HYPERSONIC POWER-LAW BODIES
OF MAXIMUM LIFT-TO-Drag RATIO

by

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Abstract

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The problem of maximizing the lift-to-drag ratio of a slender, flat-top hypersonic body is investigated under the assumptions that the pressure distribution is Newtonian and the skin-friction coefficient is constant. Direct methods are employed, and the analysis is confined to the class of bodies whose transversal contour is semicircular and whose longitudinal contour is a power law.

First, unconstrained configurations are considered, and the combination of power law exponent and the thickness ratio maximizing the lift-to-drag ratio is determined. It is found that the maximum lift-to-drag ratio $L/D = 0.360 C_f^{-1/3}$ and corresponds to a conical configuration of thickness ratio $t/l = 1.18 C_f^{1/3}$, where $C_f$ is the skin-friction coefficient.

Next, constrained configurations are considered, that is, conditions are imposed on the length, the thickness, the volume, the wetted area, and the center of pressure. For each combination of constraints, an appropriate similarity parameter is introduced, and the optimum power law exponent, thickness ratio, and the lift-to-drag ratio are determined as functions of the similarity parameter.
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1. **INTRODUCTION**

Current interest in long range vehicles capable of cruising or gliding at hypersonic speeds and maneuverable vehicles capable of reentering the earth's atmosphere from outer space missions has emphasized a definite need for configurations (bodies, wings, and wing-body combinations) with high lift-to-drag ratios. This is apparent because both the range and the maneuverability increase with the lift-to-drag ratio.

In order to determine these configurations, it is, first of all, necessary to relate the aerodynamic forces to the shapes of the bodies. At hypersonic speeds, a simple—yet quite reliable—method is the Newtonian impact theory. The basic idea of this theory is that, if the free-stream Mach number is sufficiently large, the shock wave lies so close to the body that it can be regarded to be identical with the body surface. Consequently, the pressure coefficient is determined with the assumption that the particles crossing the shock wave conserve the tangential component of the momentum but lose the normal component. With this theory, the aerodynamic characteristics were determined for particular shapes (Ref. 1 through 6), and it was concluded that, in general, the theory agrees with the experimental results quite well for sharp-nosed cones and convex configurations.

With the above assumption and the further assumption that the skin-friction coefficient is constant, Professor Miele made an intensive research on slender, flat-top, homothetic bodies at hypersonic speeds. In Ref. 7, analytical expressions were derived relating the drag, the lift, and the lift-to-drag ratio to the geometry.
of the configuration, that is, the longitudinal and transversal contours of the homothetic body. In Ref. 8, two complementary variational problems were formulated, that of optimizing the longitudinal contour for a given transversal contour and that of optimizing the transversal contour for a given longitudinal contour, the criterion of optimization being the lift-to-drag ratio. The existence of similar solutions were investigated, and it was concluded that (1) the optimum longitudinal contour of a body of arbitrary transversal contour can be determined from the known optimum longitudinal contour of a body of semicircular cross section and (2) the optimum transversal contour of a body of arbitrary longitudinal contour can be determined from the known optimum transversal contour of a conical body.

Because of Ref. 8, the general problem of finding the optimum configuration is reduced to that of determining the extremal properties of these reference bodies. Here, a body of semicircular cross section is considered, and its longitudinal contour is determined so as to yield the maximum lift-to-drag ratio, either without constraints or with constraints imposed on the length, the thickness, the volume, the wetted area, and the position of the center of pressure. Direct methods are employed, and the analysis is confined to the class of power law contours.

The hypotheses employed are as follows: (a) a plane of symmetry exists between the left-hand and right-hand sides of the body; (b) the upper surface is flat; (c) the base plane is perpendicular to both the plane of symmetry and the plane of the flat top; (d) the free-stream velocity is perpendicular to the base plane and, therefore, is parallel to the line of intersection of the plane of symmetry and the
plane of the flat top; (e) the pressure coefficient is twice the cosine squared of
the angle formed by the free-stream velocity and the normal to each surface
element; (f) the skin-friction coefficient is constant; (g) the contribution of the
tangential forces to the lift is negligible with respect to the contribution of the
normal forces; (h) the body is slender in the longitudinal sense; (i) the transversal
contour is semicircular; and (j) the longitudinal contour is represented by a
power law.
2. **FUNDAMENTAL EQUATIONS**

Consider a cylindrical coordinate system \( O\rho r \) (Fig. 1): the origin \( O \)
is the apex of the body; the \( x \)-axis is the line of intersection of the plane of
symmetry and the flat top, and positive leeward; \( r \) is the distance of any point
from the \( x \)-axis; and \( \theta \) measures the angular position of this point with respect
to the flat top.

If the hypotheses (a) through (h) are invoked and if the lower surface of
the body is represented by the relationship

\[
r = r(x, \theta)
\]

the aerodynamic quantities (drag \( D \), lift \( L \), and pitching moment \( M \) per unit
free-stream dynamic pressure \( q \)) and the geometric quantities (volume \( V \) and
wetted area \( S \)) can be written as (Ref. 7 and 8)

\[
\frac{D}{q} = \int_0^\ell \int_0^{\pi/2} \left[ \frac{4r x^2}{r^2 + r_\theta^2} \right] \sin \theta \, d\theta \, dx + 2C_f \int_0^\ell r(x, 0) \, dx
\]

\[
\frac{L}{q} = \int_0^\ell \int_0^{\pi/2} \left[ \frac{4r^2 r_\theta^2}{r^2 + r_\theta^2} \right] (r \sin \theta - r_\theta \cos \theta) \, d\theta \, dx
\]

\[
\frac{M}{q} = \int_0^\ell \int_0^{\pi/2} \left[ \frac{4xr^2 r_\theta}{r^2 + r_\theta^2} \right] (r \sin \theta - r_\theta \cos \theta) \, d\theta \, dx
\]

\[
V = \int_0^\ell \int_0^{\pi/2} r^2 \, d\theta \, dx
\]

\[
S = \int_0^\ell \int_0^{\pi/2} 2\sqrt{r^2 + r_\theta^2} \, d\theta \, dx + \int_0^\ell r(x, 0) \, dx
\]
where $t$ is the length of the body and the subscripts $x$ and $\theta$ denote partial derivatives.

Next, we invoke hypothesis (i) and observe that, if every cross section is semicircular, the function (1) degenerates into

$$r = r(x)$$

(3)

with the implication that

$$r_\theta = 0$$

(4)

everywhere. Hence, the surface integrals (2) reduce to the line integrals

$$D/q = \int_0^t r \left[ 2\pi r^3 + C_4(2 + \pi) \right] dx$$

$$L/q = \int_0^t 4rr' dx$$

$$M/q = \int_0^t 4xrr' dx$$

(5)

$$V = (\pi/2) \int_0^t r^2 dx$$

$$S = (2 + \pi) \int_0^t r dx$$

in which $r'$ denotes the total derivative $dr/dx$.

In accordance with hypothesis (j), we specialize the longitudinal contour (3) to the power law

$$r/t = (x/t)^h$$

(6)
where \( n \) is an undetermined exponent and \( t \) is the base thickness. Consequently, Eqs. (5) become

\[
\begin{align*}
\frac{D}{q} &= \left(\frac{t^4}{L^2}\right) f_1 + t \ell C_f f_2 \\
\frac{L}{q} &= \left(\frac{t^3}{L}\right) f_3 \\
\frac{M}{q} &= t^3 f_4 \\
V &= t^2 t f_5 \\
S &= t t f_2
\end{align*}
\]

in which the functions

\[
\begin{align*}
f_1(n) &= \pi n^3 / (2n - 1) \\
f_2(n) &= (2 + \pi) / (n + 1) \\
f_3(n) &= 4n^2 / (3n - 1) \\
f_4(n) &= 4n / 3 \\
f_5(n) &= \pi / 2(2n + 1)
\end{align*}
\]

depend on the power law exponent only, and are valid for \( n > 0.5 \).

Certain derived quantities can be obtained from Eqs. (7). Thus, the dimensionless distance of the center of pressure from the apex \( \xi_0 = x_0 / \ell = M/Lt \) is given by

\[
\xi_0 = \frac{f_4}{f_3}
\]
and the lift-to-drag ratio by

\[ L/D = \frac{(t^2/\ell) f_3}{(t^3/\ell^2) f_1 + \tau C_f f_2} \]  \hspace{1cm} (10)

For convenience, Eq. (10) can be rewritten as

\[ E = \frac{\tau^2 f_3}{\tau^3 f_1 + f_2} \]  \hspace{1cm} (11)

providing the lift-to-drag ratio parameter \( E \) and the thickness ratio parameter \( \tau \) are defined as

\[ E = (L/D) C_f^{1/3} \quad , \quad \tau = (t/\ell) C_f^{-1/3} \]  \hspace{1cm} (12)
3. **UNCONSTRAINED CONFIGURATION**

The first step in the analysis is to determine the maximum lift-to-drag ratio of a configuration which is unconstrained geometrically and aerodynamically. According to Eq. (11), the lift-to-drag ratio parameter depends on both the thickness ratio parameter and the power law exponent, that is, it has the form $E = E(\tau, n)$. Therefore, the optimum values of $\tau$ and $n$ are determined by the simultaneous relationships

$$
E_{\tau} = 0, \quad E_{n} = 0
$$

in which the subscripts denote partial derivatives. These relationships can be written explicitly as

$$
\tau^3 f_1 - 2f_2 = 0
$$

$$
\dot{f}_3 (\tau^3 f_1 + f_2) - f_3 (\tau^3 \dot{f}_1 + f_2) = 0
$$

with the dot sign denoting a total derivative with respect to $n$. From Eq. (14-1), it appears that the optimum thickness ratio is such that the skin-friction drag is one-third of the total drag. Furthermore, upon eliminating the thickness ratio from Eqs. (14), we obtain the relationship

$$
2g_1 + g_2 - 3g_3 = 0
$$
where

\[ g_1 = \frac{\dot{f}_1}{f_1}, \quad g_2 = \frac{\dot{f}_2}{f_2}, \quad g_3 = \frac{\dot{f}_3}{f_3} \]  \hspace{1cm} (16)

On account of the definitions (8-1) through (8-3), we see that Eq. (15) is solved by

\[ n = 1 \]  \hspace{1cm} (17)

which means that the optimum longitudinal contour is conical. With this understanding, the thickness ratio parameter (14-1) and the lift-to-drag ratio parameter (11) become

\[ \tau = (2/\pi + 1)^{1/3} \approx 1.18 \]  \hspace{1cm} (18)

\[ E = \frac{4}{3\pi} \left( \frac{2}{\pi} \right)^{2/3} (2 + \pi)^{1/3} \approx 0.360 \]

Equation (18-2) represents the upper limit to the lift-to-drag ratio which can be obtained with a flat-top configuration of semicircular cross section subjected to a flow parallel to the flat-top. Should the configuration be required to satisfy a certain number of geometric and/or aerodynamic constraints, a loss in the lift-to-drag ratio would occur with respect to that predicted by Eq. (18-2).
4. **GIVEN CENTER OF PRESSURE**

To prescribe the nondimensional distance of the center of pressure from the apex is equivalent to prescribing the power law exponent in accordance with Eq. (9). Therefore, the lift-to-drag ratio parameter can be maximized with respect to the thickness ratio parameter only, and the relevant optimum condition is represented by Eq. (13-1) implicitly or Eq. (14-1) explicitly. Because of Eq. (14-1), the optimum thickness ratio parameter is given by

\[
\tau = \left(\frac{2f_2}{f_1}\right)^{1/3}
\]

(19)

and the associated lift-to-drag ratio parameter is

\[
E = \left(\frac{f_3}{3}\right)\left(\frac{4}{f_1^2f_2}\right)^{1/3}
\]

(20)

The parametric equations (9), (19), and (20) admit solutions of the form

\[
n = P(\xi_o) , \quad \tau = Q(\xi_o) , \quad E = R(\xi_o)
\]

(21)

which are plotted in Figs. 2 through 4. For \(\xi_o = 2/3\), the body is conical, and the maximum lift-to-drag ratio parameter reaches the upper limit represented by Eq. (18-2). For any other values of \(\xi_o\), lower values of the lift-to-drag ratio parameter are obtained as shown in Fig. 4.
5. **GIVEN THICKNESS AND LENGTH**

To prescribe the thickness and the length is equivalent to prescribing the thickness ratio parameter $\tau$ in accordance with the definition (12-2). Therefore, the lift-to-drag ratio parameter (11) can be maximized with respect to the power law exponent only, and the relevant optimum condition is represented by Eq. (13-2) implicitly or Eq. (14-2) explicitly. Because of Eq. (14-2), the optimum power law exponent satisfies the relationship

$$\tau = \left( \frac{f_1}{g_2 - g_3} \right)^{-1/3} \left( \frac{f_2}{g_3 - g_1} \right)^{1/3}$$

(22)

whose solutions are such that $n > 0.789$. The associated lift-to-drag ratio parameter is given by

$$E = \left( \frac{f_1}{g_2 - g_3} \right)^{-2/3} \left( \frac{f_2}{g_3 - g_1} \right)^{-1/3} \left( \frac{f_3}{g_1 - g_2} \right)$$

(23)

The parametric equations (22) and (23) admit solutions of the form

$$n = P(\tau) \quad , \quad E = R(\tau)$$

(24)

which are plotted in Figs. 5 and 6. When the thickness ratio parameter has the value 1.18, the configuration is conical, and the associated lift-to-drag ratio parameter is $E = 0.360$. For larger values of the thickness ratio parameter, the configuration is convex, and for smaller values, it is concave.
6. **GIVEN VOLUME AND LENGTH**

If the volume and the length are given, it is convenient to rewrite Eq. (7-4)
in the form

\[ K_1 = \tau^2 f_5 \]  \hspace{1cm} (25)

where

\[ K_1 = (V/\ell)^3 C_f^{-2/3} \]  \hspace{1cm} (26)

is the volume-thickness parameter, a known quantity. The lift-to-drag ratio
parameter (11) is to be maximized with respect to the combinations of \( \tau \) and \( n \)
which ensure the constancy of the right-hand side of Eq. (25). In accordance
with Lagrange multiplier theory, we introduce an undetermined constant \( \lambda \) and
define the fundamental function

\[ F = E + \lambda \tau^2 f_5 \]  \hspace{1cm} (27)

Then, the optimum conditions are

\[ F_\tau = 0, \quad F_n = 0 \]  \hspace{1cm} (28)

which are equivalent to

\[ E_\tau + 2\lambda \tau f_5 = 0, \quad E_n + \lambda \tau^2 f_5 = 0 \]  \hspace{1cm} (29)
and, upon elimination of the Lagrange multiplier, imply that

\[ \tau g_5 E_\tau - 2E_n = 0 \]  \hspace{1cm} (30)

where

\[ g_5 = \frac{f_5}{f_5} \]  \hspace{1cm} (31)

In the light of Eq. (11), Eq. (30) can be rewritten as

\[ \tau = \left( \frac{f_1}{g_2 - g_3 + g_5} \right)^{-1/3} \left( \frac{2f_2}{2g_3 + g_5 - 2g_1} \right)^{1/3} \]  \hspace{1cm} (32)

and its solutions are such that \( n > 0.735 \). The associated lift-to-drag ratio parameter and volume-length parameter are given by

\[ E = \left( \frac{f_1}{g_2 - g_3 + g_5} \right)^{-2/3} \left( \frac{2f_2}{2g_3 + g_5 - 2g_1} \right)^{-1/3} \left( \frac{2f_3}{3g_5 - 2g_1 + 2g_2} \right) \]  \hspace{1cm} (33)

\[ K_1 = \left( \frac{f_1}{g_2 - g_3 + g_5} \right)^{-2/3} \left( \frac{2f_2}{2g_3 + g_5 - 2g_1} \right)^{2/3} f_5 \]

The parametric equations (32) and (33) admit solutions of the form

\[ n = P(K_1) \quad \tau = Q(K_1) \quad E = R(K_1) \]  \hspace{1cm} (34)
which are plotted in Figs. 7 through 9. When the volume-length parameter has
the value 0.727, the configuration is conical, with a thickness ratio parameter
$\tau = 1.18$ and a lift-to-drag ratio parameter $E = 0.360$. For larger values of the
volume-length parameter, the configuration is convex, and for smaller values,
it is concave.
7. **GIVEN VOLUME AND THICKNESS**

If the volume and the thickness are given, it is convenient to rewrite Eq. (7-4) in the form

\[ K_2 = \frac{f_5}{\tau} \]  (35)

where

\[ K_2 = \left(\frac{V}{t^3}\right) C_f^{1/3} \]  (36)

is the volume-thickness parameter, a known quantity. The lift-to-drag ratio parameter (11) is to be maximized with respect to the combinations of \( \tau \) and \( n \) which ensure the constancy of the right-hand side of Eq. (35). In accordance with Lagrange multiplier theory, we introduce an undetermined constant \( \lambda \) and define the fundamental function

\[ F = E + \lambda f_5/\tau \]  (37)

Then, the optimum conditions are

\[ F_\tau = 0 \quad , \quad F_n = 0 \]  (38)

which are equivalent to

\[ \tau^2 E_\tau - \lambda f_5 = 0 \quad , \quad \tau E_n + \lambda f_\tau = 0 \]  (39)
and, upon elimination of the Lagrange multiplier, imply that

$$\tau g_5 E_n + E_n = 0$$  \hspace{1cm} (40)

In the light of Eq. (11), Eq. (40) can be rewritten as

$$\tau = \left( \frac{f_1}{g_2 - g_3 - 2g_5} \right)^{-1/3} \left( \frac{f_2}{g_3 - g_5 - g_1} \right)^{1/3}$$  \hspace{1cm} (41)

and its solutions are such that $0.5 < n < 1.31$. The associated lift-to-drag ratio parameter and volume-thickness parameter are given by

$$E = \left( \frac{f_1}{g_2 - g_3 - 2g_5} \right)^{-2/3} \left( \frac{f_2}{g_3 - g_5 - g_1} \right)^{-1/3} \left( \frac{f_3}{3g_5 + g_1 - g_2} \right)$$  \hspace{1cm} (42)

$$K_2 = \left( \frac{f_1}{g_2 - g_3 - 2g_5} \right)^{1/3} \left( \frac{f_2}{g_3 - g_5 - g_1} \right)^{-1/3} f_5$$

The parametric equations (41) and (42) admit solutions of the form

$$n = P(K_2) \hspace{0.5cm}, \hspace{0.5cm} \tau = Q(K_2) \hspace{0.5cm}, \hspace{0.5cm} E = R(K_2)$$  \hspace{1cm} (43)

which are plotted in Figs. 10 through 12. When the volume-thickness parameter has the value 0.444, the configuration is conical, with a thickness ratio parameter $\tau = 1.18$ and a lift-to-drag ratio parameter $E = 0.360$. For larger values of the volume-thickness parameter, the configuration is convex, and for smaller values, it is concave.
8. GIVEN WETTED AREA AND LENGTH

If the wetted area and the length are given, it is convenient to rewrite

Eq. (7-5) in the form

$$K_3 = \tau f_2$$

(44)

where

$$K_3 = (S/L^2)C_f^{-1/3}$$

(45)

is the wetted area-length parameter, a known quantity. The lift-to-drag ratio parameter (11) is to be maximized with respect to the combinations of $\tau$ and $n$ which ensure the constancy of the right-hand side of Eq. (44). In accordance with Lagrange multiplier theory, we introduce an undetermined constant $\lambda$ and define the fundamental function

$$F = E + \lambda \tau f_2$$

(46)

Then, the optimum conditions are

$$F_\tau = 0 , \quad F_n = 0$$

(47)

which are equivalent to

$$E_\tau + \lambda f_2 = 0 , \quad E_n + \lambda \tau \hat{f}_2 = 0$$

(48)
and, upon elimination of the Lagrange multiplier, imply that

\[ \tau g_{\tau} E - E = 0 \quad (49) \]

In the light of Eq. (11), Eq. (49) can be written as

\[ \tau = \left( -\frac{f_1}{3g_2 - g_3} \right) ^{-1/3} \left( \frac{f_2}{g_1 - g_2 - g_3} \right) ^{1/3} \quad (50) \]

and its solutions are such that \( n > 0.719 \). The associated lift-to-drag ratio parameter and wetted area-length parameter are given by

\[ E = \left( -\frac{f_1}{3g_2 - g_3} \right) ^{-2/3} \left( \frac{f_2}{g_1 - g_2 - g_3} \right) ^{-1/3} \left( \frac{f_3}{g_1 - 4g_2} \right) \quad (51) \]

\[ K_3 = \left( -\frac{f_1}{3g_2 - g_3} \right) ^{-1/3} \left( \frac{f_2}{g_1 - g_2 - g_3} \right) ^{1/3} \left( \frac{f_2}{g_1 - g_2 - g_3} \right) \]

The parametric equations (50) and (51) admit solutions of the form

\[ n = P(K_3) \quad , \quad \tau = Q(K_3) \quad , \quad E = R(K_3) \quad (52) \]

which are plotted in Figs. (13) through (15). When the wetted area-length parameter has the value 3.03, the configuration is conical, with a thickness ratio parameter \( \tau = 1.18 \) and a lift-to-drag ratio parameter \( E = 0.360 \). For larger values of the wetted area-length parameter, the configuration is convex, and for smaller values, it is concave.
9. **GIVEN WETTED AREA AND THICKNESS**

If the wetted area and the thickness are given, it is convenient to rewrite Eq. (7-5) in the form

\[ K_4 = f_2 / \tau \]  

(53)

where

\[ K_4 = (S / t^2) C_f^{1/3} \]  

(54)

is the wetted area-thickness parameter, a known quantity. The lift-to-drag ratio parameter (11) is to be maximized with respect to the combinations of \( \tau \) and \( n \) which ensure the constancy of the right-hand side of Eq. (53). In accordance with Lagrange multiplier theory, we introduce an undetermined constant \( \lambda \) and define the fundamental function

\[ F = E + \lambda f_2 / \tau \]  

(55)

Then, the stationary conditions are given by

\[ F_\tau = 0 \quad , \quad F_n = 0 \]  

(56)

which are equivalent to

\[ \tau^2 E_\tau - \lambda f_2 = 0 \quad , \quad \tau E_n + \lambda f_2 = 0 \]  

(57)
and, upon elimination of the Lagrange multiplier, imply that

$$\tau g_2 E_{\tau} + E_n = 0$$  (58)

In the light of Eq. (11), Eq. (58) can be rewritten as

$$\tau \left[ n^3 - \left( \frac{f_1}{g_2 + g_3} \right) - 1 \left( \frac{f_2}{g_1 + g_2 - g_3} \right) \right] = 0$$  (59)

After Eqs. (11), (53) and (59) are combined, the power law exponent, the thickness ratio parameter and the lift-to-drag ratio parameter can be solved in the form

$$n = P(K_4) \quad , \quad \tau = Q(K_4) \quad , \quad E = R(K_4)$$  (60)

The functions (60) are double-valued for $K_4 < 3.31$, quadruple-valued for $3.31 < K_4 < 3.73$, and triple-valued for $K_4 > 3.37$. Therefore, one must separate relative minimum solutions from relative maximum solutions and, among the latter, one must determine the absolute maximum solution by direct comparison of the lift-to-drag ratio parameter. This absolute maximum solution is plotted in Figs. (16) through (18). The optimum longitudinal contour is conical for $K_4 \leq 3.33$ and concave for $K_4 \geq 3.33$. For $K_4 \geq 3.73$, the configuration is such that $\tau = 0$ and $n = \infty$.

The maximum lift-to-drag ratio parameter achieves its upper limit $E = 0.360$ when the wetted area-thickness parameter is 2.18, and the associated thickness ratio parameter is 1.18.
10. DISCUSSION AND CONCLUSIONS

In the previous sections, the problem of maximizing the lift-to-drag ratio of a slender, flat-top, hypersonic body is investigated under the assumptions that the pressure distribution is Newtonian and the skin-friction coefficient is constant. Direct methods are employed, and the analysis is confined to the class of bodies whose transversal contour is semicircular and whose longitudinal contour is a power law.

First, unconstrained configurations are considered, and the combination of thickness ratio and power law exponent maximizing the lift-to-drag ratio is determined. For a friction coefficient $C_f = 10^{-3}$, the maximum lift-to-drag ratio is $L/D = 3.60$ and corresponds to a half-cone of thickness ratio $t/\ell = 0.118$.

Next, constrained configurations are considered, that is, given conditions are imposed on the length, the thickness, the volume, the wetted area, and the position of the center of pressure. For each case, an appropriate similarity parameter is introduced, and the optimum power law exponent, thickness ratio, and lift-to-drag ratio are determined as functions of the similarity parameter. The lift-to-drag ratio of a constrained configuration is smaller than that of the optimum unconstrained configuration; however, for a particular value of the similarity parameter, equality is achieved.

While the longitudinal contour is conical for an unconstrained configuration, it can be either convex or concave for constrained configurations, depending on the value of the similarity parameter. Since the Newtonian pressure law has been verified experimentally for convex configurations only, the results pertaining
to concave configurations are merely indicative of qualitative trends.

To examine how closely the true optimum configurations can be approximated by power law bodies, a comparison with the results obtained in Ref. 9 using the indirect method of the calculus of variations is in order. For unconstrained configurations the solutions obtained by direct methods and those obtained by indirect methods are identical. For constrained configurations, the solutions are almost identical as long as \((t/\ell)C_f^{-1/3} \geq 1.18\) (given \(t\) and \(\ell\)), as long as \((V/t^3)C_f^{1/3} \leq 0.526\) (given \(t\) and \(V\)), and as long as \((V/t^3)C_f^{-2/3} \geq 0.727\) (given \(t\) and \(V\)). For other values of the similarity parameters, the differences are more significant. This is due to the fact that the indirect methods yield optimum bodies whose rear part is a half-cylinder.

Finally, it is of interest to compare the present lift-to-drag ratios with those characteristic of drag-optimized, flat-top configurations. The analysis is omitted for the sake of brevity, since it involves only a slight modification of that presented in Ref. 10. As expected, the lift-to-drag ratio of a minimum drag configuration is lower than that of a maximum lift-to-drag ratio configuration. The relative differences depend on the similarity parameter and are shown in Figs. 5 through 18.
REFERENCES


LIST OF CAPTIONS

Fig. 1  Coordinate system.

Fig. 2  Power law exponent (given center of pressure).

Fig. 3  Optimum thickness ratio (given center of pressure).

Fig. 4  Maximum lift-to-drag ratio (given center of pressure).

Fig. 5  Optimum power law exponent (given thickness and length).

Fig. 6  Maximum lift-to-drag ratio (given thickness and length).

Fig. 7  Optimum power law exponent (given volume and length).

Fig. 8  Optimum thickness ratio (given volume and length).

Fig. 9  Maximum lift-to-drag ratio (given volume and length).

Fig. 10 Optimum power law exponent (given volume and thickness).

Fig. 11 Optimum thickness ratio (given volume and thickness).

Fig. 12 Maximum lift-to-drag ratio (given volume and thickness).

Fig. 13 Optimum power law exponent (given wetted area and length).

Fig. 14 Optimum thickness ratio (given wetted area and length).

Fig. 15 Maximum lift-to-drag ratio (given wetted area and length).

Fig. 16 Optimum power law exponent (given wetted area and thickness).

Fig. 17 Optimum thickness ratio (given wetted area and thickness).

Fig. 18 Maximum lift-to-drag ratio (given wetted area and thickness).
Figure 6
Figure 9

V, l ≡ Given

E_{max}

D_{min}

(l/l_0) C_f^{1/3}

0.4

0.3

0.2

0.1

0.5

1.0

1.5

2.0

2.5
Figure 12
Figure 15
Figure 17

S, t = Given

D_{min}

E_{max}

\frac{(S/t^2)C_f^{1/3}}{(l/l)C_f^{-1/3}}

D_{min}, E_{max}