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A Contactless Method of Measuring the Resistivity of Semiconductor Materials

by

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ABSTRACT

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A study has been made of a contactless technique of measuring resistivity, which uses a time-varying magnetic field to induce eddy currents in a sample and measures the magnetic field produced by the eddy currents. The theoretical basis of this method is studied. An experimental system using a phase discriminating network is developed, and used to measure a voltage induced by the eddy current magnetic field. Data are reported for samples in the range from 0.1 to 11 ohm-cm. This method is predicted to be feasible for measuring resistivities in the range from $10^{-10}$ ohm-cm to $10^{+6}$ ohm-cm.
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INTRODUCTION

If one asked a number of engineers, "What would be the ideal instrument for measuring the resistivity of a material?", one would get many and varied answers. However, among the various answers one would discover that some characteristics were commonly desired: 1) the ability to measure resistivities that range from $10^{-7}$ ohm·cm to $10^{22}$ ohm·cm; 2) the versatility to be used on any type of material; 3) the means to measure spatial distribution of resistivity; 4) the advantage of determining resistivity independent of the size or geometry of materials; and, most importantly, 5) the capability of measuring resistivity without making physical electrical contact to the material. The characteristics enumerated above would be desirable in an ideal instrument, because they represent problems encountered in the use of existing instruments.

The requirement of making physical electrical contact to a material in order to determine its resistivity is a severe one. Any measurement technique demanding this is dependent on the varied surface characteristics of different materials, and is vulnerable to changes in the surface characteristics from sample to sample of one type of material. In addition, damage to the material may be induced when contact is made, i.e., crystal deformation may result from pressure ohmic contacts. In an industrial context, the time and effort necessary to achieve proper ohmic contact without inducing damage may be unacceptable.

In this thesis, a technique of measuring resistivity without making physical electrical contact to a material is thoroughly studied. Magnetic induction is the basis of this method, in which the material to be tested is stimulated with a time-
varying magnetic field, and responds by producing a magnetic field dependent on
its own resistivity. This thesis analyzes the mechanism of stimulation and response;
proposes a system for inducing the stimulation and detecting the response; and exper¬
mentally verifies the feasibility of this technique.

Many of the electrical properties of semiconductor devices are determined
by their resistivities. The widespread use of semiconductors has prompted the
refinement of existing techniques and the pursuit of new methods of resistivity measure¬
ment. Many of the most common methods have been pushed to their practical limits
by the range of resistivities of semiconductors, the geometry of samples, and their
surface characteristics. Since the need for improved techniques prompted this
investigation of a contactless method, its application to semiconductor materials
will be stressed.
BACKGROUND

To put this project in perspective it is necessary to discuss the following:
1) the definition of resistivity; 2) the importance of the resistivity of semiconductor materials for production and research; and, 3) the most common methods of measuring resistivity, their advantages and disadvantages.

The resistivity of a material derives its definition from Ohm's law. The proportionality constant which relates the current density to the electric field is called the conductivity. The reciprocal of the conductivity is the resistivity.

\[ \mathbf{J} = \sigma \mathbf{E} \]

\[ \mathbf{J} = \text{current density [amperes/cm}^2] \]
\[ \mathbf{E} = \text{electric field [volts/cm]} \]
\[ \sigma = \text{conductivity [mhos/cm]} \]
\[ \rho = \text{resistivity [ohm-cm]} \]

The resistivity of a material depends on the production of charge carriers for conduction, and on all the mechanisms which influence the motion of these carriers. Expressions for the resistivity in terms of other physical parameters differ for the various classes of materials, as the conduction processes differ. If the physical mechanisms which influence the resistivity of a material are known, then a measurement of resistivity will give valuable information about the electrical characteristics of the material.
Semiconductor materials have resistivities in the range from $10^{-2}$ to $10^9$ ohm-cm. The resistivity is highly temperature-dependent. At absolute zero there are no mobile charge carriers, whereas at room temperature a significant number are mobile. The conduction properties depend on the electron energy bands which are a consequence of the periodic nature of the crystal. The conduction and valence bands are separated by a forbidden energy gap of height $E_g$. At absolute zero, the conduction band is void of any electrons, and the valence band is filled. At room temperature, some electrons are excited from the valence band to the conduction band. Both the electrons in the conduction band and the holes left behind in the valence band contribute to the current.

When no impurities are present, or when the effect of impurities on current flow is negligible, the resistivity is known as the intrinsic resistivity, and can be expressed as

$$\rho = \frac{1}{n_e \mu_e + p_h \mu_h}$$

$n_e$ is the concentration of electrons in the conduction band; $p_h$ is the concentration of holes in the valence band; $e$ is the electronic charge; and $\mu_e$ and $\mu_h$ are the mobilities of the electrons and holes, respectively. The mobility is determined principally by the lattice scattering of the electrons and holes, and gives rise to drift velocities. The mobility is defined as the magnitude of the drift velocity per unit electric field.

$$\mu = |v|/E$$
Since the mobility arises from electron-phonon interaction, it will certainly be dependent on temperature. If impurities are present, then scattering of electrons by impurities may result, and this will affect the mobility.

The intrinsic carrier concentrations, \( n_i \) and \( \mu_i \), upon which the intrinsic resistivity is dependent, are controlled by the ratio of the energy gap to the temperature, i.e., \( \frac{E_g}{k_B T} \), where \( k_B \) is Boltzmann's constant.

The introduction of impurities into a semiconductor material will greatly change its electrical properties. The resistivity can be decreased by four or five orders of magnitude by doping procedures.

Pentavalent impurities are called donor impurities because when ionized they contribute electrons to the conduction band. Similarly, trivalent impurities are called acceptors because they can take on electrons from the valence band leaving behind holes. The ionization energies of these impurities in a semiconductor material are much less than the energy gap of the material. For many impurities, the ionization energies are of the order of \( k_B T \), at room temperature, so thermal ionization results. Incident light of the proper frequency can also be a source of ionization of these impurities.

If a semiconductor material contains a much larger concentration of donors than acceptors, the number of ionized donors will be much greater than the number of ionized acceptors. For a material at a temperature for which practically all the donors are ionized, equating the generation and recombination rates of charge carriers shows that: 1) the electron concentration in the conduction band is approximately equal to the donor atom concentration \( N_d \), i.e., \( n \approx N_d \); and, 2) the hole concentration in the valence band is small compared to the electron
concentration in the conduction band, i.e., $p \ll n$. The conduction of current will be principally by electrons. This type of material is called $n$-type, and the resistivity can be approximated by

$$\rho \approx \frac{1}{N_a e \mu_e}$$

Similarly, if a semiconductor material contains a much larger concentration of acceptors than donors, the number of ionized acceptors will be much greater than the number of ionized donors. At a temperature for which practically all the acceptors are ionized, it can be shown that: 1) the hole concentration in the valence band is approximately equal to the acceptor atom concentration $N_a$, i.e., $p \approx N_a$; and 2) the electron concentration in the conduction band is small compared to the hole concentration in the valence band, i.e., $n \ll p$. The conduction of current will be principally by holes. This type of material is called $p$-type, and the resistivity can be approximated by

$$\rho \approx \frac{1}{N_a e \mu_h}$$

As can be concluded from the above discussion, resistivity is a complicated quantity. It is sensitive to temperature, to light, and to small quantities of impurities. In studying the basic properties and mechanisms of conduction, a resistivity measurement must be coupled with information of the mobility and carrier densities to be of value.

However, for doped semiconductors, the resistivity is a measure of the impurity concentration. The characterization of material according to its impurity level is the principal use of resistivity.
In an industrial context, resistivity is used as a check on the quality of bars of "pure" material. It allows the prediction of the additional doping required to achieve a desired resistivity level. In doping and single-crystal-growing of rods, an impurity level which varies with axial position usually results. Resistivity measurements are made along the rod so it can be cut into sections having the proper resistivity ranges for different devices. For use in mass production of integrated circuits, it is desirable that the resistivity be uniform over the cross-section of the slices cut from a single crystal rod. Resistivity measurements are required to check the radial profile.

Resistivity is an important factor in determining certain characteristics of semiconductor devices. A diode's reverse breakdown voltage increases as the resistivity increases. The breakdown voltage constitutes a limitation in rectifier diodes, and defines a reference voltage for reference diodes. The electrical power production capabilities of semiconductor solar cells are dependent on resistivity. Transistor properties, such as collector breakdown and \( V_{ce} \) (saturation) are influenced by resistivity. Thus, the production of good devices with characteristics within specified tolerances requires knowledge and control of the resistivity.

There have been numerous methods used over the years to measure resistivity. Consideration will be given to some of the most common methods. Each method has advantages and disadvantages making it well suited to some application and not to others.

The most basic method is the bulk resistance method. It consists of fabricating a bar of material of cross-sectional area, \( A \), and length, \( L \). Ohmic contacts are applied which cover the ends. Then, the resistance, \( R \), between the two end
contacts is measured. The resistivity is given by

\[ \rho = R \frac{A}{\ell} \]

There are several obvious disadvantages associated with this method. It requires fabrication of the material into a special geometry. The accuracy of the measurement depends on the accuracy to which \( A \) and \( \ell \) can be constructed and measured. The application of ohmic contacts to the ends is time-consuming. This method is not applicable for non-destructive use on thin wafers of material.

The two-probe method [1] is a refinement of the bulk resistance method, for it allows the determination of changes of resistivity along the length of the bar. The cross sectional area of the bar, \( A \), must be known accurately. Ohmic contacts must be applied to the ends of the bar, and a current, \( I \), passed through the bar. Using two sharp probes, accurately spaced a distance, \( \Delta \), apart, the voltage drop between two points, \( V \), is measured. The resistivity is given by

\[ \rho = \frac{VA}{I\Delta} \]

The two-probe method possesses all of the disadvantages of the bulk method, with the addition of the problems of maintaining an accurate probe spacing, and obtaining good ohmic contacts with the probes.

A three-point probe method exists. However, its application is quite restricted. It is utilized to determine the resistivity of epitaxial films of \( n \)-type material on \( n^+ \)-type substrates, or \( p \)-type material on \( p^+ \)-type substrates [2]. One of the probes is a tungsten whisker used to form a point contact diode with the epitaxial film. The other two probes make ohmic contact to the film: one is used
to apply a voltage between itself and the whisker contact; the other is used to measure the breakdown voltage of the point-contact diode. Empirical calibration curves of breakdown voltage versus resistivity are required. The accuracy of the measurements is influenced by temperature, whisker pressure and whisker diameter. The film thickness must be large enough so that the depletion region of the point-contact diode does not reach the film-substrate interface before breakdown occurs.

The method most widely used in the semiconductor industry is the four-point probe technique. It utilizes four probes, usually placed in-line. A constant current, $I$, is passed through the two outer probes, and a voltage, $V$, is measured between the two inner probes.

Use of the four-point probe method for measuring the resistivity of semiconductor materials requires the following conditions to be met[3]: 1) the resistivity of the material is uniform in the area of measurement; 2) minority carrier injection in the semiconductor by the current-carrying probes is small; 3) the surface on which the probes rest is flat, with no surface leakage current; 4) the four probes contact the surface in a straight line and their spacing can be accurately controlled; and 5) the boundaries between the current-carrying electrodes and the bulk material are hemispherical and small in diameter compared to the probe spacings. When any of the above conditions is not fully met, error in the measurements results. There has been much effort put forth to develop instruments and procedures which meet the above conditions simultaneously.

The resistivity is given by the following expression:

$$\rho = \frac{LV}{I}$$
where \( L \) is a geometrical factor [4]. The value of \( L \) is a function of the sample size and shape. Much work has been done in determining \( L \) for different geometrical configurations and boundary conditions [4,5,6,7,8].

There are many problems encountered in the implementation of a practical system. Some elements which greatly influence reproducibility and accuracy are: the electrical conductance and resistance against wear of the probe material; the probe pressure and tip radius, which affect the contact resistance; and the probe spacing. It is the probe spacing which gives rise to the greatest source of error when attempts are made to correlate measurements taken using different probe heads.

The use of the four-point probe technique has proved to be reasonably successful and so widespread that a tremendous effort is being put forth to standardize measurement techniques. Resistivities of silicon and germanium can be measured up to 1000 ohm-cm before accuracy begins to break down. However, difficulties arise when using this technique on gallium arsenide because of its surface characteristics.

Several microwave techniques of measuring resistivity have been developed because of the increasing importance of the application of semiconductor materials to microwave devices. At microwave frequencies in the range from 9 to 60 Ghz, the skin depth is less than 10 mils for resistivities up to 1 ohm-cm. As the resistivity increases, the skin depth increases. Therefore, in certain resistivity, frequency, and thickness ranges, there is a possibility of measurements independent of thickness.

One microwave method terminates a transmission line with a sample of semiconductor material and measures the return loss [1]. The return loss, in decibels, equals minus twenty times the log to the base 10 of the reflection coefficient. At very low resistivities, the material will absorb no power and the return loss will be
zero. As resistivities become higher, power will begin to be absorbed, meaning a higher return loss. When a perfect match is achieved, the return loss should be infinite. Empirical calibration curves of return loss versus resistivity are required. The principal disadvantage of this method is poor sensitivity. The spread from low to high resistivities (.01 to 1000 ohm-cm) results in only a 4 decibel change in the return loss.

A method which gives increased sensitivity and accuracy makes use of a high-Q resonant cavity, which couples a generator to a crystal detector [1]. A portion of the slice being tested becomes a section of the cavity wall, and so the resistivity of the slice will influence the Q of the cavity. The amount of signal transferred through the cavity increases with the Q of the cavity. This technique also depends on empirical calibration. For thin slices, 5-20 mils thickness, accuracy of this method begins to break down around 10 ohm-cm, because the slope of the resistivity versus transmission loss curve increases as the resistivity increases.

There are several microwave techniques which call for inserting a semiconductor material into a waveguide. In the transmission through a slice method, the slice is placed normal to the axis of the waveguide [1, 9]. The change in the amount of transmitted power with the slice in and out of the system is measured. The resistivity is determined from empirical calibration curves for different thicknesses. Reasonable accuracy is achieved up to 60 ohm-cm.

The resistivity of a semiconductor sample can be deduced by measuring the propagation coefficient when the sample is inserted in a slotted waveguide [10]. Identical slots parallel to the axis of the guide are cut in the broad walls. Theoretical results can be predicted using the theory for inhomogeneously filled waveguides.
Correction factors are required because: 1) reflections occur at each end of the sample; 2) the slots disturb the field pattern within the waveguide; and, 3) reflections will also occur at the ends of the slotted section.

It is also possible to predict theoretically the propagation coefficient of a rectangular guide with one of the narrow walls consisting of a semiconductor material [11]. Experimentally, the change in the propagation constant with the semiconductor in place, and then with a brass plate replacing it, can be determined by measuring phase and attenuation changes. The resistivity is calculated from this change in propagation constant. One important disadvantage is that the semiconductor sample must be shaped so that it exactly fits the section of the narrow wall it is replacing. Resistivities from 5 to 25 ohm-cm have been measured by this method.
CONTACTLESS METHOD

To measure resistivity, it is necessary to cause an event to occur which is dependent on the resistivity of a specimen. Quantitative observation and comparison of events for different specimens enables the determination of the resistivity.

As stated in Ohm's law, the flow of current due to an electric field is dependent on resistivity. An electric field can be forced to exist in a material by applying a voltage across it. This may be accomplished by using ohmic contacts as is done in all the probe methods. However, an electric field can be induced without the use of ohmic contacts if magnetic interaction is used.

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1)
\]

\[
\mathbf{J} = \left( \frac{1}{\rho} \right) \mathbf{E} \quad (2)
\]

\[
\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (3)
\]

\[
\mathbf{B} = \mu \mathbf{H} \quad (4)
\]

\[
\mathbf{D} = \varepsilon \mathbf{E} \quad (5)
\]

From equation (1), one of Maxwell's equations, it is seen that a time-changing magnetic flux density, \( \mathbf{B} \), will produce an electric field, \( \mathbf{E} \). A current density, \( \mathbf{J} \), given in equation (2), will result, which is dependent on resistivity. The current density will produce a magnetic field, as can be seen from equation (3), another of Maxwell's equations. The magnetic field intensity, \( \mathbf{H} \), is related to the magnetic flux density, \( \mathbf{B} \), by the permeability of the medium, \( \mu \), in equation (4). The electric flux density, \( \mathbf{D} \), is related to the electric field intensity, \( \mathbf{E} \), by the permittivity of the medium, \( \varepsilon \), in equation (5).
Since the current density is a function of the resistivity, the magnetic field produced by it will be dependent on the resistivity. The detection of this magnetic field does not require ohmic contact to the specimen being studied. The above statements establish the basis for contactless stimulation and contactless observation of an event which is dependent on resistivity.

This principle has previously been used to measure resistivity. The techniques using this approach differ in their implementation of the stimulation, observation and comparison of the events.

A method which utilizes the interaction between a rotating magnetic field and the currents induced in a sample was proposed by Chaberski [12]. A conducting sample suspended in this field will experience a torque which can be measured. The torque acting on the sample can be calculated using the fact that the power delivered by the rotating magnetic field is equal to the mechanical torque multiplied by the angular velocity. The power delivered by the rotating magnetic field is that dissipated by the eddy currents in the sample.

The event dependent on the resistivity is the flow of eddy currents. The observation of this event is the measurement of the torque exerted on the sample. By comparing torques for different samples, the resistivity can be determined. Chaberski claims his technique can be used over the resistivity range from $10^{-8}$ to $10^{+8}$ ohm-cm, and that the measurements can be made independent of sample shape.

The remaining techniques mentioned here use the fact that when an electrical conductor is inserted into the fields of an inductor carrying alternating current, eddy currents are induced in the conductor; these currents produce magnetic fields which change the terminal impedance of the inductor.
Poehler and Liben [13] proposed a method in which a semiconductor material, usually a thin wafer, is placed at the end of an R. F. solenoid, changing its terminal impedance. The resistive component is given by $P/I^2$, where $P$ is the input power and $I$ is the rms current to the coil. The power loss consists of the resistive loss in the coil windings and the eddy current loss in the material placed at the end. The resistivity can be related by an approximate expression to the power loss due to the eddy currents in the sample. The power loss can be determined by measuring the change in $Q$ of the coil using a $Q$-meter circuit.

Stimulation of the eddy currents, which are dependent on resistivity, is by means of the end fields of a solenoid. This occurrence is observed by measuring the change in $Q$ of the coil. The resistivity is then related to the change in $Q$. The operating frequency must be such that the skin depth is greater than the sample thickness. Measurements have been made for materials in the resistivity range from 0.1 to 50 ohm-cm.

Zimmerman [14] calculated the changes in resistance and inductance when a cylindrical conductor or a spherical conductor are placed inside a solenoid. Using a bridge circuit, the experimental changes can be measured. By comparing these to the theoretical curves the resistivity can be determined. So it is by measuring the changes in lumped circuit parameters resulting from the action of the eddy currents, that the events dependent on resistivity are observed and compared.

This method was used on Cu at liquid nitrogen temperatures, where its resistivity is $0.2 \times 10^{-6}$ ohm-cm, and on Cu-Mn alloy at liquid helium temperature, where its resistivity is $0.42 \times 10^{-6}$ ohm-cm.
Myers [15] proposed a method using the variation of the output voltage of a transformer with the electric and magnetic properties of the core. Two identical air-cooled transformers with primaries connected in series and secondaries connected in series opposition are driven with an oscillator. Using a resistance-capacitance network the voltage across the secondary of one transformer is altered until the output across both secondaries is zero. When the specimen is inserted into the core of one transformer, the voltage balance is destroyed. An R-C network is then adjusted to give minimum output voltage. Myers states that the change in phase of the secondary voltage is directly proportional to the conductivity of the specimen; and that the phase change is proportional to the change in resistance required to achieve a null. This method has been used for resistivities below 1 ohm-cm.

Yosim et al. [16] applied Myers' method to the measurement of the resistivities of liquid melts at elevated temperatures for resistivities below 1 ohm-cm. They refined the theory by deriving expressions for the change in phase of the voltage induced at the secondary, when the specimen is inserted, in terms of the physical parameters of the system, one of which is the resistivity of the specimen.

To detect the change in the phase angle, an impedance bridge circuit was constructed. Two matched transformers with the primaries connected in series and the secondaries connected in series opposition were balanced by a resistance-capacitance circuit. The specimen was inserted into the core of one transformer, and the resistance-capacitance circuit readjusted to achieve balance. The changes in resistance and capacitance were used to calculate the change in phase angle.

For these last two methods, the fields in the core of the transformer stimulate an event, the production of eddy currents. This event affects the output voltage of
the secondary. The observation of the event is accomplished by measuring the change in phase angle of the secondary voltage.

The event occurring when a time-changing magnetic field impinges on an electrical conductor is the flow of current in the conductor. These currents, known as eddy currents, are dependent on resistivity.

All of the contactless methods described above stimulated eddy currents. Their observations of them are characterized by varying degrees of indirectness, i.e., measurement of a torque, a change in Q, changes in resistance and inductance, and a change in phase angle. All but the first technique involve measuring the change in an electrical circuit parameter caused by the occurrence of this event. There is a more direct means of observing the eddy currents.

The most direct method would be one which measures the eddy currents directly, but this is not possible in a contactless scheme. The next most direct method is to observe the magnetic fields of the eddy currents. The detection and measurement of these fields is the essence of this thesis.

It has been well established that the eddy currents produced by a stimulating magnetic field produce an additional magnetic field. Therefore, the resultant magnetic field outside the specimen is a vector superposition of two fields. The problem is to find a means of sensing the eddy current magnetic field while ignoring the stimulating magnetic field. The stimulating field is of much larger magnitude, and therefore, one would not expect to find a region where it has attenuated to a magnitude below that of the field produced by the eddy currents. Detecting the desired field is not just a matter of positioning a sensing coil in a special location with respect to the specimen, because any region where the eddy current magnetic field
is present, the stimulating field also appears.

The solution rests on the fact that the two magnetic fields differ in time phase. To show this, it is necessary to consider the form of the expression for the resultant magnetic field when a conductive material is placed in a time-varying magnetic field. The resultant field quantities must satisfy Maxwell's equations and the continuity equation. The exact solution of these equations subject to boundary conditions is a formidable task because of the coupling between the electric and magnetic fields. However, an exact solution is not necessary in order to obtain the desired information concerning the phase difference.

The method of successive approximations [17], presented below, eliminates the simultaneous solution of Maxwell's equations by substituting an infinite series of unidirectional couplings for the bilateral coupling resulting from the time derivatives of the electric and magnetic fields. The fields are expressed as an infinite series. The number of terms of this series representation, which is necessary to describe sufficiently the behavior of a system, depends on the dimensions of the system and the highest frequency of time variation. Thus, there are ranges of frequency in which the various higher-order terms can be neglected. However, as the frequency of the time variation increases the addition of these higher-order terms becomes necessary. When the frequency is high enough that the wavelength associated with it is comparable to the dimensions of the system, then all the higher-order terms must be included.

The solutions for the electric field, the magnetic field, the current density, and the charge density must satisfy the following equations:
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]
\[ \nabla \times \mu \mathbf{B} = \mathbf{J} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \]
\[ \nabla \cdot \mathbf{J} = -\frac{\partial \varphi}{\partial t} \]

The solutions are of the form

\[
\mathbf{E} = E_0 + E_1 + E_2 + \ldots + E_k + \ldots
\]
\[
\mathbf{B} = B_0 + B_1 + B_2 + \ldots + B_k + \ldots
\]
\[
\mathbf{J} = J_0 + J_1 + J_2 + \ldots + J_k + \ldots
\]
\[
\varphi = \varphi_0 + \varphi_1 + \varphi_2 + \ldots + \varphi_k + \ldots
\]

The zero-order terms must satisfy the following equations:

\[ \nabla \times E_0 = 0 \]
\[ \nabla \times \mu B_0 = J_0 \]
\[ \nabla \cdot J_0 = 0 \]

The first-order terms must satisfy the following equations:

\[ \nabla \times E_1 = -\frac{\partial B_0}{\partial t} \]
\[ \nabla \times \mu B_1 = J_1 + \epsilon \frac{\partial E_0}{\partial t} \]
\[ \nabla \cdot J_1 = -\frac{\partial \varphi_0}{\partial t} \]

And the kth-order terms must satisfy the following equations:

\[ \nabla \times E_k = -\frac{\partial B_{k-1}}{\partial t} \]
\[ \nabla \times \mu B_k = J_k + \epsilon \frac{\partial E_{k-1}}{\partial t} \]
\[ \nabla \cdot J_k = -\frac{\partial \varphi_{k-1}}{\partial t} \]
Now consider the problem of determining the magnetic field in a medium of zero conductivity produced by an independent current distribution $J_s(r,t)$. Let this current distribution be present on a perfect conductor, $\sigma = \infty$, surrounded by a medium of zero conductivity, $\sigma = 0$. This current distribution is actually a surface current distribution.

The zero-order terms in the medium of $\sigma = 0$ must satisfy:

$$\nabla \times E_o = 0$$
$$\nabla \times \mu B_o = J_o = J_s$$
$$\nabla \cdot J_o = \nabla \cdot J_s = 0$$

Since the electric field inside the perfect conductor is zero and $J_s$ is a surface current density, then from equation (6), the zero-order electric field in the non-conductive medium is zero, $E_o = 0$. The zero-order magnetic field $B_o$, given by equation (7), has the same spatial distribution that a static field distribution would possess, i.e., if $J_s$ were independent of time.

The first-order terms must satisfy:

$$\nabla \times E_i = -\frac{\partial B_o}{\partial t}$$
$$\nabla \times \mu B_i = 0$$
$$\nabla \times J_i = 0$$

Equation (8) predicts a first-order electric field caused by a time-varying zero-order magnetic field. Since $\sigma = 0$ and since no zero-order electric field exists, equation (9) predicts that the first-order magnetic field will be zero.
The second-order terms must satisfy:

\[ \nabla \times E_2 = 0 \]  \hspace{1cm} (10)

\[ \nabla \times \mu \nabla B_2 = \varepsilon \frac{\partial E_1}{\partial t} \]  \hspace{1cm} (11)

\[ \nabla \cdot J_2 = 0 \]

Since no first-order magnetic field exists, equation (10) predicts that the second-order electric field is zero. Equation (11) predicts that a second-order magnetic field is caused by a time-varying first-order electric field.

If this procedure is continued for high order terms, as shown schematically in Figure 1, the resulting electric and magnetic fields will be of the following form:

\[ E = E_1 + E_3 + E_5 + \ldots \]  \hspace{1cm} (12)

\[ B = B_0 + B_2 + B_4 + \ldots \]  \hspace{1cm} (13)

From equations (12) and (13), the electric field series only contains odd-order terms and the magnetic field series contains only even-order terms.

---

Figure 1. Successive Approximation Procedure for Medium of Zero Conductivity
Without considering a specific geometrical configuration it is possible to gain some information about the time dependence of the various terms. From equations (6) through (11) and Figure 1, the proportionality relations, listed in Table I, are observed between the time derivatives of \( J_s \) and the different terms of the series expressions.

\[
\begin{align*}
B_0 & \propto J_s & E_1 & \propto \frac{\partial J_s}{\partial t} \\
B_2 & \propto \frac{\partial^2 J_s}{\partial x^2} & E_3 & \propto \frac{\partial^3 J_s}{\partial x^3} \\
B_4 & \propto \frac{\partial^4 J_s}{\partial x^4} & E_5 & \propto \frac{\partial^5 J_s}{\partial x^5}
\end{align*}
\]

Table I. Proportionality Relations for \( \sigma = 0 \).

Assume that \( J_s \) has a sinusoidal time variation, i.e., \( J_s \propto \cos \omega t \). Then, the proportionality relations of Table I can be rewritten as in Table II. Then, from equations (12) and (13), it can be concluded that:

\[
\begin{align*}
E & \propto \sin \omega t \\
B & \propto \cos \omega t
\end{align*}
\]

That is, the electric and magnetic fields are 90\(^\circ\) out of time phase, as expected.

\[
\begin{align*}
B_0 & \propto \cos \omega t & E_1 & \propto \omega \sin \omega t \\
B_2 & \propto \omega^2 \cos \omega t & E_3 & \propto \omega^3 \sin \omega t \\
B_4 & \propto \omega^4 \cos \omega t & E_5 & \propto \omega^5 \sin \omega t
\end{align*}
\]

Table II. Proportionality Relations for Sinusoidal Steady State for \( \sigma = 0 \).
A quasi-static field solution is defined as the sum of the zero-order and the first-order terms. It is expected to give sufficient accuracy when the wavelength associated with the highest frequency of time variations is very much larger than the dimensions of the system.

For the above problem, the quasi-static fields are:

\[ B \approx B_0 \]
\[ E \approx E_1 \]

Notice that the quasi-static magnetic field is independent of the electric field, and has the spatial distribution of a field produced by a static current distribution. The quasi-static fields are a valid description of the resultant field distribution when the frequency is low enough that the second-order terms, whose magnitudes vary as \( \omega^2 \), are negligible compared to the zero-order and first-order terms. The relative magnitudes of the zero-, first-, and second-order terms depend on the geometrical configuration under consideration; and so, the frequency at which the quasi-static fields alone no longer give a valid description of the resultant field distribution depends on geometry.

Let \( B_s \) represent the magnetic field produced in a medium of zero conductivity by the current distribution \( J_s \). \( B_s \) can be determined to the desired accuracy by including as many terms as necessary.

Now consider the case where a material having a non-zero conductivity, \( \sigma \), is positioned near the independent current distribution \( J_s \). The form of the expressions for the fields inside the conductive material is desired. Using the method of successive approximations, the zero-order fields must satisfy:
\[ \nabla \times E_0 = 0 \quad (14) \]
\[ \nabla \times \mu B_0 = \mathbf{J}_0 = \mathbf{J}_s \quad (14) \]
\[ \nabla \cdot \mathbf{J}_0 = \nabla \cdot \mathbf{J}_s = 0 \quad (14) \]

As in the previous case, the zero-order electric field is zero, but the current distribution causes a zero-order magnetic field which has essentially the static field distribution.

The first-order fields must satisfy:

\[ \nabla \times E_1 = -\frac{\partial B_0}{\partial t} \quad (15) \]
\[ \nabla \times \mu B_1 = \sigma E_1 \quad (15) \]
\[ \nabla \cdot \mathbf{J}_1 = 0 \quad (15) \]

The time-varying zero-order magnetic field \( B_0 \) produces a first-order electric field \( E_1 \). A first-order magnetic field \( B_1 \) is produced because the material has a non-zero conductivity, i.e., \( E_1 \) causes an eddy current distribution \( \mathbf{J}_1 \) which produces a magnetic field \( B_1 \).

The second-order fields must satisfy:

\[ \nabla \times E_2 = -\frac{\partial B_1}{\partial t} \quad (16) \]
\[ \nabla \times \mu B_2 = \sigma E_2^+ + \epsilon \frac{\partial E_1}{\partial t} \quad (16) \]
\[ \nabla \cdot \mathbf{J}_2 = 0 \quad (16) \]

A second-order electric field \( E_2 \) is produced by the time-varying first-order magnetic field. The second-order magnetic field \( B_2 \) can be expressed as the sum of two components:

\[ B_2 = B_{2,1} + B_{2,2} \]
\( B_{2,1} \) is due to the flow of currents in the conductive material caused by the second-order electric field \( E_2 \), whereas \( B_{2,2} \) is due to the time-varying first-order electric field.

The third-order fields must satisfy:

\[
\nabla \times E_3 = -\frac{\partial B_2}{\partial t} = -\frac{\partial B_{2,1}}{\partial t} - \frac{\partial B_{2,2}}{\partial t} \quad (17)
\]

\[
\nabla \times \mu B_3 = \sigma E_3 + \epsilon \frac{\partial E_2}{\partial t} \quad (17)
\]

\[
\nabla \cdot J_3 = 0 \quad (17)
\]

The third-order electric field can be expressed as the sum of two components:

\[ E_3 = E_{3,1} + E_{3,2} \]

\( E_{3,1} \) is produced by the time-varying \( B_{2,1} \) and \( E_{3,2} \) is produced by the time-varying \( B_{2,2} \). The third-order magnetic field \( B_3 \) can be expressed as the sum of three components:

\[ B_3 = B_{3,1} + B_{3,2} + B_{3,3} \]

\( B_{3,1} \) and \( B_{3,2} \) result from the eddy currents in the conductive material caused by the third-order electric field components, \( E_{3,1} \) and \( E_{3,2} \) respectively. \( B_{3,3} \) results from the time-varying second-order electric field \( E_2 \).

The fourth-order fields must satisfy:

\[
\nabla \times E_4 = -\frac{\partial B_3}{\partial t} = -\frac{\partial B_{3,1}}{\partial t} - \frac{\partial B_{3,2}}{\partial t} - \frac{\partial B_{3,3}}{\partial t} \quad (18)
\]

\[
\nabla \times \mu B_4 = \sigma E_4 + \epsilon \frac{\partial E_3}{\partial t} + \epsilon \frac{\partial E_{3,1}}{\partial t} + \epsilon \frac{\partial E_{3,2}}{\partial t} \quad (18)
\]

\[
\nabla \cdot J_4 = 0 \quad (18)
\]
The fourth-order electric field \( \mathbf{E}_4 \) can be expressed as the sum of three components,

\[
\mathbf{E}_4 = \mathbf{E}_{4,1} + \mathbf{E}_{4,2} + \mathbf{E}_{4,3}
\]

resulting from the three time-varying components of \( \mathbf{B}_3 \). The fourth-order magnetic field \( \mathbf{B}_4 \) can be expressed as the sum of five components,

\[
\mathbf{B}_4 = \mathbf{B}_{4,1} + \mathbf{B}_{4,2} + \mathbf{B}_{4,3} + \mathbf{B}_{4,4} + \mathbf{B}_{4,5}
\]

\( \mathbf{B}_{4,1}, \mathbf{B}_{4,2}, \) and \( \mathbf{B}_{4,3} \) result from the eddy currents caused by \( \mathbf{E}_{4,1}, \mathbf{E}_{4,2} \) and \( \mathbf{E}_{4,3} \), respectively. \( \mathbf{B}_{4,4} \) and \( \mathbf{B}_{4,5} \) result from the time-varying third-order electric field components \( \mathbf{E}_{3,1} \) and \( \mathbf{E}_{3,2} \).

Following the above procedure, the components of all the terms of the infinite series can be calculated. The above process of deriving these components is shown schematically in Figure 2. The resultant magnetic field in the conductive medium, including up to the fourth-order fields, can be expressed as follows:

\[
\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3 + \mathbf{B}_4 + \ldots
\]

\[
\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 + \mathbf{B}_{2,1} + \mathbf{B}_{2,2} + \mathbf{B}_{3,1} + \mathbf{B}_{3,2} + \mathbf{B}_{3,3} + \mathbf{B}_{4,1} + \mathbf{B}_{4,2} + \mathbf{B}_{4,3} + \mathbf{B}_{4,4} + \mathbf{B}_{4,5} + \ldots
\]

If the conductivity of the material is decreased to zero, \( \sigma = 0 \), the effect of the eddy currents on the resultant magnetic field should disappear. From the chart of Figure 2, the effect of making \( \sigma = 0 \) is easily seen. \( \sigma \mathbf{E}_1 \) will be zero, which means that all the field quantities derived from it will be zero. Similarly,
Figure 2. Successive Approximation Procedure
\[ \sigma E_{s,t} \] will be zero and so will all the components derived from it. The zero-order magnetic field will be \( B_0 \). There will be no first-order magnetic field. The second-order magnetic field will consist only of \( B_{2,t} \) which is produced by the time-varying first-order electric field. There will be no third-order magnetic field. The fourth-order magnetic field will consist only of \( B_{4,t} \) which is produced by the time-varying third-order electric field, etc. . . . As expected, with \( \sigma = 0 \), the chart of Figure 2 reduces to the chart of Figure 1, and the resultant magnetic field is identical to that expressed in equation (13).

From equation (19), equate to \( B_s \) the summation of all the terms remaining when \( \sigma \) is set equal to zero.

\[
B_s = B_0 + B_{2,2} + B_{4,s} + \ldots
\]

(20)

\( B_s \) represents the stimulating field; it is the magnetic field which causes eddy currents in the conductive material, and is present in the absence of the eddy currents. Also from equation (19), equate the summation of all the terms, which depend on a non-zero conductivity to \( B_{ec} \).

\[
B_{ec} = B_1 + B_{3,1} + B_{3,1} + B_{3,2} + B_{3,3} + B_{4,1} + B_{4,2} + B_{4,3} + B_{4,4} + \ldots
\]

(21)

\( B_{ec} \) is the magnetic field arising from the eddy currents in the conductive material.

The total resultant magnetic field can be expressed as the sum of these two magnetic fields,

\[
B = B_s + B_{ec}
\]

(22)
From equations (14) through (21) and Figure 2, it is possible to obtain some information about the time dependence of the terms comprising the series solution for the resultant magnetic field. The proportionality relations of Table III are observed to occur between the individual terms and the time derivatives of $J_s$.

\begin{align*}
B_0 & \propto J_s \\
B_{2,1} & \propto \frac{\partial^2 J_s}{\partial x^2} \\
B_{4,0} & \propto \frac{\partial^4 J_s}{\partial x^4}
\end{align*}

Table III. Proportionality Relations for $\sigma \neq 0$.

Assume that $J_s$ has a time dependence that is sinusoidal, i.e., $J_s \propto \cos \omega t$.

Then the proportionality relations of Table III can be expressed as in Table IV. The infinite series of terms comprising the stimulating magnetic field $B_s$ can be combined into the general expression

$$B_s = A(x, \omega) \cos \omega t$$
Similarly, the infinite series of terms comprising the eddy current magnetic field \( B_{ec} \) can be combined into the general expression

\[
B_{ec} = C(x, \omega) \cos \omega t + D(x, \omega) \sin \omega t
\]

\( A, C \) and \( D \) are functions of position and frequency. Therefore, for a sinusoidal steady state case, \( B_s \) and \( B_{ec} \) are not necessarily in time phase. They are in time phase only if \( D \) is zero, and are exactly 90° out of phase only if \( C \) is zero. A general expression for \( B_{ec} \) with its time phase referenced to \( B_s \) is

\[
B_{ec} = F(x, \omega) \cos (\omega t + \phi(x, \omega))
\]

where \( \phi(x, \omega) \) is the phase difference between \( B_s \) and \( B_{ec} \).

\[
\begin{array}{c|c}
B_s & B_{ec} \\
B_o & = \cos \omega t & B_1 = \omega \sin \omega t \\
B_{2,2} & = \omega^2 \cos \omega t & B_{2,1} = \omega^2 \cos \omega t \\
B_{4,5} & = \omega^4 \cos \omega t & B_{3,3} = \omega^3 \sin \omega t \\
\end{array}
\]

Table IV. Proportionality Relations for Sinusoidal Steady State for \( \sigma \neq 0 \).
Next, some special circumstances are investigated in which some definite statements can be made about the magnitude of $B_{ec}$ and $\Phi(x, \omega)$.

Consider the case in which the wavelength associated with the frequency of the sinusoidal time variation is very large compared to the size of the system. Assume that sufficient accuracy is obtained using the quasi-static fields. That is, the resultant magnetic field is given by the zero-order and first-order terms,

$$B \approx B_0 + B_1 \quad (23)$$

Equation (23) is equivalent to saying that the stimulating field is approximately

$$B_s \approx B_0 \quad (24)$$

and the eddy current field is approximately

$$B_{ec} \approx B_1 \quad (25)$$

Then, from Table IV,

$$B_s \approx B_0 \propto \cos \omega t \quad (26)$$
$$B_{ec} \approx B_1 \propto \omega \sin \omega t \quad (27)$$

Notice that $B_s$ and $B_{ec}$ are exactly $90^\circ$ out of phase. Using a measuring system which allows the selection and measurement of fields $90^\circ$ out of phase with $B_s$, the fields due to the eddy currents can be measured. Notice also that the magnitude of $B_{ec}$ is proportional to frequency to first power. This is important because the frequency dependence of the experimental values of $B_{ec}$ can be used to determine the ranges in which successive higher order terms become significant in the series.
solution for the magnetic field.

If the frequency of the sinusoidal variation is large enough it may be necessary to include the second-order terms in the solution for the magnetic field in order to achieve the desired accuracy. The second-order magnetic field is

$$ B \approx B_o + B_i + B_z $$ (28)

From equation (19),

$$ B \approx B_o + B_i + B_{z,1} + B_{z,2} $$ (29)

The stimulating magnetic field including the second-order terms is

$$ B_s \approx B_o + B_{z,2} $$

The eddy current magnetic field including the second-order terms is

$$ B_{ec} \approx B_i + B_{z,1} $$

Using the proportionality relations of Table IV, the following relations can be written

$$ B_s \approx b_o \cos \omega t + b_{z,2} \omega^2 \cos \omega t $$ (30)

$$ B_{ec} \approx b_i \omega \sin \omega t + b_{z,1} \omega^2 \cos \omega t $$ (31)

The coefficients $b_o, b_i, b_{z,1}$ and $b_{z,2}$ are dependent on the particular geometrical configuration being used.

Note that in the second-order expressions (30) and (31), the second-order contribution to $B_s$ is in phase with the zero-order contribution. The first-order
contribution to the eddy current magnetic field is 90° out of phase with $B_S$, and the second-order contribution is in phase with $B_S$. So the resultant $B_{ec}$ is not exactly 90° out of phase with $B_{ec}$ as it was in the quasi-static case. Note also that the frequency variation of the second-order contributions is with the second power of frequency.

Now consider the third-order solution. The resultant magnetic field can be expressed as

$$B \approx B_0 + B_1 + B_2 + B_3$$

Using equations (20) and (21) and Table IV, the stimulating magnetic field can be expressed as

$$B_S \approx B_0 + B_{3,2} \quad (32)$$

$$B_S \propto b_0 \cos \omega t + b_{2,2} \omega^2 \cos \omega t \quad (32)$$

The eddy current magnetic field can be expressed as

$$B_{ec} \approx B_1 + B_{2,1} + B_{3,1} + B_{3,2} + B_{3,3} \quad (33)$$

$$B_{ec} \propto b_1 \omega \sin \omega t + b_{2,1} \omega^2 \cos \omega t + b_{3,1} \omega^3 \sin \omega t$$

$$+ b_{3,2} \omega^3 \sin \omega t + b_{3,3} \omega^3 \sin \omega t \quad (33)$$

$$B_{ec} \propto \left(b_1 \omega + (b_{2,1} + b_{2,2} + b_{2,3}) \omega^2 \right) \sin \omega t + b_{2,1} \omega^2 \cos \omega t \quad (33)$$

From equation (33), it is observed that the eddy current magnetic field can be expressed as the sum of two components: one that is in phase with $B_S$; and one that is 90° out of phase with $B_S$. Again, the resultant $B_{ec}$ is not exactly 90° out of phase with $B_S$. Note the frequency dependence of the coefficients of the
two components of $B_{ec}$. The coefficient of the term $90^\circ$ out of phase with $B_s$ has a frequency dependence of the form $(a\omega + b\omega^3)$, and the coefficient of the component in phase with $B_s$ varies with the second power of frequency.

Summarizing, if there is no conductive material present, the resultant magnetic field is just $B_s$, the stimulating magnetic field. When a conductive material is present, the resultant magnetic field in the conductive material can be expressed as the sum of two fields, the stimulating field $B_s$, and the eddy current field $B_{ec}$.

$$B = B_s + B_{ec}$$

In the sinusoidal steady state case, $B_s$ and $B_{ec}$ will in general differ in time phase. For the frequency range in which the quasi-static solution gives sufficient accuracy, $B_{ec}$ will be exactly $90^\circ$ out of phase with $B_s$. For higher frequencies, where second- or third-order terms are required, $B_{ec}$ will not be exactly $90^\circ$ out of phase with $B_s$, although $B_{ec}$ can be expressed as the sum of two components, one in phase with $B_s$, and the other $90^\circ$ out of phase with $B_s$.

The existence of the eddy current magnetic field and its phase difference with respect to the stimulating magnetic field has been established. The following section discusses a phase discriminating detection system which will allow the measurement of these fields. However, to then determine the resistivity, it is necessary to know the dependence of $B_{ec}$ on the resistivity or conductivity.

From the chart of Figure 2, the dependence on conductivity of each component of the series for the first-, second-, and third-order terms of $B_{ec}$ can be deduced and is given in Table V.
Components of $B_{ec}$:

- **First-order**
  
  $B_1 \propto \sigma$

- **Second-order**
  
  $B_{21} \propto \sigma^2$

- **Third-order**
  
  $B_{31} \propto \sigma^3$
  
  $B_{32} \propto \sigma$
  
  $B_{33} \propto \sigma^3$

**Table V.** Dependence of $B_{ec}$ on Conductivity.
EXPERIMENTAL MEASUREMENT SYSTEM

The necessary elements of an experimental measurement system using the proposed contactless method are: 1) a source of the stimulating magnetic field $B_3$; 2) a detector of the magnetic fields; and 3) a phase discriminating system allowing the separation of the information about $B_3$ and the information about $B_{ec}$ which come from the detector.

The source of the time-varying field is a sinusoidal current flowing in a ten-turn coil, called the primary coil. The detector is a ten-turn coil, called the secondary coil. The signal received from the detector is a sinusoidal voltage which is dependent on the time-rate-of-change of the flux of the magnetic field over its cross-sectional area. The two coils are arranged coaxially. The sample of material is inserted into the region between the coils, i.e., "sandwiched" between them. These coils are called the measuring coils. The coaxial arrangement is used because of its axial symmetry, which is helpful in performing otherwise cumbersome calculations. The phase discrimination is accomplished by using a Princeton Applied Research Precision Lock-In Amplifier, Model HR-8, [18].

This lock-in amplifier has many capabilities which will not be fully utilized in this experiment. It is capable of selecting a band of frequencies from a signal spectrum applied to its input, and converting the information to an equivalent bandwidth about dc. An input signal is mixed with a synchronous reference signal in a phase-sensitive demodulator to produce sum and difference frequencies. The difference frequency produced by the component of the input signal at the reference signal frequency and synchronous with it is zero. A low-pass filter at the output of the
mixer rejects the high-frequency components corresponding to the sum frequencies and passes the difference frequencies which lie within its passband. The output of the low-pass filter is due to that portion of the signal spectrum which lies within a passband determined by the low-pass filter. The output becomes a dc level proportional to the fundamental component of the input signal.

Below is a discussion of the phase discrimination capabilities of the Model HR-8, followed by the application of these capabilities to the problem of distinguishing between the voltages produced in the secondary coil by \( B_s \) and \( B_{ec} \).

![Lock-In Amplifier Block Diagram](image-url)

**Figure 3.** Lock-In Amplifier Block Diagram
Figure 3 is a block diagram representation of the lock-in amplifier operating in the selective-external mode. In this mode of operation, the reference signal is derived from a source which is synchronized with the experiment, so that the phase difference between the signal being measured and the reference signal remains fixed during the measurement.

In order to demonstrate the phase discrimination capabilities, consider the input signal and the reference signal to be sinusoidal voltages of frequency \( \omega \). Assume the input signal is of amplitude \( \sqrt{2} A \), i.e., its rms value is A. Figure 4 illustrates the voltages at different processing stages. The input signal is amplified (K is the amplification constant), and then enters the mixer. The reference signal is amplified; the desired phase shifting is applied; and it is converted to a unit square wave before entering the mixer.

The mixing operation and its output are shown in Figure 4c. The mixer output is given in equation (34), where \( \theta \) represents the phase difference between the reference and signal inputs to the mixer. The Fourier series representing the mixer output consists of a dc term, and of sinusoidal and cosinusoidal terms of the fundamental frequency and integral multiples of the fundamental frequency. If the cutoff frequency of the low-pass filter is much lower than the fundamental frequency, then the output of the filter will be just the dc component, which is given by equation (35).
Figure 4. Different Processing Stages of Lock-In Amplifier
\[ V_{\text{filter}} = \left( \frac{1}{T} \right) \int_{0}^{T} V_{\text{mixer}}(t) \, dt \]  
(35)

\[ V_{\text{filter}} = \left( \frac{1}{2\pi} \right) \left\{ \int_{0}^{\pi} \sqrt{A} \sin(\omega t) \, d\omega + \int_{\pi}^{2\pi} \sqrt{A} \sin(\omega t + \theta) \, d\omega \right\} \]  
(35)

\[ V_{\text{filter}} = \left( \frac{2}{\pi} \right) \sqrt{A} \cos \theta \]  
(35)

The output of the low-pass filter is coupled through a dc amplifier to the panel meter. If the lock-in amplifier is calibrated properly, the panel meter reading will be

\[ V_{\text{out}} = A \cos \theta \]

That is, the output voltage is equal to the rms value of the input signal times the \( \cos \theta \).

The phase discrimination capabilities are evident. The output voltage reading is dependent on the phase difference between the reference and signal inputs to the mixer, i.e., \( \theta \). \( \theta \) is controlled by phase shifting the reference signal. When the reference and signal inputs to the mixer are synchronous, i.e., \( \theta = 0 \), then the output voltage reading equals the rms value of the input signal.

\[ V_{\text{out}} = A \]

But when the reference and the signal inputs to the mixer are \( 90^\circ \) out of phase, i.e., \( \theta = 90^\circ \), the output voltage reading is zero.

\[ V_{\text{out}} = 0 \]

Now, apply this phase discrimination technique to the problem of distinguishing between the stimulating field \( B_s \) and the eddy current field \( B_{ec} \). Suppose that the voltage produced by \( B_s \) in the detector coil is \( V_i(t) \) and that produced by \( B_{ec} \)
is $v_0(x)$. For the sinusoidal steady state case of frequency $\omega$, the input signal to the lock-in amplifier is

$$v_{\text{signal}} = \sqrt{2} v_1 \sin \omega x + \sqrt{2} v_2 \sin (\omega x + \phi)$$

$v_1$ and $v_2$ are out of phase by $\phi$, and their rms values are $V_1$ and $V_2$, respectively. See Figure 5. If the reference input to the mixer is adjusted so that it is out of phase by $\theta_1$ degrees with respect to $v_1$, it is then out of phase with respect to $v_2$ by $\theta_2$ degrees, where $\theta_2$ is given by

$$\theta_2 = \phi + \theta_1$$

The output reading is then

$$V_{\text{out}} = V_1 \cos \theta_1 + V_2 \cos \theta_2$$

$$V_{\text{out}} = V_1 \cos \theta_1 + V_2 \cos (\phi + \theta_1)$$

The terms of equation (36) are plotted in Figure 6. Note that if $\theta_1 = 90^\circ$, then $V_1 \cos \theta_1 = 0$. $V_{\text{out}}$ is then given by

$$V_{\text{out}} = V_2 \cos (\phi + 90^\circ) = -V_2 \sin \phi$$
All the information about $V_1$ is eliminated. The output contains only information concerning $V_2$. The rms value of $V_2$, $V_2$, is not measured directly unless $\phi$ is known to be 90°. If $\phi$ is known and is not 90°, $V_2$ can of course be calculated from the measured value of $V_2 \sin \phi$.

Summarizing, the phase discrimination capabilities of the Model HR-8 make it possible, if the phase of the reference input to the mixer is adjusted properly, to eliminate completely the component of the input signal which is due to $B_{5c}$, and still retain information concerning $B_{ec}$.

In light of the above description of the operation and use of the lock-in amplifier, the proposed measurement system must provide a means of insuring that:

1) $\theta_1$ can be set equal to 90°; and, 2) the ratio of $V_1/V_2$ is not large.

It is obvious that $\theta_1$ must be very close to 90° or a significant portion of $V_1$ will contribute to the output reading. How does one know when $\theta_1$ is 90°? One way of determining this is to vary the amplitude of $V_1$ by some means and simultaneously adjust the phase of the reference input to the mixer until variations in the amplitude of $V_1$ no longer affect the output reading.

If the ratio of $V_1/V_2$ is large, then a slight error in the adjustment of $\theta_1$ equal to 90° may cause error in the output reading of the order of $V_2$. This is
evident from Figure 6. The greater $V_1$ is than $V_2$, the more critical becomes the adjustment of $\theta_1$. A means of decreasing $V_1$ while not affecting $V_2$ is needed.

Both of the above needs can be met by the addition of another set of coils, called the nulling coils. The nulling coils consist of a primary coil and a secondary coil which are identical to the measuring coils. The primary coils of the measuring and nulling sets are connected in series, while the secondary coils are connected in series opposition. Both sets are positioned far enough apart that there is no appreciable interaction. The primary and secondary coils of the nulling set are mounted so that the spacing between these two is variable.

With no sample present between the measuring coils, the voltage induced in the secondary of the measuring coils is due solely to the stimulating magnetic field $B_s$; denote this voltage $V_{1a}$. The voltage induced in the secondary of the nulling coils is due to an identical stimulating field $B'_s$; denote this voltage $V_{1b}$. Neglecting the resistance and capacitance of the coil windings, and any such higher order effects, the voltage appearing at the terminals of the secondary series-opposition pair is

$$V_i = V_{1a} - V_{1b}$$

Note that $B_s$ and $B'_s$, produced by the same current, are in phase; and so, $V_{1a}$ and $V_{1b}$ are also in phase.

The field distribution of a ten-turn coil varies greatly with axial position. So if the spacing between the secondary and the primary of the nulling coils is varied, the magnitude of $V_{1b}$ is varied, thus, the magnitude of $V_i$ is varied. The magnitude of $V_i$ can theoretically be made zero by adjusting the spacing so that $V_{1b}$ is exactly equal to $V_{1a}$. This measurement system now has the capability of reducing $V_i$ to
an arbitrarily small value, making it comparable to $V_2$.

Since the magnitude of $V_i$ can be controlled without disturbing $B_s$ and the configuration of the measuring coils, it is possible to set $\theta_i$ equal to $90^\circ$. The spacing of the primary and secondary nulling coils is varied while the phase of the reference input to the mixer is adjusted until the variation of the magnitude of $V_i$ no longer has an effect on the output of the lock-in amplifier. If there is no noise signal present, then the output reading is zero.

This procedure for eliminating the signal due to $B_s$ is performed without the sample between the measuring coils. Insertion of the sample between the measuring coils results in an additional magnetic field produced by the eddy currents, which differs in phase with $B_s$ by $\phi$. This eddy current magnetic field produces a voltage $V_2$ in the secondary of the measuring coils. The output reading of the lock-in amplifier will be $-V_4 \sin \phi$, which is that component of $V_2$ which is $90^\circ$ out of phase with $V_i$. That is, the output reading will be proportional to the integral of the eddy current magnetic field over the cross-sectional area of the secondary coil.

Figure 7 shows a block diagram of the experimental system. The reference signal is derived from a resistor in series with the primary coils. The reference signal is then in phase with the primary current, which is in phase with the stimulating magnetic field. Thus the reference input signal always has a fixed phase with respect to $B_s$. 
Figure 7. Experimental System
ANALYSIS OF COAXIAL GEOMETRY

Presented in this section is an analysis of the coaxial geometry based on the quasi-static field solutions. This analysis is undertaken for two reasons: 1) to determine if an optimum geometrical configuration exists for which \( V_z \) is a maximum and for which \( V_z/V_L \) is a maximum; and 2) to develop the capability of calculating the value of \( V_z \) for a specific geometrical configuration, so that the resistivity can be determined from the experimental values of \( V_z \).

The model used in this analysis consists of a one-turn primary coil of radius \( a_0 \), a circular semiconductor slice of radius \( a_1 \) and thickness \( t \), and a one-turn secondary coil of radius \( a_2 \). The planes of the primary coil, the sample and the secondary coil are all perpendicular to a common axis. See Figure 8. \( z_1 \) is the distance from the plane of the primary coil to the center plane of the sample. \( z_2 \) is the distance from the center plane of the sample to the plane of the secondary coil. The quantities to be calculated in this analysis are the voltage due to the stimulating magnetic field, \( V_L \), and the voltage due to the eddy current magnetic field, \( V_z \), produced in the secondary coil.

![Coaxial Model](image-url)

Figure 8. Coaxial Model
The stimulating magnetic field $B_s$ is equal to $B_0$, and the eddy current magnetic field $B_{ec}$ is equal to $B_1$. It is necessary to know $B_0$ and $B_1$ at the plane of the secondary coil in order to calculate $V_1$ and $V_2$.

The zero-order magnetic field is produced by the primary current $\dot{I}_p$. $B_0$ is given by the following equation:

$$\nabla \times \mu B_0 = J_s$$

where $J_s$ represents the current density of a circular current loop. If the permeability of the sample equals that of free space, which is a reasonable approximation for semiconductor materials, then $B_0$ is continuous throughout the region surrounding the primary, including the region of the sample. Thus, $B_0$ is determined by calculating the fields of a circular loop of current $\dot{I}_p$. Cylindrical coordinates are used, with the $z$-axis coinciding with the axis of the coils. Because of the symmetry of a circular loop, $B_0$ has a radial component $B_{0r}$, an axial component $B_{0z}$, but no azimuthal component $B_{0\phi}$; and, $B_{0r}$ and $B_{0z}$ are independent of $\phi$.

A first-order electric field is produced by the time-varying zero-order magnetic field. $E_1$ is given by the following equation:

$$\nabla \times E_1 = -\frac{\partial B_0}{\partial t}$$

Expressing $\nabla \times E_1$ in cylindrical coordinates and equating components, gives the following:

$$\left(\frac{1}{r}\right) \frac{\partial E_{1z}}{\partial \phi} - \frac{\partial E_{1\phi}}{\partial z} = -\frac{\partial B_{0r}}{\partial t}$$

$$\frac{\partial E_{1r}}{\partial z} - \frac{\partial E_{1z}}{\partial r} = \frac{\partial B_{0\phi}}{\partial t}$$

$$\left(\frac{1}{r}\right) \frac{\partial (rE_{1\phi})}{\partial r} - \left(\frac{1}{r}\right) \frac{\partial E_{1r}}{\partial \phi} = \frac{\partial B_{0z}}{\partial t}$$
Because $B_0$, and all derivatives with respect to $\phi$ are zero, these equations reduce to the following:

$$\frac{\partial E_{\phi}}{\partial z} = -\frac{\partial B_{0r}}{\partial t}$$

(37)

$$\frac{\partial E_{ir}}{\partial z} - \frac{\partial E_{iz}}{\partial r} = 0$$

(37)

$$\frac{(1/r) \frac{\partial (r E_{\phi})}{\partial r}}{\partial r} = -\frac{\partial B_{0z}}{\partial t}$$

(37)

Notice in equation (37), that $E_{ir}$ and $E_{iz}$ are independent of $B_0$. They can be set equal to zero. Only one component of $E_i$ remains, and that is $E_{i\phi}$. $E_{i\phi}$ is tangent to all the surfaces of a circular sample. And since the tangential component of the electric field must be continuous across the boundary, $E_{i\phi}$, and, therefore, the first-order electric field is continuous at the sample edge.

Notice that if $B_0$ had been a uniform field, i.e., $B_0 = B_{0z}$ and independent of $z$, then

$$\frac{\partial E_{i\phi}}{\partial z} = 0$$

and $E_{i\phi}$ would not vary with $z$ position. The fact that $B_0$ is not a uniform field, but a diverging field, i.e., $B_{0r}$ is non-zero and both $B_{0r}$ and $B_{0z}$ are functions of $z$, means that $E_{i\phi}$ will be a function of $z$. How rapidly $E_{i\phi}$ varies with $z$ depends on the time derivative of $B_{0r}$ and the functional dependence of $B_{0r}$ and $B_{0z}$ on $z$.

However, if the sample is thin and the frequency low enough, $E_{i\phi}$ changes very little across the thickness of the sample. This analysis is confined to the cases where can be assumed to be constant across the thickness of the sample. This restricts this analysis to the frequency and resistivity ranges where the skin depth is much larger than the thickness of the sample.
The first-order electric field in all space surrounding the primary coil has only an azimuthal component. \( E_{i\phi} \) is given by equation (38).

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r E_{i\phi} \right) / \partial r = - \frac{\partial B_{o z}}{\partial t}
\]  

Equation (39) is an equivalent statement of (38).

\[
\left( \nabla \times E_{i} \right) \cdot \hat{z} = - \left( \frac{\partial B_{o}}{\partial t} \right) \cdot \hat{z}
\]  

By Stoke's theorem, the surface integral of (39) over a plane perpendicular to the \( z \)-axis can be expressed as

\[
\oint \left( \nabla \times E_{i} \right) \cdot d\mathbf{S} = - \frac{\partial}{\partial t} \oint (B_{o}) \cdot dS
\]

Performing the integration over a surface bounded by a circular path of radius \( r \), gives

\[
E_{i\phi} = - \frac{1}{r} \frac{\partial}{\partial t} \int_{0}^{r} B_{o z} r \, dr
\]

Thus, the first-order electric field, as a function of \( r \), can be determined by integrating the \( z \)-component of \( B_{o} \) over a circular cross-section of radius \( r \).

The sample has a non-zero conductivity; conduction currents result from the drift of electrons under the influence of the first-order electric fields. This volume current density \( \mathbf{J}_{1} \), is the eddy current distribution, and is given by

\[
\mathbf{J}_{1} = \nabla \times E_{i}
\]

Once \( E_{1} \) has been calculated throughout the sample, the eddy current density is
known. Note that $\mathbf{J}_1$ has only an azimuthal component $J_{1\phi}$.

This first-order current density produces a first-order magnetic field. Since there are no surface currents and the permeability of the semiconductor material is the same as that of free space, both the normal and tangential components of $\mathbf{B}_1$ will be continuous across the sample boundary. So the first-order magnetic field inside and outside the sample is given by

$$\nabla \times \mu_0 \mathbf{B}_1 = \mathbf{J}_1$$

The procedure used to calculate the first-order magnetic field at the plane of the detector involves approximating the current density distribution by a series of concentric rings of current. The approximate current for the ring of radius $r_n$ is determined by equation (40).

$$i_n = \pi \int_{r_n - \Delta r/2}^{r_n + \Delta r/2} J_{\phi} \, dr \quad (40)$$

The integral of equation (40) represents the area under the curve of $J_{\phi}(r)$ versus $r$ over the range of $r$ from $r_n - \Delta r/2$ to $r_n + \Delta r/2$. As shown in Figure 9, this area can be approximated by a rectangle of height $J_{\phi}(r_n)$ and width $\Delta r$. Then, the appropriate current of the ring of radius $r_n$ is

$$i_n = J_{\phi}(r_n) \cdot \Delta r \cdot \pi$$

Figure 9. Rectangular Approximation for Calculation of Current in the nth Ring.
The calculation of the fields of the eddy current distribution is replaced by the calculation of the fields of a series of equivalent current rings. And so, the first-order magnetic field $B_1$ at the plane of the secondary coil can be determined by the superposition of the fields of each equivalent current ring.

The voltage $V_1(t)$ induced in the secondary coil due to the stimulating field is easily calculated by performing the surface integral of $\frac{dB_s}{dt}$ over the circular cross-section of the secondary coil. Since the plane of the secondary is perpendicular to the z-axis, it is only the z-component of $B_s$ which contributes. Therefore, $V_1(t)$ is given by the following equation:

$$V_1(t) = -\frac{\delta}{\delta t} \int_{S_2} B_s \cdot \frac{\partial S_2}{\partial z} = -\frac{\delta}{\delta t} \int_0^{a_2} B_{0z} z \pi r \, dr$$

Similarly, the voltage $V_2(t)$ induced in the secondary coil due to the eddy current magnetic field can be calculated. $V_2(t)$ is given by the following equation:

$$V_2(t) = -\frac{\delta}{\delta t} \int_{S_2} B_{ec} \cdot \frac{\partial S_2}{\partial z} = -\frac{\delta}{\delta t} \int_0^{a_2} B_{1z} z \pi r \, dr$$

The procedure described above for calculating the voltages $V_1(t)$ and $V_2(t)$ is implemented into a computer program. Figure 10 shows this process in block diagram form. A copy of the program is included in Appendix I.

The computer program which was developed allows the study of the effect of the variation of the geometrical parameters: $a_0$, $a_1$, $a_2$, $z_1$, and $z_2$. (See Figure 8.) The results are displayed in terms of these quantities, normalized with respect to $a_0$, the radius of the primary. The normalized parameters are:

$$A1 = \frac{a_1}{a_0}$$
$$A2 = \frac{a_2}{a_0}$$
Figure 10. Block Diagram of Computer Program

- Calculation of $B_S$ from ring of current $i_x$
- Calculation of $E_i$ in region of sample
- Calculation of eddy current distribution $J_i$
- Calculation of equivalent current rings
- Calculation of fields of each equivalent current ring
- Superposition of fields to form $B_{eq}$
- Calculation of $V_2$
- Calculation of $V_1$
- Calculation of $V_2/V_1$
### Mathematical Expressions

\[ \frac{z_1}{\alpha_0} \]

\[ \frac{z_2}{\alpha_0} \]

The voltage \( v_z \) is represented by a normalized quantity \( V_2 \), where

\[ V_2 = V_2 \left[ \frac{8}{(\pi \omega \alpha_o \mu_o \zeta \sigma I_p)} \right] \]

\( V_z \) and \( I_p \) are the rms values of the eddy current induced voltage and the primary current, respectively. All other quantities have been previously defined.

The ratio of \( v_z \) to \( v_i \) is represented by the quantity \( V_21 \), which is equal to

\[ V_21 = \left( \frac{V_z}{V_i} \right) \left[ \frac{4}{(\pi \omega \mu_o \alpha_0 \zeta \sigma)} \right] \]

\( V_i \) is the rms value of voltage induced in the secondary by the stimulating field. The results of this analysis are summarized in Figures 11 through 16.

### Figures

Figure 11 consists of a series of plots of \( V_2 \), for constant \( \bar{z}_1 \) and \( \bar{z}_2 (\bar{z}_1 = \bar{z}_2 = 0.1) \), as \( A_1 \) and \( A_2 \) are varied. For small \( A_1 \), there are peaks in the \( V_2 \) versus \( A_2 \) curves; there are specific values of \( A_2 \) which maximize \( V_2 \). For larger \( A_1 \), the peaks in the \( V_2 \) versus \( A_2 \) curves become broader; the value of \( A_2 \) must be larger than a certain minimum value in order to maximize \( V_2 \), but then it is not extremely critical.

Figure 12 consists of a series of plots of \( V_2 \), for constant \( A_1 \) and \( \bar{z}_2 (A_1 = 2.0, \bar{z}_2 = 0.1) \), as \( A_2 \) and \( \bar{z}_1 \) are varied. As \( \bar{z}_1 \) is increased, \( V_2 \) decreases; the peaks in \( V_2 \) occur at larger values of \( A_2 \), and their heights are greatly reduced.

Figure 13 consists of a series of plots of \( V_2 \), for constant \( A_1 \) and \( \bar{z}_1 (A_1 = 2.0, \bar{z}_1 = 0.1) \), as \( A_2 \) and \( \bar{z}_2 \) are varied. In general, as \( \bar{z}_2 \) is increased, \( V_2 \) decreases;
the peaks in $V_2$ occur at larger values of $A_2$; the peaks are broadened; and their heights are greatly reduced.

Summarizing, the smaller $z_1$ and $z_2$ are, the larger $V_2$ becomes. For small $z_1$ and $z_2$, the values of $A_1$ and $A_2$ become critical. The larger $A_1$ is, the broader becomes the peak in $V_2$, and the less critical $A_2$ is; $A_2$ should be slightly smaller than $A_1$.

Figure 14 illustrates the dependence of $V_{21}$ on $A_1$ and $A_2$. For small $A_1$, the best $A_2$ for a maximum $V_{21}$ is a small value. However, as $A_1$ becomes larger, an additional peak in $V_{21}$ appears, which changes the value of $A_2$ which maximizes $V_{21}$, to a value slightly less than $A_1$.

Figure 15 illustrates the dependence of $V_{21}$ on $z_1$. As $z_1$ is increased, the peak in $V_{21}$ versus $A_2$ is flattened out; the value of $A_2$ becomes less critical, as long as it is less than $A_1$.

Figure 16 illustrates the dependence of $V_{21}$ on $z_2$. As $z_2$ is increased, the value of $A_2$ is no longer critical.

Summarizing, for large $z_1$ and $z_2$ the values of $A_2$ and $A_1$ are not extremely critical for there are no sharp maxima in $V_{21}$. However, for small $z_1$ and $z_2$, definite maxima and minima begin to occur in $V_{21}$, and the values of $A_1$ and $A_2$ become critical; $A_2$ should be approximately equal to $A_1$.

In conclusion, the geometrical arrangement of the coaxial system, which results in the largest eddy current induced voltage in the secondary coil, is that in which the primary and the secondary coils are as close as possible to the sample; and, that in which the radius of the sample is larger than the radius of the primary coil, and in which the radius of the secondary coil is slightly smaller than the radius.
of the sample. This geometrical configuration is also one in which the ratio of the eddy current induced voltage to the stimulating field voltage is close to its maximum value.
Figure 11. $V_2(A_2, A_1)$

$z_1 = 0.1$

$z_2 = 0.1$

Figure 11. $V_2$ as a Function of $A_1$ and $A_2$ for Constant $z_1$ and $z_2$. 
Figure 12. \( V_2(A_2, z_1) \)

\[ A_1 = 2.0 \]

\[ A_2 = 0.1 \]
Figure 13. $V_2(A_2, Z_2)$ for $A_1 = 2.0$ and $Z_1 = 0.1$.
Figure 14. $V_{21}$ as a Function of $A_1$ and $A_2$ for Constant $\bar{z}_1$ and $\bar{z}_2$. 
Figure 15. $V_{Z1}$ as a Function of $A_2$ and $Z_1$ for Constant $A_1$ and $A_2$. 

$V_{Z1}(A_2, Z_1)$

$A_1 = 2.0$

$Z_2 = 0.1$
Figure 16. $V_2 I(A_2, Z_2)$ for constant $A_1$ and $Z_1$. 

$A_1 = 2.0$

$Z_1 = 0.1$
The experimental procedure involved in measuring the eddy current voltage is briefly outlined here:

1. Select frequency of operation.
2. Tune reference channel of lock-in amplifier.
3. Calibrate signal channel of lock-in amplifier.
4. Connect secondary series opposition voltage to signal channel input.
5. Select primary current.
6. Adjust reference signal to appropriate level.
7. Position measuring coils for desired \( z_1 \) and \( z_2 \).
8. Position nulling coils so spacing is approximately \( z_1 + z_2 \).
9. Select desired sensitivity scale and time constant setting.
10. Adjust phase of reference signal until variation in spacing of nulling coils causes no variation in panel meter reading.
11. Set offset voltage to null panel meter to zero.
12. Insert sample and take reading.

The value of the voltage measured using the above procedure is denoted \( V_{ze} \); and as shown in the discussion of the "Experimental System,"

\[
V_{ze} = V_{out} = -V_z \sin \phi
\]

Recall that \( \phi \) is the phase difference between the voltage produced by the stimulating magnetic field and the voltage produced by the eddy current magnetic field. Only if
The normalized secondary radius is

\[ A_2 = \frac{a_2}{a_o} = 3.63 \]

<table>
<thead>
<tr>
<th>Primary (inches)</th>
<th>Secondary (inches)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner radius</td>
<td>.125</td>
</tr>
<tr>
<td>Outer radius</td>
<td>.155</td>
</tr>
<tr>
<td>Mean radius</td>
<td>.139</td>
</tr>
<tr>
<td></td>
<td>.485</td>
</tr>
<tr>
<td></td>
<td>.525</td>
</tr>
<tr>
<td></td>
<td>.505</td>
</tr>
</tbody>
</table>

Table VI. Radii of Measuring Coils

The samples of semiconductor materials for which data are reported in this section have various radii, thicknesses, and resistivities. See Table VII. The resistivities listed
are those determined from four-point probe measurements. Samples No. 5, No. 9, and No. 11 are circular slices of silicon. Sample No. 12 is a rather thick slice of germanium, and is not of circular cross-section because of the nature of the zone refining process used in producing doped germanium ingots. Its shape is similar to a slice of bread. However, this "slice of bread shape" has been approximated by a circular cross-section with the radius listed in Table VII for sample No. 12.

<table>
<thead>
<tr>
<th>Sample</th>
<th>Radius (inches)</th>
<th>Thickness (mils)</th>
<th>Resistivity (ohm-cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. 5</td>
<td>.738</td>
<td>8.1</td>
<td>0.198</td>
</tr>
<tr>
<td>No. 9</td>
<td>.752</td>
<td>8.0</td>
<td>8.3</td>
</tr>
<tr>
<td>No. 11</td>
<td>.5</td>
<td>10.8</td>
<td>11.13</td>
</tr>
<tr>
<td>No. 12</td>
<td>.525</td>
<td>122.5</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Table VII: Sample Parameters

The results presented below fall into four categories: 1) the initial observation of an eddy current induced voltage; 2) the dependence of $V_{ze}$ on frequency; 3) the comparison of the results from the experiment and the quasi-static computer solution; and, 4) the measurement of $V_{ze}$ from samples of different resistivity.

**Initial Observations**

Figure 17 is a plot of $V_{ze}$ as a function of the rms value of the primary current for samples No. 5 and No. 12. The frequency of operation was 10 kc/s. These curves are linear with $I_p$, as expected. The slope of the curve for Sample No. 12 is a factor of 10.5 larger than that for sample No. 5. This is due to the fact that No. 12 has a lower resistivity and a much greater volume than No. 5. Their volumes are .106 in$^3$ and .0014 in$^3$, respectively. Therefore, the amount of eddy currents in
Figure 17. $V_{ze}$ versus $I_p$ for Samples No. 12 and No. 5

<table>
<thead>
<tr>
<th></th>
<th>No. 12</th>
<th>No. 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_1$</td>
<td>2.16</td>
<td>1.75</td>
</tr>
<tr>
<td>$z_2$</td>
<td>2.48</td>
<td>2.06</td>
</tr>
<tr>
<td>$A_1$</td>
<td>3.78</td>
<td>5.32</td>
</tr>
<tr>
<td>slope</td>
<td>2.93 $\text{nv/ma}$</td>
<td>0.28 $\text{nv/ma}$</td>
</tr>
</tbody>
</table>
sample No. 12 is much larger; the magnetic fields produced by them much stronger; and so, a larger value of \( V_{Ze} \) is induced for a given \( I_p \). These two curves display the properties expected.

**Frequency Dependence**

Because of the difficulty of solving the field equations exactly, the relative magnitudes of the different-order terms in the solution for the eddy current magnetic field are not known. However, the general form of the frequency dependence of each term is known. Observing the frequency response experimentally indicates the frequency ranges in which additional higher-order terms become significant.

The eddy current induced voltage produced by sample No. 5 was measured over the frequency range from 1 kc/s to 10 kc/s. Measurements were limited to this frequency range because: 1) the magnitude of the eddy current induced voltage at 1 kc/s approaches the limit of the sensitivity of the lock-in amplifier, approximately 0.2 nanovolts; and 2) the upper limit of the frequency range of the signal channel preamplifier is 10 kc/s.

The data is summarized in Table XII in Appendix II. Each value entered in the table is the average of three consecutive readings. Between each reading, the sample is removed from its position between the coils; the phase setting and panel meter zero are rechecked; and the sample is reinserted.

Figure 18 consists of a plot of \( \log V_{Ze} \) versus \( \log f \) for three different values of primary current, i.e., \( I_p = 150 \) m. a., 300 m. a., and 600 m. a. Since \( V_{Ze} \) should be linearly related to the primary current, and the three curves are for currents which differ by a factor of 2, their separation should be \( \log 2 \), which is the case.
Figure 18. $V_{2e}$ versus Frequency for Primary Current Equal to 150 m. a., 300 m. a., and 600 m. a.
Figure 19 consists of three similar curves for three other values of primary current which differ by a factor of 2, i.e., \( I_p = 200 \text{ m. a.}, 400 \text{ m. a.}, \) and \( 800 \text{ m. a.} \).

It is obvious that the set of curves for all the different values of primary current generate a family of curves with identical shape. Table VIII lists the initial slope, i.e., the slope at 1 kc/s, and the final slope, i.e., the slope at 10 kc/s, for the six curves plotted. The average initial slope is 2.05 and the average final slope is 3.95. If these slopes are taken as asymptotic values of the slope, then at low frequencies \( V_{ze} \) is approximately proportional to \( f^2 \) and at high frequencies \( V_{ze} \) is proportional to \( f^4 \). The break frequency defined by the intersection of the asymptotes is approximately 6 kc/s.

<table>
<thead>
<tr>
<th>Primary Current (m. a.)</th>
<th>Initial Slope</th>
<th>Final Slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>150</td>
<td>2.06</td>
<td>3.87</td>
</tr>
<tr>
<td>200</td>
<td>2.14</td>
<td>3.86</td>
</tr>
<tr>
<td>300</td>
<td>2.05</td>
<td>3.84</td>
</tr>
<tr>
<td>400</td>
<td>2.02</td>
<td>4.06</td>
</tr>
<tr>
<td>600</td>
<td>2.06</td>
<td>3.91</td>
</tr>
<tr>
<td>800</td>
<td>1.98</td>
<td>4.17</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td><strong>2.05</strong></td>
<td><strong>3.95</strong></td>
</tr>
</tbody>
</table>

Table VIII. Initial and Final Slopes of \( \log V_{ze} \) versus \( \log f \) Curves.

Since \( V_{ze} \) is proportional to the time derivative of the component of the eddy current magnetic field which is \( 90^\circ \) out of phase with the stimulating magnetic field, this component of the magnetic field must be proportional to \( f \) at low frequencies and \( f^3 \) at high frequencies. This change in frequency dependence indicates that the next higher-order term of the eddy current magnetic field expression has become
Table 19. $V_{2e}$ versus Frequency for Primary Current Equal to 200 m. a., 400 m. a., and 800 m. a.
significant. [See Table IV, p. 30, and equation (33), p. 33.] What this means physically is that as the frequency is increased: 1) the time-varying eddy current magnetic fields themselves induce eddy currents, which in turn produce magnetic fields; 2) the time-varying eddy current magnetic field produces a time-varying electric field which alters its field distribution; and 3) the field distribution of the primary coil is no longer just the field distribution of a dc current loop, but is altered by the coupling of the time-varying electric and magnetic fields. These three occurrences give rise to $B_{3,1}$, $B_{3,2}$ and $B_{3,3}$ of equation (33), p. 33. $B_{3,1}$ depends on $\sigma^3$, but both $B_{3,2}$ and $B_{3,3}$ depend on $\sigma$.

The resulting behavior of $V_{z,e}$ with frequency appears very advantageous. Since $V_{z,e}$ decreases with increasing resistivity, there is, for a given frequency, a limit to the highest resistivity sample from which a value of $V_{z,e}$ can be measured. However, if the frequency of operation is in the range where $V_{z,e}$ is proportional to $f^2$, increasing the frequency by a factor of 10 increases $V_{z,e}$ by a factor of 100. And similarly, if in the range where $V_{z,e}$ is proportional to $f^4$, increasing the frequency by a factor of 10 increases $V_{z,e}$ by a factor of 10,000. This appears very promising.

It is desirable for calibration reasons that $V_{z,e}$ is proportional to $\sigma$ over any useable frequency range. Certainly in the range where $V_{z,e}$ is proportional to $f^2$, $V_{z,e}$ is linearly dependent on $\sigma$. In the range where $V_{z,e}$ is proportional to $f^4$, the term $B_{3,1}$ is proportional to $\sigma^3$, and all other terms are proportional to $\sigma$. Thus, $V_{z,e}$ will be linearly dependent on $\sigma$ in this frequency range only if the contribution of this term is negligible compared to $B_{3,2}$ and $B_{3,3}$. The condition, which forces the use of higher frequencies in performing measurements, is high
resistivity, i.e., low conductivity so that \( \sigma < 1 \) and \( \sigma^2 \ll \sigma \). And so it appears very probable that by the appropriate choices of compatible frequency and resistivity ranges, \( V_{2e} \) will be proportional to \( \sigma \).

**Comparison of Quasi-Static Theory and Experiment**

The model implemented in the computer program for calculating \( V_z \) is based on the quasi-static field solutions, and predicts \( V_z \) varies as \( f^2 \). Therefore, this model should only give accurate results in the low frequency range.

Comparison of the results of the computer solution to the experimental results is most easily done by comparison of the quantity \( V_2 \). Recall that

\[
V_2 = V_z \left[ \frac{8}{\pi r^2} \frac{\alpha^2}{\omega^2} \right] (41)
\]

The experimental results to be compared are those for sample No. 5 listed in Table XII of Appendix II. Figure 20 and Figure 21 show \( V_{2e} \) plotted against \( I_p \) for the various frequencies. The slope of the curve for each frequency gives an average of the ratio of \( V_{2e} / I_p \) for that frequency, and is listed in Table IX. Substituting the thickness of sample No. 5, the primary radius and the permeability of air, equation (41) reduces to

\[
V_2 = \left( V_z / I_p f^2 \right) (3.7 \times 10^4)
\]

The value of \( V_2 \) from the experimental data for each frequency is labeled \( V_{2e} \) in Table IX.

\( V_2 \), predicted by the computer solution, is plotted as a function of \( A2 \) in Figure 22. The value of \( V_2 \) corresponding to \( A2 = 3.6 \), which is the normalized
Figure 20. $V_{ze}$ versus $I_p$ for Various Frequencies from Sample No. 5
Figure 21. $V_{2e}$ versus $I_p$ for Various Frequencies for Sample No. 5
Figure 22. Computer Prediction of $V_2$ as a Function of $A_2$ for Sample No. 5.

$Z_1 = 1.8$
$Z_2 = 2.0$
$A_1 = 5.3$
<table>
<thead>
<tr>
<th>Frequency (kc/s)</th>
<th>$\frac{V_{2e}}{I_p}$ (volts/amp)</th>
<th>$V_{2e}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1.01 \times 10^{-9}$</td>
<td>.324</td>
</tr>
<tr>
<td>2</td>
<td>$3.97 \times 10^{-9}$</td>
<td>.318</td>
</tr>
<tr>
<td>3</td>
<td>$9.44 \times 10^{-9}$</td>
<td>.336</td>
</tr>
<tr>
<td>4</td>
<td>$17.35 \times 10^{-9}$</td>
<td>.347</td>
</tr>
<tr>
<td>5</td>
<td>$29.4 \times 10^{-9}$</td>
<td>.376</td>
</tr>
<tr>
<td>6</td>
<td>$47.6 \times 10^{-9}$</td>
<td>.423</td>
</tr>
<tr>
<td>7</td>
<td>$75.6 \times 10^{-9}$</td>
<td>.490</td>
</tr>
<tr>
<td>8</td>
<td>$121.0 \times 10^{-9}$</td>
<td>.610</td>
</tr>
<tr>
<td>9</td>
<td>$195.7 \times 10^{-9}$</td>
<td>.772</td>
</tr>
<tr>
<td>10</td>
<td>$283.6 \times 10^{-9}$</td>
<td>.906</td>
</tr>
</tbody>
</table>

Table IX. Experimental Results for $\frac{V_{2e}}{I_p}$ and $V_{2e}$ for Sample No. 5

secondary radius used for taking the data on sample No. 5, is

$$V_2 = 0.345$$

Figure 23 shows $V_2$ and $V_{2e}$, the theoretical and experimental values of the normalized voltage, plotted against frequency. $V_{2e}$ is within 7.8% of $V_2$ for frequencies equal to or less than 4 kc/s. Obviously, higher order effects become important at higher frequencies, and so the experimental results deviate from that predicted by the quasi-static model.

Using the experimentally measured values of $V_{2e}$, at frequencies equal to or less than 4 kc/s and the theoretical value of $V_2$, the resistivity of sample No. 5 can be predicted. These values are listed in Table X. Four point probe measurements in the center of the sample yielded a resistivity of .198 ohm-cm. Experimental values are within 8.6% of this value.

There are many possible sources of error: 1) the measurement of the geometrical parameters $Z_1$, $Z_2$, $A_1$ and $A_2$; 2) the actual coil windings occupying a finite
Table 23. Comparison of the Values of $V_2$, Measured Experimentally ($V_{2e}$), and Those Predicted from the Quasi-Static Computer Solution.
Table X. Experimentally Predicted Values of Resistivity.

Volume, and the computer model replacing them by an infinitesimally thin loop of current of single radius; 3) the effective radius of sample No. 5 being slightly reduced from that given in Table VII because of the flat indexing edge; 4) the possibility of the sample being positioned slightly off axis; 5) the computer solution approximation of the eddy current density by equivalent current rings; and 6) variations in resistivity throughout the sample.

Different Resistivities

Table XI lists the values of $V_{2e}$ measured for the three samples No. 5, No. 9 and No. 11 at a frequency of 10 kc/s for a primary current of 500 m. a. The spacings between the primary coil and sample, and sample and secondary were held constant,

<table>
<thead>
<tr>
<th>Sample</th>
<th>$V_{2e}$ (nanovolts)</th>
<th>AI</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. 5</td>
<td>530</td>
<td>5.04</td>
</tr>
<tr>
<td>No. 9</td>
<td>27.5</td>
<td>5.14</td>
</tr>
<tr>
<td>No. 11</td>
<td>23.0</td>
<td>3.41</td>
</tr>
</tbody>
</table>

Table XI. Eddy Current Induced Voltage for Three Samples of Different Resistivities
i.e., \( \frac{\pi}{2} = .37, \frac{\pi}{2} = .42 \).

Samples No. 5 and No. 9 are approximately of the same diameter and thickness, and so the decrease in the value of \( V_{z\epsilon} \) for No. 9 is due mainly to the increase in resistivity. The difference in the values of \( V_{z\epsilon} \) for No. 5 and No. 9 and the value for No. 11 is due partially to an increase in resistivity and partially to the change in geometry. No. 11 has a smaller diameter, but is about 25% thicker than No. 5 or No. 9. The significant fact here is that an eddy current induced voltage has been easily measured from thin slices with resistivities in the range from 0.2 ohm-cm to 11 ohm-cm.
CONCLUSIONS

Resistivity is used to characterize semiconductor materials, and therefore measurement techniques for determining it play a very important role in the production of semiconductor devices. The need for a contactless method exists.

A contactless method, which interacts with the sample by means of magnetic fields, has been studied. A time-varying magnetic field stimulates eddy currents in the sample, which depend on resistivity. The eddy currents in turn produce a magnetic field, which can be detected.

The successive approximation method of solving Maxwell's equations explains the separation of the resultant magnetic field into the stimulating magnetic field and the eddy current magnetic field. The eddy current magnetic field has components in phase and 90° out of phase with the stimulating magnetic field.

An experimental system has been developed which enables the detection of information due to the eddy current magnetic field while ignoring the stimulating magnetic field. It uses a phase-discriminating network to measure the voltage induced in a detection coil which is due to the component of the eddy current magnetic field that is 90° out of phase with the stimulating magnetic field.

The eddy current induced voltage was successfully measured. The frequency dependence of this voltage was $f^2$ at low frequencies and $f^4$ at higher frequencies. This dependence agrees with the successive approximation theory.

A computer solution based on the quasi-static fields predicted an optimum coaxial geometry for maximizing the eddy current induced voltage. Agreement was obtained between the computer solution values of $V_2$ and the experimental values
of $V_{ze}$, in the appropriate frequency range. Thus, the capability of absolute
determination of the resistivity has been developed, i.e., empirical calibration is
not necessary in this frequency range.

Eddy current induced voltages have been easily measured at 10 kc/s for
samples of resistivity up to 11 ohm-cm. The possibility of extending this method to
higher resistivities seems very promising because of the $f^q$ dependence of the eddy
current induced voltage.

The main factors restricting the use of this technique are the frequency and
sensitivity ranges of the phase sensitive network. Typical specifications for currently
available equipment list a sensitivity range from 1 nanovolt to 5 millivolts, and a
frequency range from 10 c/s to 150 kc/s. Based on these specifications and the
frequency response data for sample No. 5 (Table XII, Appendix II), the range of
resistivity producing a measureable $V_{ze}$ can be predicted.

For the high resistivity limit, the minimum value of $V_{ze}$ that can be measured
is approximately 1 nanovolt. Using the value of $V_{ze}$ for a primary current of 500
m. a. at 10 kc/s, and the fact that $V_{ze} \propto \sigma f^q$ in this frequency range, the highest
resistivity of a sample for which a 1 nanovolt $V_{ze}$ will result at a frequency of 150
kc/s is approximately

$$\rho_{\text{max}} = 1.44 \times 10^6 \text{ ohm-cm}$$

For the low resistivity limit, the maximum value of $V_{ze}$ is 5 millivolts.
Using the value of $V_{ze}$ for a primary current of 500 m. a. at 1 kc/s, and the fact
that $V_{ze} \propto \sigma f^k$ in this resistivity and frequency range, the lowest resistivity of a
sample for which $V_{ze}$ equals 5 millivolts is approximately
\[ \rho = 0.2 \times 10^{-7} \text{ ohm-cm} \]

However, this is really an artificial restriction on the lower limit of resistivity, because \( V_{2e} \) can always be attenuated to the 5 millivolt level if it is too large. This can be done by decreasing the primary current, by increasing the spacing of the coils, and/or by decreasing the frequency. Decreasing the frequency from 1 kc/s to 10 c/s decreases \( V_{2e} \) by a factor of \( 10^{-4} \). The only real lower limit on the resistivity arises from the skin depth. For the skin depth to be of the order of the thickness of sample No. 5, i.e., 10 mils, at a frequency of 10 c/s, the resistivity would have to be

\[ \rho_{\text{min}} = 2.54 \times 10^{-10} \text{ ohm-cm} \]

So the projected usable resistivity range of this technique is from \( 10^{-10} \) ohm-cm to \( 10^{+6} \) ohm-cm. Note the resistivity of silver is \( 1.62 \times 10^{-6} \) ohm-cm, and the resistivity of intrinsic silicon at room temperature is \( 2.25 \times 10^{5} \) ohm-cm. This contactless method can, therefore, be used to measure resistivities of materials from metals to semiconductors.

An important fact to remember concerning this method is that the eddy current induced voltage measured at the secondary gives no indication of the spatial variation of the resistivity in the sample. From this voltage, an average value of the resistivity can be deduced. Whether this is an advantage or disadvantage depends on the particular application.

The eddy current induced voltage is very geometry-dependent. Only if the geometrical parameters are the same for different samples can any information about the resistivity be deduced from the values of the eddy current induced voltages.
Calculations for absolute measurement of the resistivity can be performed for a coaxial system. However, there are many parameters which must be controlled when implementing such a system: 1) coaxial alignment of the sample and coils in parallel planes; 2) the spacings of the coils; 3) thickness and diameter of the sample; and 4) the diameters of the coils.

One advantage of this contactless method is that the size of the sample is not limited to a minimum size because of contact area, e.g., in the four-point probe method the dimensions of the sample must be larger than the spacing of the current probes. Also boundary effects do not become predominant as the sample size is reduced. This method has been shown to be feasible for measuring the resistivity of thin slices. Another advantage of the contactless property is that this method can be used on a material that has surface characteristics which make pressure ohmic contacts difficult, e.g., gallium arsenide.

A contactless method of measuring resistivity would appear to be useful in areas other than semiconductor technology. Use in determining the resistivity of liquids or amorphous materials seems very possible, which could be of value to the fields of bioengineering and chemical engineering.
APPENDIX I

Computer Program for Calculating $V_1$, $V_2$ and $V_2$. 
VP USE INCREMENT SUE IS ONE TENTH USE WHENEVER A? IS larger THAN 0.5
KK MUST ALWAYS BE ODD
LL MUST ALWAYS BE ODD
GSF TAKES NO ACTION WHEN NDIM IS LESS THAN 3
KK > 3
LL > 3
LLL > 3

*START OF 5

DIMENSION BC(61,61),
2ARG(61,61),CDI(61,61),CD(61,61),
3BN(61),SQ(61),BU(61,2),BCC(61,61),BS(61,61),
4BCR(61,61),BSK(61,61),V1(40,40),V2(40,40)

START OF 5

BC PACKAGE

DO 100 K=1,61
READ 102,(BC(I,K),I=1,61)
102 FORMAT (5E15.5)
100 CONTINUE

C MASTER DO LOOP S
Z1MAX=2
Z1MIN=2
Z1STP=2

DO 1000 II=Z1MIN,Z1MAX,Z1STP
Z1=(II-1)*0.1
Z2MIN=2
Z2MAX=2
Z2STP=2

DO 2000 JJ=Z2MIN,Z2MAX,Z2STP
A1MIN=II
A1MAX=II
A1STP=2

DO 3000 KK=A1MIN,A1MAX,A1STP
KKK=KK/2.0+1.0
A2MIN=61
A2MAX=61
A2STP=2

DO 4000 LL=A2MIN,A2MAX,A2STP
LLL=LL/2.0+1.0
PRINT 4001,KKK,LLL
4001 FORMAT(2X,"KKK = "13,5X,"LLL = "13)

DD 200 I=1,61
XN=(I-1)*0.05
ARG(I,II)=XN*BC(I,II)
200 CONTINUE

CALL GSF(0.05,ARG(1,1),CDI(1,1),KK)
T=TIME(2)/50
PRINT 506,T
506 FORMAT(" PROCESSOR TIME USED IN SECONDS ="F7.2)
1
PRINT 1
1 FORMAT(1H1,2X,"Z1",6X,"XN",11X,"ARG",14X,"CDI",14X,"CD")
DD 200 L=2,KKK
I=2*L-1
XN=(L-1)*0.1
CD(L,II)=-CDI(I,II)/XN
PRINT ?,?1,AN,ARU(,II),CDI(II),CU(II),I1
FORMAT(,6,3,2X,F6,3,2X,E15.5,2X,E15.5,2X,E15.5,2X,E15.5,5X,13)
CONTINUE

C

1

PI=3.141593
B=TAN(P1/8.0)
Z2=(JJ-1)0.1
DO 300 L=2*KK
ZN=Z2/0.1*(L-1)
BN(L)=Z0.0/((Sqrt(1,0+7N*ZN)**3,0)
DO 20 J=1,LLL
XN=FLOA1((1-1)/(L-1)
DO 30 J=1,5
YN=(J-1)*XN
IF (1-L)22*2122
21 IF (JJ-1)23*2023
23 B1=0
GU TO 24
22 A1=1.0-XN
IF (A1)41,44,41
41 C1=COS(ATAN(ZN/A1))
IF (C1)45,44,45
45 D1=A1/C1
IF (D1)46,44,46
44 B1=0
GU TO 24
48 B1=(C1/1.0*D1)*(SIN(CATAN((1.0*YN)/(ABS(D1))))+SINCATAN((YN+1.0)/(ABS(D1))))
1ABS(D1)))
24 A2=1.0*YN
IF (A2)67,66,67
67 C2=COS(ATAN(ZN/A2))
IF (C2)68,66,68
68 D2=A2/C2
IF (D2)69,66,69
66 B2=0.0
GU TO 80
89 B2=(C2/1.0*D2)*(SIN(CATAN((1.0*XN)/(ABS(D2))))+SINCATAN((1.0-XN)/(1ABS(D2)))))
1ABS(D2)))
80 A3=1.0*XN
C3=COS(ATAN(ZN/A3))
D3=A3/C3
B3=(C3/(PI*D3))*(SIN(CATAN((YN+1.0)/(ABS(D3))))+SINCATAN((YN+1.0)/(1ABS(D3))))
A4=1.0*YN
C4=COS(ATAN(ZN/A4))
D4=A4/C4
B4=(C4/(PI*D4))*(SIN(CATAN((1.0*XN)/(ABS(D4))))+SINCATAN((1.0+XN)/(1ABS(D4)))))
1ABS(D4)))
81 BZ=B1+B2+B3+B4
80 (1,1)=32
IF (1-1)52*51*52
51 IF (JJ=1)53*53*52
53 R5=BN(L)/B0(1,1)
52 B0(1,J)=3W(1,J)*RS
30 CONTINUE
20 CONTINUE
90 I=1,LLL
XN=FLOA1((1-1)/(L-1)
DO 91 J=1,5
YN=(J-1)*B*XN
IF (1-I)26*27*26
IF (JJ-1)²*J*28
R1=0.0
GU TO 24
A1=1.0-XN
IF (A1)¹2,11,12
C1=COS(atan(ZN/A1))
IF (C1)¹5,11,13
D1=A1/C1
IF (D1)¹11,14
B1=0.0
GU TO 29
B1=(C1/(PI*U1)) *(sin(atan((B-YN)/(abs(D1))))+sin(atan((Y+B)/(abs(D1)))))
A2=1.0-YN
IF (A2)¹2,11,12
C2=cos.atan(ZN/A2))
IF (C2)¹5,11,13
D2=A2/C2
IF (D2)¹64,11,64
B2=0.0
GU TO 65
B2=(C2/(PI*D2)) *(sin(atan((B+XN)/(abs(D2))))+sin(atan((B-XN)/(abs(D2)))))
A3=1.0+XN
C3=COS(atan(ZN/A3))
D3=A3/C3
B3=(C3/(PI*D3)) *(sin(atan((Y+B)/(abs(D3))))+sin(atan((Y-B)/(abs(D3)))))
A4=1.0+YN
C4=COS(atan(ZN/A4))
D4=A4/C4
B4=(C4/(PI*D4)) *(sin(atan((B+XN)/(abs(D4))))+sin(atan((B-XN)/(abs(D4)))))
A5=1.41421-YN-XN/1.41421
IF (A5)¹72,71,72
C5=COS(atan(ZN/A5))
IF (C5)¹73,71,73
D5=A5/C5
IF (D5)¹74,71,74
B5=0.0
GO TO 75
H5=0.5*(XN+YN+1.41421)
G5=H5+YN-XN
E5=SQRT((H5-1.0)*(H5-1.0)+(G5-B)*(G5-B))
F5=SQRT((H5-B)*(H5-B)+(G5-1.0)*(G5-1.0))
S5=F5/SORT((XN-B)*(XN-B)+(YN-1.0)*(YN-1.0)+(ZN*ZN)))
S5=E5/SORT((XN-1.0)*(XN-1.0)+(YN-B)*(YN-B)+(ZN*ZN)))
IF (H5-1.0)¹35,35,34
S5=SF5
S5=(C5/(PI*U5))*(SF5+S5)
A6=(XN+1.41421-YN)/1.41421
C6=COS(atan(ZN/A6))
D6=A6/C6
H6=0.5*(YN+XN+1.41421)
G6=H6+YN-XN
E6=SQRT((H6+H)*(H6+H)+(G6-1.0)*(G6-1.0))
F6=SQRT((H6+1.0)*(H6+1.0)+(G6-1.0)*(G6-1.0))
S6=F6/SORT((XN+B)*(XN+B)+(YN-1.0)*(YN-1.0)+(ZN*ZN)))
S6=F6/SORT((XN+1.0)*(XN+1.0)+(YN-B)*(YN-B)+(ZN*ZN)))
IF (H6+U)¹37,37,36
S6=SF6
37 $d6 = \left(\frac{C_{fc}}{\pi \cdot D6J}\right) \left(SF_0 + \frac{r}{J}\right)$

$A7 = (Y_N + A > U1.41421)/1.41421$

$C7 = \cot(\pi / 2)$

$87$

$A7s(Y_N+A>U1.41421)/1.41421$

$C7sCObf omitted$

$H7 = 0.ii^XNi-YS-l.41421)$

$G7 = M_7 + X E7-SGKT((H7M.O>*H7+i.O>+(G7+b)*(G7+B)>)$

$F7 = \sqrt{((H7 + 3)*CH7 + b) + (G7+1.0)*(G7+1.0))}$

$SE7 = E7/\sqrt{(XN-B)*CXN-1.0+1.0)+E7+(YN+B)*(YN+B)+(ZN*ZN)>)_67.}$

$SF_7 = F7/\sqrt{(XN-B)*CXN-1.0+1.0)+E7+(YN+B)*(YN+B)+(ZN*ZN)>)_67.$

$Ab = (YN-X'J + 1,41421)/1.41421$

$Bb=0,0$

$I0 = Y_N(1.41421)$

$C0 = SCRT((H0-1.0)*(H0-1.0)+(G8+1.0)*(G8+1.0))$

$F8 = \sqrt{((H8-l .0)*(H6-1.0)+(G8 + b)*(G8 + B)>)}

$SE8 = E8/\sqrt{(XN-B)*CXN-1.0+1.0)+E7+(YN+B)*(YN+B)+(ZN*ZN)>)_68.}$

$SF_8 = F8/\sqrt{(XN-B)*CXN-1.0+1.0)+E7+(YN+B)*(YN+B)+(ZN*ZN)>)_68.$

$Hb = 0.!?*C YNFXN + 1.41421)$

$B8 = (C8/(\pi *D8))*(SF8+sE8)

$B0 = (B1+32*33+B4+B5+B6+B7+B8)

$DO 10 I=1,LLL

$XN=(I-1)*0.1$

$IF (I-L)47,46,47$

$DO 30 L=1,LLL

$SUM = SUM + S0M + 8CC(I#L)*CD(L,in/(L-U

$DO 92 I=1,LLL

$XN = (I-1)*0.1$

$IF (I-L)47,46,47$

$DO 30 L=1,LLL

$SUM = SUM + S0M + 8CC(I#L)*CD(L,in/(L-U

$DO 92 I=1,LLL

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$DO 92 I=1,LLL

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$DO 92 I=1,LLL

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$DO 92 I=1,LLL

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$SUM = SUM + S0M + 8CC(I#L)*CD(L,in/(L-U

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$DO 30 L=1,LLL

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$SUM = SUM + S0M + 8CC(I#L)*CD(L,in/(L-U

$DO 92 I=1,LLL

$XN = (I-1)*0.1$

$IF (I-L)47,46,47$
BS(1, JJ) = SUM
XN = (J = 1) X 0.1
PRINT 8, 2, XN, BS(1, JJ)
8 FORMAT (F6.3, 2X, F6.3, 2X, E15.5)
301 CONTINUE
T = TIME(2) / 60
PRINT 503, T
508 FORMAT (" PROCESSOR TIME USED IN SECONDS =", F7.2)
C C V1, V2 PACKAGE
C C CALCULATION OF INDUCED VOLTAGES V1, V2 IN SECONDARY COIL
C C V1 = INDUCED VOLTAGE DUE TO THE PRIMARY COIL
C C V2 = INDUCED VOLTAGE DUE TO THE EDDY CURRENTS IN SAMPLE
C
DU 410 I = 1, I
C MULTIPLICATION OF COIL FIELD BY K
410 BCK(I, J, JJ-1) = BCK(I, I + JJ-1) X 0.05 * (I = 1)
NDIM = ILL
H = 0.05
CALL USF(H, BCK(I, I + JJ-1), V1(I, I + JJ-1), NDIM)
DU 420 I = 1, I
C MULTIPLICATION OF SAMPLE FIELD BY R
420 BSR(I, JJ) = BS(I, JJ) X 0.1 * (I = 1)
NDIM = ILL
H = 0.1
CALL USF(H, BSR(I, JJ), V2(I, JJ), NDIM)
PRINT 490
1"V21")
A1 = (KK-1) X 0.05
DO 430 I = 2, ILL
A2 = (J = 1) X 0.1
V21 = V2(I, JJ) / V1(2 * I = 1, I + JJ-1)
PRINT 491, 71, Z2, A1, A2, V1(2 * I = 1, I + JJ-1), V2(I, JJ), V21
430 CONTINUE
T = TIME(2) / 60
PRINT 502, T
502 FORMAT (" PROCESSOR TIME USED IN SECONDS =", F7.2)
4000 CONTINUE
T = TIME(2) / 60
PRINT 503, T
503 FORMAT (" PROCESSOR TIME USED IN SECONDS =", F7.2)
3000 CONTINUE
T = TIME(2) / 60
PRINT 504, T
504 FORMAT (" PROCESSOR TIME USED IN SECONDS =", F7.2)
2000 CONTINUE
T = TIME(2) / 60
PRINT 505, T
505 FORMAT (" PROCESSOR TIME USED IN SECONDS =", F7.2)
1000 CONTINUE
8001 STOP; END
SUBROUTINE USKH(Y,Z,NDIM)

SUBROUTINE QSF

PURPOSE
TO COMPUTE THE VECTOR OF INTEGRAL VALUES FOR A GIVEN
EQUIDISTANT TABLE OF FUNCTION VALUES.

USAGE
CALL QSF (H,Y,Z,NDIM)

DESCRIPTION OF PARAMETERS
H - THE INCREMENT OF ARGUMENT VALUES.
Y - THE INPUT VECTOR OF FUNCTION VALUES.
Z - THE RESULTING VECTOR OF INTEGRAL VALUES, Z MAY BE
IDENTICAL WITH Y.
NDIM - THE DIMENSION OF VECTORS Y AND Z.

REMARKS
NO ACTION IN CASE NDIM LESS THAN 3.

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
NONE

METHOD
BEGINNING WITH Z(1)=0, EVALUATION OF VECTOR Z IS DONE BY
MEANS OF SIMPSON'S RULE TOGETHER WITH NEWTON'S 3/8 RULE OR A
COMBINATION OF THESE TWO RULES. TRUNCATION ERROR IS OF
ORDER H**5 (I.E. FOURTH ORDER METHOD). ONLY IN CASE NDIM=3
TRUNCATION ERROR OF Z(2) IS OF ORDER H**4.

FOR REFERENCE, SEE
(1) F.B.HILDEBRAND, INTRODUCTION TO NUMERICAL ANALYSIS,
MCGRAW-HILL, NEW YORK/TORONTO/LONDON, 1956, PP.71-76.
(2) R.ZURMUEHL, PRAKTISCHE MATHEMATIK FUR INGENIEURE UND
PHYSIKER, SPRINGER, BERLIN/GOETTINGEN/HEIDELBERG, 1963.

DIMENSION Y(NDIM),Z(NDIM)

HT=33333333H
IF(NDIM GT 5)7,8,1

NDIM IS GREATER THAN 5, PREPARATIONS OF INTEGRATION LOOP
1 SUM1=Y(2)+Y(2)
SUM1=SUM1+SUM1
SUM1=HT*(Y(1)+SUM1+Y(3))
AUX1=Y(4)+Y(4)
AUX1=AUX1+AUX1
AUX1=SUM1+HT*(Y(3)+AUX1+Y(5))
AUX2=HT*(Y(1)+3.6/5*(Y(2)+Y(5))+2.625*(Y(3)+Y(4))+Y(6))
SUM2=Y(5)+Y(5)
SUM2=SUM2+SUM2
SUM2=AUX2+HT*(Y(4)+SUM2+Y(6))
Z(1)=0.

...
AUX = Y(3) + Y(4)
AUX = AUX + Y(5)
Z(2) = SUM2 - 41 * Y(2) + AUX + Y(4)
Z(3) = SUM1
Z(4) = Y(4)
IF (NDIM = 5) THEN Z(3) = SUM2

C INTEGRATION LOOP
2 DO 1 = 1, NDIM
SUM1 = AUX1
SUM2 = AUX2
AUX1 = Y(I-1) + Y(I-1)
AUX1 = AUX1 + AUX1
AUX1 = SUM1 + HT * (Y(I-1) + AUX1 + Y(I-1))
Z(I-2) = SUM1
IF (I = NDIM) THEN SUM1 = 0
AUX2 = Y(I) + Y(I)
AUX2 = AUX2 + AUX2
AUX2 = SUM2 + HT * (Y(I-1) + AUX2 + Y(I-1))
Z(I-1) = SUM2
Z(NDIM) = AUX1
RETURN
6 Z(NDIM) = SUM2
Z(NDIM) = AUX1
RETURN
C END OF INTEGRATION LOOP

C 7 IF (NDIM = 3) THEN
NDIM IS EQUAL TO 4 OR 5
8 SUM2 = 1.5 * HT * (Y(1) + Y(2) + Y(2) + Y(3) + Y(3) + Y(3) + Y(4))
SUM1 = Y(2) + Y(2)
SUM1 = SUM1 + SUM1
SUM1 = HT * Y(1) + SUM1 + Y(3)
Z(1) = 0.
AUX1 = Y(3) + Y(3)
AUX1 = AUX1 + AUX1
AUX1 = SUM2 + HT * (Y(2) + AUX1 + Y(4))
IF (NDIM = 5) THEN YP 9
AUX1 = Y(4) + Y(4)
AUX1 = AUX1 + AUX1
Z(5) = SUM1 + HT * (Y(3) + AUX1 + Y(5))
10 Z(3) = SUM1
Z(4) = SUM2
RETURN
C NDIM IS EQUAL TO 3
11 SUM1 = HT * (1.25 * Y(1) + Y(2) + Y(2) + 4.5 * Y(3))
SUM2 = Y(2) + Y(2)
SUM2 = SUM2 + SUM2
Z(3) = HT * (Y(1) + SUM2 + Y(3))
Z(1) = 0.
Z(2) = SUM1
12 RETURN
END

SEGMENT
APPENDIX II

Table XII lists $V_{ze}$ in nanovolts as a function of rms primary current in m.a. and frequency in kc/s. Data were taken using sample No. 5. The normalized geometrical parameters are:

$A_1 = 5.32$
$A_2 = 3.63$
$Z_1 = 1.75$
$Z_2 = 2.06$
<table>
<thead>
<tr>
<th>Frequency (kc/s)</th>
<th>$V_{ze}$ (mv)</th>
<th>Primary Current (m. a.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100</td>
<td>150</td>
</tr>
<tr>
<td>1</td>
<td>--</td>
<td>.15</td>
</tr>
<tr>
<td>2</td>
<td>.3</td>
<td>.57</td>
</tr>
<tr>
<td>3</td>
<td>.88</td>
<td>1.45</td>
</tr>
<tr>
<td>4</td>
<td>1.8</td>
<td>2.55</td>
</tr>
<tr>
<td>5</td>
<td>2.97</td>
<td>4.47</td>
</tr>
<tr>
<td>6</td>
<td>4.83</td>
<td>7.16</td>
</tr>
<tr>
<td>7</td>
<td>7.5</td>
<td>11.2</td>
</tr>
<tr>
<td>8</td>
<td>12</td>
<td>18.2</td>
</tr>
<tr>
<td>9</td>
<td>19.3</td>
<td>29</td>
</tr>
<tr>
<td>10</td>
<td>28</td>
<td>42</td>
</tr>
</tbody>
</table>

Table XII. Data showing $V_{ze}$ Dependent on Frequency and Primary Current
REFERENCES


