RICE UNIVERSITY

TRANSIENT RESPONSE OF A HARMONIC OSCILLATOR
WITH QUADRATIC DAMPING

by

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ABSTRACT

An analytical investigation is made of the transient response of a harmonic oscillator with damping proportional to the square of the velocity. The input motions considered are the half-cycle velocity pulse and the half-cycle displacement pulse. The response quantities studied are the maximum displacement, velocity, and acceleration of the system. Results were obtained by integrating the equation numerically and are summarized in the form of response spectra. The results are compared to those obtained with linear damping; a method for obtaining equivalent linear damping coefficients is given.

The study indicates that quadratic damping is most effective in reducing responses in the range of frequencies for which the responses are the largest. For lightly damped systems, quadratic damping was found to be equally effective in reducing each of the three responses studied. But for heavily damped systems it is most effective in reducing velocities.
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NOTATION

c = linear damping proportionality constant

d = quadratic damping proportionality constant

\( F(t) = F_0 f(t) \) = forcing function

\( F_0 \) = maximum value of the forcing function

\( f(t) \) = characteristic shape of the forcing function

\( f \) = undamped natural frequency, in cycles per unit time

\( k \) = spring constant

\( m \) = mass of the system

\( p = \sqrt{k/m} \) = undamped circular natural frequency

\( r = T_0 f \) = frequency parameter

\( T_0 \) = characteristic time associated with the forcing function

\( t \) = time

\( U \) = maximum absolute deformation of the spring

\( V = pU \) = psuedo-velocity

\( x \) = displacement of the mass from equilibrium position

\( y \) = input ground displacement

\( y_o \) = maximum ground displacement

\( x_o = F_0/k \) = characteristic displacement

\( \beta = c/2mp \) = linear damping factor

\( \gamma = dx_o/m \) = quadratic damping factor

\( \eta = x/x_o \) = dimensionless displacement

\( \tau = t/T_0 \) = dimensionless time
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Fig. 9b  Ratios of Energy Absorbed During the Half-Cycle Displacement Pulse in the Equivalent Linear Damper to the Energy Absorbed in the Quadratic Damper
I. INTRODUCTION

The object of this study is to investigate the transient response of single-degree-of-freedom systems in which the damping is proportional to the square of the velocity, such as might be the case with hydrodynamic damping. In particular, a study is made of the relation between quadratically damped systems and the much-studied linearly damped systems.

Jacobsen (1)* has examined the steady-state solution to quadratically damped systems using an approximate procedure. He introduced the concept of an equivalent linear damping factor with a value determined by equating the energies absorbed by each damper acting alone. His results have been verified by Rosenberg and Wang (2) for several steady-state excitations.

Some analytical work has been done for the unforced oscillator (3, 4), but transient behavior has received comparatively little attention. Jacobsen and Ayre (5) proposed a graphical procedure which could be applied to the forced case but gave no specific results.

In this study the governing differential equation of motion is integrated numerically for two excitations, the half-cycle velocity pulse and the half-cycle displacement pulse. The concept of an equivalent linear damping factor is extended to transient problems in an exact way.

All symbols used are defined where they are first introduced, and are assembled at the front for convenient reference.

* The numbers appearing in parentheses refer to the number of the reference listed in the bibliography.
II. METHOD OF ANALYSIS

2.1 System Considered

The system analyzed is shown in Figure 1. It consists of a rigid mass, m, connected to a base by a linear spring of constant k and dampers with resistance proportional to the velocity (proportionality constant c) and the square of the velocity (proportionality constant d). The displacement from equilibrium is denoted by x and the forcing function by F(t).

The governing differential equation of motion is

\[ \frac{d^3 x}{dt^3} + \frac{c}{m} \frac{dx}{dt} + \frac{d}{m} \frac{d^2 x}{dt^2} + kx = F(t) = -m\ddot{y}(t) \] (1)

It may be noted that by interpreting the forcing function in terms of the ground acceleration, \( \ddot{y}(t) \), and x as the deformation of the spring, the system is the same as that of a single-degree-of-freedom system subjected to a ground motion. Notice that the absolute value in the quadratic term merely preserves the sign so that the damping force always opposes the motion. The equation is shown in this form even though the systems considered were not ones in which both linear and quadratic damping forces were present.

Equation (1) may be put in the following dimensionless form:

\[ \frac{d^2 \eta}{d \tau^2} + 4\pi \beta \tau \frac{d \eta}{d \tau} + \gamma \frac{d \eta}{d \tau} \frac{d^2 \eta}{d \tau^2} + (2\pi \tau)^2 \eta = (2\pi \tau)^2 f(\tau) \] (2)

where

\[ \tau = \frac{t}{T_o}, \quad T_o = \text{characteristic time associated with the forcing function (See section 2.2)} \]

\[ \beta = T_o f = \text{frequency parameter, } f = \text{natural frequency (cps)} \]

\[ \gamma = \frac{x}{x_o}, \quad x_o = \text{characteristic displacement} = \frac{F_o}{k} \]
2. 2 **Excitation Considered**

Two simple pulses were used as excitations, the half-cycle velocity pulse and the half-cycle displacement pulse. Displacement, velocity, and acceleration diagrams associated with each input motion are given in Figure 2 as well as the characteristic times associated with each. $T_0$ in the case of the half-cycle velocity is taken to be the duration of the pulse and half the duration in the case of the half-cycle displacement. In both cases the excitations are piecewise linear.

2. 3 **Numerical Solution Procedure**

A modified third order predictor-corrector method as described by Hamming (6) was used to obtain the solution. The method is not self-starting, but requires knowledge of one previous point. The first time step was taken with a fourth order Runge-Kutta procedure as outlined by Romanelli (7) before proceeding to the main calculation.

The predictor-corrector procedure is presented below. Suppose values are known for displacement, velocity, and acceleration at points $i-1$ and $i$ a small time increment, $h$, apart. $h$ is a dimensionless increment and dots above the variables are used to denote derivatives with respect to $\tau$.

Step 1. Predict at point $i+1$:

\[
f(\tau) = \frac{F(\tau)}{F_0}, \quad -m\ddot{y}(\tau), \quad F_0 = \text{maximum value of the forcing function}
\]

\[
\beta = \frac{c}{2mp} = \text{per cent of critical damping}
\]

\[
\gamma = \frac{dx_0}{m} = \text{quadratic damping factor}
\]
\[ \eta_p = \eta_{i-1} + 2h\dot{\eta}_i \]
\[ \ddot{\eta}_p = \ddot{\eta}_{i-1} + 2h\dddot{\eta}_i \]

Step 2. Calculate a predicted acceleration by solving the differential equation for acceleration and substituting the predicted values from step 1.

\[ \ddot{\eta}_p = f(\eta_p, \dot{\eta}_p, \tau) \]

Step 3. Calculate a corrected value of displacement and velocity:

\[ \eta_c = \eta_i + \frac{h}{2}(\ddot{\eta}_p + \ddot{\eta}_i) \]
\[ \dot{\eta}_c = \dot{\eta}_i + \frac{h}{2}(\dddot{\eta}_p + \dddot{\eta}_i) \]

Step 4. The error in step 1 is \( \frac{1}{3}h^3\dddot{\eta}_i + \frac{1}{60}h^5\eta(0_1) \) where \( \theta_1 \) is some mean value in the interval. The error in step 3 is \( -\frac{1}{12}h^3\dddot{\eta}_i - \frac{1}{24}h^4\dddot{\eta}(\theta_2) \) for some \( \theta_2 \) in the interval. Hence most of the error can be "mopped-up" by taking

\[ \eta_{i+1} = \frac{4}{5}\eta_c + \frac{1}{5}\eta_p \]
\[ \dot{\eta}_{i+1} = \frac{4}{5}\dot{\eta}_c + \frac{1}{5}\dot{\eta}_p \]

The procedure is thus a fourth order method. A value for \( \dddot{\eta}_{i+1} \) is obtained in the same manner as in step 2 by solving the differential equation for acceleration. The procedure can be shown to be stable. (See Appendix A.)

No iteration was performed. Accuracy was determined experimentally by taking a time step sufficiently small to obtain four significant figures. This accuracy was found by comparison with the results of Bielak (9) who determined response curves for these pulses with linear damping.
2.4 Response Quantities

The response quantities studied were the absolute maximum values of the displacement, velocity, and acceleration. Results were generated by first computing each of the three quantities for a wide range of $\beta$'s and a specific value of $T_0$, retaining all values in a table. Then a set of responses corresponding to some value of $\gamma$ was calculated. Each response was then used to obtain an equivalent $\beta$ by interpolating in the table of linearly damped responses. Hence for each $\gamma$ there are in general at least three equivalent $\beta$'s, one for each response quantity considered. The $\gamma$ and $\beta$ are then equivalent in the sense that the responses corresponding to the two kinds of damping are the same.

In addition, the energy absorbed in the linear and quadratic dampers was calculated so that the validity of the equivalent energy concept for transient motion might be evaluated. The energies were calculated using the trapezoidal rule which, in dimensionless form, are

$$E_\beta = \frac{E_L}{(2kx_0 \omega_0^2)} = \frac{2\beta}{\pi \tau} \left[ \frac{\hbar}{2} (\hat{h}_1^2 + \hat{h}_{i+1}^2) \right]$$

and

$$E_\gamma = \frac{E_Q}{(2kx_0 \omega_0^2)} = \frac{\gamma}{2(\pi \tau)^2} \left[ \frac{\hbar}{2} (|\hat{h}_i| |\hat{h}_{i+1}|^2 + |\hat{h}_{i+1}| |\hat{h}_{i+1}|) \right]$$

The derivation of these formulas is presented in Appendix B.

2.5 Block Diagram

A computer program was written to carry out the calculations. The program can handle any single-degree-of-freedom system with linear and/or quadratic
damping. The general layout of the program and the order in which computations are done are indicated in the block diagram in Diagram A. The complete program is given in Appendix C.
BLOCK DIAGRAM

Read system parameters and forcing function and h

\[ t = 0 \]

Calculate η and ̇η using a Runge-Kutta procedure

Are the response values calculated in this step max. in absolute value?

Yes → Store maximum values

Increment t by amt. h

No

Has the pulse ended?

Yes

Calculate responses for this time step using the predictor-corrector method

No → Is t greater than pulse duration plus one natural period?

Yes

Store all maximum responses

Select a new β

No → Have all β problems been done?

Yes

β = 0

Select a new γ value

No → Is β = 0?

Yes

Interpolate to obtain the equivalent β's

Interpolate to obtain energy corresponding to equivalent β

Find ratio of energy absorbed in equivalent linear damper to that in quadratic damper

Diagram A
III. PRESENTATION AND DISCUSSION OF RESULTS

3.1 Response Spectra

Using the information from the computer output, two types of graphical representation were prepared, the first being response spectra. (See Figures 3a, 3b, 4a, 4b.) For these plots the problem is interpreted as a ground motion problem with the principal response taken as the maximum absolute deformation of the spring. The deformation is non-dimensionalized with respect to the maximum ground displacement and plotted on a four-way log scale as is frequently done in earthquake engineering (8). The vertical scale is the so-called pseudo-velocity of the system defined by $pU$ where $p$ is the circular natural frequency of the system and $U$ is the absolute maximum spring deformation. It is plotted non-dimensionalized with respect to $\gamma_0$, the maximum ground velocity.

3.2 Equivalent Linear Damping Coefficients

The second type of graph represents results of the calculation of equivalent linear damping coefficients. By obtaining the responses for some particular $\gamma$ and $T_0f$ and using a second degree polynomial interpolation among the responses with linear damping, equivalent $\beta$'s can usually be found. The $\beta$'s which are obtained in this way will of course vary with both $\gamma$ and $T_0f$. The interdependence of these three parameters is displayed in Figures 7a-8c. Each curve represents a single value of $\gamma$. Hence if $\gamma$ and $T_0f$ are known, a $\beta$ for which one response remains unchanged may be obtained from the graphs. The dotted lines which appear on these plots represent special circumstances which are explained in the next section.
3.3 Difficulties Encountered

The dotted lines on the plots of equivalent linear damping coefficients (Figures 7a-8c) represent areas in which an equivalent $\beta$ either did not exist or was multivalued. To see how this difficulty can arise, consider the half-cycle velocity pulse in the vicinity of $T_0f = 4.0$. The effect of small amounts of linear damping is to smooth out the kinks in the response spectra causing a slightly higher response value for lightly damped systems than for undamped systems. A detail of the response spectra in the vicinity of $T_0f = 4.0$ is shown in Figure 5. As the amount of damping is increased the response eventually drops below the response of the undamped system as it should.

Similar behavior occurs in the case of quadratic damping and it is in some cases even more pronounced. E.g., at $T_0f = 4.0$, responses for certain values of $\gamma$ exceed those for any $\beta$. (See Figures 5 and 6.) Hence if a response corresponding to some $\gamma$ is carried into a table of $\beta$ responses, it may be that there is more than one equivalent $\beta$ available or no $\beta$ at all.

When the determination of an equivalent $\beta$ is multivalued, frequently none of its values fit into the equivalent $\beta$ curves, i.e. there are often points lying both far above and far below the otherwise smooth curves of equivalent $\beta$.

When this occurs and when there is no $\beta$ available, the point is represented by a dashed line and another interpretation is proposed.

The effects just described are small. Among the values calculated, the increases above the undamped displacement response as plotted in the response spectra never exceed ten per cent of the undamped value. The same percentage does not hold for the true acceleration response which may increase as much.
as fifty per cent in the acceleration sensitive region of the spectrum. (e.g., at $T_0f = 4.0$, half-cycle velocity) But the effects are still small since by moving short distances in either direction along the abscissa (i.e. varying $T_0f$) from a point at which a valley occurs, true accelerations can be found which are five to ten times the maximum at that point. Hence if a response value is needed for design, it is suggested to smooth out the undamped response curve in the region of difficulty. The values obtained will as a rule be greater than any damped response (linear or quadratic) in that range.

3.4 Results of Energy Calculations

As the response quantities were being calculated, a side calculation of the energy absorbed in each damper was made according to equations 3 and 4. Once an equivalent $\beta$ was calculated, an energy absorbed in the damper at the end of the pulse and corresponding to that $\beta$ was obtained by interpolating among stored values. A ratio of the energy absorbed in the equivalent linear damper to that absorbed in the quadratic damper was taken. The averages of these ratios were plotted in Figures 9a and 9b along with the highest and lowest ratio at each $T_0f$. No distinction was made in these graphs as to which points correspond to which $\gamma$'s or which response was used in obtaining the equivalent $\beta$'s. Each average represents 24 points. No average was taken at points where the equivalent $\beta$ was multivalued or non-existent.
IV. CONCLUSIONS

4.1 Comparison of Linear and Quadratic Damping

One of the objectives of this study was to establish some kind of relationship between linear and quadratic damping. With a few exceptions, given a quadratic damping factor, $\gamma$, an equivalent linear damping factor, $\beta$, can be obtained from Figures 7a-8c for which any one of three responses is unchanged. The shapes of these curves show that quadratic damping is most effective in the central portion of the graphs, in the range of frequencies for which the responses are the largest. In fact the shapes of the curves resemble the response curves themselves, being more flattened for heavier amounts of damping. It may also be noticed that for small amounts of damping the general shape and magnitude of the curves do not vary greatly with either the type of response considered or the type of input motion considered. However for high amounts of damping, the quadratically damped systems are most effective in reducing velocities and least effective in reducing accelerations.

4.2 Applicability of Energy Considerations

A ratio of the energy absorbed in the equivalent linear damper to the energy absorbed in the quadratic damper during the pulse was taken as described in section 3.4. The ratio (See Figures 9a and 9b.) was found to fluctuate from about 0.3 to about 2.0 with most values falling near 1.0. The mean values of the ratios fell between .90 and 1.02. It is concluded that the concept of equating the energies as described is only a rough approximation for transient behavior and not entirely reliable.
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REFERENCES


APPENDIX A

Stability of the Numerical Procedure

This derivation shows that the propagation error of this method does not grow with time, i.e., that the errors in previous steps are not magnified during present and future steps.

The procedure used as described in the text and applied to the calculation of values at point i+1 is:

\[
\eta_{i+1} = \frac{4}{\delta} \left[ \eta_i + \frac{h}{2} (\hat{\eta}_\rho + \hat{\eta}_i) \right] + \frac{1}{\delta} (\eta_{i-1} + 2h \hat{\eta}_i)
\]

\[
\hat{\eta}_{i+1} = \frac{4}{\delta} \left[ \hat{\eta}_i + \frac{h}{2} (\hat{\eta}_\rho + \hat{\eta}_i) \right] + \frac{1}{\delta} (\hat{\eta}_{i-1} + 2h \hat{\eta}_i)
\]  \(\text{(A1)}\)

Let the error in the calculated value of \(\eta_i\) be given by \(e_i\), and the error in \(\hat{\eta}_i\) be \(e_{\hat{\eta}_i}\). Since the differential equation can be solved for \(\hat{\eta}_i\), let the error in \(\hat{\eta}_i\) be

\[
\Delta \hat{\eta}_i = \hat{\eta}_{\text{true}} - \hat{\eta}_i = \frac{\partial f(\eta, \hat{\eta}, \tau)}{\partial \eta} \Delta \eta_i + \frac{\partial f(\eta, \hat{\eta}, \tau)}{\partial \hat{\eta}} \Delta \hat{\eta}_i
\]

\[
= -2Ae_i - Be_i
\]  \(\text{(A2)}\)

where \(\hat{\eta} = f(\eta, \hat{\eta}, \tau)\) is the differential equation. \(A\) and \(B\) are assumed to be approximately constant over a single time step.

The errors in the predicting formulas then are given by

\[
e_{\rho} = e_{\hat{\eta}_i} + 2h (e_i)
\]

\[
e_p = e_{\hat{\eta}_i} + 2h (-2Ae_i - Be_i)
\]  \(\text{(A3)}\)

and the errors in the final calculation at point i+1 which result from the propagation of previous errors are

\[
e_{i+1} = \frac{4}{\delta} [e_i + \frac{h}{2} (-2Ae_p - Be_p - 2Ae_i - Be_i)] + \frac{1}{\delta} [e_{i-1} + 2h (-2Ae_i - Be_i)]  \\
e_{\hat{\eta}_{i+1}} = \frac{4}{\delta} [e_i + \frac{h}{2} (e_\rho + e_i)] + \frac{1}{\delta} [e_{i-1} + 2h e_i]
\]  \(\text{(A4)}\)

Upon substitution of the errors for the predicting formulas, the expressions (A4) become, after rearranging,
\[ \varepsilon_{i+1} - \frac{4}{5} (1 - 2Ah - Bh^2 + 4A^2 h^4) \varepsilon_i - \frac{1}{5} (1 - 4Ah) \varepsilon_{i-1} \]
\[ - \frac{4}{5} (2AB h^2 - Bh) \varepsilon_i + \frac{2}{5} Bh \varepsilon_{i-1} = 0 \]

and

\[ \varepsilon_{i+1} + \frac{4}{5} (Bh^2 - 1) \varepsilon_i - \frac{1}{5} \varepsilon_{i-1} + \frac{4}{5} (2Ah^2 - h) \varepsilon_i - \frac{2}{5} h \varepsilon_{i-1} = 0 \quad (A5) \]

If now we assume solutions of the form

\[ \varepsilon_i = \beta^i, \quad \varepsilon_i = \mu \beta^i \quad (\mu = \text{constant}) \]

and write equations (A5) at \( i = 1 \), then

\[ \mu \beta^3 - \frac{4}{5} (1 + 4A^2 h^2 - Bh^2 - 2Ah) \mu \beta - \frac{1}{5} (1 - 4Ah) \mu \]
\[ - \frac{4}{5} (2AB h^2 - Bh) \beta + \frac{2}{5} Bh = 0 \]

and

\[ \frac{4}{5} (2Ah^2 - h) \mu \beta - \frac{2}{5} \mu + \beta^3 + \frac{4}{5} (Bh^2 - 1) \beta - \frac{1}{5} = 0 \]

Eliminating \( \mu \) between these equations gives the following quartic in \( \beta \).

\[ - \beta^4 + \frac{8}{5} (1 - Ah + (2A^2 - B)h^3) \beta^3 - \frac{1}{5} \left[ -\frac{6}{5} + \frac{12}{5} Ah + \frac{16}{5} (B - 4A^2) h^2 + \frac{32}{5} AB h^2 + \frac{16}{5} (-B^3) h^4 \right] \beta^2 + \frac{4}{25} \left[ -2 + 6Ah - 2(B + 2A^2) h^2 + 4AB h^2 \right] \beta + \frac{1}{25} (-1 + 4Ah - 4Bh^2) = 0 \quad (A6) \]

The solution of this equation will give a measure of the propagation error, but it is difficult to solve in general terms. To find the nature of the solution, \( \beta \) is expanded in a power series in \( h \). Then,

\[ \beta = \beta_0 + \beta_1 h + \beta_2 h^2 + \ldots \]
\[ \beta^3 = \beta_0^3 + 3 \beta_0 \beta_1 h + \beta_2 \beta_0 h + \ldots \]
\[ \beta^4 = \beta_0^4 + 4 \beta_1 \beta_0 \beta_0^2 h + \ldots \]

Substituting these into equation (A6) for \( \beta \) and retaining only \( h^3 \) terms and lower

\[ - \beta_0^4 - 4 \beta_1 \beta_0 \beta_0^3 h - (6 \beta_1 \beta_0 \beta_0^2 + 4 \beta_2 \beta_0^3) h^2 + \frac{8}{5} \left( \beta_0^3 + 3 \beta_2 \beta_0 \beta_0^2 h + 3 (6 \beta_1 \beta_0 \beta_0^2 + 4 \beta_2 \beta_0^3) h^2 + \ldots \right) \]
\[ + 3 (\beta_2 \beta_0^2 + \beta_0 \beta_1 h)(2A^2 - B) \beta_0^3 h^2 + \frac{12}{5} A (\beta_3 h + 2 \beta_1 \beta_0 \beta_0^2 h + \beta_2 \beta_0^3 h^2) \]
\[ + \frac{16}{5} (B - 4A^2) \beta_0 h^4 - \frac{8}{25} [-1 - 4Ah - 4Bh^2] \beta_0 h^3 + \frac{1}{25} [-1 + 4Ah - 4Bh^2] = 0 \quad (A7) \]
Setting the coefficients of the powers of $h$ equal to zero successively, the following four solutions are obtained:

$$
\beta_{A,B} = 1 + (-A \pm \sqrt{A^2 - B}) h + O(h^2 + ...)
$$

$$
\beta_{c,0} = -\frac{1}{3} + \frac{1}{3} (A \pm \sqrt{A^2 - B}) h + O(h^2 + ...)
$$

(A9)

These expressions have an equivalent exponential form as follows:

$$
(1 + (-A \pm \sqrt{A^2 - B}) h + ...)^i = \exp \left[ ( -A \pm \sqrt{A^2 - B} ) i h + ... \right]
$$

(A10)

Since the total propagation error at the $i$th step is the complete solution of (A6) or

$$
c_i \beta_A^i + c_2 \beta_B^i + c_3 \beta_c^i + c_4 \beta_p^i
$$

then by dropping the higher order terms in (A10) the error is

$$
c_i e^{(-A + \sqrt{A^2 - B}) \tau} + c_2 e^{(-A - \sqrt{A^2 - B}) \tau} + c_3 \left( \frac{1}{3} \right) e^{(A + \sqrt{A^2 - B}) \tau} + c_4 \left( -\frac{1}{3} \right) e^{(-A - \sqrt{A^2 - B}) \tau}
$$

(A11)

If $A$ and $B$ are now evaluated from (A2) and (2),

$$
B = (2 \pi r)^2 \quad A = 2 \pi r \beta + \gamma \left| \frac{\partial n}{\partial r} \right|
$$

Both $A$ and $B$ are always positive real numbers. It should be noted that the first two terms of (A11) behave in the same manner as the solution of the differential equation. This can be seen by examining the linear system,

$$
c_i e^{(-2 \pi r \beta + 2 \pi r \sqrt{\beta^2 - 1}) \tau} + c_2 e^{(-2 \pi r \beta - 2 \pi r \sqrt{\beta^2 - 1}) \tau}
$$

in which the expression becomes identical in form with the true solution, that is, the product of a decaying exponential and an oscillation for $\beta < 1$ and a decaying exponential for $\beta \geq 1$.

The last two terms in (A11) behave in the same manner as the extraneous solution to the difference equation and die out rapidly with time.

Since the error terms do not increase more rapidly than the solution itself, the procedure is numerically stable.
The expressions used in calculating energies dissipated in each damper are now derived. For the linear damper

\[ E_L = - \int F \, dx = \int_0^t \left( c \frac{dx}{dt} \right) \frac{dx}{dt} \, dt = c \int_0^t \left( \frac{dx}{dt} \right)^2 \, dt \]

or in dimensionless form

\[ E_\beta = \frac{E_L}{(\frac{1}{2} k x_0^2)} = \frac{2 \beta}{\pi r} \int_0^\tau \left( \frac{d\eta}{d\tau} \right)^2 \, d\tau \]

Evaluating the integral using the trapezoidal rule as done in the study,

\[ E_\beta = \frac{2 \beta}{\pi r} \left[ \frac{h}{2} \left( \dot{\eta}_i^2 + \dot{\eta}_{i+1}^2 \right) \right] \]

For the quadratic damper

\[ E_Q = - \int F \, dx = \int_0^t \left( d \left| \frac{dx}{dt} \right| \frac{dx}{dt} \right) \frac{dx}{dt} \, dt = d \int_0^t \left| \frac{dx}{dt} \right| \left( \frac{dx}{dt} \right)^2 \, dt \]

or in dimensionless form

\[ E_\gamma = \frac{E_Q}{(\frac{1}{2} k x_0^2)} = \frac{\gamma}{2 (\pi r)^2} \int_0^\tau \left| \frac{d\eta}{d\tau} \right| \left( \frac{d\eta}{d\tau} \right)^2 \, d\tau \]

Calculations were again made using the trapezoidal rule

\[ E_\gamma = \frac{\gamma}{2 (\pi r)^2} \left[ \frac{h}{2} \left( |\dot{\eta}_i| \dot{\eta}_i^2 + |\dot{\eta}_{i+1}| \dot{\eta}_{i+1}^2 \right) \right] \]
APPENDIX C

A listing of the statements in the main part of the computer program used in this study follows along with a listing of the principal subroutines. The program is written in Fortran for use on the IBM 7040 at Rice University.
JERING, QUADRATIC DAMPING
ISN SOURCE STATEMENT
FORTRAN SOURCE LIST
05/09/6:

0 $IBFT MAINPR DECK
C MAIN PROGRAM, SDF SYSTEM WITH QUADRATIC DAMPING
C NRS = NO. OF SETS OF DATA TO BE INPUT
C ETA0, ETADD, ARE INITIAL VALUES OF DISPLACEMENT AND VELOCITY
C C = NO. OF TIME STEPS PER FREQUENCY PARAMETER, R
C CT = NO. OF TIME STEPS AFTER WHICH CURRENT VALUE IS PRINTED, IF
C THE VALUE OF CT=0, THEN NO TIME HISTORY WILL BE PRINTED
C D = DURATION OF THE PULSE
C JJJ = NO. OF VALUES IN PIECEWISE LINEAR FORCING FUNCTION
C KB = NO. OF VALUES IN THE TABLE OF BETA RESPONSES FROM CARDS
C KC = NO. OF VALUES IN THE TABLE OF BETA RESPONSES
C KD = 0. ....00 DO NOT PUNCH BETA RESPONSE TABLE, OTHERWISE KD
C KA = 0.0. DO NOT CALCULATE A TABLE OF BETA RESPONSES, OTHERWISE DO
C TAU(J), FT(J) = COORDINATES OF TABULATED FORCING FUNCTION
C BINC, NBINC = INCREMENTS AND NO. OF INCREMENTS OF BETA, ..IF INCRE
C MENTS ARE NOT USED THEN BINC = 0, AND NBINC BECOMES THE NO. OF
C BETAS TO BE READ IN ON SEPARATE CARDS
C SIMILARLY WITH RINC, NRINC, GINC, NGINC
1 DIMENSION TAU(50), FT(50), R(20), G(20)
2 DIMENSION ORD1(50), ORD2(50), ORD3(50), ORD4(50), ORD5(50)
3 DIMENSION TEMP1(5), TEMP2(5), TEMP3(5), TEMP4(5), TEMP5(5), TEMP6(5)
4 DIMENSION FORCE[14]
5 DIMENSION Y[2,5], X[2,5]
6 READ(5, 116) NRS
7 116 FORMAT(2I14)
8 98 READ(5,100) ETA0, ETA0, C, CT, J, JJJ, KB, KC, KD
9 100 FORMAT(2E11.4, 3F10.4, 4I3)
10 MAIN0010
11 READ(5, 111)(ORDER(I), I=1, 14)
12 111 FORMAT(13A6, A2)
13 MAIN0020
14 READ(5, 101)(TAU(J), FT(J), I=1, JJJ)
15 101 FORMAT(2E11.4)
16 IF (1.KA .EQ. 0) GO TO 102
17 102 FORMAT(2E11.4)
18 READ(5, 115) K
19 115 FORMAT(F16.8)
20 READ(5, 103)(B(J), I=1, JJJ)
21 103 FORMAT(5E16.8)
22 NID = KC
23 READ(5, 104)(ORDS(I), I=1, KC)
24 104 FORMAT(16.8)
25 KK = 0
26 KN = 0
27 KP = 0
28 KG = 0
29 IE = 1
30 NTMP = 0
31 CALL PAGER(0, 0, IND)
32 IF (D123 .LT. 213) GO TO 225
33 213 CALL PAGER(2, 0, IND)
34 WRITE(6, 215)
35 215 FORMAT(10H THIS IS AN UNFORCED PROBLEM.)
36 GO TO 225
37 CALL PAGER(4, 0, IND)
38 WRITE(6, 216)
39 216 FORMAT(10H FORCING FUNCTION . . . , A6, A2)
40 WRITE(6, 220)
41 MAIN0210
42 MAIN0220
43 MAIN0230
44 MAIN0240
45 MAIN0250
46 MAIN0260
47 MAIN0270
48 MAIN0280
49 MAIN0290
50 MAIN0300
51 MAIN0310
52 MAIN0320
53 MAIN0330
54 MAIN0340
55 MAIN0350
56 MAIN0360
57 MAIN0370
RING, QUADRATIC DAMPING
FORTRAN SOURCE LIST MAINPR

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101 220 FORMAT(H10,24H TIME(DML) TIME(J))
102 DO 258 J=1,JJJ
103 CALL PAGER(1,2,IND)
104 WRITE(6,221)TAU(J),FT(J)
105 221 FORMAT(H10,E11.4,*4,TIME(J))
106 258 CONTINUE
110 225 GAMMA=0.
111 IF(KB.EQ.0)GO TO 112
114 NBINC=0
115 KA=1
116 GO TO 246
117 112 READ(5,222)BETA,BINC,NBINC,KA
118 222 FORMAT(E11.4,F10.4,I3)
119 IF(NBINC.LE.1)GO TO 246
120 READ(5,242)(B(K),K=2,NBINC)
121 DO 182 IA=1,NBINC
125 BETA=B(A)
126 IF(KB.EQ.0)GO TO 246
127 READ(5,249)(R(R),M=2,NRINC)
128 GO TO 237
129 IF(KA.GT.0)GO TO 241
130 IF(KA.GT.0)GO TO 113
131 GO TO 237
132 240 B(1)=BETA
133 IF(NBINC.LE.1)GO TO 246
134 READ(5,246)(B(K),K=2,NBINC)
135 242 FORMAT(E11.4)
136 246 NBIN=NBINC+1
137 DO 182 IA=1,NBINC
138 IF(KB.EQ.0)GO TO 113
139 NRINC=1
140 RR(l)=R
141 GO TO 237
142 113 IF(A-NBINC.EQ.1)GO TO 96
143 IF(KA.GT.0)GO TO 182
144 IF(NBINC.EQ.1)GO TO 246
145 IF(KA.GT.0)GO TO 246
146 KA=1
147 GO TO 96
148 241 NBINC=NBINC+1
149 DO 259 INN=1,NBINC
150 XINN=INN
151 259 B(INN)=BETA+(XINN-U)*BINC
152 GO TO 246
153 IF(KA.GT.0)GO TO 246
154 IF(NRINC)GO TO 234
155 RR(l)=R
156 GO TO 246
157 GO TO 246
158 242 FORMAT(E11.4)
159 DO 246 INN=1,NBINC
160 XINN=INN
161 259 B(INN)=BETA+(XINN-U)*BINC
162 GO TO 246
163 96 IF(KA.GT.0)GO TO 237,237
164 249 FORMAT(E11.4,F10.4,I3)
165 231 IF(NRINC)GO TO 237
166 RR(l)=R
167 234 IF(NRINC.EQ.1)GO TO 237
168 READ(5,235)(R(R),M=2,NRINC)
169 RR(l)=R
170 IF(NRINC)GO TO 237
171 READ(5,235)(R(R),M=2,NRINC)
172 235 FORMAT(E11.4)
173 GO TO 237
174 246 DO 182 IA=1,NRINC
175 XM=M
176 R=R+(XM-U)*RINC
177 IF(KA.GT.0)GO TO 246
178 IF(KA.GT.0)GO TO 246
179 IF(KA.GT.0)GO TO 246
180 IF(KA.GT.0)GO TO 246
181 IF(KA.GT.0)GO TO 246
182 260 HT=HACT
183 220 KK=1
184 250 IF(KA.GT.0)GO TO 260
RING, QUADRATIC DAMPING

ISN
SOURCE STATEMENT

224  READ(5,248) GAMMA,GINC,NGINC
226  248 FORMAT(E11.4,F10.4,I3)
227    NGINC=NGINC
230    KN=1
231    IF(NGINC).LT.262,262,255
232    255 KK=2
233    NGINC=1
234    GO TO 260
235    260 IF(GINC).GT.0 GO TO 88
236    G(IN)=0.
237    IF(KA.GT.0)GO TO 70
238    IF(KA-NBINC.EQ.1)GO TO 86
242    READ(5,245)(G(I),L=2,NGINC)
244    245 FORMAT(E11.4)
247    IF(KA.GE.1)GO TO 63
250    63 NTEMP=NGINC
253    IF(KA-NBINC.EQ.1)GO TO 86
256    86 GA=0.
257    IF(KA.GT.0)GO TO 70
258    IF(KA-NBINC.EQ.1)GO TO 86
260    GO TO 88
261    BETA=0.
262    IF(KA.GE.2)GO TO 182
264    GAMMA=G(IN)
265    BE=0.
266    IF(KA.GE.2)GO TO 182
268    182 NTEMP=NGINC
269    NGINC=1
270    GO TO 88
271    BETA=0.
272    IF(KA.GE.2)GO TO 182
274    GAMMA=G(IN)
277    BE=0.
279    IF(KA.GE.2)GO TO 182
282    DO 257 IN=1,NGINC
285    257 G(IN)=GAMMA*(XIN-1.)*GINC
291    258 IN=IN+1
292    GO TO 257
295    BETA=0.
296    IF(KA.GE.2)GO TO 182
299    BETA=0.
302    IF(KA.GE.2)GO TO 182
305    GO TO 224
308    224 NTEMP=NGINC
311    244 NGINC=NGINC+1
312    DO 257 IN=1,NGINC
314    257 G(IN)=GAMMA*(XIN-1.)*GINC
318    258 IN=IN+1
321    GO TO 257
324    87 IF(KB.GT.0)GO TO 78
327    78 IF(KP.GE.1)GO TO 85
330    85 BETA=0.
333    224 NTEMP=NGINC
336    182 NTEMP=NGINC
339    185 BETA=0.
342    74 B(1D)=(XID-1.)*0.25
345    GO TO 224
348    GO TO 224
351    GO TO 224
354    GO TO 224
357    GO TO 224
360    GO TO 224
363    GO TO 224
366    GO TO 224
369    GO TO 224
372    GO TO 224
375    GO TO 224
378    GO TO 224
381    GO TO 224
384    GO TO 224
387    GO TO 224
390    GO TO 224
393    GO TO 224
396    GO TO 224
399    GO TO 224
402    GO TO 224
405    GO TO 224
408    GO TO 224
411    GO TO 224
414    GO TO 224
417    GO TO 224
420    GO TO 224
423    GO TO 224
426    GO TO 224
429    GO TO 224
432    GO TO 224
435    GO TO 224
438    GO TO 224
441    GO TO 224
444    GO TO 224
RING, QUADRATIC DAMPING

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FORTRAN SOURCE LIST MAINPR

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345  45 IF(XID.GT.15.1) GO TO 75
350  B(ID)=.25*(XID-11.)*.05
351  BETA=B(ID)
352  GO TO 224
353  75 B(ID)=.50*(XID-16.)*.1
354  BETA=B(ID)
355  GO TO 224
356  78 BETA=0.
357  GAMMA=G(IE)
360  KG=2
361  224 CALL PAGER(3,0,IND)
362  WRITE(6,211)
363  211 FORMAT(1HO,91LINEAR DAMP. FACTOR QUAD. DAMPING FACTOR RATIO INITIAL CONDITIONS,23X,9HTIME STEP) MAIN=1600
364  WRITE(6,210)
365  210 FORMAT(6H BETA=,E11.4,11H GAMMA=,E11.4,7H R = ,E11.4,1HMAIN=1620
366  ETAMX=ETAO
367  ETDMX=ETADO
368  N=2
369  NN=2
370  ETAX=ETAO
371  ETAD=ETADO
372  ACC=FNCT(R,BETA,GAMMA,ETAD,ETAO,N,TAU,FT,0.)
373  ACCM=ACC
374  XNACC=ACC
375  XNMX=ETAD
376  XNMX=ETAD
377  ETAC=ETADO
378  ETAC=ETADO
379  ACCE=ETAC
380  EX=ETADX
381  Z=2.*3.14159265*R
382  EQDML=EQDML*GAMMA*H*ABS(EX)*ABS(EC)*EC*EC/(Z*Z)
383  ELDML=ELDML*BETA*H*(EX*EC+EC*EC)*Z/Z
384  IF(EQDML.GE.EQDML) GO TO 204
385  205 ETAMX=ETAC
386  ETAD=ETAC
387  EL=ELDML
388  EQI=EQDML
389  TI=H
390  IF(EQDML.GE.EQDML) GO TO 207
391  206 ETDMX=ETAD
392  T2=H
393  IF(AACC.GE.XNACC) GO TO 190
394  190 XNACC=AACC
395  T6=T6
396  191 XNMX=ETAC
397  ETAC=ETAC
398  176 XNMX=ETAC
RING, QUADRATIC DAMPING
ISN SOURCE STATEMENT
FORTRAN SOURCE LIST MAINPR
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444       ETD4=ETADC
445       FL4=ELDML
446       EQ4=EQDML
447       T4=H
450       177 IF(ETADC.GE.XNDMX) GO TO 179
453       178 XNDMX=ETADC
454       T5=H
455       179 IF(ACCMX.GE.ACCC) GO TO 181
460       180 ACCMX=ACCC
461       T7=H
462       181 CALL NUMINT(ETAX,ETAX,ETAC,ETADC,ACCC,H,HT,R,BETA,GAMMA,
463       N,TAU,FT,MAIN2110)
466       CELEHL*EQDML,ETAMX,ETDMX,ACCMX,
467       XNACC,XNMX,XNDMX,D,PSVEL,
468       EL1,EQ1,EL3MAIN
470       IF(KA.LE.O)GO TO 182
477       IF(KG*GE.2)GO TO 77
483       IF(KP.GT.1)GO TO 83
487       GO TO 87
495       83 IF(KQ.GT.0)GO TO 82
498       0RD1(ID)=PSVEL
500       0RD2(ID)=EL1
502       0RD3(ID)=EL3
504       0RD4(ID)=ABS(ETDMX)
506       0RD5(ID)=ABS(ACCMX)
507       ID=ID*1
509       IF(XID.LT.30.1)GO TO 79
513       NID=ID-1
517       WRITE(7,106)ETA0,ETADO,C,CT,D,JJJ,NID
519       WRITE(7,107)(TAU(I),FT(I),1=1,14)
521       WRITE(7,108)(B(I),ORD1(I),ORD2(I),ORD3(I),QRD4(I),I=1,NID)
523       WRITE(6,52)
525       52 FORMAT(100,59H THERE ARE NO EQUIV. BETAS WITHIN RANGE OF VALUES CALCU
527       LATED)
529       IQ=0
531       CALL TAB(NID,ORD1,B,PSVEL,IG,IN,TEMP1)
534       ETDMX=ABS(ETDMX)
537       CALL TAB(NID,ORD4,B,ETDMX,IG,IN,TEMP4)
539       ACCMX=ABS(ACCMX)
542       CALL TAB(NID,ORD5,B,ACCMX,IG,IP,TEMP5)
544       I=1+I+IP
546       IF(IQ.GT.0)GO TO 51
547       CALL PAGE(2,0,IND)
550       WRITE(6,52)
RING, QUADRATIC DAMPING
ISN SOURCE STATEMENT FORTRAN SOURCE LIST MAINPR 05/09/67

572 51 CALL PAGER(2,0,IND)
573 WRITE(6,50)GAMMA
574 50 FORMAT(10x,32HPARABOLIC INTERPOLATION(Gamma = ,E10.4,1H))
575 IF(IN,EQ.0)GO TO 58
576 DO 57 II=1,1H
577 57 TEMP1=11H
578 IF(EQ.0)GO TO 58
579 DO 57 II=1,1H
580 IF(EQ.0)GO TO 58
581 CALL PAGER(1,0,IND)
582 WRITE(6,49)TEMP1,1H
583 MAIN2530
584 IF(EQ.0)GO TO 58
585 DO 55 11 = 1,IN
586 TEM1=TEMP4(II)
587 EPR2(II)=TBFNK(NID,B,ORD3,TEMI/FEQ3
588 CALL PAGER(1,0,IND)
589 WRITE(6,48)TEMP4(II),EPR2(II)
590 MAIN2610
591 DO 53 II=1,IP
592 TEM1=TEMP5(II)
593 EPR3(II)=TBFNK(NID,B,ORD3,TEMI/FEQ3
594 CALL PAGER(1,0,IND)
595 WRITE(6,47)TEMP5(II),EPR3(II)
596 MAIN2690
597 IF(IE-NGIN)59,182,182
598 IE=IE+1
599 GO TO 78
600 END
RING, QUADRATIC DAMPING

FORTRAN SOURCE LIST

ISN SOURCE STATEMENT

0 $16FTC NUMINT DECK
1 SUBROUTINE NUMINT(ETAX,ETADX,ETAC,ETADC,ACC,N,H,BETA,GAMMA)
2 CAU,FT,ELDML,EOQML,ETAMX,ETDMX,ACCX,KNMX,KNMX,D,PSVEL,EL1,ENUMI)
3 =
4 CO1,EL3,EQ3,K1)
5 NUMI -
6 C BELOW ARE LISTED IN ORDER THE ARGUMENTS OF SUBROUTINE NUMINT
7 C ETAX AND ETADX = OLD VALUES OF THE VARIABLE AND ITS FIRST DERIV.
8 C ETAC AND ETADC = NEW VALUES OF THE VARIABLE AND ITS FIRST DERIV.
9 C ACCC = CURRENT VALUE OF THE SECOND DERIV. OF THE VARIABLE.
10 C EQDML = ENERGY DISSIPATED IN QUAD. DAMPER IN FIRST STEP.
11 C ETAMX AND ETDMX = MAX VALUES OF VARIABLE AND ITS FIRST DERIV.
12 C D = DURATION OF THE PULSE.
13 C H,HT ARE TIME INCREMENT AND SPACING OF TIME HISTORY
14 C R,BETA,GAMMA, ARE THE SYSTEM PARAMETERS
15 C N = NO. OF VALUES IN FORCING FUNCTION TABLE
16 C TAU,FT, ARE VALUES IN THE FORCING FUNCTION TABLE
17 C ELDML = DIMENSIONLESS ENERGY DISSIPATED BY THE LINEAR DAMPER
18 C EOQML = DIMENSIONLESS ENERGY DISSIPATED IN THE QUADRATIC DAMPER
19 C THE NEXT SIX ARGUMENTS ARE THE MAX. POS. AND NEG. VALUES OF THE
20 C VARIABLE AND ITS DERIVATIVES
21 C D = DURATION OF THE PULSE
22 C PSVEL = PSUEDO VELOCITY NONDIMENSIONIZED WITH RESPECT TO MAX
23 C GROUND VELOCITY
24 C THE NEXT FOUR ARGUMENTS ARE ENERGIES ABSORBED AT END PULSE AND
25 C TOTAL ENERGIES ABSORBED
26 DIMENSION TAU(150),FT(50)
27 NUMI0020
28 NUMI0030
29 NUMI0040
30 NUMI0050
31 NUMI0060
32 NUMI0070
33 NUMI0080
34 NUMI0090
35 NUMI0100
36 NUMI0110
37 NUMI0120
38 NUMI0130
39 NUMI0140
40 NUMI0150
41 NUMI0160
42 NUMI0170
43 NUMI0180
44 NUMI0190
45 NUMI0200
46 NUMI0210
47 NUMI0220
48 NUMI0230
49 NUMI0240
50 NUMI0250
51 NUMI0260
52 NUMI0270
53 NUMI0280
54 NUMI0290
55 NUMI0300
56 NUMI0310
RING, QUADRATIC DAMPING

FORTRAN SOURCE LIST NUMINT

ISN

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42      EQ1=EQDML
43      T1=T
44      104 IF(ETDMX.GE.ETADC) GO TO 161
45      106 ETDMX=ETADC
50      T2=T
51      161 IF(ACCX.GE.XNACC) GO TO 155
54      192 XNACC=ACCX
55      T6=T
56      155 IF(ETACX.GE.XNMX) GO TO 157
61      156 XNMX=ETAC
62      ET04=ETADC
63      EL4=ELDML
64      EQ4=EQDML
65      T6=T
66      157 IF(ETADCX.GE.XNDMX) GO TO 159
71      158 XNDMX=ETADC
72      T5=T
73      159 IF(ACCMX.GE.ACCC) GO TO 107
76      160 ACCMX=ACCC
77      T7=T
100     107 IF(T>D)143,108,108
101     108 IF(T009,120,120
102     109 EL3=EMLDML
103     EQ3=EQDML
104     120 IF(T=D-1./R)143,121,121
105     121 Z=2.*3.14159265*R
106     EN1TO=ELDML+EQDML+ETACXETADCXETADC/(Z*Z)
107     IF(ABS(XNMX).GT.ABS(XNMX)) GO TO 230
112     ACCMX=XNACC
113     T7=T
114     238 IF(ABS(ETDMX).GT.ABS(XNDMX)) GO TO 239
117     ETDMX=XNDMX
120     T2=T5
121     239 IF(ABS(ETAMX).GT.ABS(XNMX)) GO TO 240
124     ETAMX=XNMX
125     T1=T4
126     EL1=EL4
127     EQ1=EQ4
130     240 IF(N.GE.2) GO TO 242
133     PSEVE=0
134     GO TO 243
135     242 CALL G1NDMX(IN,TAU,FT,AMAX,VMAX,DMAX)
136     PSEVE=ABS(ETAMX)/(Z*VMAX)
137     243 IF(KA.GT.0)GO TO 99
142     CALL PAGE7(7,0,19)
143     WRITE(6,162)
144     162 FORMAT(1H0,6X,5HTOTAL,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8HMAB,6X,8H...
WRITE(6,165) T1,T2,T7
WRITE(6,241)
WRITE(6,166)T,ETAMX,ETDMX, ACCMX,EL1,EQ1,EL3,EQ3,PSVEL
WRITE(6,167)T,E TAC,ET ADC, ACCC, ELM, ELM NUMI
GO TO 140
END NUMI
SUBROUTINE RUNKUT(Y, XK, HN, BETA, GAMMA, K, N, TAU, FT)

RUNKUT IS A SINGLE STEP 4TH ORDER RUNGE-KUTTA PROCEDURE

DIMENSION Y(2,5), XK(2,5), TAU(50), FT(50), A(4), B(4), C(4), Q(2,5)

A(1) = .5
A(2) = 292893
A(3) = 1.707107
A(4) = 1.6666667
B(1) = 2.
B(2) = 1.
B(3) = 1.
B(4) = .2.
C(1) = .5
C(2) = 292893
C(3) = 1.707107
C(4) = .5

DO 110 J = 1, 4
  XJ = J
  T = (XJ - 1.) * HN * .25
  DO 111 I = 1, NN
    XK(I, J) = FUNC(I, J, Y, BETA, GAMMA, K, N, TAU, FT, T)
  111 CONTINUE
  DD 110 J = 1, 4
  Q(1, 1) = 0.
  Y(I, J+1) = Y(I, J) + HN * (A(J) * (XK(I, J) - B(J) * Q(I, J)))
  DO 110 I = 1, NN
  110 CONTINUE
RETURN
END
SUBROUTINE TAB(N,TAU,FT,T,NDF,II,F)
C TAB INTERPOLATES EITHER LINEARLY OR QUADRATICALLY. MULTIPLE VALUES
C ARE PERMITTED. TAB FINDS THEM ALL (UP TO 5).
C N=NO. OF VALUES IN TABLE
C TAU=TABULATED ABCISSA
C FT=TABULATED VALUES OF THE FUNCTION
C T=ABCissa FOR WHICH A VALUE IS DESIRED
C NDF=NO. OF DEGREE OF FIT DESIRED=1 OR 2
C II=NO. OF VALUES IF FUNCTION IS MULTIVALUED=1 FOR SINGLE VALUE
C F=VALUE(S) OF INTERPOLATION
2 DIMENSION TAU(50),FT(50),F(5)
3 II=0
4 J=2
5 IF(TAU(J)-T)14,9,9
6 14 IF(J=N)11,2,2
7 11 IF(TAU(J).GT.TAU(J-1))GO TO 13
8 15 J=J+1
9 GO TO 14
10 13 IF(T>TAU(J))16,16,15
11 16 JJ=J
12 10 IF(NDF-1)127,127,104
13 9 IF(J=N)12,2
14 1 IF(TAU(J).LT.TAU(J-1))GO TO 3
15 5 J=J+1
16 GO TO 9
17 2 RETURN
18 3 IF(TAU(J).GT.T)GO TO 5
19 GO TO 16
20 127 II=II+1
21 33 F(II)=(T-TAU(JJ-1))*(FT(JJ)-FT(JJ-1))/((TAU(JJ)-TAU(JJ-1))*FT(JJ-1)
22 34 GO TO 132
23 35 104 II=II+1
24 36 IF(J.LT.3)JJ=3
25 41 F(II)=(T-TAU(JJ-1))*(T-TAU(JJ))**FT(JJ-2)/((TAU(JJ)-TAU(JJ-2))*FT(JJ-2)
26 42 36 132 IF(TAU(J)-TAU(J-1))14,14,9
43 END
FUNCTION FNCT(ETA,ETAD,TAU,FT,T,A,B,Gamma)

DIMENSION TAU(50),FT(50)
PI=3.14159265

XX=TBF NK(N,TAU,FT,T,1)
XP=4.*PI*PI*R*R

FNCT=XP*XX-4.*PI*B*TA*ETAD-Gamma*ABS(ETAD)*ETAD-XP*ETA

RETURN

END
FUNCTION FUNC(I,J,Y,BETA,GAMMA,R,N,TAU,FT,T)
DIMENSION Y(2,5),TAU(50),FT(50)
IF(I-I)200,200,201
200 FUNC=Y(2,J)
RETURN
PI=3.14159265
JJJ=N
XP=4.*PI*PI*R*R
FUNC=XP*TBENK(N,TAU,FT,T,1)-4.*PI*BETA*R*Y(2,J)-GAMMA*ABS(Y(2,J))*G0ER0840
CY(2,J)=XP*Y(1,J)
RETURN
END
FUNCTION TBFNK(N, TAU, FT, T, NDF)
DIMENSION TAU(50), FT(50)
100 IF(NDF-i) 100, 100, 101
120 JJ=1
130 IF(T-TAU(JJ)) 127, 133, 129
129 JJ=JJ+1
10 GO TO 130
127 TBFNK=(T-TAU(JJ-1))*FT(JJ-1)/(TAU(JJ)-TAU(JJ-1)+FT(JJ-1)
12 GO TO 132
131 TBFNK=FT(N)
14 132 RETURN
15 133 TBFNK=FT(JJ)
16 RETURN
17 101 IF(T-TAU(N)) 102, 103, 103
20 102 JJ=3
21 107 IF(T-TAU(JJ)) 104, 104, 106
22 106 JJ=JJ+1
23 GO TO 107
25 RETURN
26 103 JJ=N
27 GO TO 104
30 END