RICE UNIVERSITY

LINEAR, INCOMPRESSIBLE HYPOFLUENT LUBRICANTS IN CONTINUOUS-SLEEVE JOURNAL BEARINGS

by

George Joseph Fix, III

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

Master of Science

Thesis Director's signature:

Paul R. Parley

Houston, Texas

May, 1965

ABSTRACT

A perturbation procedure is used to find the first effects of elasticity in the lubricating media. This is achieved by considering a linear incompressible hypofluent type material with a small elastic effect in a journal. The parameter which measures the extent of the elastic influence on the solution is proportional to

$$\frac{(\text{viscosity}) \qquad (\text{speed of rotation})}{(\text{elastic shear modulus}) \qquad (\text{clearance})}$$

The relationship between the external load on a journal and the above elastic parameter, as well as the relationship between the driving torque and the elasticity parameter is given.

For most liquid lubricants in standard practicing situations, the effects of elasticity in the incompressible fluid are negligible. The solution shows that the classical normality between the eccentric displacement and lateral loading expected for an incompressible fluid is not preserved when the fluid has elastic effects.

George Fix

ACKNOWLEDGEMENTS

The author wishes to express his appreciation to Professor Paul R. Paslay for his suggestion of this thesis topic and for his invaluable assistance and advice during its preparation.

Gratitude is also extended to my wife, Linda, without whose moral support and patience the completion of this thesis would have not been possible.

TABLE OF CONTENTS

NOMENCLATURE

x_j	j<u>th</u> coordinate
$\dot{x}_j$	j<u>th</u> component of velocity tensor
u	radial physical component in cylindrical coordinates of velocity tensor
v	tangential physical component in cylindrical coordinates of velocity tensor
$\dot{x}_{i,j}$	covariant derivative of i<u>th</u> component of the velcoity tensor with respect to j<u>th</u> coordinate
d_{ij}	$\frac{1}{2}(\dot{x}_{i,j} + \dot{x}_{j,i})$ = ij component of deformation rate tensor
ω_{ij}	$\frac{1}{2}(\dot{x}_{i,j} - \dot{x}_{j,i})$ = ij component of vorticity tensor
g_{ij}	ij component of metric tensor
t^i_j	ij mixed component of stress tensor
σ_{ij}	physical component of $s^i_j = \sqrt{\frac{g_{ij}}{g_{jj}}}\, s^i_j$
p	$-\frac{1}{3}\left(\sum_{\ell=1}^{3} s^\ell_\ell\right)$ = mean normal stress
δ^i_j	Kronecker delta
s^i_j	$t^i_j + p\,\delta^i_j$ = deviatoric stress tensor
$\dot{s}^i_j$	$\frac{\partial s^i_j}{\partial t} + \sum_{\ell=1}^{3} s^i_{j,\ell}\,\dot{x}^\ell + \sum_{\ell=1}^{3} s^i_\ell\,\omega^\ell_j - \sum_{\ell=1}^{3} \omega^i_\ell\, s^\ell_j$ = Jaumann stress rate[6]

R	outside bearing radius
R_o	inside bearing radius
e	bearing eccentricity
δ	$R - R_o$
L	bearing width
η	viscosity coefficient
G	elastic shear modulus
ω	angular velocity of inner bearing
V	$R_o \omega$
λ_1	$\frac{\eta V}{G\delta}$ = elastic perturbation parameter
λ_2	$\frac{e}{R_o}$ = perturbation parameter associated with bearing eccentricity
W_x	horizontal component of bearing load
W_y	vertical component of bearing load
φ	$\tan^{-1} \frac{W_y}{W_x}$
$W\left(\frac{\eta V}{G\delta}\right)$	$\sqrt{W_x^2 + W_y^2}$

I. INTRODUCTION

In recent years many different phenomena occuring in lubrication practice have been investigated analytically[1]. However, no satisfactory analytical results have been published concerning the influence of the elasticity of the lubricating medium, an effect which is known to occur at high shearing rates[2]. To investigate this effect a continuous journal, fully filled, is considered in which cavitation is ignored.

In Section II, an appropriate lubricant, the Linear Hypofluent material, is proposed. This material, while being primarily a viscous fluid, has in addition some elastic properties. Also in Section II the tensorial constitutive equation defining the Linear Hypofluent material is represented in cylindrical coordinates.

The assumption that the elastic effects are small, the case most applicable to the Linear Hypofluent material, and that the bearing eccentricity is small in comparison with the bearing radius, makes a perturbation solution possible. In Section III the governing field equations and boundary conditions are expanded in terms of the parameters $\lambda_1 = \eta V/G\delta$ and $\lambda_2 = e/R_o$, the first being a measure of the elastic effects and the second a measure of the bearing eccentricity.

In Section IV the zero order solution is given; in Sections V and VI the first order solutions are given, while in Sections VII and VIII the second order solutions are given.

The two most important results needed in bearing practice are the expressions for the load carrying capacity, W, and the driving torque, T. The expressions are developed in Section IX, and graphs are obtained which show the dependance of T and W on the elastic pertubation parameter, $\lambda_1 = \eta V/G\delta$.

II. DEVELOPMENT OF GOVERNING EQUATIONS

There is a wide class of materials which can be generally classified as elastic-viscous, or more commonly visco-elastic. The simplest of which, and the one best suited for this particular investigation where elastic effects are small, is the Linear Hypofluent material (sometimes called the linear Maxwell material). The material is defined by the following constitutive equation

$$d^i_j = \frac{1}{2\eta} S^i_j + \frac{1}{2G} \dot{S}^i_j \qquad \text{(II.1)}$$

An elegant discussion of this material and its generalizations is given in a paper by Noll[3]. Simple problems illustrating the one-dimensional behavior of this material are given in 4 and 5.

Figure I shows a cross section of the journal configuration under consideration with certain geometric relations. Neglecting the end effects for this long bearing leads to the following velocity field for a cylindrical coordinate system with origin at 0:

$$\begin{aligned} \dot{x}^r &= u(r,\theta) \\ \dot{x}^\theta &= v(r,\theta)/r \qquad \text{(II.2)} \\ \dot{x}^z &= 0 \end{aligned}$$

The stress field may be represented as follows in terms of physical components:

$$\begin{aligned}
\sigma_{rr} &= \sigma_{rr}(r,\theta) & \sigma_{r\theta} &= \sigma_{r\theta}\ (r,\theta) \\
\sigma_{\theta\theta} &= \sigma_{\theta\theta}(r,\theta) & \sigma_{rz} &= 0 \\
\sigma_{zz} &= \frac{1}{2}\ (\sigma_{rr} + \sigma_{\theta\theta}) & \sigma_{r\theta} &= 0
\end{aligned} \tag{II.3}$$

The boundary conditions for this problem are (referring to Figure 1)

$$\begin{aligned}
&\text{at } r = R_o & u &= 0 \\
& & v &= R_o\ \omega \equiv V \\
&\text{at } r = R + e\cos\theta & u &= 0 \\
& & v &= 0
\end{aligned} \tag{II.4}$$

For this velocity field the constitutive equations can be shown to be:

$$\left.\begin{aligned}
2\frac{\partial u}{\partial r} &= \frac{1}{2\eta}\ (\sigma_{rr}-\sigma_{\theta\theta}) + \frac{u}{2G}\ \frac{\partial}{\partial r}\ (\sigma_{rr} - \sigma_{\theta\theta}) \\
&\quad + \frac{v}{2Gr}\ \frac{\partial}{\partial\theta}\ (\sigma_{rr}-\sigma_{\theta\theta}) + \frac{r\sigma_{r\theta}}{G}\left(\frac{\partial}{\partial r}\left(\frac{v}{r}\right) - \frac{1}{r^2}\ \frac{\partial u}{\partial\theta}\right) \\
\frac{\partial u}{\partial\theta} + r^2\frac{\partial}{\partial r}\left(\frac{v}{r}\right) &= \frac{1}{\eta}\ r\sigma_{r\theta} + \frac{ur}{G}\frac{\partial}{\partial r}\ \sigma_{r\theta} + \frac{v}{G}\ \frac{\partial}{\partial\theta}\ \sigma_{r\theta} \\
&\quad + \frac{1}{2G}\ (\sigma_{rr} - \sigma_{\theta\theta})\left(\frac{\partial u}{\partial\theta} - r^2\ \frac{\partial}{\partial r}\ \left(\frac{v}{r}\right)\right) \\
\frac{\partial u}{\partial r} + \frac{1}{r}\ \frac{\partial v}{\partial\theta} + \frac{u}{r} &= 0
\end{aligned}\right\} \tag{II.5}$$

The only non-trivial equilibrium equations are given below where the inertia terms have been neglected:

$$\frac{\partial}{\partial r}\sigma_{rr} + \frac{1}{r}\frac{\partial}{\partial\theta}\sigma_{r\theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0$$

$$\frac{\partial}{\partial r}(r\sigma_{r\theta}) + \frac{\partial}{\partial\theta}(\sigma_{\theta\theta}) + \sigma_{r\theta} = 0$$

III. PERTURBATION EXPANSIONS

Since the elastic effects on the lubricant are small and since for most bearings in practice the eccentricity is small in comparison with the journal radius, a double perturbation on the parameters

$$\lambda_1 = \frac{\eta V}{G\delta}, \qquad \lambda_2 = \frac{e}{R_o}$$

is possible. The velocities and the stresses are expanded as indicated below:

$$\left.\begin{aligned}
u &= u_o + \lambda_1 u_1 + \lambda_2 u_2 + \lambda_1\lambda_2 u_{12} + \lambda_1^2 u_{11} \\
v &= v_o + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_1\lambda_2 v_{12} + \lambda_1^2 v_{11} \\
\sigma_{rr} &= \sigma_{rr_o} + \lambda_1 \sigma_{rr_1} + \lambda_2 \sigma_{rr_2} + \lambda_1\lambda_2 \sigma_{rr_{12}} + \lambda_1^2 \sigma_{rr_{11}} \\
\sigma_{\theta\theta} &= \sigma_{\theta\theta_o} + \lambda_1 \sigma_{\theta\theta_1} + \lambda_2 \sigma_{\theta\theta_2} + \lambda_1\lambda_2 \sigma_{\theta\theta_{12}} + \lambda_1^2 \sigma_{\theta\theta_{11}} \\
\sigma_{r\theta} &= \sigma_{r\theta_o} + \lambda_1 \sigma_{r\theta_1} + \lambda_2 \sigma_{r\theta_2} + \lambda_1\lambda_2 \sigma_{r\theta_{12}} + \lambda_1^2 \sigma_{r\theta_{11}}
\end{aligned}\right\} \quad \text{(III.1)}$$

The reader should note that the first order effects of both eccentricity and elasticity are included while the only second order terms included are the cross product effect ($\lambda_1\lambda_2$ term) and the elastic effect (λ_1^2 term). These terms were felt adequate for the purposes of the present investigation.

The above expansions (III.1) are substituted into the governing field equations and, successively, coefficients of $\lambda_1, \lambda_2, \lambda_1^2, \lambda_{12}$ are set to zero. The resulting equations are:

Zero order solution

(i) constitutive equations

$$\left.\begin{aligned} 2\frac{\partial}{\partial r}\left(\frac{u_o}{V}\right) &= \frac{(\sigma_{rr_o}-\sigma_{\theta\theta_o})}{2\eta V} \\ \frac{\partial}{\partial\theta}\left(\frac{u_o}{V}\right) + r^2\frac{\partial}{\partial r}\left(\frac{v_o}{rV}\right) &= \frac{r\sigma_{r\theta_o}}{\eta V} \end{aligned}\right\} \qquad \text{(III.2)}$$

(ii) equilibrium equations

$$\left.\begin{aligned} \frac{\partial\sigma_{rr_o}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta_o}}{\partial\theta} + \frac{1}{r}(\sigma_{rr_o} - \sigma_{\theta\theta_o}) &= 0 \\ \frac{\partial}{\partial r}(r\sigma_{r\theta_o}) + \frac{\partial}{\partial\theta}\sigma_{\theta\theta_o} + \sigma_{r\theta_o} &= 0 \end{aligned}\right\} \qquad \text{(III.2)}$$

First order elasticity solution (λ_1)

(i) constitutive equations

$$\begin{aligned} 2\frac{\partial}{\partial r}\frac{u_1}{V} = {} & \frac{(\sigma_{rr_1}-\sigma_{\theta\theta_1})}{2\eta V} + \frac{u_o\delta}{2V}\frac{\partial}{\partial r}\left(\frac{\sigma_{rr_o}-\sigma_{\theta\theta_o}}{\eta V}\right) \\ & + \frac{v_o\delta}{2rV}\frac{\partial}{\partial\theta}\left(\frac{\sigma_{rr_\theta} - \sigma_{\theta\theta_o}}{\eta V}\right) + \frac{\partial}{\partial r}\left(\frac{v_o}{rV}\right)\frac{r\sigma_{r\theta_o}\delta}{\eta V} \\ & - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{u_o}{V}\right)\left(\frac{\sigma_{r\theta_o}\delta}{\eta V}\right) \end{aligned} \qquad \text{(III.3)}$$

$$\frac{\partial}{\partial\theta}\left(\frac{u_1}{V}\right) + r^2\frac{\partial}{\partial r}\left(\frac{v_1}{rV}\right) = \frac{r\sigma_{r\theta_o}}{\eta V} + \frac{u_o}{V}\frac{\partial}{\partial r}\left[\frac{r\sigma_{r\theta_o}\delta}{\eta V}\right] - \frac{u_o}{V}\left(\frac{\sigma_{r\theta_o}\delta}{\eta V}\right)$$

$$+\left(\frac{v_o}{\sigma V}\right)\frac{\partial}{\partial\theta}\left(\frac{\sigma_{r\theta_o}\delta}{\eta V}\right) + \left[\frac{\partial}{\partial\theta}\left(\frac{u_o}{V}\right) - r^2\frac{\partial}{\partial r}\left(\frac{v_o}{rV}\right)\right]$$

$$\frac{\left(\sigma_{rr_o} - \sigma_{\theta\theta_o}\right)}{2\eta V}$$

(ii) Equilibrium equations

$$\left.\begin{aligned} &\frac{\partial\sigma_{rr_1}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta_1}}{\partial\theta} + \frac{1}{r}\left(\sigma_{rr_1} - \sigma_{\theta\theta_1}\right) = 0 \\ &\frac{\partial}{\partial r}\left(r\sigma_{r\theta_1}\right) + \frac{\partial}{\partial\theta}\sigma_{\theta\theta_1} + \sigma_{r\theta_1} = 0 \end{aligned}\right\} \quad (III.3)$$

First order viscous solution (λ_2)

(i) constitutive equations

$$\left.\begin{aligned} &2\frac{\partial}{\partial r}\left(\frac{u_2}{V}\right) = \frac{\left(\sigma_{rr_2} - \sigma_{\theta\theta_2}\right)}{2\eta V} \\ &\frac{\partial}{\partial\theta}\frac{u_2}{V} + r^2\frac{\partial}{\partial r}\left(\frac{v_2}{rV}\right) = \frac{r\sigma_{r\theta_2}}{\eta V} \end{aligned}\right\} \quad (III.4)$$

(ii) equilibrium equations:

$$\left.\begin{aligned} &\frac{\partial\sigma_{rr_2}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta_2}}{\partial\theta} + \frac{1}{r}\left(\sigma_{rr_2} - \sigma_{\theta\theta_2}\right) = 0 \\ &\frac{\partial}{\partial r}\left(r\sigma_{r\theta_2}\right) + \frac{\partial}{\partial\theta}\sigma_{\theta\theta_2} + \sigma_{r\theta_2} = 0 \end{aligned}\right\} \quad (III.4)$$

Second order elasticity solution (λ_1^2)

(i) constitutive equations

$$2\frac{\partial u_{11}}{\partial r} = \frac{\left(\sigma_{rr_{11}} - \sigma_{\theta\theta_{11}}\right)}{2\eta} + \frac{u_o\delta}{\eta V}\frac{\partial}{\partial r}\frac{\left(\sigma_{rr_1} - \sigma_{\theta\theta_1}\right)}{2}$$

$$+ \frac{u_1\delta}{\eta V}\frac{\partial}{\partial r}\frac{\left(\sigma_{rr_o} - \sigma_{\theta\theta_o}\right)}{2} + \frac{v_o\delta}{r\eta\nabla}\frac{\partial}{\partial\theta}\frac{\left(\sigma_{rr_1} - \sigma_{\theta\theta_1}\right)}{2}$$

$$+ \frac{v_1\delta}{r\eta\nabla}\frac{\partial}{\partial\theta}\frac{\left(\sigma_{rr_o} - \sigma_{\theta\theta_o}\right)}{2} + \frac{\sigma_{r\theta_o}r\delta}{\eta V}\frac{\partial}{\partial r}\left(\frac{v_1}{r}\right)$$

$$+ \frac{\sigma_{r\theta_1}r\delta}{\eta V}\frac{\partial}{\partial r}\left(\frac{v_o}{r}\right) - \frac{\sigma_{r\theta_o}\delta}{r\eta V}\frac{\partial}{\partial\theta}u_1 - \frac{\sigma_{r\theta_1}\delta}{r\eta V}\frac{\partial u_o}{\partial\theta}$$

$$\frac{\partial u_{11}}{\partial\theta} + r^2\frac{\partial}{\partial r}\left(\frac{v_{11}}{r}\right) = \frac{r}{\eta}\sigma_{r\theta_{11}} + \frac{u_{o\delta}}{\eta V}\frac{\partial}{\partial r}\left(r\sigma_{r\theta_1}\right) \qquad \text{(III.5)}$$

$$+ \frac{u_{1\delta}}{\eta V}\frac{\partial}{\partial r}\left(r\sigma_{r\theta_o}\right) - \frac{u_{o\delta}}{\eta V}\sigma_{r\theta_1} - \frac{u_1\delta}{\eta V}\sigma_{r\theta_o}$$

$$+ \frac{v_o\delta}{\eta V}\frac{\partial\sigma_{r\theta_1}}{\partial\theta} + \frac{v_1\delta}{\eta V}\frac{\partial\sigma_{r\theta_o}}{\partial\theta} \quad \frac{\left(\sigma_{rr_o} - \sigma_{\theta\theta_o}\right)}{2\eta V}\frac{\partial u_1}{\partial\theta}$$

$$+ \frac{\left(\sigma_{rr_1} - \sigma_{\theta\theta_1}\right)}{2\eta V}\frac{\partial u_o}{\partial\theta} - \frac{r^2\left(\sigma_{rr_o} - \sigma_{\theta\theta_o}\right)}{2\eta V}\frac{\partial}{\partial r}\left(\frac{v_1}{r}\right)$$

$$- \frac{r^2\left(\sigma_{rr_1} - \sigma_{\theta\theta_1}\right)}{2\eta V}\frac{\partial}{\partial r}\left(\frac{v_o}{r}\right)$$

(ii) equilibrium equations

$$\frac{\partial\sigma_{rr_{11}}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta_{11}}}{\partial\theta} + \frac{1}{r}\left(\sigma_{rr_{11}} - \sigma_{\theta\theta_{11}}\right) = 0$$

$$\frac{\partial}{\partial r}\left(r\sigma_{r\theta_{11}}\right) + \frac{\partial\sigma_{\theta\theta_{11}}}{\partial\theta} + \sigma_{r\theta_{11}} = 0 \qquad \text{(III.5)}$$

Second order elastic-viscous solution $(\lambda_1 \lambda_2)$

(i) constitutive equations

$$
\left.
\begin{aligned}
& 2\frac{\partial u_{12}}{\partial r} = \frac{\left(\sigma_{rr_{12}} - \sigma_{\theta\theta_{12}}\right)}{2\eta} + \frac{u_o \delta}{nV} \frac{\partial}{\partial r} \left(\frac{\sigma_{rr_2} - \sigma_{\theta\theta_2}}{2}\right) \\
& + \frac{u_2 \delta}{nV} \frac{\partial}{\partial r} \left(\frac{\sigma_{rr_o} - \sigma_{\theta\theta_o}}{2}\right) + \frac{v_o \delta}{r\eta V} \frac{\partial}{\partial \theta} \left(\frac{\sigma_{rr_2} - \sigma_{\theta\theta_2}}{2}\right) \\
& + \frac{v_{2\delta}}{r\eta V} \frac{\partial}{\partial \theta} \left(\frac{\sigma_{rr_o} - \sigma_{\theta\theta_o}}{2}\right) + \frac{\sigma_{r\theta_o} r \delta}{\eta V} \frac{\partial}{\partial r} \left(\frac{v_2}{r}\right) \\
& + \frac{\sigma_{r\theta_2} r \delta}{\eta V} \frac{\partial}{\partial r} \left(\frac{v_o}{r}\right) - \frac{\sigma_{r\theta_o} \delta}{r\eta V} \frac{\partial u_2}{\partial \theta} - \frac{\sigma_{r\theta_2} \delta}{r\eta V} \frac{\partial u_o}{\partial \theta} \\
& \frac{\partial u_{12}}{\partial \theta} + r^2 \frac{\partial}{\partial r} \left(\frac{v_{12}}{r}\right) = \frac{r\sigma_{r\theta_{12}}}{\eta} + \frac{u_o \delta}{\eta V} \frac{\partial \left(r\sigma_{r\theta_2}\right)}{\partial r} \\
& + \frac{u_{2\delta}}{\eta V} \frac{\partial \left(r\sigma_{r\theta_o}\right)}{\partial r} - \frac{u_{o\delta}}{\eta V} \sigma_{r\theta_2} - \frac{u_2 \delta}{\eta V} \sigma_{r\theta_2} \\
& + \frac{v_o \delta}{\eta V} \frac{\partial \sigma_{r\theta_2}}{\partial \theta} + \frac{v_2 \delta}{\eta V} \frac{\partial \sigma_{r\theta_o}}{\partial \theta} + \frac{\left(\sigma_{rr_o} - \sigma_{\theta\theta_o}\right)\delta}{2\eta V} \frac{\partial u_2}{\partial \theta} \\
& + \frac{\left(\sigma_{rr_2} - \sigma_{\theta\theta_2}\right)\delta}{2\eta V} \frac{\partial u_o}{\partial \theta} - \frac{r^2\left(\sigma_{rr_o} - \sigma_{\theta\theta_o}\right)\delta}{2\eta V} \left(\frac{v_2}{r}\right) \\
& - \frac{r^2\left(\sigma_{rr_2} - \sigma_{\theta\theta_2}\right)\delta}{2\eta V} \frac{\partial}{\partial r} \left(\frac{v_o}{r}\right)
\end{aligned}
\right\} \quad \text{(III.6)}
$$

(ii) equilibrium equations

$$
\left.
\begin{aligned}
& \frac{\partial \sigma_{rr_{12}}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{ro_{12}}}{\partial \theta} + \frac{1}{r} \left(\sigma_{rr_{12}} - \sigma_{\theta\theta_{12}}\right) = 0 \\
& \frac{\partial \left(r\sigma_{r\theta_{12}}\right)}{\partial r} + \frac{\partial \sigma_{\theta\theta_{12}}}{\partial \theta} + \sigma_{r\theta_{12}} = 0
\end{aligned}
\right\} \quad \text{(III.6)}
$$

Similarily the boundary conditions can be written

$$\left.\begin{aligned} &\text{at } r = R_o \quad u(r,\theta) = 0,\ v(r,\theta) = V \\ &\text{at } r = R + e\cos\theta \quad u(r,\theta) = 0,\ v(r,\theta) = 0 \\ &\quad = R + \lambda_2 R_o \cos\theta \end{aligned}\right\} \quad \text{(III.7)}$$

$v_o(R + \lambda_2 R_o \cos\theta, \theta), v_1(R+\lambda_2 R_o \cos\theta, \theta)$, etc., can be expanded in power series about R. Neglecting higher order terms than the second we obtain

$$\begin{aligned} v(R + \lambda_2 R_o \cos\theta; \theta) = &\left\{ v_o(R,\theta) + \frac{\partial v_o}{\partial r}\bigg|_R \lambda_2 R_o \cos\theta \right\} \\ &+ \left\{ v_1(R,\theta) + \frac{\partial v_1}{\partial r}\bigg|_R \lambda_2 R_o \cos\theta \right\} \lambda_1 \\ &+ \left\{ v_2(R,\theta) \right\} \lambda_2 + \left\{ v_{11}(R,\theta) \right\} \lambda_1^2 + \left\{ v_{12}(R,\theta) \right\} \lambda_1 \lambda_2 \end{aligned} \quad \text{(III.8)}$$

A similar expansion may be obtained for $u(R + \lambda_2 R_o \cos\theta, \theta)$

Putting $v_o(R_o; \theta) = V$ we obtain for the boundary conditions

$$\text{zero order:} \quad \left.\begin{aligned} u_o(R_o,\theta) = 0,\ u_o(R,\theta) = 0 \\ v_o(R_o,V) = V,\ v_o(R,\theta) = 0 \end{aligned}\right\} \quad \text{(III.9)}$$

$$\text{first order}, \lambda_1: \quad \left.\begin{aligned} u_1(R_o,\theta) = u_1(R,\theta) = 0 \\ v_1(R_o,\theta) = v_1(R,\theta) = 0 \end{aligned}\right\} \quad \text{(III.10)}$$

$$\begin{aligned} \text{first order}, \lambda_2: \quad &u_2(R_o,\theta) = 0,\ u_2(R,\theta) = -R_o\cos\theta \frac{\partial u_o}{\partial r}\bigg|_R \\ &v_2(R_o,\theta) = 0,\ v_2(R,\theta) = -R_o\cos\theta \frac{\partial v_o}{\partial r}\bigg|_R \end{aligned} \quad \text{(III.11)}$$

$$\left.\begin{aligned} \text{second order}, \lambda_1^2:\ & u_{11}(R_o, \theta) = u_{11}(R, \theta) = 0 \\ & v_{11}(R_o, \theta) = v_{11}(R, \theta) = 0 \end{aligned}\right\} \qquad \text{(III.12)}$$

$$\text{second order}, \lambda_1\lambda_2:\quad u_{12}(R_o, \theta) = 0,\ u_{12}(R, \theta) = -R_o \cos\theta \left.\frac{\partial u_1}{\partial r}\right|_R$$

$$v_{12}(R_o, \theta) = 0,\ v_{12} = -R_o \cos\theta \left.\frac{\partial v_1}{\partial r}\right|_R$$

(III.13)

IV. ZERO ORDER SOLUTION

From symmetry the stress and velocity field will be for this case

$$\left.\begin{aligned} \sigma_{rr_o} &= \sigma_{rr_o}(r) & u_o &= 0 \\ \sigma_{r\theta_o} &= \sigma_{r\theta_o}(r) & v_o &= v_o(r) \end{aligned}\right\} \qquad \text{(IV.2)}$$

The boundary conditions are obtained from Page 11

$$\left.\begin{aligned} &\text{at } r = R_o & u_o &= 0, & v_o &= V \\ &\text{at } r = R & u_o &= 0, & v_o &= 0 \end{aligned}\right\} \qquad \text{(IV.2)}$$

Therefore the constitutive equations become

$$\begin{aligned} -\left(\sigma_{\theta\theta_o} + p_o\right) &= \left(\sigma_{rr_o} + p_o\right) = 0 \\ \sigma_{r\theta_o} &= \eta r \frac{d}{dr}\left(\frac{v_o}{r}\right) \end{aligned} \qquad \text{(IV.3)}$$

where $p_o = 1/2(\sigma\theta\theta_o + \sigma rr_o)$. The equilibrium equations are

$$\left.\begin{aligned} \frac{d\sigma_{rr_o}}{dr} &= 0 \\ \frac{d\left(r\sigma_{r\theta_o}\right)}{dr} + \frac{1}{r}\sigma_{r\theta_o} &= 0 \end{aligned}\right\} \qquad \text{(IV.4)}$$

combining

$$\frac{d^2 v_o}{dr^2} + \frac{1}{r}\frac{dv_o}{dr} - \frac{1}{r^2} v = 0 \qquad \text{(IV.5)}$$

this is easily integrated to give

$$\left.\begin{aligned} v_o &= \frac{R_o V}{R^2-R_o^2}\left[\frac{R^2}{r} - r\right] \\ \text{and} \quad \sigma_{r\theta_o} &= \frac{-2\eta\, R_o V}{R^2 - R_o^2}\left(\frac{R^2}{r}\right) \\ \sigma_{rr_o} &= \text{constant} \end{aligned}\right\} \qquad \text{(IV.6)}$$

Seeking the linear form of these equations, we use the variable $t = r-R_o$, expanded $(R_o + t)^{-1}$ in a binominal series keeping only first and zero order terms in t, and use the approximation $R \cong R_o$. This gives

$$\left.\begin{aligned} u_o &= 0 \\ v_o &= V\,(1-t/\delta) \\ -(\sigma_{\theta\theta_o} + p_o) &= \sigma_{rr_o} + p_o = 0 \\ p_o &= \text{constant} \\ \sigma_{r\theta_0} &= \frac{nV}{\delta} \end{aligned}\right\} \qquad \text{(IV.7)}$$

V. FIRST ORDER ELASTICITY SOLUTION

The stress and velocity fields are from symmetry

$$\left.\begin{aligned} \sigma_{rr_1} &= \sigma_{rr_1}(r) & u_1 &= 0 \\ \sigma_{r\theta_1} &= \sigma_{r\theta_1}(r) & v_1 &= v_1(r) \end{aligned}\right\} \qquad (V.1)$$

The boundary conditions are

$$\left.\begin{aligned} &\text{at} \quad r = R_o & u_1 &= v_1 = 0 \\ &\text{at} \quad r = R & u_1 &= v_1 = 0 \end{aligned}\right\} \qquad (V.2)$$

Therefore the constitutive equations are

$$\left.\begin{aligned} 0 &= \frac{1}{\eta}\left(\sigma_{rr_1} + p_1\right) + \frac{r\sigma_{r\theta_o}}{\eta}\left[\frac{d}{dr}\left(\frac{v_o}{rV}\right)\right] \\ r^2 \frac{d}{dr}\left(\frac{v_1}{r}\right) &= \frac{r}{\eta}\sigma_{r\theta_1} \end{aligned}\right\} \qquad (V.3)$$

and the equilibrium equations are

$$\left.\begin{aligned} \frac{d\sigma_{r\theta_1}}{dr} + \frac{\sigma_{r\theta_1}}{r} &= 0 \\ \frac{d\sigma_{rr_1}}{dr} + \frac{2}{r}\left(\sigma_{rr_1} + p_1\right) &= 0 \end{aligned}\right\} \qquad (V.4)$$

combining second of IV.3 and the first of IV.4 gives

$$\frac{d^2 v_1}{dr^2} + \frac{1}{r}\frac{dv_1}{dr} - \frac{v_1}{r^2} = 0 \qquad (V.5)$$

which because of the boundary conditions has only a trivial solution. Therefore

$$v_1 = 0$$
$$\sigma_{r\theta_1} = 0 \qquad (V.6)$$

The first of V.3 becomes

$$\sigma_{rr_1} + p_1 = \frac{4\eta\delta R_o^{\,2} V}{(R^2 - R_o^{\,2})} \left(\frac{R^4}{r}\right) \qquad (V.7)$$

Using the same linearization technique as was given in III, (V.7) becomes

$$\sigma_{rr_1} + p_1 = \frac{-\eta V}{R_o \delta} \left[R_o - 4t\right]$$

Combining IV.4 and IV.8 and recalling $\frac{t}{R_o} << 1$,

$$p_1 \cong \text{constant}$$

In conclusion

$$u_1 = v_1 = 0$$
$$\sigma_{r\theta_1} = 0$$
$$\left(\sigma_{rr_1} + p_1\right) = - \left(\sigma_{\theta\theta_1} + p_1\right) = \frac{-nV}{R_o \delta} \left[R_o - 4t\right]$$
$$p_1 \cong \text{constant}$$

VI. FIRST ORDER VISCOUS SOLUTION WITH ECCENTRICITY

The velocity and stress fields will be

$$
\begin{aligned}
u_2 &= u_2\,(r,\theta) & \sigma_{rr_2} &= \sigma_{rr_2}(r,\theta) \\
v_2 &= v_2\,(r,\theta) & \sigma_{r\theta_2} &= \sigma_{r\theta_2}(r,\theta) \\
& & \sigma_{\theta\theta_2} &= \sigma_{\theta\theta_2}(r,\theta)
\end{aligned}
\qquad \text{(VI.1)}
$$

The constitutive equations are

$$
\begin{aligned}
2\,\frac{\sigma u_2}{\partial r} &= \frac{1}{\eta}\left(\sigma_{rr_2} + p_2\right) \\
\frac{\partial u_2}{\partial \theta} + r^2\,\frac{\partial}{\partial r}\left(\frac{v_2}{r}\right) &= \frac{r}{\eta}\,\sigma_{r\theta_2}
\end{aligned}
\qquad \text{(VI.2)}
$$

The equilibrium equations are

$$
\begin{aligned}
\frac{\partial}{\partial r}\,\sigma_{rr_2} + \frac{1}{r}\,\frac{\partial \sigma_{r\theta_2}}{\partial \theta} + \frac{2}{r}\left(\sigma_{rr_2} + p_2\right) &= 0 \\
\frac{\partial\left(r\sigma_{r\theta_2}\right)}{\partial r} - \frac{\partial \sigma_{rr_2}}{\partial \sigma} + \sigma_{r\theta} &= 0
\end{aligned}
\qquad \text{(VI.3)}
$$

The boundary conditions are

$$
\begin{aligned}
&\text{at } r = R_o & u_2 &= v_2 = 0 \\
&\text{at } r = R & u_2 &= 0, \; v_2 = \frac{R_o V}{\delta}\,\cos\theta
\end{aligned}
\qquad \text{(VI.4)}
$$

An approximate solution for this case has been established for many years (see (1)). But in the classical approaches, a "wedge-type" solution in Cartesian coordinates was used to obtain all the stresses except for the mean normal stress

$p = 1/2(\sigma_{rr} + \sigma_{\theta\theta})$. As these results are unsatisfactory for the purposes of this problem, an additional perturbation method will be used.

Expand the velocities as follows

$$\left.\begin{aligned} u_2 &= A_1(\theta) + B_1(\theta)\left(\frac{t}{R_o}\right) + C_1(\theta)\left(\frac{t}{R_o}\right)^2 \\ u_2 &= A_2(\theta) + B_2(\theta)\left(\frac{t}{R_o}\right) + C_2(\theta)\left(\frac{t}{R_o}\right)^2 \end{aligned}\right\} \quad \text{(VI.5)}$$

Applying the boundary conditions and combining gives

$$\left.\begin{aligned} u_2 &= B_1(\theta)\,\frac{t}{R_o} + \frac{B_1(\theta)R_o}{\delta}\left(\frac{t}{R_o}\right)^2 \\ v_2 &= B_2(\theta)\left(\frac{t}{R_o}\right) + \left\{\frac{R_o^3 V}{\delta^3}\cos\theta - \frac{B_2(\theta)R_o}{\delta}\right\}\left(\frac{t}{R_o}\right)^2 \end{aligned}\right\} \quad \text{(VI.6)}$$

Substituting these into the linearized constitutive equations gives the expansions for $\sigma_{r\theta_{12}}$ and $(\sigma_{rr_{12}} + p_{12})$

$$\left.\begin{aligned} \frac{1}{\eta}\left(\sigma_{rr_2} + p_2\right) &= 2\,\frac{B_1(\theta)}{R_o} - \frac{4B_1(\theta)}{\delta}\left(\frac{t}{R_o}\right) \\ \frac{1}{\eta}\,\sigma_{r\theta_2} &= \left[\frac{B_2(\theta)}{R_o}\right] + \left[\frac{B_1^{\nabla}(\theta)}{R_o} - \frac{2B_2(\theta)}{\delta} + \frac{2R_o^2 V\cos\theta}{\delta^3}\right]\left(\frac{t}{R_o}\right) - \left[\frac{B_2(\theta)}{\delta} + \frac{R_o V}{\delta^3}\cos\theta\right]\left(\frac{t}{R_o}\right) \end{aligned}\right\} \quad \text{(VI.7)}$$

Further as in the classical solutions, it will be assumed that

$$P_2 = P_2(\theta)$$

and hence it and its derivatives will be of the zero order in $\frac{t}{R_o}$. Substituting the above into the $\widehat{r\theta}$ equilibrium equation and equating powers of (t/R_o), allows for the determination of $B_1(\theta)$ and $B_2(\theta)$.

The final results are

$$\left.\begin{aligned}
u_2 &= \frac{R_o V}{2\delta^2}\, t(1-\frac{t}{\delta}) \quad \sin\theta \\
v_2 &= \frac{R_o^2 V}{\delta^3}\, t(1-\frac{t}{\delta}) \quad \cos\theta \\
-\left(\sigma_{\theta\theta_2} + p_2\right) &= +\left(\sigma_{rr_2} + p_2\right) = -\frac{\eta R_o V}{\delta^2}\left[1-\frac{2t}{\delta}\right] \sin\theta \\
\sigma_{r\theta_2} &= \frac{\eta R_o^{\,2} V}{\delta^3}\left[1-\frac{2t}{\delta}\right] \cos\theta \\
p_2 &= \frac{-2\eta R_o^2 V}{\delta^3} \sin\theta + \text{constant}
\end{aligned}\right\} \quad \text{(VI.8)}$$

VII. SECOND ORDER ELASTIC SOLUTION

As with the first order solution we have for the stress and velocity fields

$$u_{11} = 0 \qquad \sigma_{rr_{11}} = \sigma_{rr_{11}}(r)$$
$$v_{11} = v_{11}(r) \qquad \sigma_{r\theta_{11}} = \sigma_{r\theta_{11}}(r) \qquad \text{(VII.1)}$$
$$\sigma_{\theta\theta_{11}} = \sigma_{\theta\theta_{11}}(r)$$

The constitutive equations become

$$0 = \frac{1}{\eta}\left(\sigma_{rr_{11}} + p_{11}\right)$$
$$r^2 \frac{d}{dr}\left(\frac{v_{11}}{r}\right) = \frac{r}{\eta}\sigma_{r\theta_{11}} - r^2 \frac{\left(\sigma_{rr_1} + p_1\right)\delta}{\eta}\frac{d}{dr}\left(\frac{v_o}{rV}\right) \qquad \text{(VII.2)}$$

The equilibrium equations are

$$\frac{d}{dr}\sigma_{rr_{11}} = 0, \quad \frac{d\left(r\sigma_{r\theta_{11}}\right)}{dr} + \sigma_{r\theta_{11}} = 0 \qquad \text{(VII.3)}$$

The boundary conditions are

$$\text{at } r = R_o \qquad u_{11} = v_{11} = 0$$
$$\text{at } r = R \qquad u_{11} = v_{11} = 0 \qquad \text{(VII.4)}$$

Combining, we obtain

$$\frac{d^2v_{11}}{dr^2} + \frac{1}{r}\frac{dv_{11}}{dr} - \frac{v_{11}}{r^2} = \frac{+32\, V R_o^3 R^6 \delta^2}{(R^2 - R_o^2)^3}\left(\frac{1}{r^7}\right) \qquad \text{(VII.5)}$$

Since $R_o \cong R$, $R^2 - R_o^2 \cong 2\,\delta\, R_o$ this becomes

$$\frac{d^2 v_{11}}{dr^2} + \frac{1}{r}\frac{dv_{11}}{dr} - \frac{v_{11}}{r^2} = \frac{4VR_o^6}{\delta}\left(\frac{1}{r^7}\right) \qquad \text{(VII.6)}$$

The complementary solution of this equation is of the form

$$v_{11_c} = ar + \frac{b}{r} \qquad \text{(VII.7)}$$

where a and b are constants.

For a particular solution, the method of variation of parameters will be used. In this method we assume a particular solution of the form

$$v_{11_1} = a(r)r + \frac{b(r)}{r} \qquad \text{(VII.8)}$$

where a(r) and b(r) are arbitrary functions satisfying

$$r\frac{da(r)}{dr} + \frac{1}{r}\frac{db(r)}{dr} = 0 \qquad \text{(VII.9)}$$

Substituting back into (VII.6) gives

$$\frac{da(r)}{dr} - \frac{1}{r^2}\frac{db(r)}{dr} = \frac{+16VR_o^6}{\delta r^7} \qquad \text{(VII.10)}$$

(VII.9) and (VII.10) can be easily solved for a(r) and b(r). The complete solution turns out to be

$$v_{11} = v_{11_c} + v_{11_p} = c_1 r + \frac{c_2}{r} + \frac{2VR_o^6}{\delta r^5} \qquad \text{(VII.11)}$$

where c_1 and c_2 are constants of integration determined from the boundary conditions.

Application of the boundary conditions gives

$$v_{11} = +\frac{8}{6}\left[\frac{V\delta^2 R^2}{\left(R^2-R_o^2\right)^4 R_o}\right]\left[(R^4-R_o^4)r - \left(\frac{R^6-R_o^6}{r}\right) + \frac{R_o^4 R^4 (R^2-R_o^2)}{r^5}\right] \quad \text{(VII.12)}$$

Since $R \cong R_o$, we have $R^\eta - R_o^\eta = \eta\delta R_o^{n-1}$ where η is a positive integer. Hence:

$$v_{11} = \frac{+V}{6\delta}\left[2r - \frac{3R_o^2}{r} + \frac{R_o^6}{r^5}\right] \quad \text{(VII.13)}$$

Further from (VII.2)

$$\sigma_{r\theta_{11}} = \eta r \frac{d}{dr}\left(\frac{v_{11}}{r}\right) + \frac{\eta V R_o^6}{\delta r^6} \quad \text{(VII.14)}$$

To linearize (VII.13) and (VII.14), let $r=R_o+t$, and expand the various terms $(R_o+t)^{-\eta}$ in binomoninal series keeping only zero and first order terms in t. This gives

$$v_{11} \cong 0$$

$$\sigma_{r\theta_{11}} = \frac{+\eta V}{\delta R_o}\left[R_o - 2t\right] \quad \text{(VII.15)}$$

also from (VII.1), (VII.2), and (VII.3)

$$u_{11} = 0$$

$$\left(\sigma_{rr_{11}} + p_{11}\right) = -\left(\sigma_{\theta\theta_{11}} + p_{11}\right) = 0$$

$$p_{11} = \text{constant}$$

VIII. SECOND ORDER ELASTIC-VISCOUS SOLUTION

The velocity and stress fields will be

$$
\begin{aligned}
u_{12} &= u_{12}(r,\theta) & \sigma_{rr_{12}} &= \sigma_{rr_{12}}(r,\theta) \\
v_{12} &= v_{12}(r,\theta) & \sigma_{\theta\theta_{12}} &= \sigma_{\theta\theta_{12}}(r,\theta) \\
 & & \sigma_{r\theta_{12}} &= \sigma_{r\theta_{12}}(r,\theta)
\end{aligned}
\qquad \text{(VIII.1)}
$$

The constitutive equations are:

$$
\begin{aligned}
\frac{\partial u_{12}}{\partial \theta} + r^2 \frac{\partial}{\partial r} \frac{v_{12}}{r} &= \frac{r\sigma_{r\theta_{12}}}{\eta} + \frac{u_2}{\eta} \frac{\partial}{\partial r}\left(\frac{r\sigma_{r\theta_o}\delta}{\eta}\right) \\
&\quad -r^2 \frac{\left(\sigma_{rr_2}+p_2\right)\delta}{\eta} \frac{\partial}{\partial r}\left(\frac{v_o}{rV}\right) - \frac{u_2}{V}\left(\frac{\sigma_{r\theta_o}\delta}{\eta}\right) + \frac{v_o}{V}\frac{\partial}{\partial \theta}\left(\frac{\sigma_{r\theta_2}\delta}{\eta}\right)
\end{aligned}
\qquad \text{(VIII.2)}
$$

$$
\begin{aligned}
2\frac{\partial u_{12}}{\partial r} &= \frac{\left(\sigma_{rr_{12}} + p_{12}\right)}{\eta} + \frac{1}{r}\left(\frac{v_o}{V}\right)\frac{\partial}{\partial \theta}\left(\sigma_{rr_2} + p_2\right)\frac{\delta}{\eta} \\
&\quad + \frac{r\sigma_{r\theta_o}\delta}{\eta}\frac{\partial}{\partial r}\left(\frac{v_2}{rV}\right) - \frac{1}{r}\frac{\sigma_{r\theta_o}\delta}{\eta}\frac{\partial}{\partial \theta}\left(\frac{u_2}{V}\right) + \frac{r\sigma_{r\theta_2}\delta}{\eta}\frac{\partial}{\partial r}\left(\frac{v_o}{rV}\right)
\end{aligned}
$$

The equilibrium equations are

$$
\begin{aligned}
\frac{\partial \sigma_{rr_{12}}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta_{12}}}{\partial \theta} + \frac{2}{r}\left(\sigma_{rr_{12}} + p_{12}\right) &= 0 \\
\frac{\partial\left(r\sigma_{r\theta_{12}}\right)}{\partial r} - \frac{\partial \sigma_{rr_{12}}}{\partial \theta} + \sigma_{r\theta_{12}} &= 0
\end{aligned}
\qquad \text{(VIII.3)}
$$

The boundary conditions are

$$
\begin{aligned}
&\text{at } r = R_o, \quad u_{12} = 0, \quad v_{12} = 0 \\
&\text{at } r = R, \quad u_{12} = \left.\frac{\partial u_1}{\partial r}\right|_R \cos\theta = 0, \; v_{12} = \left.\frac{\partial v_1}{\partial r}\right|_R \cos\theta = 0
\end{aligned}
\qquad \text{(VIII.4)}
$$

The linearized solutions are substituted in (VIII.2), and then the resulting equations are linearized in t. The result is

$$\left.\begin{aligned}
&\frac{\partial u_{12}}{\partial r} - \frac{1}{\eta}\left(\sigma_{rr_{12}} + p_{12}\right) = \frac{2R_o^2 V}{\delta^3}\cos\theta - \frac{2R_o^2 V}{\delta^4}\, t \sin\theta \\
&\frac{\partial u_{12}}{\partial \theta} + r^2\left(\frac{\partial}{\partial r}\right)\frac{v_{12}}{r} = \frac{r\sigma_{r\theta_{12}}}{\eta} - \frac{2R_o^2 V}{\delta^2}\sin\theta \\
&+ \frac{R_o^2 V}{\delta^3}\, t \cos\theta
\end{aligned}\right\} \quad \text{(VIII.5)}$$

The above are solved by exactly the same perturbation method given in V. The results are

$$\left.\begin{aligned}
&u_2 = \frac{R_o^2 V}{3\delta^3} + \left(1 - \frac{t}{\delta}\right)\cos\theta \\
&v_R \quad \frac{-4R_o^2 V}{3\delta^3}\, t\left(1 - \frac{t}{\delta}\right)\sin\theta \\
&-\left(\sigma_{\theta\theta_{12}} + p_{12}\right) = \left(\sigma_{rr_{12}} + p_{12}\right) = \frac{-nR_o^2 V}{3\delta^3}\left[1 - \frac{2t}{\delta}\right]\cos\theta \\
&\sigma_{r\theta_{12}} = \frac{-4\eta R_o^2 V}{\delta^3}\left(1 - \frac{2t}{\delta}\right)\sin\theta \\
&p_{12} = \frac{5\eta R_o^2 V}{3\delta^2}\cos\theta + \text{constant}
\end{aligned}\right\} \quad \text{(VIII.6)}$$

IX. LOAD AND DRIVING TORQUE EQUATIONS

In lubrication practice the relationship between the eccentricity and external load on the shaft is of primary interest. Now from Figure 1

$$\left.\begin{aligned} W_x &= +\int_0^{2\pi}\left\{\sigma_{rr}\Big|_{r=R_o}\sin\theta + \sigma_{r\theta}\Big|_{r=R_o}\cos\theta\right\} LR_o\,d\theta \\ W_y &= \int_0^{2\pi}\left\{\sigma_{rr}\Big|_{r=R_o}\cos\theta - \sigma_{r\theta}\Big|_{r=R_o}\sin\theta\right\} LR_o\,d\theta \end{aligned}\right\}\quad\text{(IX.1)}$$

Substituting, the non-zero terms become

$$\left.\begin{aligned} W_x &= +\int_0^{2\pi}\left\{\sigma_{rr_2}\Big|_{r=R_o}\sin\theta + \sigma_{r\theta_2}\Big|_{r=R_o}\cos\theta\right\}\left(\frac{e}{R_o}\right) LR_o\,d\theta \\ &\cong +\int_0^{2\pi}\left(\frac{e}{R_o}\right)\left\{\frac{2\eta R_o^2 V}{\delta^3}\sin^2\theta - \frac{\eta R_o^2 V}{\delta^3}\cos^2\theta\right\} LR_o\,d\theta \\ &= \frac{+2\pi\eta R_o^3 LV}{\delta^3}\left(\frac{e}{R_o}\right) \\ W_y &= \int_0^{2\pi}\left\{\sigma_{rr_{12}}\Big|_{r=R_o}\cos\theta - \sigma_{r\theta_{12}}\Big|_{r=R_o}\sin\theta\right\} LR_o\,d\theta \\ &= \left(\frac{e}{R_o}\right)\left(\frac{nV}{G\delta}\right)\int_0^{2\pi}\left\{+\frac{4\eta R_o^2 V}{3\delta^3}\cos^2\theta + \frac{4\eta R_o^2 V}{\delta^3}\sin^2\theta\right\} LR_o\,d\theta \\ &= \frac{16\eta\pi R_o^3 LV}{3\delta^3}\left(\frac{nV}{G\delta}\right)\left(\frac{e}{R_o}\right) \end{aligned}\right\}\quad\text{(IX.2)}$$

When $\frac{\eta V}{G\delta}$ vanishes, W_x is the load calculated from the usual lubrication theory after linearization with respect to e/R_o.

The angle, φ, of inclination of the external force to the normal to the eccentricity is given by

$$\varphi = \tan^{-1} \frac{W_y}{W_x} = \tan^{-1} \left(+ \frac{8}{3} \frac{\eta V}{G\delta}\right) \cong +\frac{8}{3} \left(\frac{\eta V}{G\delta}\right) \qquad \text{(IX.3)}$$

Figure 2 shows the dependence of φ on $\left(\frac{\eta V}{G\delta}\right)$. It should be noted that when $\left(\frac{\eta V}{G\delta}\right)$ vanishes and there is no elastic effect, this angle becomes zero which is to be expected from the usual theory of lubrication for viscous materials (see (1)).

Another measure of the elastic effect which is of practical importance is the change in the net load W where

$$W \left(\frac{\eta V}{G\delta}\right) = \sqrt{W_x^2 + W_y^2} \qquad \text{(IX.4)}$$

A convenient ratio to study is, for a given eccentricity, the ratio of $W\left(\frac{\eta V}{G\delta}\right) - W(0)$ to $W(0)$. This ratio measures the increase in the load carrying capacity due to the development of W_y. Recalling that $\left(\frac{\eta V}{G\delta}\right)$ is small compared to 1

$$\frac{W \left(\frac{\eta V}{G\delta}\right) - W(0)}{W(0)} \cong \frac{32}{9} \left(\frac{\eta V}{G\delta}\right)^2 \qquad \text{(IX.5)}$$

This relationship is shown in Figure 3. This figure shows that a value of .17 is required to increase the load carrying capacity by 10%.

Another important expression in lubrication practice is the equation for the driving torque. This is given by:

$$T = \int_0^{2\pi} \sigma_{r\theta}\Big|_{r=R_o} LR_o^2 d\theta \tag{IX.6}$$

The non-zero contributions are

$$T = \int_0^{2\pi} \left\{ \sigma_{r\theta_o}\Big|_{r=R_o} + \sigma_{r\theta_{11}}\Big|_{r=R_o} \frac{RV^2}{G\delta} \right\} LR_o^2 d\theta \tag{IX.7}$$

$$= \int_0^{2\pi} \left\{ -\frac{\eta V}{\delta} + \frac{\eta V}{\delta}\left(\frac{\eta V}{G\delta}\right)^2 \right\} LR_o d\theta \tag{IX.8}$$

$$T = \frac{-2\pi\eta LR_o^2 V}{\delta} \left\{ 1 - \left(\frac{nV}{G\delta}\right)^2 \right\}$$

as with the external load we have

$$\frac{T\left(\frac{nV}{G\delta}\right) - T(0)}{T(0)} = -\left(\frac{nV}{G\delta}\right)^2 \tag{IX.9}$$

The relation is shown in Figure 4.

Although there is as yet no reliable data available for the elastic modui of standard lubricants, it is estimated that the values of $\left(\frac{nV}{G\delta}\right)$ currently encountered are less than .01 even in extreme operating conditions. Figures 3 and 4 then show that for standard lubricants currently in common usage elastic effects of the type considered here are negligible.

It should also be noted that as G, the elastic shear modulus, is reduced, the ratio $\dfrac{T\left(\frac{\eta V}{G\delta}\right) - T(0)}{T(0)}$ decreases. This result is physically reasonable since a reduction in G corresponds to a reduction in the stiffness of the material.

Further, as G is reduced, the ratio $\dfrac{W\left(\frac{\eta V}{G\delta}\right) - W(0)}{W(0)}$ increases, a result which on first notice seems not to be so physically appealing. That this ratio decreases follows from the fact that $W = \sqrt{W_x^2 + W_y^2}$, where W_2 is independent of G and W_y is proportional to 1/G. When the material is purely viscous, $\frac{1}{G} = 0$, the only force is the lateral load W_x, a result predicted by the classical theory. As small elastic effects are introduced, $|W_y|$ becomes strictly greater than zero while W_x remains the same, and hence W increases.

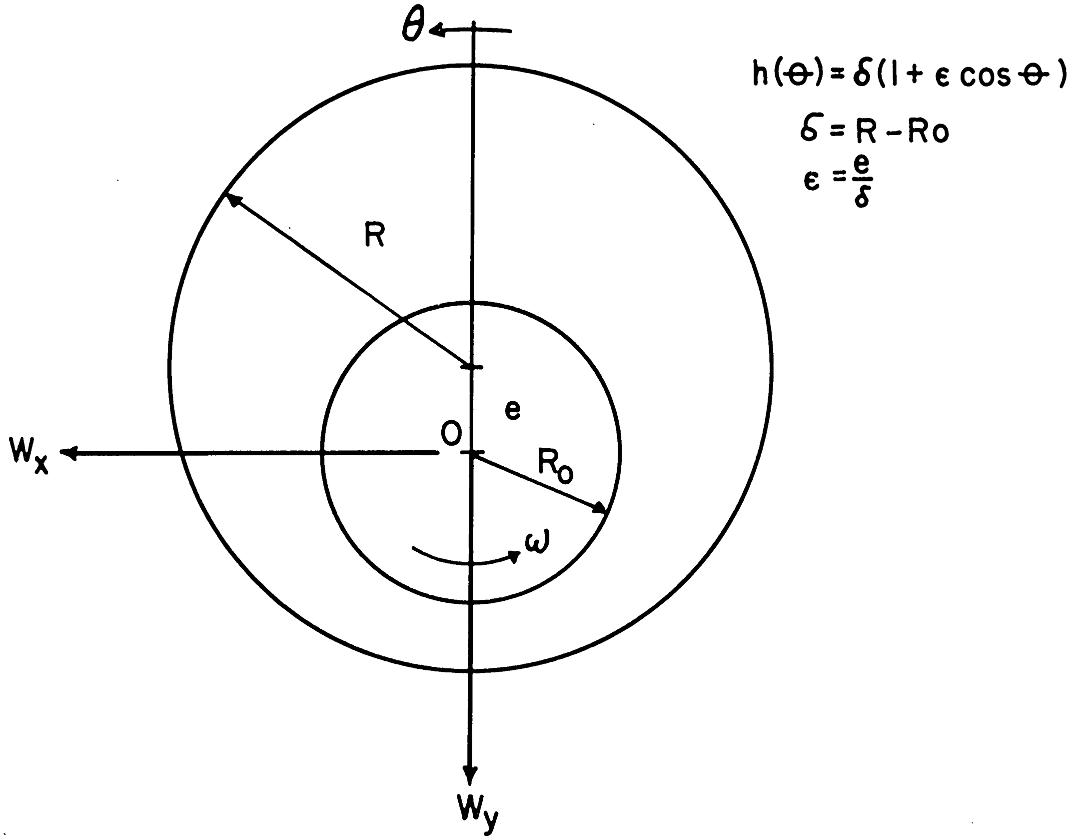

FIGURE 1 - CONFIGURATION OF JOURNAL BEARING, SHOWING THE DIMENSIONS $R, R_o, e, h(\theta)$; THE ANGLE θ; THE ANGULAR VELOCITY ω, AND THE EXTERIOR LOADING ON THE SHAFT, W_x AND W_y

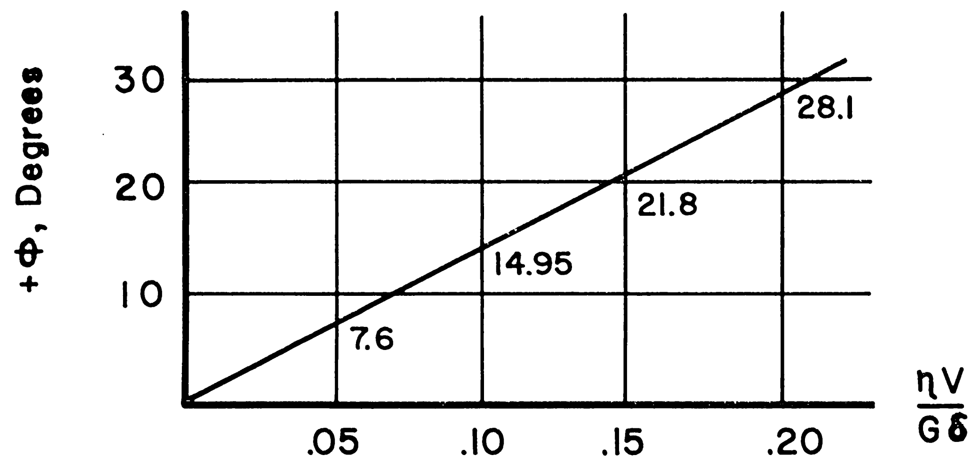

FIGURE 2 – DEPENDENCE OF THE ANGLE Φ WHICH MEASURES THE DEVIATION OF THE INCLINATION OF THE EXTERNAL LOAD FROM THE NORMAL TO THE ECCENTRICITY ON THE PARAMETER $\frac{\eta V}{G\delta}$ WHICH INDICATES MAGNITUDE OF THE ELASTIC EFFECT

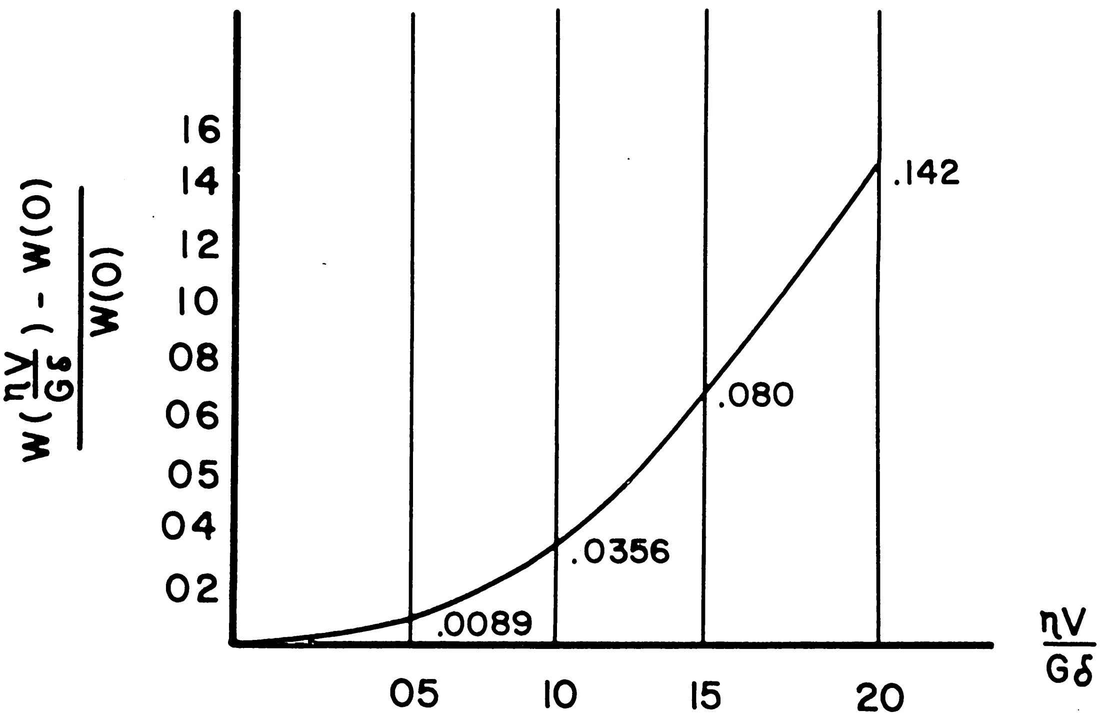

FIGURE 3 - RATIO OF LOAD INCREASE DUE TO ELASTIC EFFECT TO LOAD WITHOUT ELASTIC EFFECT VS. $\frac{\eta V}{G\delta}$ THE PARAMETER MEASURING THE ELASTIC EFFECT

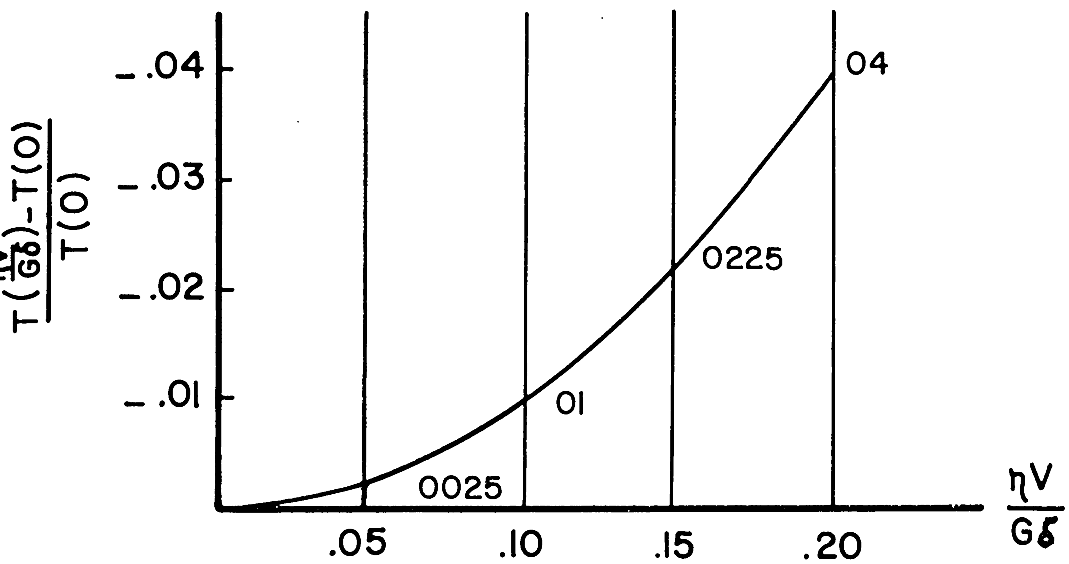

FIGURE 4 - DIMENSIONLESS TORQUE CHANGE DUE TO ELASTIC EFFECTS VERSUS $\frac{\eta V}{G\delta}$ THE PARAMETER MEASURING THE ELASTIC EFFECT

REFERENCES

1. Sternlicht, B. and Pinkus, W., Theory of Hydrodynamic Lubrication, McGraw-Hill, 1961

2. Lodge, A. S., Elastic Liquids, Academic Press, 1964.

3. Noll, W.,"On the Continuity of the Solid and Fluid and Fluid States", Journal Rational Mechanics and Analysis, 4, No. 1, (1955) Pages 3-81.

4. Kolsky, H., Stress Waves in Solids, Dover, 1953.

5. Bland, D. R., Theory of Linear Viscoelasticity, Pergamon Press, 1960.

6. Prager, W.,"An Elementary Discussion of Definations of Stress Rate", Quarterly of Applied Mathematics, 18, (1961) Pages 403-407.