sigma \leq 2 \]
\[ \phi(\sigma) = -3(\sigma - 5) \quad 2 \leq \sigma \leq 3 \]

Fig. A1 shows that near global asymptotic stability is obtained

2-2

\[ \phi(\sigma) = 0 \quad \text{if} \quad \sigma \leq 1 \quad \text{or} \quad \sigma \geq 6 \]
\[ \phi(\sigma) = 3(\sigma - 1) \quad 1 \leq \sigma \leq 2 \]
\[ \phi(\sigma) = 3 \quad 2 \leq \sigma \leq 5 \]
\[ \phi(\sigma) = -3(\sigma - 2) \quad 5 \leq \sigma \leq 6 \]

Theorem 3 is not satisfied and near asymptotic stability may not be obtained since \( U \) is a real stable focus (see Fig. A2)

2-3

\[ \phi(\sigma) = 0 \quad \text{if} \quad \sigma \leq 1 \]
\[ \phi(\sigma) = 3(\sigma - 1) \quad 1 \leq \sigma \leq 2 \]
\[ \phi(\sigma) = -3(\sigma - 1) \quad 2 \leq \sigma \leq 3 \]
\[ \phi(\sigma) = 3/4 \quad 3 \leq \sigma \leq 6 \]

Condition (1) of theorem 4 is violated and the separatrix \( S_1 \) approaches a stable limit cycle surrounding the unstable focus \( U \) (see Fig. A3)
DUTERTRE  FORTRAN SOURCE LIST

ISN SOURCE STATEMENT

0 S18FTC CKJAC
1 DOUBLE PRECISION DEP(39)
2 COMMON DEP/G/N,T,DELT
3 C N=DIMENSION OF THE SYSTEM
4 C T=TIME
5 C DELT=STEP OF INTEGRATION
6 C XI,X2,X3=STATE VARIABLES
7 C DEP(I),I=1,N INITIAL CONDITIONS
8 C INITIAL TIME=0
9 3 DIMENSION XI(800),X2(800)
10 N =2
11 DEP(1)=-2.25001
12 DEP(2)=0.
13 DELT=0.05
14 T=0.
15 CALL NIODES(1)
16 XI(1)=DEP(1)
17 X2(1)=DEP(2)
18 DO 10 J=2,800
19 CALL NIODES(3)
20 XI(J)=DEP(1)
21 10 X2(J)=DEP(2)
22 CALL PLTREX(X1,X2,800)
23 CALL PLOT(X1,X2,800)
24 STOP
25 END
```
SUBROUTINE NIODES(I ENTER)
C GENERAL 'NUMERICAL INTEGRATION' SUBROUTINE FOR 'SYSTEMS OF FIRST'
C ORDER ORDINARY DIFFERENTIAL EQUATIONS USING AN ADAMS METHOD WITH
C A RUNGE-KUTTA STARTER

DIMENSION P(39), TE(39), YPR(39,4)
DIMENSION DV(39)
DOUBLE PRECISION DEPI39, NEP39
COMMON DEP/G/N, T, DELT...
GO TO (1, 2, 3), I ENTER

DATA A1, A2, A3, A4, B1, B2, B3, B4, B5 = 0.375000,
      1.541666, -2.4583333, 2.291666, -0.26388889E-1,
      2.01472222, -0.36666667, 0.89722222, 0.343611117

1 M1 = 4
11 M2 = 1
12 M3 = 2
13 M4 = 3
14 IK = 1
15 KOUNT = 0
C*** LEFT END OF INTERVAL RUNGE-KUTTA  (LOC=6)
C*** SET UP INITIAL CONDITIONS FOR RUNGE-KUTTA
C*** SIZE OF INTEGRATION STEP EQUALS 'DELT'
16 DELBY6 = DELT * 1.6666667
17 DELBY2 = DELT * 0.5000
20 DU 17, J = 1, N;
21 17 DV(J) = DEP(J)
23 CALL DERIV(T, DV, P)
24 DO 18 J = 1, N
25 18 YPR(J, M1) = P(J)
27 RETURN
30 3 V = T
31 T = V + DELT
32 GO TO (2000, 1000), IK
C*** LEFT END OF INTERVAL ADAMS-MOULTON  (LOC=1)
33 1000 DO 140 J = 1, N
34 TE(J) = B5 * YPR(J, M1) + B3 * YPR(J, M2) + B2 * YPR(J, M3) + B1 * YPR(J, M4)
35 140 DV(J) = DEP(J) + DELT *(A4 * YPR(J, M1) + A3 * YPR(J, M2) + A2 * YPR(J, M3) + A1 * YPR(J, M4))
37 CALL DERIV(T, DV, P)
C*** RIGHT END OF INTERVAL ADAMS-MOULTON  (LOC=2)
40 DO 150 J = 1, N
41 150 DV(J) = DEP(J) + DELT *(B5 * P(J) + TE(J))
43 CALL DERIV(T, DV, P)
44 DO 151 J = 1, N
45 YPR(J, M4) = P(J)
46 151 DEP(J) = DEP(J) + DELT *(B5 * P(J) + TE(J))
50 160 CONTINUE
51 5000 MO = M4
52 M4 = M3
53 M3 = M2
54 M2 = M1
55 M1 = MO
56 RETURN
57 2000 VV = V + DELBY2
60 DO 260 J = 1, N
```
DUTERTRE FORTRAN SOURCE LIST

SOURCE STATEMENT

61 260  DV(J) = DEP(J) + YPR(J,M1)*DELB2

*** CENTER OF INTERVAL (1) RUNGE-KUTTA  (LOC=3)

63  CALL DERIV(VV,DV,P)
64  DO 261 J=1,N
65 261  DV(J) = DEP(J) + P(J)*DELB2

*** CENTER OF INTERVAL (2) RUNGE-KUTTA  (LOC=4)

67  CALL DERIV(VV,DV,TE)
70  DO 262 J=1,N
71 262  DV(J) = DEP(J) + TE(J)*DELT
72 262  TE(J) = 2*Q*(TE(J)+P(J))

*** RIGHT END OF INTERVAL RUNGE-KUTTA  (LOC=5)

74  CALL DERIV(T,DV,P)
75  DO 263 J=1,N
76 263  DV(J) = DEP(J) + DELB6*(P(J)+TE(J)+YPR(J,M1))

101  CALL DERIV(T,DV,P)
102  DO 264 J=1,N
103 264  YPR(J,M4) = P(J)

105 250  CONTINUE
106  KOUNT = KOUNT + 1
107  IF(KOUNT.LT.3) GO TO 5000
112  IK = 2
113  GO TO 5000
114  END
SUBROUTINE DERIV(T, DV, P)

DIMENSION DV(39), P(39)

C DEFINE SIGMA AND PHI(SIGMA)

SIGMA = -DV(1) + DV(2)

IF(SIGMA.LE.0) PHI = 0

IF(SIGMA.LE.20. AND. SIGMA.GE.1o) PHI = 30*(SIGMA-10)

IF(SIGMA.LE.2.4 AND. SIGMA.GE.20) PHI = -30*(SIGMA-30)

C WRITE DOWN THE STATE VARIABLE EQUATIONS

P(1) = DV(2)

P(2) = DV(1) - DV(2) - PHI

RETURN

END
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