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ON THE STABILITY OF LINEAR PERIODIC TIME-VARYING SYSTEMS

by

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ABSTRACT

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Stability in a linear system having periodic time variable elements in the feedback path is determined using the Theorem of Floquet and the second method of Liapunov. Approximate conditions for asymptotic stability are derived from a finite determinant which is a truncated approximation to an infinite set of equations. The stability problem is thus reduced to one of solving the characteristic equation of a linear system with constant coefficients and classical methods such as the Nyquist plot, Routh-Hurwitz criterion and root-locus method can be applied. Examples have been given showing the approximate boundaries of stability.
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Chapter 1. INTRODUCTION

One of the most fundamental problems associated with studying
automatic control is the investigation of system stability. For linear,
time-invariant systems, it is sufficient to test for stability by using
classical methods such as Nyquist criterion, Bode diagram, Routh-Hurwitz
criterion or root-locus method, but for time-varying systems, the above
mentioned criteria are no longer valid. The problem here is to determine
exact if possible and certainly approximate boundaries of stability for
a linear, periodic, time-varying system by using the theory of Floquet
in a Liapunov formulation. In this chapter, theorems which are actually
involved such as Liapunov's stability theorem and the theory of Floquet
are briefly described.

Liapunov's Theorem on Stability

Consider the system

\[ \dot{x} = f(x, t), \quad f(0, t) = 0 \tag{1-1} \]

as an n-vector system where \( f(x, t) \) is continuous and satisfies a
Lipschitz condition in \( x \) for all \( x \) and \( t \). For the system described by
equation (1-1), an important stability theorem by Liapunov [1,2] states
if there exists a differentiable function \( V(x, t) \) (called the Liapunov
function) such that

\[ W_1(x) \geq V(x, t) \geq W_2(x) > 0, \quad x \neq 0, \tag{1-2} \]

\[ V(0, t) = 0, \quad t \geq 0, \tag{1-3} \]

and

\[ \dot{V}(x, t) \leq 0, \tag{1-4} \]

where \( W_1(x) \) and \( W_2(x) \) are positive definite, then the system is stable.
If in addition to equations (1-2) and (1-3),
where $\mathbb{W}_3(x)$ is positive definite, then the system is asymptotically stable.

**Floquet Theory**

Consider the $n$th order linear differential equation with periodic coefficients of the form

$$\dot{x} = F(t) x, \quad (1-6)$$

where $F(t)$ is continuous and periodic with period $T$, i.e., $F(t+T) = F(t)$. Let $L(t)$ be a nonsingular transformation with bounded coefficients of period $T$ such that

$$y = L(t) x. \quad (1-7)$$

It is seen by using equation (1-7) that equation (1-6) yields

$$\dot{y} = R y, \quad (1-8)$$

where

$$R = L L^{-1} + L F L^{-1}. \quad (1-9)$$

According to a theorem by Liapunov [2], there exists a nonsingular transformation $L(t)$ such that $R$ is a constant matrix. This means that a linear system with periodic coefficients is reducible to a linear system with constant coefficients through a nonsingular transformation. Now since $L(t)$ is nonsingular, equation (1-8) may be examined for stability instead of equation (1-6). The characteristic equation of system (1-8) is

$$\det(R - \lambda I) = 0 \quad (1-10)$$

where $I$ is the unit matrix and $\lambda$ is the eigenvalue.

For the system given by equation (1-8), if the real part of each eigenvalue of $R$ is negative, then the system is asymptotically
stable [3]. If one of the eigenvalues has a positive real part, then
the system is unstable. Here, the eigenvalues of \( \mathbf{A} \) determine the
stability.
Chapter 2. LINEAR SYSTEM WITH A SINGLE TIME-VARIABLE PARAMETER

Mathematical Description of the System

The problem under consideration is that of a linear feedback system with a linear time-invariant system having a transfer function $G(s)$ in the forward path and a gain $f(t)$ in the feedback path. This system is shown in Fig. 1.

\[
\begin{align*}
\text{zero} & \quad + \quad u \quad G(s) \quad \sigma \\
\text{input} & \quad - \quad i(t) \\
\end{align*}
\]

Fig. 1. Time-varying system.

The system may be represented as

\[
\begin{align*}
\dot{x} &= F \dot{x} + G u \\
u &= -f(t) \sigma \\
\sigma &= h^t \dot{x} \\
\end{align*}
\]

(2-1)

where

\[
\begin{align*}
G(s) &= h^t(s I - F)^{-1} F \\
\end{align*}
\]

(2-2)

These equations may also be expressed as

\[
\begin{align*}
\dot{x} &= \hat{F} \dot{x} \\
u &= -f(t) \sigma \\
\sigma &= h^t \dot{x} \\
\end{align*}
\]

(2-3)

where

\[
\begin{align*}
\hat{F} &= F - G h^t \\
\end{align*}
\]

(2-4)

The system may also be shown in a vector flow diagram as in Fig. 2.
The gain \( l(t) \) in the feedback path is assumed to be
\[
l(t) = \delta (e^{i2\omega t} + e^{-i2\omega t}) = 2 \delta \cos 2\omega t.
\] (2-5)

More general periodic time-varying parameters will be treated in chapter 3. From chapter 1, \( L(t) \) in equation (1.7) must be periodic. Thus \( L(t) \) can be written as
\[
L(t) = \sum_{n=-\infty}^{\infty} L_n e^{in\omega t}.
\] (2-6)

According to the theorem of Floquet that was mentioned in chapter 1, there exists a nonsingular \( L(t) \) so that \( R \) in equation (1.8) is a constant matrix. Substitute equations (2-5) and (2-6) into (1.9), the following expression is obtained.
\[
\sum_{n=-\infty}^{\infty} L_n (in\omega)e^{in\omega t} + \left( \sum_{n=-\infty}^{\infty} L_n e^{in\omega t} \right) \left[ \overline{F} - \delta (e^{i2\omega t} + e^{-i2\omega t}) E' h' \right] = \sum_{n=-\infty}^{\infty} R L_n e^{in\omega t}.
\] (2-7)

This must hold for all \( n \). Thus
\[
L_n (in\omega + \overline{F}) e^{in\omega t} - \delta (L_{n-2} + L_{n+2}) E' h' = R L_n \quad \text{(all } n) \tag{2-8}
\]

where \( R \) and \( L_n \) are unknowns. This equation leads to two sets of independent equations, one set for \( n \) even and one set for \( n \) odd.
System stability may be determined by using either odd set or even set.

Using the odd set, equation (2-8) may be written as

\[ L_{2j-1}[i(2j-1)\omega + \varepsilon] - \delta(L_{2j-3} + L_{2j+1})^j h' = R L_{2j-1} \quad (\text{all } j) \]

(2-9)

Rather than working on the infinite set of equations, it is more convenient to consider an approximation to the infinite set of equations formed by truncation. Assume all \( L_s \) to be zero for \( s > N \). Since the odd set is being used, \( N \) must be an odd number. By writing out all the individual equations in equation (2-9), the following set of equations is obtained.

\[
\begin{align*}
2j-1=-N: & \quad L_{-N}[i(2N+1)\omega + \varepsilon] - \delta(L_{-N+2} + L_{-N-2})^N h' = R L_{-N} \\
2j-1=-N+2: & \quad L_{-N+2}[i(-N+2)\omega + \varepsilon] - \delta(L_{-N+4} + L_{-N})^N h' = R L_{-N+2} \\
2j-1=-N+4: & \quad L_{-N+4}[i(-N+4)\omega + \varepsilon] - \delta(L_{-N+6} + L_{-N+2})^N h' = R L_{-N+4} \\
\vdots & \quad \vdots \\
2j-1=-3: & \quad L_{-3}[i(3)\omega + \varepsilon] - \delta(L_{-5} + L_{-1})^3 h' = R L_{-3} \\
2j-1=-1: & \quad L_{-1}[i(-1)\omega + \varepsilon] - \delta(L_{-3} + L_{1})^1 h' = R L_{-1} \\
2j-1=1: & \quad L_{1}[i(1)\omega + \varepsilon] - \delta(L_{-1} + L_{3})^1 h' = R L_{1} \\
2j-1=3: & \quad L_{3}[i(3)\omega + \varepsilon] - \delta(L_{-3} + L_{5})^3 h' = R L_{3} \\
\vdots & \quad \vdots \\
2j-1=N-4: & \quad L_{N-4}[i(N-4)\omega + \varepsilon] - \delta(L_{N-6} + L_{N-2})^{N-4} h' = R L_{N-4} \\
2j-1=N-2: & \quad L_{N-2}[i(N-2)\omega + \varepsilon] - \delta(L_{N-4} + L_{N})^{N-2} h' = R L_{N-2} \\
2j-1=N: & \quad L_{N}[iN\omega + \varepsilon] - \delta(L_{N-2} + L_{N+2})^N h' = R L_{N}
\end{align*}
\]

Write equation (2-10) in the matrix form

\[ E A = R \& \]

(2-11)
where

\[ L = \begin{bmatrix}
L_{-N} & L_{-N+2} & L_{-N+4} & \cdots & L_{-3} & L_{-1} & L_1 & L_3 & \cdots & L_{N-4} & L_{N-2} & L_N
\end{bmatrix} \]  

(2-12)

and

\[
A = \begin{bmatrix}
-\omega I^+ + \delta h' & 0 & \cdots & 0 \\
-\delta h' & i(-N+2)\omega I^+ + \delta h' & \cdots & 0 \\
0 & -\delta h' & i(-N+4)\omega I^+ + \delta h' & \cdots & 0 \\
0 & 0 & -\delta h' & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
(2-13)
\]

It is important to look for stability for the given system, hence the eigenvalues of \( A \). Stability can be assured if all the eigenvalues of \( A \) have negative real parts. If one of the eigenvalues has a positive real part, then the system is unstable. Now for \( A \), from equation (1-8)

\[ \dot{V} = A V. \]  

(2-14)

A Liapunov function of the quadratic form

\[ V = y^+ P y \]  

(2-15)

is assumed for the system (2-14), where \( + \) means complex conjugate transpose. Thus

\[ \dot{V} = -y^+ Q y, \]  

(2-16)
where \( -\mathbf{Q} = \mathbf{R}^T\mathbf{P} + \mathbf{P}\mathbf{R} \) \hspace{1cm} (2-17)

and \( \mathbf{P} \) and \( \mathbf{Q} \) are hermitian. Now for asymptotic stability, a necessary and sufficient condition is that any positive definite \( \mathbf{Q} \) gives a positive definite \( \mathbf{P} \) [1]. Multiply by \( \mathbf{X}^T \) and \( \mathbf{X} \) in equation (2-17) to obtain

\[
-\mathbf{X}^T\mathbf{Q}\mathbf{X} = \mathbf{X}^T\mathbf{R}^T\mathbf{P}\mathbf{X} + \mathbf{X}^T\mathbf{P}\mathbf{R}\mathbf{X}.
\]

Using equation (2-11), equation (2-18) becomes

\[
-\mathbf{X}^T\mathbf{Q}\mathbf{X} = (\mathbf{X}^T\mathbf{A})^T\mathbf{P}\mathbf{X} + \mathbf{X}^T\mathbf{P}\mathbf{A}\mathbf{X}
\]

\[
= \mathbf{A}^T(\mathbf{X}^T\mathbf{P}\mathbf{X}) + (\mathbf{X}^T\mathbf{P}\mathbf{X})^T\mathbf{A},
\]

or

\[
-\mathbf{Q} = \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A},
\]

where \( \mathbf{P} = \mathbf{X}^T\mathbf{P}\mathbf{X} \) \hspace{1cm} (2-20)

and \( \mathbf{Q} = \mathbf{X}^T\mathbf{Q}\mathbf{X} \) \hspace{1cm} (2-21)

\( \mathbf{P} \) and \( \mathbf{Q} \) can be positive definite if and only if \( \mathbf{X}^T\mathbf{P}\mathbf{X} \) and \( \mathbf{X}^T\mathbf{Q}\mathbf{X} \) are positive definite, and stability can be assured if and only if any positive definite \( \mathbf{Q} \) gives a positive definite \( \mathbf{P} \). This can happen if and only if all the eigenvalues of \( \mathbf{A} \) are in the left half plane. Thus look at

\[
\dot{\mathbf{G}} = \mathbf{0}
\]

where \( \mathbf{G} = \lambda\mathbf{I} - \mathbf{A} \) \hspace{1cm} (2-23)

Substituting the expression of \( \mathbf{A} \) into equation (2-24), \( \mathbf{G} \) can thus be written as
\[
C = \begin{bmatrix}
\delta_{gh}^* & 0 & 0 & 0 & 0 & 0 \\
\delta_{gh}^* & [\lambda - i(\omega + 2)]I - F & \delta_{gh}^* & 0 & 0 & 0 \\
0 & \delta_{gh}^* & [\lambda - i(\omega + 4)]I - F & \delta_{gh}^* & 0 & 0 \\
0 & 0 & \delta_{gh}^* & (\lambda + i\omega)I - F & \delta_{gh}^* & 0 \\
0 & 0 & 0 & \delta_{gh}^* & (\lambda - i\omega)I - F & 0 \\
0 & 0 & 0 & 0 & \delta_{gh}^* & [\lambda - i(\omega - 4)]I - F & \delta_{gh}^* \\
0 & 0 & 0 & 0 & 0 & \delta_{gh}^* & (\lambda - i\omega)I - F \\
\end{bmatrix}
\]
Define a \((\ell+1)\times(\ell+1)\) diagonal matrix \(B\) such that

\[
B = \begin{bmatrix}
(\lambda+1\omega)I - F & 0 \\
0 & [\lambda-1(-\ell+2)\omega]I - F & 0 \\
& 0 & [\lambda-2(-\ell+4)\omega]I - F & 0 \\
& & 0 & (\lambda+1)I - F
\end{bmatrix}
\]

\[(2-26)\]
and

\[
\mathbf{H}^{-1} = \begin{bmatrix}
\frac{1}{(\lambda + \lambda \omega)\mathbb{I} - \mathcal{E}} & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{(\lambda - \lambda (-1+\omega)\mathbb{I} - \mathcal{E})} & 0 & \cdots & 0 \\
0 & 0 & \frac{1}{(\lambda + \lambda (3\omega)\mathbb{I} - \mathcal{E})} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \frac{1}{(\lambda - \lambda (-2n+\omega)\mathbb{I} - \mathcal{E})} \\
0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]

(2.27)
Now,

$$\mathbf{U} \mathbf{U}^{-1} = \mathbf{I} \quad \text{(2-28)}$$

Multiply \( \mathbf{U} \) by \( \mathbf{U}^{-1} \), thus

$$\det \mathbf{U} = \det (\mathbf{U} \mathbf{U}^{-1}) \mathbf{U} = (\det \mathbf{U})(\det \mathbf{U}^{-1}) \mathbf{U} \quad \text{(2-29)}$$

Now the eigenvalues of \( \mathbf{U} \) are those of \( \mathbf{F} \) but shifted by \( 1(2j-1)\omega \). Thus if \( \mathbf{F} \) is stable, the eigenvalues of \( \mathbf{U} \) are in the left half plane. Hence it is only necessary to examine

$$\det \mathbf{U}^{-1} = 0 \quad \text{(2-30)}$$

or

$$\mathbf{I} \delta[(\lambda + j\omega)\mathbf{I} - \mathbf{F}]^{-1} - \mathbf{I} = 0 \quad \text{(2-31)}$$

$$\delta[(\lambda - 1(-j+2)\omega)\mathbf{I} - \mathbf{F}]^{-1} \mathbf{I} \delta[(\lambda - 1(-j+2)\omega)\mathbf{I} - \mathbf{F}]^{-\mathbf{I}} \quad \text{(2-32)}$$
This is equivalent to the examination of the following equation for a nontrivial solution.

\[ \mathbf{H}^{-1} \mathbf{b} = 0 \]  \hspace{1cm} (2-32)

where

\[ \mathbf{b} = \begin{bmatrix} b_{-n} \\ b_{-n+2} \\ \vdots \\ b_{-3} \\ b_{-1} \\ b_1 \\ b_3 \\ \vdots \\ b_{n+2} \\ b_n \end{bmatrix} \]  \hspace{1cm} (2-33)

A nontrivial \( \mathbf{b} \), a solution other than \( b_{-n} = b_{-n+2} = \cdots = b_{-3} = b_{-1} = b_1 = 0 = b_3 = \cdots = b_{n-2} = b_n \), will exist if and only if the determinant of the coefficients is equal to zero, i.e., \( \mathbf{H}^{-1} \mathbf{c} = 0 \). Thus

\[
\delta\left[ (\lambda - i(2j-1)\omega)\mathbf{I} - \mathbf{F} \right]^{-1} \mathbf{h}' b_{2j-3} + \mathbf{i} b_{2j-1} \\
+ \delta\left[ (\lambda - i(2j-1)\omega)\mathbf{I} - \mathbf{F} \right]^{-1} \mathbf{h}' b_{2j+1} = 0 \hspace{1cm} (2j-1 \leq n) \]  \hspace{1cm} (2-34)

Assuming complete controllability and observability of the system [6], \( \mathbf{h} \neq 0 \). Multiply equation (2-34) by \( \mathbf{h}' \), the following equation is obtained.

\[
\delta \mathbf{h}' \left[ (\lambda - i(2j-1)\omega)\mathbf{I} - \mathbf{F} \right]^{-1} \mathbf{h}' b_{2j-3} + \mathbf{h}' b_{2j-1} \\
+ \delta \mathbf{h}' \left[ (\lambda - i(2j-1)\omega)\mathbf{I} - \mathbf{F} \right]^{-1} \mathbf{h}' b_{2j+1} = 0 \hspace{1cm} (2j-1 \leq n) \]  \hspace{1cm} (2-35)

From equation (2-2), define
\[ h^t[(\lambda-1(2j-3)\omega I-E)^{-1} = 0[\lambda-1(2j-3)\omega] = G_{-2j+3} \]
\[ h^t[(\lambda-1(2j-1)\omega I-E)^{-1} = 0[\lambda-1(2j-1)\omega] = G_{-2j+1} \]
\[ h^t[(\lambda-1(2j+1)\omega I-E)^{-1} = 0[\lambda-1(2j+1)\omega] = G_{-2j+1} \]

Equation (2-35) can be written as
\[ \delta G_{-2j+1} h^t b_{2j-3} + h^t b_{2j-1} + \delta G_{-2j+1} h^t b_{2j+1} = 0 \quad (2j-1 \leq N) \]

This set of equations can also be written in the matrix form

\[
\begin{bmatrix}
1 & \delta G_{-N} & 0 \\
\delta G_{-N-2} & 1 & \delta G_{-N-2} \\
0 & \delta G_{-N-4} & 1 \\
& \ddots & \ddots & \ddots \\
& & \delta G_{-1} & 1 & \delta G_{-1} \\
& & 0 & \delta G_{-1} & 1 & \delta G_{-1} \\
& & 0 & 0 & \delta G_{-3} & 1 \\
& & & \ddots & \ddots & \ddots & \ddots \\
& & & & 1 & \delta G_{-N+4} & 0 \\
& & & & \delta G_{-N+2} & 1 & \delta G_{-N+2} \\
& & & & 0 & \delta G_{-N} & 1
\end{bmatrix}
\begin{bmatrix}
b_{-N} \\
b_{-N+2} \\
b_{-N+4} \\
\vdots \\
b_{-3} \\
b_{-1} \\
b_{1} \\
\vdots \\
b_{N-4} \\
b_{N-2} \\
b_{N}
\end{bmatrix} = 0.
\]

In order to have nontrivial solutions for \( h \), the determinant of its coefficients must be zero, i.e.,
For $N = 1$, equation (2-39) becomes

\[
\begin{bmatrix}
1 & \delta g_1 \\
\delta g_{-1} & 1
\end{bmatrix} = 0. 
\]

(2-40)

For $N = 3$, equation (2-39) becomes

\[
\begin{bmatrix}
1 & \delta g_3 & 0 & 0 \\
\delta g_1 & 1 & \delta g_1 & 0 \\
0 & \delta g_{-1} & 1 & \delta g_{-1} \\
0 & 0 & \delta g_{-3} & 1
\end{bmatrix} = 0. 
\]

(2-41)
For \( N = \infty \), equation (2-39) will then become the exact determinant. The \( \lambda_1 \) can be determined from equation (2-39), using \( N \) as large as is required by the problem. In the case where

\[
G_3 = G(\lambda + 13\omega) \ll 1
\]

and

\[
G_{-3} = G(\lambda - 13\omega) \ll 1,
\]

then \( N = 1 \) will provide useful stability information. The conditions under which the system is asymptotically stable are that the \( \lambda_1 \) have only negative real parts. In order to have \( \lambda_1 \) satisfying equation (2-39) and having negative real parts, classical stability methods can be applied. Thus the problem is reduced to a problem of solving a characteristic equation.

**Example — Mathieu Equation**

The above proposed procedure can well be illustrated by an example. Consider a second order system governed by the well-known Mathieu equation with damping

\[
x + 2\zeta \omega_n x + \omega_n^2 (1 - 2\zeta \cos 2\omega t) x = 0.
\]

The equation can be represented as in equation (2-1) by taking \( x_1 = x \), and \( x_2 = \dot{x} \). Thus

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\omega_n^2 (1 - 2\zeta \cos 2\omega t) x_1 - 2\zeta \omega_n x_2
\end{align*}
\]

or

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-\omega_n^2 & -2\zeta \omega_n
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
\omega_n^2 2\zeta \cos 2\omega t x_1
\end{bmatrix}
\]

\[
= 
\begin{bmatrix}
0 & 1 \\
-\omega_n^2 & -2\zeta \omega_n
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
\omega_n^2 \zeta \omega_n
\end{bmatrix} u
\]

(2-45)
where \[ u = -t(t)\sigma = 2\delta \cos 2\omega t x_1 \] (2.46)

Let \[ t(t) = -2\delta \cos 2\omega t \] (2.47)

then \[ \sigma = x_1 = h^t x = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \] (2.48)

where \[ h^t = \begin{bmatrix} 1 & 0 \end{bmatrix} \] (2.49)

Here,

\[ G(s) = h^t(sI - \Xi)^{-1} x \]

\[ = \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2} \] (2.50)

This system can also be represented as in Fig. 3.

![Fig. 3. A second order system](image)

Using the first approximation, i.e., \( n = 1 \), equation (2.40) becomes

\[ \delta^2 G_1 G_{-1} = 1 \] (2.51)

Now,

\[ G(\lambda) = \frac{\omega_n^2}{\lambda^2 + 2\zeta \omega_n \lambda + \omega_n^2} \] (2.52)

\[ G_1 = G(\lambda+i\omega) = \frac{\omega_n^2}{(\lambda + i\omega)^2 + 2\zeta \omega_n (\lambda + i\omega) + \omega_n^2} \]

\[ = \frac{\omega_n^2}{(\lambda^2 + 2\zeta \omega_n \lambda + \omega_n^2 - \omega^2) + i2\omega(\lambda + \zeta \omega_n)} \] (2.53)
$G_{-1} = G(\lambda - j\omega) = \frac{\omega_n^2}{(\lambda - j\omega)^2 + 2\zeta\omega_n(\lambda - j\omega) + \omega_n^2}$

$$= \frac{\omega_n^2}{(\lambda^2 + 2\omega_n\lambda + \omega_n^2 - \omega^2) - j2\omega(\lambda + \zeta\omega_n)}$$  \hspace{1cm} (2-54)

Substituting the expressions for $G_1$ and $G_{-1}$ into equation (2-51), the following equation is obtained:

$$\lambda^4 + 4\zeta\omega_n\lambda^3 + (4\zeta^2\omega_n^2 + 2\omega_n^2 + 2\omega^2)\lambda^2 + (4\zeta^2\omega_n^3 + 4\zeta\omega_n^2)\lambda$$

$$+ (\omega_n^4 - 2\omega_n^2 + \omega^4 + 4\zeta^2\omega_n^2\omega^2 - \delta^2\omega_n^4) = 0.$$  \hspace{1cm} (2-55)

The problem is now reduced to a problem of solving the characteristic equation. The question is this: "Under what conditions are all the characteristic roots in the left half plane?"

Now apply the Routh's stability criterion [4] to equation (2-55). The necessary conditions for $\lambda$ to be in the left half plane are the following:

(a) $1 > 0$

(b) $4\zeta\omega_n > 0$

(c) $4\zeta^2\omega_n^3 + 4\zeta\omega_n\omega^2 = 4\zeta\omega_n(\omega_n^2 + \omega^2) > 0$

(d) $\omega_n^4 - 2\omega_n^2 + \omega^4 + 4\zeta^2\omega_n^2\omega^2 - \delta^2\omega_n^4 > 0$.

Conditions (a) and (c) are satisfied automatically. Condition (b) is the same as (d), which implies $\zeta$ and $\omega_n$ must be in the same sign. Since $\omega_n$ is positive, $\zeta$ must also be positive. From condition (c),

$$\delta^2 < 1 - 2(\omega/\omega_n)^2 + (\omega/\omega_n)^4 + 4\zeta^2(\omega/\omega_n)^2$$

or

$$\delta < \left\{1 - (\omega/\omega_n)^2 + [2\zeta(\omega/\omega_n)]^2\right\}^{1/2}.$$  \hspace{1cm} (2-56)

For sufficient conditions, the coefficients in equation (2-56) can be arranged in a Routhian array:
\[
\begin{align*}
1 & \quad 4\zeta^2 \omega_n^2 + 2\omega_n^2 + 2\omega^2 \\
4\zeta \omega_n & \quad 4\zeta \omega_n (\omega_n^2 + \omega^2) \\
4\zeta \omega_n^2 + \omega_n^2 + \omega^2 & \quad \omega_n^4 - 2\omega_n^2 \omega_n^2 + \omega^4 + 4\zeta \omega_n^2 \omega_n^2 - 2\omega_n^4 \\
4\zeta \omega_n (4\zeta \omega_n^2 + 4\omega_n^2 \omega_n^2 + 2\omega_n^4) & \quad 4\zeta \omega_n^2 + \omega_n^2 + \omega^2 \\
\end{align*}
\]

It is necessary only to inspect the signs in the first column. If all terms have the same sign, then all roots have negative real parts. Examining the first column, it is obvious that all the terms have the same positive sign. Hence there are no roots with positive real parts. Therefore, the system is stable if and only if \( \zeta \) is positive and equation (2-56) be satisfied. The stability boundaries are plotted in Fig. 4, and are compared to the exact boundaries [5,7].
Fig. 4. Stability boundaries found by the approximation method.
Chapter 3. LINEAR SYSTEMS WITH GENERAL PERIODIC TIME VARYING ELEMENTS IN THE FEEDBACK PATH

General Formulation

In Chapter 2, a single time varying parameter of the form \( t(t) = 2b \cos 2\omega t \) in the feedback path was assumed. The approximate method can be extended to linear systems with more general periodic time-varying parameters. If the time varying element is represented in a general form

\[
L(t) = \sum_{k=-m}^{n} \delta_k e^{ikt} \quad (k \neq 0),
\]

equation (2-7) will then become

\[
\sum_{n=-\infty}^{\infty} L_n(i\omega) e^{in\omega} + \left( \sum_{n=-\infty}^{\infty} L_n e^{in\omega} \right) \left[ F - \left( \sum_{k=-m}^{n} \delta_k e^{ikt} \right) h' \right] = \sum_{n=-\infty}^{\infty} L_n e^{in\omega}
\]

or

\[
L_n(i\omega + F) - \left( (L_{n-1} \delta_1 + L_{n+1} \delta_{-1}) + (L_{n-2} \delta_2 + L_{n+2} \delta_{-2}) + \cdots \right)
\]

\[
+ (L_{n-m} \delta_m + L_{n+m} \delta_{-m}) \right) h' = \sum_{n=0}^{\infty} L_n .
\]

Consider truncation as before by assuming

\[
L_0 = 0, \quad \text{for} \quad a > N.
\]

Write down all the equations in equation (3-2):
\[ n=N: \quad L_{n+1}(\Delta \omega + \vec{F}) = (\delta_{n-N+1} L_{n-N+1} + \cdots + \delta_{n-2} L_{n-2} + \delta_1 L_{n-1} + \delta_{n-N+1}) h^* = R \ L_{n+1} \]

\[ n=N+1: \quad L_{N+1}(\Delta (N+1) \omega + \vec{F}) = (\delta_{N-N+1} L_{N-N+1} + \cdots + \delta_{N-2} L_{N-2} + \delta_1 L_{N-1}) h^* = R \ L_{N+1} \]

\[ \vdots \]

\[ n=0: \quad L_0(\Delta 0 \omega + \vec{F}) = (\delta_{0-N+1} L_{0-N+1} + \cdots + \delta_{0-2} L_{0-2} + \delta_1 L_{0-1}) h^* = R \ L_0 \] (3-5)

\[ n=1: \quad L_1(\Delta \omega + \vec{F}) = (\delta_{1-N+1} L_{1-N+1} + \cdots + \delta_{1-2} L_{1-2} + \delta_1 L_{1-0}) h^* = R \ L_1 \]

\[ \vdots \]

\[ n=N-1: \quad L_{N-1}(\Delta (N-1) \omega + \vec{F}) = (\delta_{N-N+1} L_{N-N+1} + \cdots + \delta_{N-2} L_{N-2} + \delta_1 L_{N-1}) h^* = R \ L_{N-1} \]

\[ n=N: \quad L_N(\Delta N \omega + \vec{F}) = (\delta_{N-N+1} L_{N-N+1} + \cdots + \delta_{N-2} L_{N-2} + \delta_1 L_{N-1}) h^* = R \ L_N \]

Put equation (3-5) into the matrix form

\[ \mathbf{L} \ \mathbf{A} = \mathbf{R} \ \mathbf{L} \] (3-6)

where

\[ \mathbf{L} = \begin{bmatrix} L_{-N} & L_{-N+1} & \cdots & L_{-1} & L_0 & \cdots & L_{N-1} & L_N \end{bmatrix} \] (3-7)
and

\[
A = \begin{pmatrix}
(-i\omega_1 + E) & -\delta_{gh}^1 & 0 & \cdots & 0 \\
-\delta_{gh}^1 & [i(-N+1)\omega_1 + F] & -\delta_{gh}^1 & \cdots & 0 \\
-\delta_{gh}^1 & -\delta_{gh}^1 & (-i\omega_1 + F) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\delta_{gh}^1 & -\delta_{gh}^1 & -\delta_{gh}^1 & \cdots & [i(N-1)\omega_1 + F] \\
0 & -\delta_{gh}^1 & \cdots & \cdots & -\delta_{gh}^1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\end{pmatrix}
\]
Following the same procedures as described in Chapter 2, define

\[
\begin{pmatrix}
\delta_{gh}^1 & 0 & 0 & \cdots & 0 \\
0 & \delta_{gh}^2 & 0 & \cdots & 0 \\
0 & 0 & \delta_{gh}^3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \delta_{gh}^N
\end{pmatrix}
\]

where

\[
\Omega = \lambda I - A =
\begin{pmatrix}
(\lambda + i\omega)I - F & \delta_{gh}^1 \\
\delta_{gh}^1 & [\lambda - \lambda_1(-1)]I - F \\
\delta_{gh}^1 & \delta_{gh}^1 & (\lambda + \omega)I - F \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \delta_{gh}^1 & \delta_{gh}^1 & \cdots & [\lambda - \lambda_N(-1)]I - F
\end{pmatrix}
\]

(3-9)
\[
\mathbf{H} = \begin{bmatrix}
(\lambda + 2i\omega)I - E & 0 \\
0 & [\lambda - 2(-N+1)\omega]I - E \\
0 & 0 & (\lambda + 2\omega)I - E \\
0 & 0 & 0 & \lambda I - E \\
0 & 0 & \cdots & 0 & [\lambda - 2(N-1)\omega]I - E \\
0 & 0 & \cdots & 0 & 0 & (\lambda - 2N\omega)I - E
\end{bmatrix}
\]

and

\[
\mathbf{H}^{-1} = \begin{bmatrix}
\{\lambda + 2i\omega\}I - E \]^{-1} & 0 \\
0 & \{[\lambda - 2(-N+1)\omega]I - E \}^{-1} \\
0 & 0 & \{\lambda + 2\omega\}I - E \]^{-1} \\
0 & 0 & 0 & \lambda I - E \]^{-1} \\
0 & 0 & \cdots & 0 & [\lambda - 2(N-1)\omega]I - E \]^{-1} \\
0 & 0 & \cdots & 0 & 0 & \{[\lambda - 2N\omega]I - E \}^{-1}
\end{bmatrix}
\]

(3-10)
Finally, for stability it is only necessary to examine the following determinant:

\[
\begin{bmatrix}
\delta_{m} & \delta_{m+1} & \delta_{m} & \delta_{m-1} & \ldots & \delta_{m-(m-2)} & \ldots & \delta_{m-(m-1)} & \ldots & \delta_{m-m} & \ldots & \delta_{m-(m+2)} & \ldots
\end{bmatrix}
\]

\[= 0 \quad (3.12)\]
The $\lambda_1$ can be determined from the determinant, and the approximate stability boundaries can hence be obtained. Exact solution can be obtained by letting $n = \infty$.

**Special Cases**

Special cases can be derived from equation (3-12).

Case (1): Assume $\ell(t)$ be represented by a series of the cosine harmonics, i.e., $\delta_k = \delta_{-k}$, and $\delta_k$ is a real number. Equation (3-12) reduces to

\[
\begin{align*}
\begin{bmatrix}
\delta_{m+2} & 0 & 0 & 0 & 0 \\
0 & \delta_{m+1} & 0 & 0 & 0 \\
0 & 0 & \delta_m & 0 & 0 \\
0 & 0 & 0 & \delta_{m-1} & 0 \\
\delta_{m-2} & 0 & 0 & 0 & \delta_{m-3} \\
\end{bmatrix}
\begin{bmatrix}
\delta_1 \\
\delta_2 \\
\delta_3 \\
\delta_4 \\
\delta_5 \\
\end{bmatrix}
&= 0. \quad (3-13)
\end{align*}
\]
If only the odd set is used to determine stability, equation (3-13) will then become

\[
\begin{bmatrix}
\delta_6 g_9 & \delta_6 g_7 & \cdots \\
\delta_4 g_5 & \delta_4 g_3 & \cdots \\
\delta_2 g_1 & \delta_2 g_1 & \cdots \\
1 & \delta_2 g_1 & \cdots \\
\end{bmatrix}
= 0.
\tag{3-14}
\]

If \( \delta_k \) are taken in such a way that \( \delta_k = \delta_{-k} = \delta_2 = \delta_4 \), and

\[ \cdots = \delta_{-3} = \delta_{-1} = \delta_1 = \delta_3 = \delta_4 = \cdots = 0, \]

equation (3-14) is then reduced to

\[
\begin{bmatrix}
1 & \delta_3 & 0 & 0 & \cdots \\
\delta_3 & 1 & \delta_3 & 0 & \cdots \\
0 & \delta_3 & 1 & \delta_3 & \cdots \\
0 & 0 & \delta_3 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
= 0, \tag{3-15}
\]

which is exactly equation (2-39).
Case (2): Assume \( f(t) \) be represented only by a series of sine harmonics, i.e., \( \delta_{-k} = -\delta_k \), and \( \delta_k \) are purely imaginary numbers. Equation (3-12) becomes

\[
\begin{align*}
\begin{bmatrix}
\delta_{m} G_{m+2} & 0 & 0 & 0 & 0 \\
\delta_{m} G_{m+1} & 0 & 0 & 0 & 0 \\
\delta_{m} G_{m} & 0 & 0 & 0 & 0 \\
\delta_{m} G_{m-1} & 0 & 0 & 0 & 0 \\
\delta_{m} G_{m-2} & 0 & 0 & 0 & 0 \\
\delta_{m} G_{m-3} & 0 & 0 & 0 & 0 \\
\delta_{m} G_{m-4} & 0 & 0 & 0 & 0 \\
\delta_{m} G_{m-(m-2)} & 0 & 0 & 0 & 0 \\
\delta_{m} G_{m-(m-1)} & 0 & 0 & 0 & 0 \\
\delta_{m} G_{m-3} & 0 & 0 & 0 & 0 \\
\delta_{m} G_{m-4} & 0 & 0 & 0 & 0 \\
\delta_{m} G_{m-5} & 0 & 0 & 0 & 0 \\
\delta_{m} G_{m-(m-2)} & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\delta_{1} G_{1} \\
\delta_{2} G_{1} \\
\delta_{3} G_{1} \\
\delta_{4} G_{1} \\
\delta_{5} G_{1} \\
\delta_{6} G_{1} \\
\delta_{7} G_{1} \\
\delta_{8} G_{1} \\
\delta_{9} G_{1} \\
\delta_{10} G_{1} \\
\delta_{11} G_{1} \\
\delta_{12} G_{1} \\
\delta_{13} G_{1} \\
\delta_{14} G_{1} \\
\end{bmatrix}
= 0.
\end{align*}
\]
Case (3): Assume $i(t)$ is represented both by cosine and sine harmonics, i.e., $\delta_k = \delta_k^*$ and $\delta_k$ is a complex number. For stability it is only necessary to look at

\[
\begin{bmatrix}
1 & \delta_{1G}^* & \delta_{2G}^* & \delta_{3G}^* & \delta_{4G}^* \\
\delta_{1G} & 1 & \delta_{1Gb}^* & \delta_{2Gb}^* & \delta_{3Gb}^* \\
\delta_{2G} & \delta_{1Gb}^* & 1 & \delta_{1Gb}^* & \delta_{2Gb}^* \\
\delta_{3G} & \delta_{2Gb}^* & \delta_{1Gb}^* & 1 & \delta_{1Gb}^* \\
\delta_{4G} & \delta_{3Gb}^* & \delta_{2Gb}^* & \delta_{1Gb}^* & 1 \\
\end{bmatrix} = 0 \quad (3-17)
\]
An Illustrative Example

The application of equation (3-17) for the approximated stability boundaries can be illustrated by a second example. Consider the system governed by the equation

$$\ddot{x} + 2\zeta_n \dot{x} + \omega_n^2 (1 - 2\delta' \cos 2\omega t - 2\delta'' \sin 2\omega t)x = 0$$

(3-18)

with

$$\Sigma = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta_n \omega_n \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix},$$

(3-19)

and

$$\epsilon(t) = -(2\delta' \cos 2\omega t + 2\delta'' \sin 2\omega t)$$

$$= -(\delta' - i\delta'') e^{i2\omega t} - (\delta' + i\delta'') e^{-i2\omega t},$$

(3-20)

so

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta_n \omega_n s + \omega_n^2}.$$  

(3-21)

This system can also be represented as in Fig. 5.

Fig. 5. A second order system with two time-varying elements in the feedback path.

Here,

$$\delta_2 = -(\delta' - i\delta''), \quad \delta_{-2} = -(\delta' + i\delta''),$$

(3-22)

hence

$$\delta_{-2} = \delta_2^*.$$  

(3-23)
Using the first approximation, equation (3-17) becomes

\[ \delta_2 \delta_2 G_1 G_1 = 1. \]  (3-24)

Using equations (2-53) and (2-54), equation (3-24) becomes

\[ \lambda^4 + 4(\omega_n \lambda^3 + (4\xi_n^2 \omega_n^2 + 2\omega_n^2 + 2\omega^2) \lambda^2 + (4\xi_n^3 + 4\xi_n \omega_n^2)\lambda \\
+ [\omega_n^4 - 2\omega_n^2 \omega_n^2 + \omega^4 + 4\xi_n^2 \omega_n^2 - (\delta^2 + \delta''^2) \omega_n^4] = 0 \]  (3-25)

This is the same as equation (2-55) except \( \delta \) is now changed to \( (\delta^2 + \delta''^2)^{1/2} \). The necessary and sufficient condition for \( \lambda \) to have negative real parts is

\[ (\delta^2 + \delta''^2) < [1 - (\omega/\omega_n)^2]^2 + [2\xi(\omega/\omega_n)]^2 \]  (3-26)

For stability, equation (3-26) must be satisfied. The result is obviously the same as in Example 1. If \( t(t) \) in equation (3-20) is written into the form

\[ t(t) = -(2\delta^1 \cos 2\omega t + 2\delta^2 \sin 2\omega t) \]

\[ = -2\delta^1 \cos(2\omega t - \phi) \]  (3-27)

where \[ \phi = \tan^{-1}(\delta''/\delta') \]  (3-28)

and \[ \delta = (\delta^2 + \delta''^2)^{1/2} \]  (3-29)

then the problem is the same as in Example 1, since the phase angle \( \phi \) does not affect stability.
Chapter 4. CONCLUSION

The approach developed in this thesis for stability analysis of linear periodic time-varying systems is to reduce the problem to a characteristic equation which can be analyzed by classical stability criteria and two examples have been given showing the approximate boundaries of stability. In order to have more accurate boundaries, it is necessary to use a higher order truncated determinant rather than a first order one. But as the order of the truncated determinant gets larger, the order of the characteristic equation obtained from it increases rapidly, and hence there are considerable difficulties in solving the equation. The problem has also been considered in [7] for a single harmonic but a more general formulation has been presented here.

It is conceivable that this approach can be extended to the piece-wise periodic systems. In general, periodic functions such as square wave could be expressed in a Fourier series represented by cosine and sine terms. If its average value is zero, then the series can be represented exactly by equation (3-1). The system stability can then be analyzed by the method given in Chapter 3.
REFERENCES


