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A GEOMETRIC SOLUTION OF ROTATIONAL FLOW FIELDS

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ABSTRACT

A GEOMETRIC SOLUTION OF ROTATIONAL FLOW FIELDS

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This paper presents a geometric method of solving rotational flow fields. As a prior condition for this method to be applicable, streamlines must be known along the two boundaries of the flow region in question. This method is an extension of a geometric method of solving potential flows developed by F. O. Ringleb.

The method is based on the piecewise approximation of streamlines and their orthogonal trajectories by circular arcs. For both potential and rotational flows, only two-dimensional and axisymmetric flows may be solved, but the fluid may be compressible or incompressible.

Examples are worked where curved shock waves have induced rotational flow. Both axisymmetic and two-dimensional flows are treated in the examples.

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NOMENCLATURE

A = area

C = velocity of sound

C_v = specific heat at constant volume

g = acceleration of gravity

h = elevation

H = total head, $f(w,h,p)$

H = total head, $f(w,h,I)$

I = enthalpy

M = Mach number

n = coordinate along orthogonal to a streamline

p = pressure

r = distance from axis of symmetry

s = coordinate along streamline

S = entropy

T = absolute temperature

u = internal energy

v = component of $\bar{w}$ in the direction of $d\bar{\sigma}$

w = velocity

γ = ratio of specific heats

Γ = circulation

δ = streamline deflection angle through a shock wave referenced to the pre-shock streamline

Δ = first finite difference

θ = shock wave inclination angle referenced to the pre-shock streamline

ρ = density

σ = direction of a curve

φ = velocity potential in potential flow, or orthogonal trajectory in rotational flow

ψ = stream function

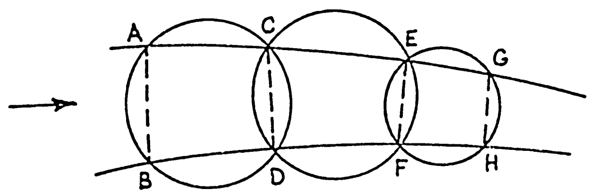
ω = vorticity

INTRODUCTION

This paper presents a geometric method of solving rotational flow fields. As a prior condition for this method to be applicable, streamlines must be known along the two boundaries of the flow region in question. If the boundary streamlines are known, it is then possible to construct the streamlines within the field and solve for the flow conditions.

This method of solving rotational flow fields is an extension of a geometric method of constructing potential flow fields developed by F. O. Ringleb. Ringleb's method is based on the piecewise approximation of the natural coordinates, formed by the streamlines and equipotential lines, by circular arcs. The construction of this circular arc network utilizes a theorem in geometry which states that "the corners of a rectangle formed by circular arcs are situated on a circle." The governing differential equations are evaluated along these nets and integrated numerically.

The procedure for the numerical integration is as follows. A fluid is flowing between two streamlines, AG and BH, as shown in the diagram below.



In addition to the two streamlines AG and BH, if the location of one of the equipotential lines AB, or an orthogonal trajectory, together with the magnitude of the velocity at points A and B are known, then the velocity distribution along AG and BH may be determined as follows: Through points A and B, A circle is drawn, of arbitrary radius, that intersects the streamlines at points C and D. This circle of arbitrary radius that is drawn should not be too large in that the streamline arcs it encloses have to be approximations to circular arcs. Passing through C and D, a circular arc is drawn normal to arcs AC and BD. This construction is possible according to the theorem in geometry as mentioned above, provided that the arcs may be approximated by circular arcs. The arcs of the rectangle are then measured and the equations in Section I are used to find the velocities at C and D. Now with the velocities at C and D known, another circle can be drawn and the procedure continued along the streamlines.

This paper deals with the case when the fluid is not a potential one, as it was in Ringleb's case. Section II derives the equations to be used with the geometric procedure when the flow is rotational. Since the flow behind a curved shock wave is rotational, this type of flow is the main concern of this paper.

Section III gives examples of rotational flows behind curved shocks. Two slender-bodies with attached curved shocks are solved, as is a blunt-body with detached shock.

Section IV discusses the results and meaning of the examples, and presents areas of possible further extension of this method.

For both potential and rotational flows, only two-dimensional and axisymmetric flows may be solved, but the fluid may be compressible or incompressible.

I.

POTENTIAL FLOW

This section treats potential flow fields, which consist of equipotential lines and streamlines. The equations for solving potential flows are obtained from the equations of continuity and irrotationality.

The first part of this section derives the basic equations for potential flow, including compressible and incompressible fluids for two-dimensional and axisymmetric flows. The second part changes the basic equations into a finite difference form that can be used in numerical integration. The last part consists of a geometric method of constructing initial equipotential lines when no analytic definition is available for the equipotential line.

1. FORMULATION OF THE BASIC EQUATIONS

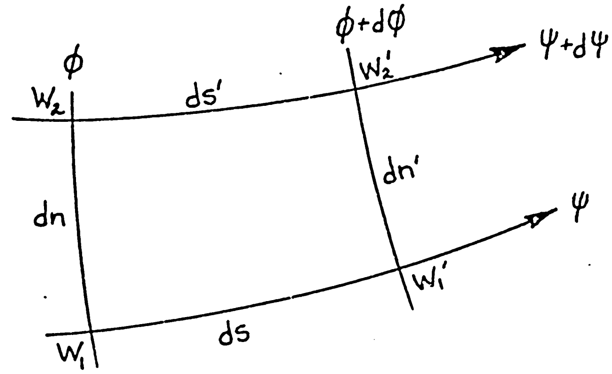


Figure 1: Natural Coordinates

The formulation of the basic equations for potential flow is presented by F. O. Ringleb in Ref. (1). The following is a synopsis of that work.

First, consider a steady two-dimensional potential flow of an incompressible fluid. Two neighboring streamlines are denoted by ψ and $\psi + d\psi$ in Figure 1, where ψ is Stokes' stream function. The distance between these streamlines, along an orthogonal trajectory, is shown as dn. If the average velocity of the flow through dn is $\bar{w}$, the flux through dn is wdn, assuming unit distance in depth. Now the flux between two streamlines is constant, so the value of $d\psi$ is given by the continuity equation:

$$d\psi = wdn \qquad (1)$$

φ-values are assigned to the orthogonal trajectories, equipotential lines, so that φ and $\varphi + d\varphi$ denote two neighboring trajectories. They form with the streamlines ψ and $\psi + d\psi$ a rectangle of generally curved arcs that can be approximated by circular arcs. This approximation of arcs to circular arcs is the essence of this method, since the arcs must be circular in order to meet one of the two requirements of the theorem which states that "the corners of a rectangle formed by circular arcs are situated on a circle". The other requirement is that these circular arcs must be orthogonal, and this requirement is met when orthogonal trajectories are specified, as they were above. The arc lenth along the streamline between φ and $\varphi + d\varphi$ is denoted by ds.

The circulation of the flow around a point p is defined in the following way (see Fig. 2). An arbitrary simple curve, C, encloses the point p.

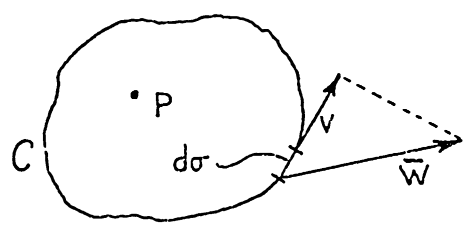


Figure 2: Circulation

The arc length of an element of C is denoted by $d\sigma$. The velocity vector $\overline{W}$ at a point on C is projected in the direction of $d\sigma$, and the length of the projection is denoted by V. Then, the circulation of the flow around point p is defined by the sum:

$$\oint v d\sigma = \Gamma \qquad (2)$$

The flow is called irrotational if $\Gamma = 0$ for every curve C enclosing p. If C is the quadrangle formed by the arc φ, $\varphi + d\varphi$, ψ, $\psi + d\psi$, the condition for an irrotational flow is that the product wds is constant along the streamlines between φ and $\varphi + d\varphi$. This result follows from the fact that v = 0 along dn. If, therefore, $d\varphi$ is identified by wds, the condition for irrotational flow becomes:

$$d\varphi = w\,ds \qquad (3)$$

Equations (2) and (3) are the natural equations of the two-dimensional steady potential flow of an incompressible fluid. They depend only on natural coordinate values and do not depend on any other arbitrarily chosen coordinate system.

In the same way, the natural equations for steady two-dimensional potential flow of a compressible fluid are obtained in the form:

$$d\varphi = w\,ds \qquad d\psi = \rho w\,dn \qquad (4)$$

where ρ is the variable mass density of the fluid. For the case of isentropic gas flow, these equations can be written:

$$d\left(\frac{\varphi}{C_o}\right) = \frac{w}{C_o}\,ds \qquad d\left(\frac{\psi}{\rho_o C_o}\right) = \frac{\rho}{\rho_o}\frac{w}{C_o}\,dn \qquad (5)$$

where C_o is used in order to relate the velocity to the Mach number, and

$$\frac{\rho}{\rho_o} = \left[1 - \frac{\gamma-1}{2}\left(\frac{w}{C_o}\right)^2\right]^{\frac{1}{\gamma-1}} \qquad (6)$$

$$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma} \qquad (7)$$

$$\frac{T}{T_o} = \left(\frac{C}{C_o}\right)^2 = \left(\frac{\rho}{\rho_o}\right)^{\gamma-1} \qquad (8)$$

$$C_o = \left(\frac{\gamma p_o}{\rho_o}\right)^{1/2} \qquad \gamma = \frac{C_p}{C_v} \tag{9}$$

are the isentropic gas relations. The subscript zero denotes the values of these variables when the gas is at rest. The velocity ratio $\frac{w}{C_o}$ is related to the Mach number M through the relation:

$$\frac{T}{T_o} = \frac{1}{1+[(\gamma-1)/2]M^2} \tag{10}$$

and Equations (8) and (6).

In the case of a steady axially symmetric potential flow of an incompressible fluid, the distance r of a point from the axis of symmetry enters the natural equations of the flow which take the form:

$$d\varphi = w\,ds \qquad d\psi = rw\,dn\ . \tag{11}$$

Finally, in the case of a steady axially symmetric isentropic gas flow, the natural equations are:

$$d\left(\frac{\varphi}{C_o}\right) = \frac{w}{C_o}\,ds \qquad d\left(\frac{\psi}{\rho_o C_o}\right) = r\,\frac{\rho}{\rho_o}\,\frac{w}{C_o}\,dn \tag{12}$$

2. FORMULATION OF THE FINITE EQUATIONS

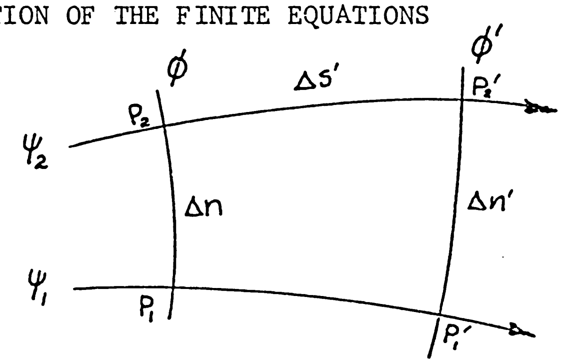


Figure 3: Natural coordinates with finite distances

The formulation of the finite equations for potential flow is presented in Refs. (1) and (2).

It has been shown in the previous part that in this method all streamlines and equipotential lines are approximated piecewise by circular arcs. Thus, each net consists of two pairs of circular arcs; one pair is streamlines, and the other pair is equipotential lines. Also, these pairs are perpendicular to each other because of the orthogonality condition between streamlines and equipotential lines. In applying the finite difference method, the infinitesimal distances dn and ds are replaced by finite distances Δn and Δs (Fig. 3); and $d\varphi$ and $d\psi$ are replaced by $\Delta\varphi$ and $\Delta\psi$ where:

$$\Delta\varphi = \varphi' - \varphi \qquad \Delta\psi = \psi_2 - \psi_1 \; . \tag{13}$$

Now it is a matter of substituting into Eq. (13) the values from the mesh for different cases of flow.

For the two-dimensional incompressible steady potential flow:

$$\Delta\varphi = \frac{w_1 + w_1'}{2} \Delta s = \frac{w_2 + w_2'}{2} \Delta s' \tag{14}$$

and

$$\Delta\psi = \frac{w_1 + w_2}{2} \Delta n = \frac{w_1' + w_2'}{2} \Delta n' \tag{15}$$

Since the arc lengths Δs, Δn, $\Delta s'$, $\Delta n'$, considered as circular arcs, are known and can be found by construction; and since w_1 and w_2 are also known, the unknown velocities w_1' and w_2' can be computed from Equations (14) and (15).

$$w_1' = \frac{[w_2 + (w_1 + w_2)(\Delta n/\Delta n')]\Delta s' - w_1 \Delta s}{\Delta s' + \Delta s} \tag{16}$$

and

$$w_2' = (w_1+w_2)\ \Delta n/\Delta n' - w_1' \quad . \tag{17}$$

Now for the case of two-dimensional steady potential compressible flow,

$$\Delta\left(\frac{\varphi}{C_o}\right) = \frac{w_1+w_1'}{2C_o}\ \Delta s = \frac{w_2+w_2'}{2C_o}\ \Delta s' \tag{18}$$

$$\Delta\left(\frac{\psi}{\rho_o C_o}\right) = \frac{\rho_{12}}{\rho_o}\ \frac{w_1+w_2}{2C_o}\ \Delta n = \frac{\rho_{12}'}{\rho_o}\ \frac{w_1'+w_2'}{2C_o}\ \Delta n' \tag{19}$$

Using a graph for $(\rho/\rho_o)(w/C_o)$ as a function of w/C_o, and letting $(w_1'+w_2')/_2 = w_2'$, then the solution for w_1' and w_2' will be

$$w_1' = \frac{(w_2+2w')\Delta s' - w_1\Delta s}{\Delta s' + \Delta s} \tag{20}$$

and

$$w_2' = 2w' - w_1' \quad . \tag{21}$$

The case of an axially symmetric potential flow of an incompressible fluid gives:

$$\Delta\varphi = \frac{w_1+w_1'}{2}\ \Delta s = \frac{w_2+w_2'}{2}\ \Delta s' \tag{22}$$

$$\Delta\psi = \frac{r_1+r_2}{2}\ \frac{w_1+w_2}{2}\ \Delta n = \frac{r_1'+r_2'}{2}\ \frac{w_1'+w_2'}{2}\ \Delta n' \tag{23}$$

and therefore,

$$w_1' = \frac{\{[(r_1+r_2)/(r_1'+r_2')](\Delta n/\Delta n')(w_1+w_2)\}\ \Delta s' - w_1\Delta s}{\Delta s' + \Delta s} \tag{24}$$

and

$$w_2' = \frac{r_1+r_2}{r_1'+r_2'}\ \frac{\Delta n}{\Delta n'}(w_1+w_2) - w_1' \quad . \tag{25}$$

Finally, in the axially symmetric case of compressible flow, the equations are:

$$\Delta\left(\frac{\varphi}{C_o}\right) = \frac{w_1+w_1'}{2C_o}\,\Delta s = \frac{w_2+w_2'}{2C_o}\,\Delta s' \tag{26}$$

and

$$\Delta\left(\frac{\psi}{\rho_o C_o}\right) = \frac{r_1+r_2}{2}\,\frac{\rho_{12}}{\rho_o}\,\frac{w_1+w_2}{2C_o}\,\Delta n =$$

$$= \frac{r_1'+r_2'}{2}\,\frac{\rho_{12}'}{\rho_o}\,\frac{w_1'+w_2'}{2C_o}\,\Delta n' \;. \tag{27}$$

From the graph of ρw versus w, with $(w_1'+w_2')/2 = w'$, the above gives:

$$w_1' = \frac{w_2+2w')\Delta s' - w_1\Delta s}{\Delta s' + \Delta s} \tag{28}$$

and

$$w_2' = 2w' - w_1' \;, \tag{29}$$

The preceding equations can be used for the stepwise constructions of solutions for two general types of initial conditions. First, the case where two streamlines and one of their equipotentials are known, along with the velocities at the intersections of the streamlines and the equipotential line. Secondly, the similar case where two equipotential lines intersect a streamline and the velocities are known at the intersection. Other cases may be found in Ref. 1.

3. A GEOMETRIC METHOD OF CONSTRUCTING THE INITIAL EQUIPOTENTIAL LINE

One of the disadvantages of this method of solution to fluid flows is that it is necessary to have an equipotential line given to begin the problem. The following is an outline of a method to approximate the equipotential line when no analytic description is available. This method of constructing equipotential lines is applicable to converging two-dimensional and axisymmetric nozzles. In describing this method, an axisymmetric nozzle is used with a wall of radius R where

$$R = (1 + 0.2x^2)^{1/2} \quad .$$

The nozzle wall and the axis of symmetry are taken as the streamlines. The following description is illustrated in Fig. 4.

Let it be assumed that it is necessary to construct the initial equipotential line at point A on the nozzle wall. First, a normal to the wall is drawn at point A and extended until it intersects the axis. This intersection is represented by point C in the diagram. With C as a center, an arc is drawn from A to B. B is the point where the arc's tangent is parallel to the axis at point C.

Since three points, A, B, and C, have now been constructed, it is possible to draw a circle passing through these points. This circle with its center at point D will intersect the axis at point E. Now if a tangent to the wall at point A is drawn, and this tangent line is extended until it intersects the axis, then the circular arc AE may be drawn using the tangent line - axis intersection as the center.

In constructing an equipotential line for a two-dimensional nozzle, the same method is used as outlined above except that the two walls will be the streamlines. This means that when point E is found, a tangent to the wall at E is drawn and extended until it intersects the tangent from point A on the other wall. Then the circular arc AE may be drawn with the intersection of the two tangents as a center.

It can be seen in the diagram that the circle is now enclosing a rectangle of circular arcs (A,B,C,E,), which is the necessary condition for this geometric solution. The line BC is an equipotential line in parallel uniform flow, and the velocities at points B and C will be equal. Using the method previously described, it will then be possible to find the velocities at points A and E. So the equipotential line and its velocities have been found.

The reason why this geometric construction works is simply because equipotential lines must be approximately circular arcs and othogonal to the streamlines for this method of flow solution. The approximate equipotential lines presented here are constructed to meet these requirements which the actual equipotential lines are only assumed to meet.

The solid line AD is the equipotential line from two-dimensional incompressible flow in the same nozzle contour, which has the equation:

$$\frac{R^2}{(C^2 \cosh^2 \varphi)} + \frac{x^2}{(C^2 \sinh^2 \varphi)} = 1$$

where $C = 2.4492$ and $\varphi = 0.7961$. This line was shown to be extremely good in that the solution to this problem using line AD was very accurate.[2] The analytic equipotential line AD can be seen to be nearly identical to the constructed line AE.

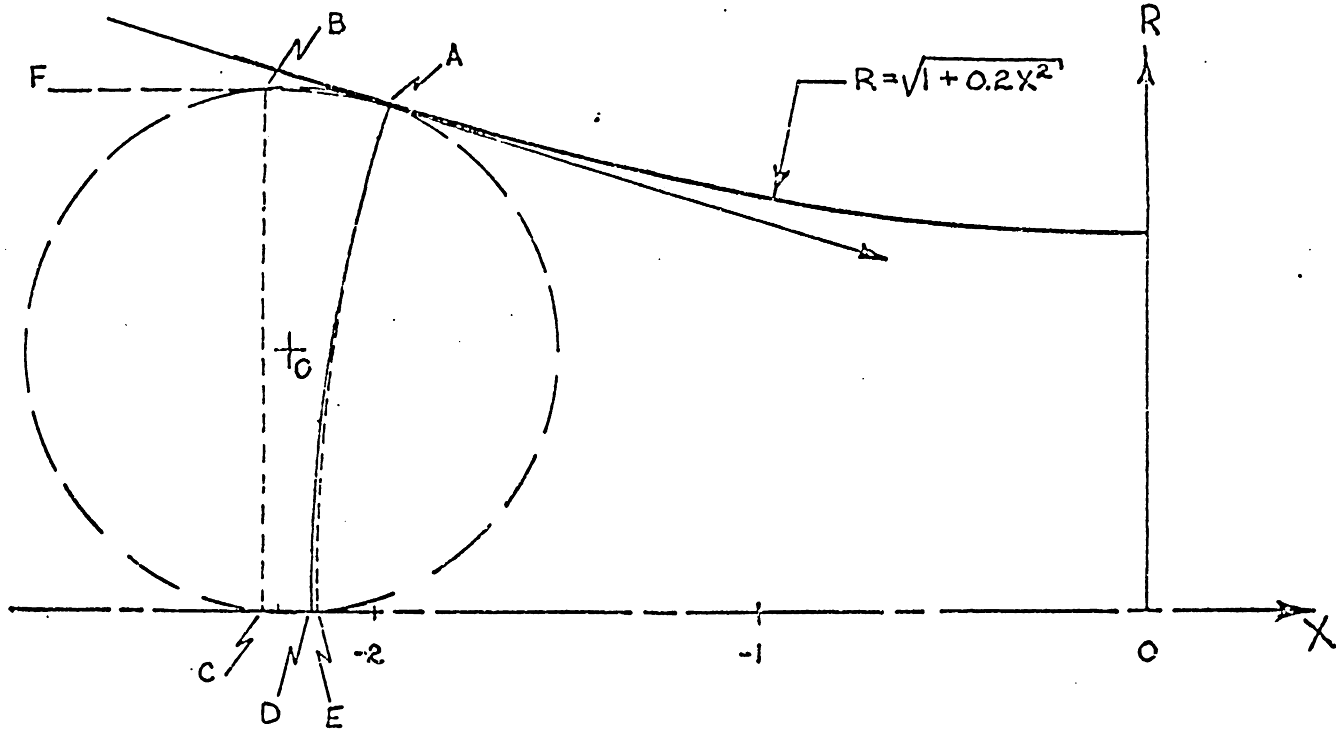


FIG. 4: INITIAL EQUIPOTENTIAL LINE CONSTRUCTION

II.

ROTATIONAL FLOW

Rotational flow differs from potential flow in one respect: for potential flow the circulation is equal to zero, while for rotational flow the circulation is different from zero. However, Helmholtz has shown that in perfect fluids the vortex lines, once created, persist in the fluid, and are material lines.[3] This leads to the fact that circulation is constant along material lines, where material lines are streamlines in this case. So the circulation, once created, is constant along a given streamline. In Part 2 of this section, the formulation of the finite difference equation is based on the circulation remaining constant along streamlines.

It should be noted that in rotational flow there are no equipotential lines. Instead, orthogonal trajectories to the streamlines are used to make up the circular arc rectangle.

In the derivation of circulation in Part 1, the condition of strictly adiabatic flow is used. This means that there is no heat conduction between the streamlines.

This paper is limited mainly to rotational flows produced by a curved shock wave, and to that end, Part 3 shows a relationship between entropy and shock angle.

1. MATHEMATICAL DERIVATION OF CIRCULATION

Since the present method requires an expression for the circulation, the first step is to derive the most general form of this expression. The following procedure is from Reference 3.

Starting with a vector form of Newton's equation

$$\frac{\partial \bar{w}}{\partial t} + (\text{curl } \bar{w} \times \bar{w}) = - \text{ grad } (gH) \qquad (1)$$

where $\bar{w}$ = velocity

g = acceleration of gravity

$H = \frac{w^2}{2g} + h \div \frac{P}{g}$ = total head (1')

h = elevation

p = pressure

and for steady flow this reduces to

$$\text{grad } (gH) = - (\text{curl } \bar{w} \times \bar{w}) \ . \qquad (2)$$

From the First and Second Laws of Thermodynamics, one can obtain

$$T \frac{\partial S}{\partial p} = \frac{\partial u}{\partial p} \quad \text{and} \quad \frac{T \, \partial S}{\partial \rho} = \frac{\partial u}{\partial \rho} - \frac{p}{\rho^2} \qquad (3)$$

where

T = absolute temperature

S = entropy

u = internal energy (3')

ρ = density

and from the definition of enthalpy (I),

$$I = u + \frac{p}{\rho} \quad . \tag{4}$$

Now, combining equations (4) and (3) gives

$$dI = du - \frac{p}{\rho^2} d\rho = T\ dS + \frac{dp}{\rho} \ . \tag{5}$$

But the total derivative of a scalar point function, f, may be expressed as

$$df = (\text{grad } f) \cdot d\bar{r} \tag{5'}$$

where $\bar{r}$ is a position vector not normal to the direction of grad f.

Using the above expression changes Eq. (5) to

$$(\text{grad } I) \cdot d\bar{r} = T(\text{grad } S) \cdot d\bar{r} + \frac{1}{\rho} (\text{grad } P) \cdot d\bar{r} \tag{5''}$$

which reduces to

$$d\bar{r} \cdot (\frac{1}{\rho} \text{ grad } P + T \text{ grad } S - \text{grad } I) = 0 \ . \tag{5'''}$$

So for the actual changes occuring in the flow, it is seen that

$$\frac{1}{\rho} \text{ grad } P = \text{grad } I - T \text{ grad } S \quad . \tag{6}$$

In Equation (2), grad P (a term of H) must now be replaced by grad I - T grad S. The equation then reads

$$\text{curl } \bar{w} \times \bar{w} = T \text{ grad } S - \text{grad } gH \tag{7}$$

where

$$H = \frac{w^2}{2g} + h + \frac{I}{g} \quad . \tag{7'}$$

In strictly adiabatic steady flow both H and S are constant on each streamline, and it is known that their variation from streamline to streamline is determined by the boundary conditions in any particular problem. If the motion is not only steady but two-dimensional or axisymmetric, one may introduce a stream function $\psi(x,y)$ to satisfy the equation of continuity.

$$y^{\epsilon} \rho \, w_x = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad y^{\epsilon} \rho \, w_y = - \frac{\partial \psi}{\partial x}$$

with $\epsilon = \begin{cases} 0 & \text{for two-dimensional flow} \\ 1 & \text{for axisymmetric flow} \end{cases}$

Since ψ remains constant along any given streamline, as do H and S, then it may be said that both H and S are functions of ψ alone. Also, for two-dimensional and axisymmetric flows, the vortex vector, curl $\bar{w}$, has only a Z-component, ω. So Equation (7) therefore reduces to a scalar equation along the normal to the streamline:

$$\omega = \frac{1}{w} \left(T \frac{\partial s}{\partial n} - g \frac{\partial H}{\partial n} \right) \tag{9}$$

and since circulation is defined:

$$\Gamma = \oint_c w \cdot dr \tag{10}$$

so with Stokes' theorem

$$\oint_c \bar{w} \cdot dr = \int_s \bar{n} \, (\nabla \times \bar{w}) \, dA = \int_s (\bar{n} \cdot \bar{\omega}) \, dA \tag{10'}$$

The above equation is valid only for simply connected regions. Therefore:

$$d\Gamma = (n \cdot w) \, dA = \omega \, dA \tag{11}$$

with

$$n \cdot \omega = \text{comp}_n\ \omega \ = \ \omega \tag{12}$$

Now using the finite difference method and applying it to Equations (9) and (11),

$$\omega \ = \ \frac{1}{w}\left(T\frac{\Delta S}{\Delta n} - g\ \frac{\Delta H}{\Delta n}\right) \tag{13}$$

and

$$\Delta\Gamma = \omega\ \Delta A = \omega\ \Delta\bar{n}\ \Delta\bar{s} \quad . \tag{14}$$

Substituting for ω in Equation (14) with Equation (9)

$$\Delta\Gamma = \frac{1}{w}\left(T\ \frac{\Delta S}{\Delta\bar{n}} - g\ \frac{\Delta H}{\Delta\bar{n}}\right)\ \Delta\bar{n}\ \Delta\bar{s}$$

$$\Delta\Gamma = \frac{1}{w}\left(T\ \Delta S\ \Delta\bar{s} - g\ \Delta H\ \Delta\bar{s}\right) . \tag{15}$$

Equation (15) shows the dependence of circulation on entropy and enthalpy. If enthalpy, or total head, does not vary between streamlines, then circulation is solely a function of the entropy change normal to the streamlines. Flows with a total head, or total enthalpy, constant throughout the field are called homenergic, while flows with entropy constant throughout the field are called homentropic. The former case, homenergic, is the only case dealt with in this paper, but it is extremely important in gas dynamics.

The homenergic case is important mainly because when a shock wave is curved, the flow after it will not be homentropic since the values of p and ρ for each particle will depend on the slope of the shock wave at whatever point the particle reaches the transition.

However, the total head will still be constant throughout the field behind a curved shock.[3]

2. FORMULATION OF THE FINITE EQUATIONS FOR HOMENERGIC FLOWS

In the preceding part it was noted that the flow field behind a curved shock is homenergic and non-homentropic. This means that $\Delta H = 0$, and Equation (15) now reads:

$$\Delta\Gamma = \frac{T}{w} \Delta S \; \Delta\bar{s} \quad , \tag{16}$$

In order to numerically work with $\Delta\Gamma$ in a problem, it is necessary to know T, W, and ΔS. This is in addition to the measured values of Δs, $\Delta s'$, Δn, $\Delta n'$, and the known velocities w_1 and w_2.

Looking at Figure 5 below, the point P is the reference point about which the circulation will be found.

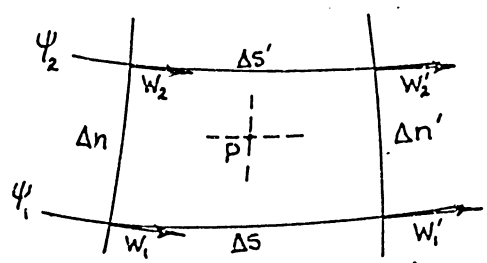


Figure 5: Reference point for circulation

Looking at Equation (16) and Figure 5, the terms in this equation refer to the values at point p, and they may be approximated in the following way:

$$T_p = \frac{T_1 + T_2}{2} \qquad W_p = \frac{w_1 + w_2 + w_1' + w_2'}{4} \tag{17}$$

$$\Delta\bar{s} = \frac{\Delta s + \Delta s'}{2}$$

where the assumption has been made that the temperature remains constant along the streamline distances Δs and $\Delta s'$ for a given mesh, and the physical size of the orthogonal trajectories does not vary too greatly in that mesh.

Substituting the terms of Equation (17) into Equation (16):

$$\Delta\Gamma = \frac{T_2 + T_1}{w_1 + w_2 + w_1' + w_2'} \cdot \Delta S \; (\Delta s + \Delta s') \tag{18}$$

Now going back to the original definition of circulation, Equation (I-2), it can be seen that the velocity will have no component in the orthogonal trajectories' direction. This means that the circulation will be entirely due to the velocity along the streamline distances Δs and $\Delta s'$. So we can say that the circulation will be given by:

$$\Delta\Gamma = \frac{w_1 + w_1'}{2} \Delta s - \frac{w_2 + w_2'}{2} \Delta s' \; . \tag{19}$$

Now it is possible to equate Equation (18) to Equation (19) to give the expression:

$$\frac{w_1 + w_1'}{2} \Delta s - \frac{w_2 + w_2'}{2} \Delta s' = \frac{(T_2 + T_1) \; \Delta S \; (\Delta s + \Delta s')}{w_1 + w_2 + w_1' + w_2'} \tag{20}$$

Dividing Eq. (20) by C_p, which changes the velocities to Mach numbers, the expression becomes, after rearrangement,

$$\left(\frac{w_1+w_1'}{2C_p} \Delta s - \frac{w_2+w_2'}{2C_p} \Delta s' \right)\left(\frac{w_1+w_2+w_1'+w_2'}{C_p} \right) =$$

$$= \frac{(T_2+T_1)\ \Delta s\ (\Delta s + \Delta s')}{C_p^{\ 2}} \tag{20'}$$

where,

C_p = local speed of sound in air = $\sqrt{\gamma R T_p}$

Eq. (20') can be expressed as

$$\left[\left(\frac{C_1}{C_p} \right) \frac{w_1+w_1'}{2C_1} \Delta s - \left(\frac{C_2}{C_p} \right) \frac{w_2+w_2'}{2C_2} \Delta s' \right]\left(\frac{w_1+w_2+w_1'+w_2'}{C_p} \right)$$

$$= \frac{(T_2+T_1)\ \Delta S\ (\Delta s + \Delta s')}{\gamma R T_p} . \tag{20''}$$

In the course of the investigation it was found that for small mesh spacing

$$\frac{C_1}{C_p} = \frac{C_2}{C_p} \approx 1$$

where C_1 and C_2 are the sonic velocities on the ψ_1 streamline and the ψ_2 streamline, respectively. So Eq. (20") can be reduced to

$$\left(\frac{w_1+w_1'}{2C_1} \Delta s - \frac{w_2+w_2'}{2C_2} \Delta s' \right)\left(\frac{w_1+w_2+w_1'+w_2'}{C_p} \right)$$

$$= \frac{2\Delta S\ (\Delta s + \Delta s')}{\gamma R} \tag{21}$$

After rearrangement, Eq. (21) appears in the form

$$\left(\frac{w_1+w_1'}{2C_1}\Delta s - \frac{w_2+w_2'}{2C_2}\Delta s'\right)\left(\frac{w_1+w_2+w_1'+w_2'}{C_p}\right)\frac{1}{(\Delta s + \Delta s')}$$

$$= \frac{2\ \Delta s}{\gamma(\gamma-1)\ C_v} \tag{22}$$

since,

$R = (\gamma-1)\ C_v$

C_v = specific heat at constant volume

Equation (22) gives a relationship between the unknowns w_1' and w_2' (as Mach numbers) and the known or computed quantities. Another relation involving the two unknown velocities is the equation of mass flow. Since the mass flow is constant between streamlines, it is possible to say: for two-dimensional flow,

$$\rho_{12}\ \frac{w_1+w_2}{2C}\ \Delta n = \rho_{12}'\ \frac{w_1'+w_2'}{2C}\ \Delta n' \tag{23}$$

for axisymmetric flow,

$$\frac{r_1+r_2}{2}\ \rho_{12}\ \frac{w_1+w_2}{2C}\ \Delta n = \frac{r_1'+r_2'}{2}\ \rho_{12}'\ \frac{w_1'+w_2'}{2C}\ \Delta n' \tag{24}$$

where,

$$\rho_{12} = \frac{\rho_1+\rho_2}{2} \quad \text{and} \quad \rho_{12}' = \frac{\rho_1'+\rho_2'}{2}\ .$$

By using a graph for ρw as a function of w, and letting $(w_1'+w_2')/_2 = w'$, the solution can be found by using the fact that w' is known from either Equation (23) or (24), whichever is applicable. Since

$$\frac{w'}{C} = \frac{w_1'+w_2'}{2C}$$

then

$$\frac{w_1'}{C} = \frac{2w'}{C} - \frac{w_2'}{C} \quad \text{or} \quad \frac{w_2'}{C} = \frac{2w'}{C} - \frac{w_1'}{C} \qquad (25)$$

Now by substituting one of the expressions of Equation (25) into Equation (22), one obtains an expression for an unknown in terms of measured or computed quantities. For example:

$$w_1' = \frac{2\Delta S}{\gamma(\gamma-1)\, C_v} \; \frac{1}{\left[\dfrac{(w_1+w_2) + 2w'}{C}\right]} + \frac{\left\{\left[2w'+w_2\right] \Delta s' - w_1\, \Delta s\right\}}{\Delta s + \Delta s'} \qquad (26)$$

The only term not known in Equation (26) is ΔS, and the next part will explain how to find it.

3. DERIVATION OF ENTROPY DISTRIBUTION BETWEEN STREAMLINES BEHIND A CURVED SHOCK WAVE

In the last section, it was explained why it is necessary to know the difference in entropy between two streamlines. Since the case of most interest with varying streamline entropy is behind a curved shock, assume that the flow is uniform and homentropic throughout the flow field before the shock. Now use the relation:

$$S_2 - S_1 = C_c \; \mathrm{LN} \left[\left(\frac{2\gamma}{\gamma-1} M_1^2 \sin^2\theta - \frac{\gamma-1}{\gamma+1}\right)\left(\frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1) M_1^2 \sin^2\theta}\right)^{\gamma}\right] \qquad (27)$$

with θ being the shock angle relative to the incoming flow, and the subscripts 1 and 2 signifying flow before and after the shock respectively.

The above Equation (27) is true only for flow along one streamline through a shock, so for the difference in entropy between two streamlines, a and b, after the shock:

$$(S_2 - S_1)_a - (S_2 - S)_b = \Delta S_{a-b} = S_{2a} - S_{2b} \quad . \qquad (28)$$

And, since the flow is uniform and homentropic before the shock,

$$S_{1a} = S_{1b} \quad \text{and} \quad M_{1a} = M_{1b} \quad . \qquad (29)$$

So, therefore in solving for the entropy difference:

$$\Delta S_{a-b} = C_v \; LN \; \left\{ \left(\frac{A \sin^2 \theta_a - B}{A \sin^2 \theta_b - B} \right) \left(\frac{B + \frac{C}{\sin^2 \theta_a}}{B + \frac{C}{\sin^2 \theta_b}} \right)^{\gamma} \right\} \qquad (30)$$

where,

$$A = \frac{2\gamma}{\gamma - 1} M_1^{\,2} \; , \quad B = \frac{\gamma - 1}{\gamma + 1} \; , \quad C = \frac{2}{(\gamma+1) \; M_1^{\,2}} \quad . \qquad (31)$$

Now expressions for every quantity in Equation (26) have been found, and problems may be solved.

III.

EXAMPLES OF ROTATIONAL FLOW

In this section, three bodies with curved shock waves are solved by the method described in Section II.

The first example is a slender open-nose body of revolution with $M_\infty = 1.525$ (see Fig. 6). The external surface of this body was solved for velocity and pressure distributions.

The second example involves the solution of the flow field inside of a slender open-nosed body of revolution. An approximation is made in that the streamlines are taken to be straight lines. A comparison of the calculated sonic line placement with the method of characteristics is also made.

The third example is the solution of a blunt-body detached shock flow field. A circular cylinder causes the shock and a combination of the method of flux analysis and the present method gives the flow field.

The first and second examples are axisymmetric flows, while the third example is two-dimensional flow.

1. (a) AXISYMMETRIC FLOW ON AN EXTERNAL SURFACE

In this problem the external surface of an inlet diffuser is solved for its velocity profile, and then the velocities are changed to their related pressures.

The shock wave, body shape, and streamline are given in Figure 6.[8] Using the axisymmetric equations, the body surface and one streamline are solved for their velocity distributions. It should be noted that in obtaining a velocity distribution, the average velocity along each mesh side should always be used. This is a result of the circulation being an average of the velocity along the sides of each mesh. Otherwise, when using the velocities found at the corner points of each mesh, the velocity distribution will vary alternately higher and lower than the average velocity. By using the average velocities between the corner points, plotting a curve of the velocity distribution will become more accurate and easier. In plotting the velocities in Figure 7, this method was used.

Since this problem is an open-nosed cone with swallowed internal shock, the cone may be approximated by a wedge of the same angle close to the shock wave. However, the limiting velocity along the body will be that of the cone. This same rule applies to the pressure distribution.

In Figure 8, the pressure ratio along the surface, p/p_{∞}, has been plotted as a function of the relative distance, x/D. The rotational characteristics method curve and the small-distur-

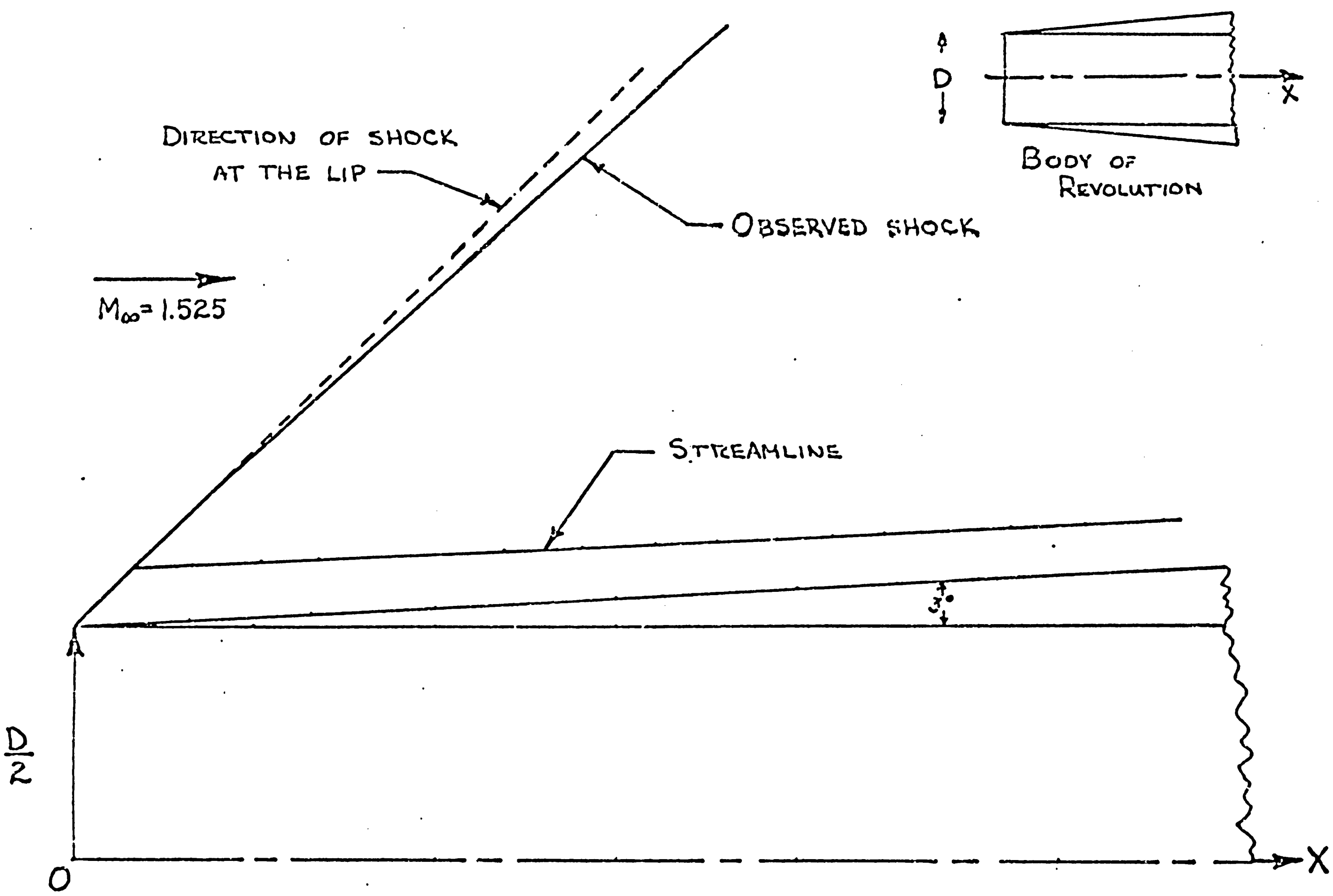


FIG. 6 : EXTERNAL SURFACE

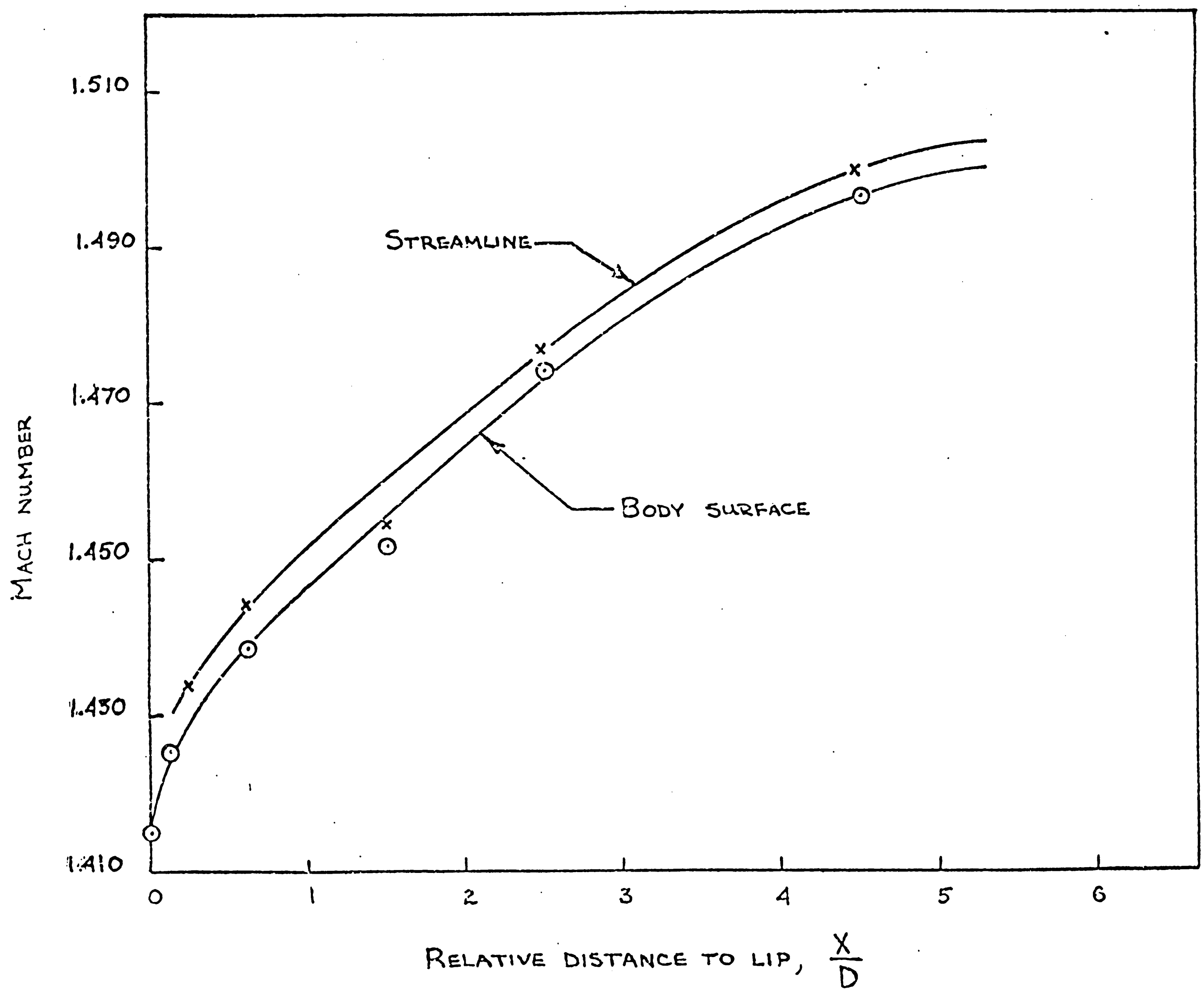


FIG. 7: VELOCITY DISTRIBUTION

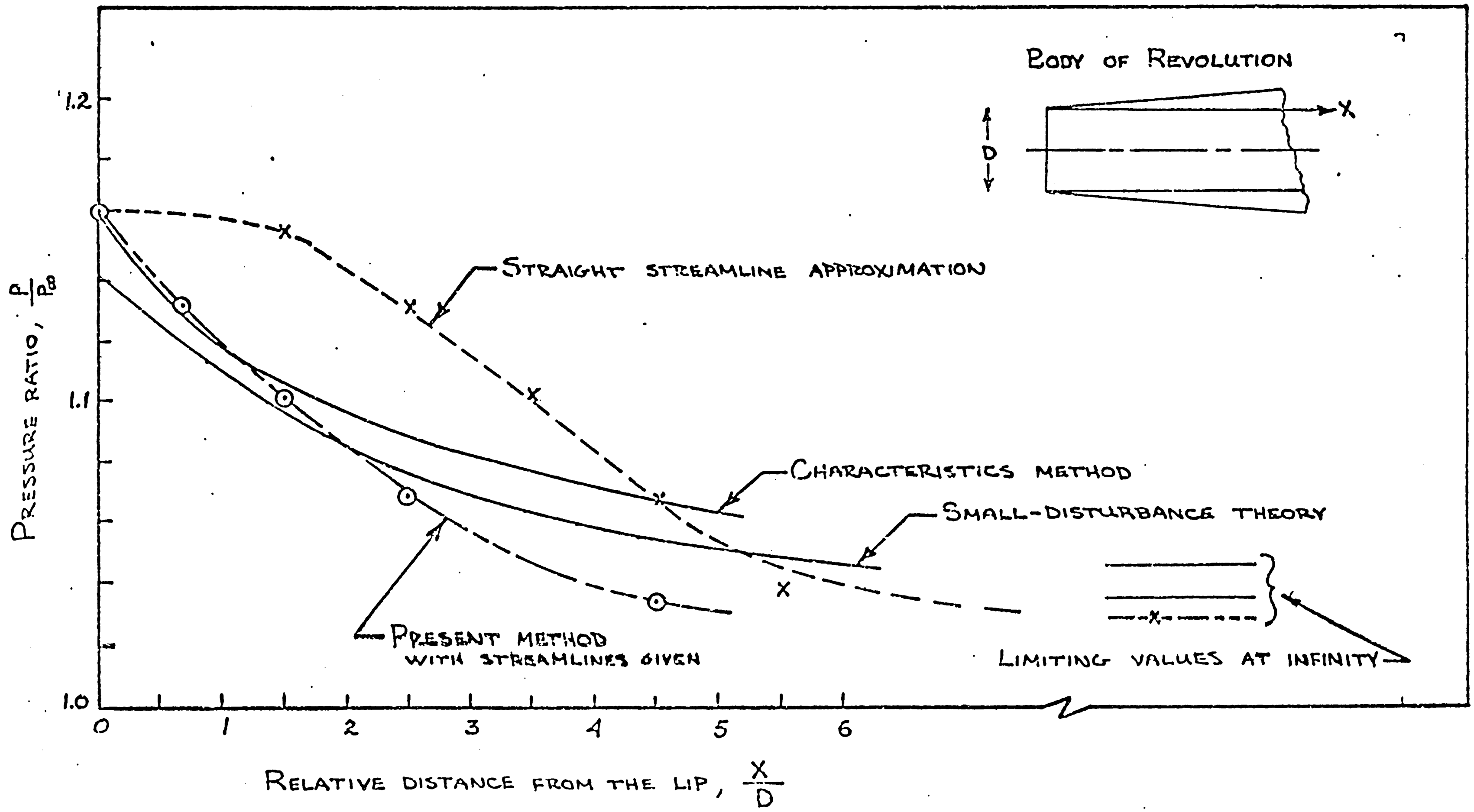


FIG. 8: PRESSURE DISTRIBUTION

bance theory curve are plotted for comparison.[8]

If an approximation to the streamline is made using a straight line, given the shock shape and the incoming velocity which determine the deflection angle, then the pressure ratio will be as shown in Figure 8. At worst, this approximation to the pressure ratio deviates by only 5 1/2% from the values found by the more rigorous methods. Using the same straight line streamline approximation would make the velocity err by only 2% at its greatest variance.

Since the straight streamline approximation seems so workable, the next problem is solved using this approximation.

1. (b) AXISYMMETRIC FLOW OF AN INTERNAL SURFACE

This problem deals with the flow inside a diffuser. In Figure 9, the body of revolution and the shock wave are given.[8] The streamlines are approximated by straight lines drawn at an angle (δ) corresponding to the incoming flow (M_∞) and the shock angle (θ) at the point where the streamline crosses the shock. To do this is possible since it may be assumed that at the shock, wedge-type flow is present.

The principle cause of inaccuracy of this geometrical method is the inability to have the velocities known at two points on an orthonormal. Looking at Figure 9, the velocities are known on each streamline immediately behind the shock, but no two of the velocities lie on the same orthogonal trajectory. A method

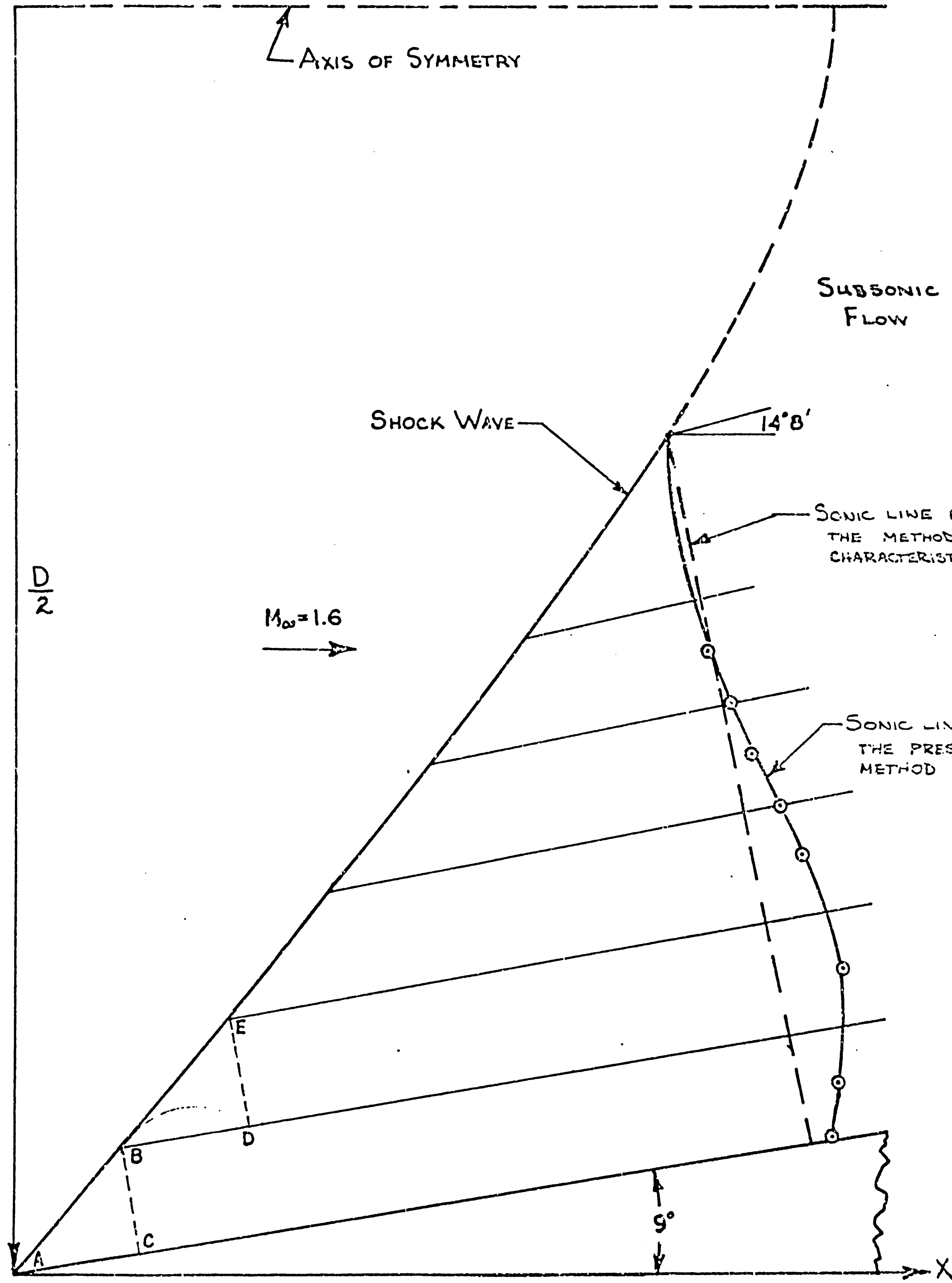


FIG. 9: INTERNAL SURFACE

for surmounting this problem is to construct the orthogonal trajectory BC in Figure 9. Then one simply assumes that the velocity at C is the same as the velocity at A. After that, the velocities may be solved completely along all streamlines.

Using the axisymmetric equations, the sonic line placement was solved. In comparison with the sonic line from the rotational method of characteristics, the sonic line from the present method appears to be fairly similar.

2. TWO-DIMENSIONAL FLOW

In Figure 10 a circular cylinder is producing a detached bow shock wave at $M_\infty = 2$. The problem of finding the shock shape and location, plus the flow field behind the shock resulted in Uchida and Yasuhara publishing the method of flux analysis.[10] In this example, the flow field from the method of flux analysis has been "forced" to fit the observed shock wave shape. Figure 10 shows the shock wave shape from the method of flux analysis and the observed shock shape.[10] Two streamlines are shown after they were "forced" to fit the observed shock. This "forcing" is simply taking the flow field from the method of flux analysis and expanding it graphically until it fits the observed shock wave. Then the present method is used on the expanded flow field.

Figure 10 shows the sonic line calculated from flux analysis compared with the sonic line found by using the observed shock shape and the "forced" streamlines with the present method.

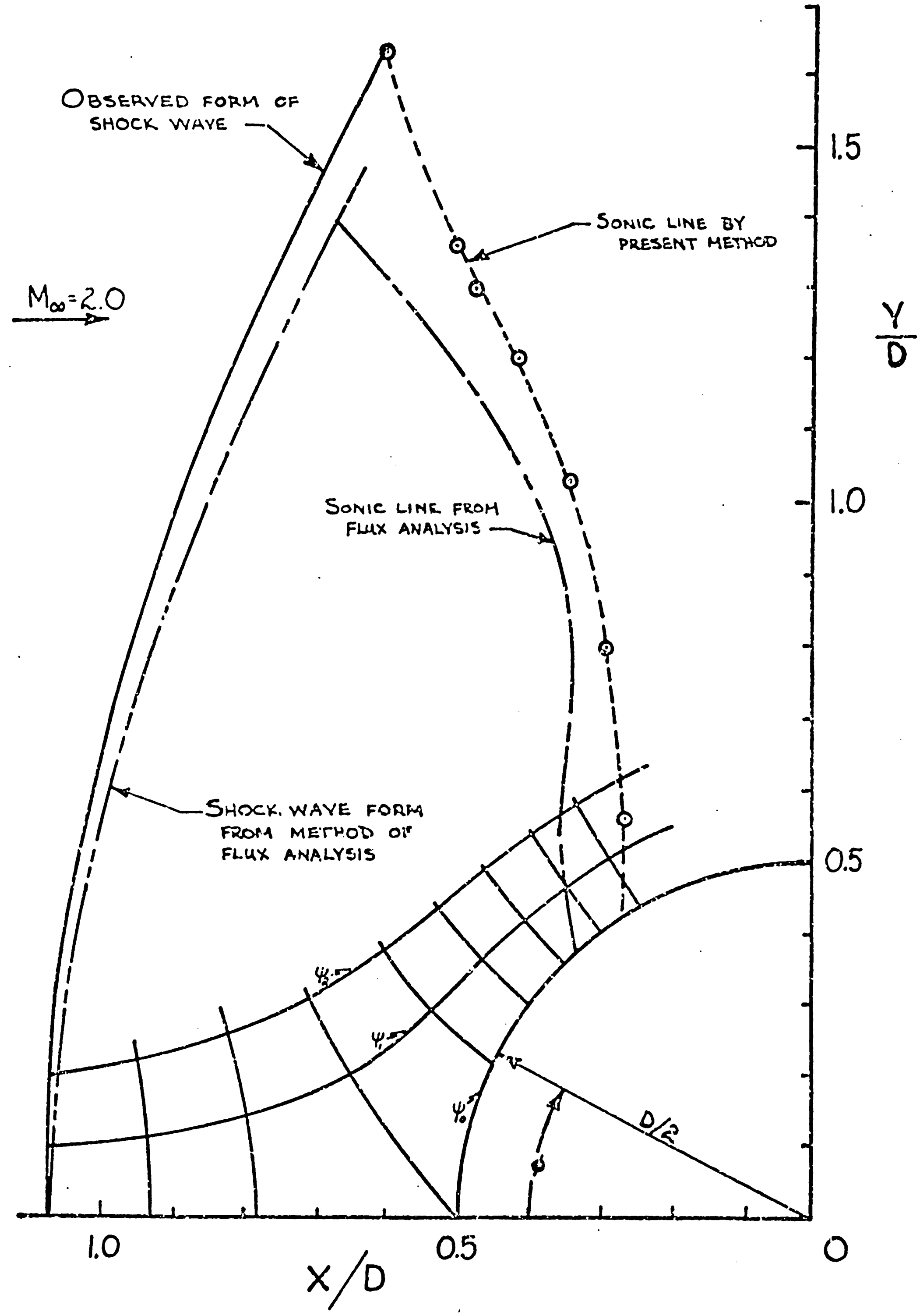


FIG. 10: CIRCULAR CYLINDER WITH
DETACHED BOW WAVE

Figure 11 is a plot of the pressure coefficient distribution on the cylinder.

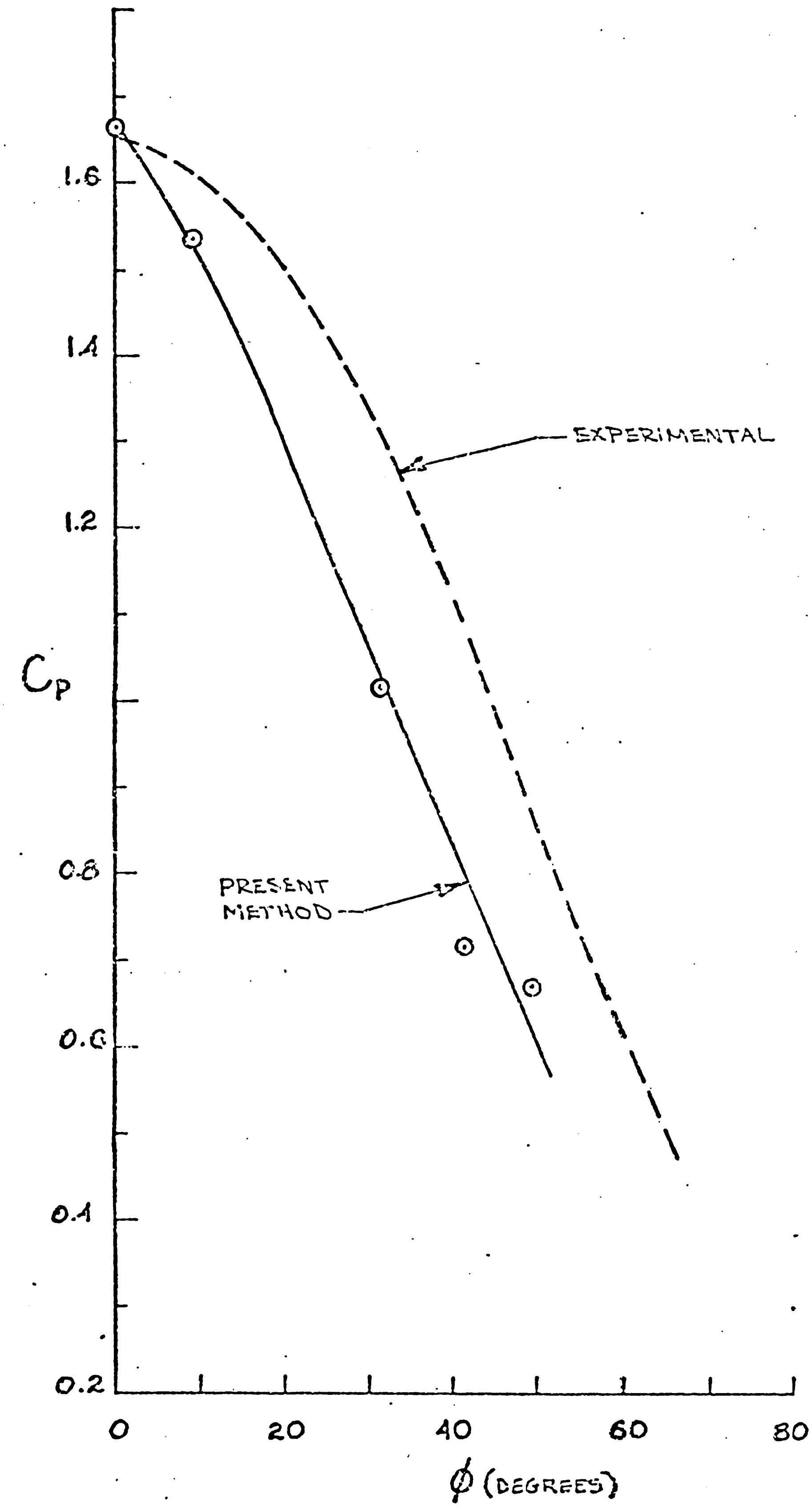


FIG. 11: PRESSURE COEFFICIENT ON THE CYLINDER

IV

RESULTS AND CONCLUSIONS

The examples show that this method of flow field solution is heavily dependent on being given accurate streamlines. This is as expected since in the Introduction it was noted that the streamlines must be known to begin this method. However, in most flows, the streamlines are not known except along the surface of the body. It was in answer to this problem of not knowing the streamlines that the approximations were made: the straight streamline for the slender body case, and the 'expanded' flow net for the blunt body case.

The straight streamline approximation gave fairly good results for both the internal and external flow problems. It should also be noted that when the streamlines are given, this method is extremely good, as is seen in Fig. 8. Also, the sonic line placement for the internal flow using the straight streamline approximation can be seen in Fig. 9 to be very close to the sonic line given by the method of characteristics.

The 'expanded' flow net approximation for the blunt body problem gave a sonic line placement that is closer to actuality than is the sonic line by the method of flux analysis. However, in solving for the pressure distribution around the body, it can be seen in Fig. 11 that this method gives pressures that are lower than those found by experiment.

From the results of the preceding examples, this method with the straight streamline approximation seems to be suited for finding the location of the sonic line for internal flow, such as in inlet diffusers.

Another possible use for this method, which was not treated in this paper, might be as follows below. In Section II-1, the equation for circulation was a function of both entropy and enthalpy. Even though this paper dealt with homenergetic flows, it is easy to see where non-homenergetic flows can be important, as are the solutions of their flow fields. One of the most obvious cases would be in the boundary layer where both entropy and enthalpy might vary between streamlines. This boundary layer problem could prove to be truly interesting because this rotational method is so easy to perform that it would not be much trouble to make several quick iterations to locate the boundary layer.

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