FORCED LATERAL MOTION OF DEEP-WATER DRILL STRINGS

by

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Thanks are extended to the Rice University Mechanical Engineering Department and to the American Petroleum Institute for enabling the author to spend the summer of 1962 studying this problem.
An equation of motion governing lateral deflection, considering elastic, dynamic, and drag forces is derived. This equation is applied to three different classes of drill string behavior: a beam with constant axial tension, a perfectly flexible cable, and a beam with linearly varying axial tension. Each equation is solved for the case of simple harmonic motion.

A typical deep-water drill string is synthesized by subdivision into short beam sections at the top and bottom, joined by a flexible cable in the center. The lateral deflection of the drill string is obtained by joining the beam and cable solutions, subject to boundary conditions at each junction. The drill string considered is displaced harmonically at the surface by the ship and is built-in at the ocean floor.

The deflection mode shapes are plotted and analyzed. The bending stresses at the top of the drill string are computed and compared with values observed under similar conditions; the bending stresses in the vicinity of the ocean floor are also plotted.
Introduction

The motivation for this study was twofold. The first goal was to devise an improved method for determining the bending stresses which are concentrated at the ocean surface and at the ocean floor. A common method of analyzing this problem is to treat the entire drill string as a flexible cable, and to compute the bending stresses on the basis of the resulting deflection curve. While it is true that the major portion of the drill string is flexible, beam action is nevertheless involved at the ocean surface, where the slope of the drill string is imposed by the drilling ship. Some beam action is also involved at the ocean floor, since the ocean bed offers resistance to changes in the slope of the drill string. Consequently, it is most desirable to incorporate localized beam behavior at the top and bottom of the drill string when the bending stresses are sought.

The second object was to investigate the dynamic effects as thoroughly as possible. Of particular interest here is the effect of the drag force which the water exerts on a drill string in lateral motion. The inertia forces of the drilling fluid as well as of the pipe itself also deserve attention, along with the buoyant forces exerted on the drill string.

The results obtained with the refined beam-string-beam model of the drill string are distinctly superior in some respects to the results from the simple string model. The bending stresses derived from the string model are of the order of 1,000 psi at the surface for the motion studied here; the stresses obtained from the refined model are 12,300 psi under the same conditions of off-hole displacement. Values observed
in test drilling with the same drill string but under unspecified conditions of off-hole displacement are of the order 25,000 psi, as quoted in Reference 2. This indicates that the value 12,000 psi obtained in the refined model is a considerable improvement. The bending stresses obtained for the model drill string at the ocean floor (60,000 psi) are considerably higher than actually experienced. The explanation is that a drill string operating into the ocean floor is not really built-in there. Some change of slope is possible, and has been observed (see Reference 2).

The dynamic effects are by no means trivial. The drag forces induce bending stresses of 1,800 psi at the surface and 2,400 psi at the ocean floor.

The effect of the buoyant force has considerable influence on the entire problem in that it reduces the axial tension at the floor by a large amount.
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SYMBOLS

(1) Quantities which depend upon section considered. The subscript \( i \) refers to the section number.

\[ T_i \] = axial tension
\[ T_{0i} \] = axial tension at top of section
\[ V_i \] = shearing force on a horizontal plane
\[ M_i \] = bending moment
\[ F_1 \] = weight of drill pipe
\[ F_2 \] = lateral drag force plus all lateral D'Alembert inertia forces
\[ \rho_i \] = mass, per unit length, of drill pipe
\[ \rho'_i \] = mass, per unit length, of drill pipe and enclosed fluid
\[ \rho''_i \] = mass, per unit length, of enclosed fluid = \( \rho'_i - \rho_i \)
\[ u_i \] = downward velocity of enclosed drilling fluid
\[ d_i \] = outer diameter of drill pipe (constant)
\[ l_i \] = inner diameter of drill pipe (constant)
\[ l_i \] = length of \( i \)th section
\[ x_i \] = position of a point on axis of drill pipe measured positive downwards from top of \( i \)th section
\[ y_i \] = deflection of drill pipe axis away from vertical line through point of entry at ocean floor
\[ I_i \] = area moment of inertia of drill pipe cross-section

(2) Other quantities.

\[ g \] = gravitational acceleration
\[ c_D \] = drag coefficient for lateral motion of drill pipe in sea water = 1.2
\[ A \] = projection of external surface area of pipe onto a plane containing drill pipe axis
\[ U = \text{lateral velocity of drill pipe relative to surrounding water} \]

\[ E = \text{modulus of elasticity of drill pipe} = 30 \times 10^6 \text{ psi} \]

\[ t = \text{time} \]

\[ f = \text{circular frequency of ship's motion (rad/sec).} \]

\[ y_o = \text{amplitude of harmonic displacement imposed at surface by ship's motion} \]
I. EQUATION OF MOTION

Consider an incremental length \( \Delta x \) of the drill pipe bounded by two horizontal planes (parallel to ocean surface) and let it be assumed that slopes and deflections (relative to overall length) are small.

![Diagram of drill pipe forces](image)

Figure 1

The forces acting on the drill pipe include the vertical tension \( T \), the horizontal shear \( V \), the bending moment \( M \), the pipe weight \( F^w \), and the lateral dynamic forces \( F^l \). The bending and lateral translation take place in the plane of the ship motion. The effects of the rotation of the drill pipe are not considered in this paper.

The weight \( F^w \) of an element of drill pipe is

\[
F^w = \rho g \Delta x
\]  

A vertical force balance, neglecting vertical acceleration, gives

\[
(T + \Delta T) - T + \rho g \Delta x = 0
\]  

or, in the limit as \( \Delta x \to 0 \),

\[
\frac{\partial T}{\partial x} = - \rho g
\]  

which gives the equation of vertical equilibrium
\[ T = T_0 - \rho g x \]  

Summation of moments yields
\[ T \Delta y - V \Delta x + (M + \Delta M) - M = 0 \]  

or
\[ V = \frac{\partial M}{\partial x} + T \frac{\partial \phi}{\partial x} \]  

The bending of the beam is assumed to obey the ordinary relation
\[ M = -EI \frac{\partial^2 \phi}{\partial x^2} \]  

so that the expression for the shear force (6) becomes
\[ V = -EI \frac{\partial^2 \phi}{\partial x^2} + (T - \rho g x) \frac{\partial \phi}{\partial x} \]  

By its definition, \( F_D \) is the sum of three forces: the drag force \( F_D \) and the lateral D'Alembert inertia forces of the drill pipe and of the enclosed drilling fluid. Streeter\(^1\) states that the drag force \( F_D \) experienced by a uniform infinite cylinder whose axis is perpendicular to a uniform incompressible flow is given by
\[ F_D = -C_D \frac{1}{2} \rho U |U| \]  

so that the drag force per unit length \( \frac{F_D}{L} \) is
\[ \frac{F_D}{L} = -C_D \frac{1}{2} \rho U |U| \]  

This definition will be used for the incremental free-body of Figure 1. Due to the extreme length of a deep-water drill string and the assumption of small slopes and deflections, the infinite length condition is approximately satisfied. Throughout most of its depth, measurements\(^2\) have shown that average current speeds are negligible so that
\[ U \approx \frac{\partial \phi}{\partial t} \]  

\(^1\) Reference 1, p. 171  
\(^2\) Reference 2, p. 168
Then

\[ \frac{F_2}{L} = -\frac{\rho_0 d C_o}{2} \frac{\partial w}{\partial t} \left| \frac{\partial w}{\partial x} \right| \]  

(12)

is the drag force per unit length.

The lateral inertia force, per unit length, of the drill pipe is

\[-\rho \frac{\partial^2 y}{\partial t^2}\]  

(13)

The lateral inertia force of the enclosed drilling fluid is

\[-\rho'' \left( \frac{\partial^2 y}{\partial t^2} + v^2 \frac{\partial^2 y}{\partial x^2} + 2v \frac{\partial^3 y}{\partial x^3} \right)\]  

(14)

The first term represents direct lateral acceleration; the second arises from the curvature of the drill pipe itself; while the third is the Coriolis acceleration.

Then \(F_2\) is given by

\[F_2 = -\left[ \frac{\rho_0 d C_o}{2} \frac{\partial w}{\partial t} \frac{\partial^2 y}{\partial x^2} | \frac{\partial w}{\partial x} | + \rho'' \left( \frac{\partial^2 y}{\partial t^2} + v^2 \frac{\partial^2 y}{\partial x^2} + 2v \frac{\partial^3 y}{\partial x^3} + \rho \frac{\partial^3 y}{\partial x^3} \right) \right] \Delta x \]  

(15)

or, since \(\rho'' + \rho = \rho'\),

\[F_2 = -\left[ \frac{\rho_0 d C_o}{2} \frac{\partial w}{\partial t} \frac{\partial^2 y}{\partial x^2} | \frac{\partial w}{\partial x} | + \rho' \frac{\partial^2 y}{\partial t^2} + \rho'' v^2 \frac{\partial^2 y}{\partial x^2} + 2\rho'' v \frac{\partial^3 y}{\partial x^3} \right] \Delta x \]  

(16)

Now referring to Figure 1, a lateral force balance yields

\[-v + (v + \Delta v) - \left[ \frac{\rho_0 d C_o}{2} \frac{\partial w}{\partial t} \frac{\partial^2 y}{\partial x^2} | \frac{\partial w}{\partial x} | + \rho' \frac{\partial^2 y}{\partial t^2} \\
+ \rho'' v^2 \frac{\partial^2 y}{\partial x^2} + 2\rho'' v \frac{\partial^3 y}{\partial x^3} \right] \Delta x = 0\]  

(17)
or

\[
\frac{\partial \mathbf{V}}{\partial \mathbf{x}} - \left[ \frac{\mathbf{R} \cdot \mathbf{C} \cdot \mathbf{D} \cdot \mathbf{E} \cdot \mathbf{F} \cdot \mathbf{G} \cdot \mathbf{H} \cdot \mathbf{I}}{\partial \mathbf{x}} + \mathbf{J} \cdot \mathbf{K} \cdot \mathbf{L} + \mathbf{M} \cdot \mathbf{N} \cdot \mathbf{O} \cdot \mathbf{P} \cdot \mathbf{Q} \cdot \mathbf{R} \cdot \mathbf{S} \cdot \mathbf{T}
\right] = 0
\]  

(18)

Substituting the shear force equation (8) yields

\[
-EI \frac{\partial^2 \mathbf{V}}{\partial \mathbf{x}^2} + \frac{\partial}{\partial \mathbf{x}} \left[ \left\{ (T_0 - \mathbf{F}^2) - \mathbf{G} \cdot \mathbf{H} \right\} \frac{\partial \mathbf{V}}{\partial \mathbf{x}} \right] \\
-2 \mathbf{F} \cdot \mathbf{N} \cdot \frac{\partial^2 \mathbf{V}}{\partial \mathbf{x} \partial \mathbf{t}} - \frac{\mathbf{R} \cdot \mathbf{S} \cdot \mathbf{T}}{2} - \mathbf{U} \cdot \frac{\partial^2 \mathbf{V}}{\partial \mathbf{t}^2} = 0
\]

(19)

as the equation governing lateral motion.

In order to obtain analytic solutions, the drag force term \( \frac{\mathbf{R} \cdot \mathbf{S} \cdot \mathbf{T}}{2} \) must be linearized. This is done by finding a positive constant \( \mathbf{F} \) such that \( \frac{\mathbf{R} \cdot \mathbf{S} \cdot \mathbf{T}}{2} \) may be replaced by \( \mathbf{F} \cdot \mathbf{H} \cdot \mathbf{I} \). For this purpose only, the drill string is treated as if it were experiencing a simple harmonic lateral displacement

\[
\mathbf{y}(\mathbf{x}, \mathbf{t}) = \mathbf{y}_0 \left[ 1 - \frac{\mathbf{L}}{2} \right] \sin \mathbf{ft}
\]

(20)

where \( \mathbf{y}_0 \) = amplitude of ship's displacement

\( \mathbf{L} \) = ocean depth

\( \mathbf{x} \) = distance below ocean surface

The energy dissipated per quarter-cycle of motion via the velocity-squared drag force is equated to the energy dissipated per quarter-cycle via the linearized drag force, which yields \( \mathbf{F} \).

Let \( \mathbf{W}_1 \) = energy dissipated via the velocity-squared drag force over the first quarter-cycle of motion, \( 0 \leq \mathbf{ft} \leq \frac{\pi}{2} \).

\( \mathbf{W}_2 \) = energy dissipated via the linear drag force over the same time interval.

\( \mathbf{R} \) = velocity-squared drag force per unit length

\[
= -\frac{\mathbf{R} \cdot \mathbf{S} \cdot \mathbf{T}}{2} \left( \frac{\mathbf{L}}{2} \right)^2 \quad \text{for} \quad 0 \leq \mathbf{ft} \leq \frac{\pi}{2}.
\]
\[ F = \text{linear drag force} = -\frac{p^* dC_p}{2} \beta \frac{d^2 \xi}{dt^2} \]

Then

\[
W_1 = \int_0^L \int_0^{\pi / 2} R d\xi d\nu = \int_0^L \int_0^{\pi / 2} -\frac{p^* dC_p}{2} \left( \frac{d^2 \xi}{dt^2} \right)^2 \frac{d\xi}{dt} d\nu
\]

\[
W_2 = \int_0^L \int_0^{\pi / 2} F d\xi d\nu = \int_0^L \int_0^{\pi / 2} -\frac{p^* dC_p}{2} \beta \left( \frac{d^2 \xi}{dt^2} \right)^2 \frac{d\xi}{dt} d\nu
\]

(21)

Using the motion (20) and equating \( W_1 \) and \( W_2 \) yields

\[
\beta = \frac{\int_0^L \int_0^{\pi / 2} -\frac{p^* dC_p}{2} \{ y_0 \left[ \frac{\xi}{L} \right] \cos ft \}^3 \frac{d\xi}{dt} d\nu}{\int_0^L \int_0^{\pi / 2} -\frac{p^* dC_p}{2} \left( y_0 \left[ \frac{\xi}{L} \right] \cos ft \right)^2 \frac{d\xi}{dt} d\nu}
\]

(22)

The absolute value sign is used to make \( \beta \) positive. Performing the integration yields

\[
\beta = \frac{2}{\pi} y_0 f
\]

(23)

Replacing \( \frac{p^* dC_p}{2} \frac{d^2 \xi}{dt^2} \) by \( \frac{p^* dC_p}{2} \beta \frac{d^2 \xi}{dt^2} \) with \( \beta \) given by (23) yields

\[-EI \frac{d^2 \xi}{dt^2} + \frac{\partial}{\partial x} \left[ \left\{ (L - f'' x^2) - \rho g x \right\} \frac{d\xi}{dt} \right]

- \frac{2 f'' v}{\pi \nu} \frac{d^2 \xi}{dt^2} - \frac{p^* dC_p y_0 f}{\pi} \frac{d^2 \xi}{dt^2} - f' \frac{d^2 \xi}{dt^2} = 0
\]

(24)
In problems of lateral motion, it is expected that ordinarily the effect of the Coriolis acceleration of the enclosed fluid (represented by the term $\frac{\partial^2 y}{\partial x^2} \frac{1}{\partial t}$) will be quite small; in this case, the equation of motion to be used is

$$-EI \frac{\partial^4 y}{\partial x^4} + \frac{3}{8x} \left\{ (T_0 - \rho \nu^2) - \rho g x \right\} \frac{\partial^2 y}{\partial x^2}$$

$$- \frac{\rho d \epsilon_0 y f}{\eta} \frac{\partial y}{\partial t} - \rho' \frac{\partial^2 y}{\partial t^2} = 0 \quad (25)$$

The Coriolis inertia force of the water will be estimated from the solution of (25) to check the validity of ignoring this effect.
II. LATERAL MOTION DUE TO SIMPLE HARMONIC TRANSLATION AT THE OCEAN SURFACE

In order to obtain a solution to the equation of motion which can satisfy boundary conditions of slope and displacement at the top and at the bottom of the drill string, it is necessary to form a composite solution consisting of different solutions for different sections of the drill string.

The top portion of the drill string, which extends downward from the ocean surface to a depth of up to about one hundred feet can, for a deep-water drill string, be treated as a beam with constant axial tension. The bottom portion, which extends upward from the ocean floor to a height of about one hundred feet can be treated exactly as a beam with variable axial tension, or as a beam with constant tension. The center portion of the drill string which lies between the top and bottom portions can be treated as a cable under varying axial tension; that is, its bending stiffness will be neglected.

The justification for adopting this procedure is as follows. If the entire drill string is treated as a cable, the deflection curve gives poor values for the bending stresses which occur at the top and at the bottom; also, the cable solution is undefined if the axial tension at the bottom vanishes. Further, the cable solution alone cannot satisfy boundary conditions of both slope and displacement at the ocean surface and at the ocean floor. The constant-tension solution is clearly not sufficient for the entire drill string. The variable-tension solution is obtained by a power series method, and possesses serious convergence problems of a practical nature when applied to the entire drill string,
although it converges satisfactorily when applied to a short section under low tension at the bottom. Finally, a drill string is not a perfectly uniform cylinder, but usually consists of several long uniform cylinders joined together, so that a composite solution is usually involved at any rate.

The deflection curve will be given by a composite solution involving three different functions, one for each of the three types of drill string behavior. Each solution can be applied to a uniform member; over each uniform section, the appropriate function is used to define the deflection curve. The top and/or bottom section of the drill string may be long enough that the drill string's behavior changes within that section; in this case, the change in behavior is assumed to occur abruptly, and the section is subdivided into two sections, a beam and a cable, even though the cross-section dimensions do not change. Boundary conditions will be applied at each junction (abrupt change in section and/or behavior) and at the ocean surface and floor, in order to evaluate the constants in the solution of the equation of motion. With all constants determined, the lateral deflection and its derivatives, as well as bending stresses, may be calculated at any point.

**Solution for Cable with Variable Tension**

Neglecting the beam stiffness represented by \( EI \frac{\partial^2 y}{\partial x^2} \), the equation of motion (25) becomes

\[
\frac{\partial}{\partial x} \left[ \left\{ (T_0 - p^2 v^2) - p g x \right\} \frac{\partial^2 y}{\partial x^2} \right] - \rho A C_n \frac{d^2 y}{d t^2} \frac{\partial^2 y}{\partial x^2} - p' \frac{\partial^2 y}{\partial t^2} = 0
\]

(26)
The dimensionless variables

\[ \eta = \frac{y}{l} \]
\[ \bar{\xi} = \frac{(T_0 - \frac{g}{\rho} v^2) - \rho \phi x}{g \rho l} \]
\[ \tau = \sqrt{\frac{g}{\rho \ell}} t \]

transform the equation of motion (26) to

\[ \frac{\partial}{\partial \bar{\xi}} (\bar{\xi} \frac{\partial \eta}{\partial \bar{\xi}}) - a_0 \frac{\partial^2 \eta}{\partial \tau^2} - \frac{\partial^2 \eta}{\partial \tau^2} = 0 \]

(27)

where

\[ a_0 = \frac{g \rho c_0 y_0 \mu l}{\pi \rho g} \sqrt{g} \]

(29)

A solution to the equation of motion (28) which is harmonic in time is found by assuming

\[ \eta(\bar{\xi}, \tau) = \text{Re} \left[ \bar{X}(\bar{\xi}) e^{i \bar{\omega} \tau} \right] \]

(30)

where \( i = \sqrt{-1} \), \( \bar{X}(\bar{\xi}) \) is a complex function of \( \bar{\xi} \) only, and Re denotes the real part of the complex quantity \( \bar{X}(\bar{\xi}) e^{i \bar{\omega} \tau} \). According to the dimensionless variables (27),

\[ \bar{\omega} = f \sqrt{\frac{g \ell}{\rho g}} \]

(31)

so that

\[ \bar{\omega} \tau \equiv ft \]

(32)
Substituting the assumed solution (30) into the equation of motion yields

$$\frac{d}{d\theta} \left( \bar{\omega} \frac{d\bar{X}}{d\theta} \right) + (\bar{\omega}^2 - i \alpha \bar{\omega}) \bar{X} = 0$$

(33)

This equation can be solved in terms of Bessel functions of a complex variable, and the solution is

$$\bar{X}(\bar{\theta}) = M J_0(\alpha \sqrt{\bar{\theta}}) + N Y_0(\alpha \sqrt{\bar{\theta}})$$

(34)

where $J_0(\alpha \sqrt{\bar{\theta}})$ and $Y_0(\alpha \sqrt{\bar{\theta}})$ are Bessel functions of the first and second kinds, respectively, of zero order. $M$ and $N$ are complex constants, and $\alpha$ is given by

$$\alpha = 2 \sqrt{\omega^2 - i \alpha \bar{\omega}} = 2 \sqrt{f^2 \left( \frac{\rho^2}{\rho g} \right) - i \frac{\rho \frac{dC_p y_o f^2 L}{\rho g}}{\rho g}}$$

(35)

or

$$\alpha = \sqrt{2} \left[ \sqrt{\left( f^2 \frac{\rho^2}{\rho g} \right)^2 + \left( \frac{\rho \frac{dC_p y_o f^2 L}{\rho g}}{\rho g} \right)^2} + f^2 \frac{\rho^2}{\rho g} \right]$$

$$- i \sqrt{\left( f^2 \frac{\rho^2}{\rho g} \right)^2 + \left( \frac{\rho \frac{dC_p y_o f^2 L}{\rho g}}{\rho g} \right)^2} - f^2 \frac{\rho^2}{\rho g}$$

(36)

Numerical values of $J_0(\alpha \sqrt{\bar{\theta}})$ and $Y_0(\alpha \sqrt{\bar{\theta}})$ are tabulated in the form $J_0(r e^{i \phi})$ and $Y_0(r e^{i \phi})$, where $(r, \phi)$ are complex polar co-ordinates into which the cartesian values $\alpha \sqrt{\bar{\theta}}$ can be readily transformed according to the relations

3 Reference 3, p. 423
4 References 4 and 5
\[ r^2 = \left[ \text{Re} \left( \alpha \sqrt{E} \right) \right]^2 + \left[ \text{Im} \left( \alpha \sqrt{E} \right) \right]^2 = (\alpha_R^2 + \alpha_I^2) E \]

\[
\phi = \arctan \left[ \frac{\text{Im} \left( \alpha \sqrt{E} \right)}{\text{Re} \left( \alpha \sqrt{E} \right)} \right] = \arctan \frac{\alpha_I}{\alpha_R}
\]

where

\[ \text{Re} \left( \alpha \sqrt{E} \right) = \text{real part of } \alpha \sqrt{E} \]

\[ \text{Im} \left( \alpha \sqrt{E} \right) = \text{imaginary part of } \alpha \sqrt{E} \]

\[ \alpha_R = \text{real part of } \alpha \]

\[ \alpha_I = \text{imaginary part of } \alpha \]

Now let

\[ M = M_R + i M_I \]

\[ N = N_R + i N_I \]

\[ \mathcal{J}_R(\alpha \sqrt{E}) = \text{Re} \left\{ \mathcal{J}_0(\alpha \sqrt{E}) \right\} \]

\[ \mathcal{J}_I(\alpha \sqrt{E}) = \text{Im} \left\{ \mathcal{J}_0(\alpha \sqrt{E}) \right\} \]

\[ Y_{OR}(\alpha \sqrt{E}) = \text{Re} \left\{ \mathcal{Y}_0(\alpha \sqrt{E}) \right\} \]

\[ Y_{OI}(\alpha \sqrt{E}) = \text{Im} \left\{ \mathcal{Y}_0(\alpha \sqrt{E}) \right\} \]

where \( M_R, M_I, N_R \) and \( N_I \) are real constants. Then the solution (30) becomes

\[
\eta(\phi, \tau) = \left[ M_R \mathcal{J}_R(\alpha \sqrt{E}) - M_I \mathcal{J}_I(\alpha \sqrt{E}) \right] + \left[ N_R \mathcal{Y}_R(\alpha \sqrt{E}) - N_I \mathcal{Y}_I(\alpha \sqrt{E}) \right] \cos \omega \tau
\]

\[
+ \left[ -M_I \mathcal{J}_R(\alpha \sqrt{E}) - M_R \mathcal{J}_I(\alpha \sqrt{E}) \right] \sin \omega \tau
\]

The slope \( \frac{\partial \eta}{\partial \phi} \) is given by
\[
\frac{\partial^2 \eta}{\partial \theta^2} (\theta, \tau) = \frac{1}{2\sqrt{E}} \left[ - (M_R \alpha_R - M_I \alpha_I) J_{R} (\alpha \sqrt{E}) + (M_I \alpha_R + M_R \alpha_I) J_{I} (\alpha \sqrt{E}) - (N_R \alpha_R - N_I \alpha_I) Y_{R} (\alpha \sqrt{E}) + (N_I \alpha_R + N_R \alpha_I) Y_{I} (\alpha \sqrt{E}) \right] \cos \omega \tau
\]

\[
+ \frac{1}{2\sqrt{E}} \left[ (M_R \alpha_R - M_I \alpha_I) J_{I} (\alpha \sqrt{E}) + (M_I \alpha_R + M_R \alpha_I) J_{R} (\alpha \sqrt{E}) + (N_R \alpha_R - N_I \alpha_I) Y_{I} (\alpha \sqrt{E}) + (N_I \alpha_R + N_R \alpha_I) Y_{R} (\alpha \sqrt{E}) \right] \sin \omega \tau
\]

where

\[
J_{R} (\alpha \sqrt{E}) = Re \{ J_{I} (\alpha \sqrt{E}) \}
\]
\[
J_{I} (\alpha \sqrt{E}) = Im \{ J_{I} (\alpha \sqrt{E}) \}
\]
\[
Y_{R} (\alpha \sqrt{E}) = Re \{ Y_{I} (\alpha \sqrt{E}) \}
\]
\[
Y_{I} (\alpha \sqrt{E}) = Im \{ Y_{I} (\alpha \sqrt{E}) \}
\]

and \( J_{I} (\alpha \sqrt{E}) \) and \( Y_{I} (\alpha \sqrt{E}) \) are Bessel functions of the first and second kinds, respectively, of the first order. Numerical values of \( J_{I} (\alpha \sqrt{E}) \) and \( Y_{I} (\alpha \sqrt{E}) \) are tabulated\(^5\) in terms of the polar co-ordinates \((r, \theta)\) in Equation (37). If dynamic effects are negligible, the solution to equation (33) is

\[
\eta = (K ln \sqrt{E} + L) \cos \omega \tau + (M ln \sqrt{E} + N) \sin \omega \tau
\]

where \((K, L, M, N)\) are all real constants.

\[\text{(2) Solution for Beam at Top Under Constant Tension}\]

The equation of motion for a beam under constant tension is obtained by setting

\[T = \text{constant} = S\]

Then using \(M = -EI \frac{\partial^2}{\partial \theta^2} \), the shear force equation (6), and the drag force linearization factor (23) in equation (18) gives

\[\text{References 4 and 5}\]
The dimensionless variables

\[ \eta = \frac{y}{I} \]

\[ \bar{\xi} = \frac{x}{L} \]

\[ \tau = \sqrt{\frac{pL}{EI}} t \]

transform equation (44) to

\[ \frac{\partial^4 \eta}{\partial \bar{\xi}^4} - K_3 \frac{\partial^2 \eta}{\partial \bar{\tau}^2} + K_2 \frac{\partial \eta}{\partial \bar{\tau}} + K_o \frac{\partial^3 \eta}{\partial \bar{\tau}^3} = 0 \]  

with

\[ K_3 = \frac{L^2}{EI} (S - p'' v^2) \]

\[ K_2 = \frac{p \sqrt{EI}}{\pi} \left( \sqrt{\frac{pL}{EI}} \sqrt{p \bar{L}} \right) \]

\[ K_o = \frac{EI}{L^4} \]

A harmonic solution to the equation of motion (46) is obtained by assuming

\[ \eta = Re \left[ \gamma(\bar{\xi}) e^{i\bar{\omega} \bar{\tau}} \right] \]

where

\[ \bar{\omega} = f \sqrt{\frac{EI}{pL}} \quad (\bar{\omega} \bar{\tau} = ft) \]

This yields

\[ \frac{d^4 \gamma}{d \bar{\xi}^4} - K_3 \frac{d^2 \gamma}{d \bar{\tau}^2} - \lambda \gamma = 0 \]
where
\[ \lambda = (K_0 \omega^2) - i(K_2 \omega) \] (52)

The four solutions \( Y_j \) of (51) are found by setting
\[ Y_j = e^{P_j x} \quad j = 1, 2, 3, 4 \] (53)

with each \( P_j \) a constant. This yields a quartic equation in \( P_j \)
\[ P_j^4 - K_3 P_j^2 - \lambda = 0 \] (54)

The four roots \( P_j \) are
\[ P_{j,2} = \pm (a - i b) \]
\[ P_{j,4} = \pm (a + i b) \] (55)

where
\[
\begin{align*}
    a &= \frac{1}{2} \sqrt{K_3} \left[ \sqrt{(C_0+1)^2 + D_0^2} + (C_0+1) \right] \\
    b &= \frac{1}{2} \sqrt{K_3} \left[ \sqrt{(C_0+1)^2 + D_0^2} - (C_0+1) \right] \\
    c &= \frac{1}{2} \sqrt{K_3} \left[ \sqrt{(C_0-1)^2 + D_0^2} - (C_0-1) \right] \\
    d &= \frac{1}{2} \sqrt{K_3} \left[ \sqrt{(C_0-1)^2 + D_0^2} + (C_0-1) \right]
\end{align*}
\] (56)

and
\[
\begin{align*}
    C_0 &= \frac{1}{\sqrt{2}} \sqrt{\left( \frac{4 K_0 \omega^2}{K_3^2} + 1 \right)^2 + \left( \frac{4 K_2 \omega}{K_3^2} \right)^2} + \left( \frac{4 K_0 \omega^2}{K_3^2} + 1 \right) \\
    D_0 &= \frac{1}{\sqrt{2}} \sqrt{\left( \frac{4 K_0 \omega^2}{K_3^2} + 1 \right)^2 + \left( \frac{4 K_2 \omega}{K_3^2} \right)^2} - \left( \frac{4 K_0 \omega^2}{K_3^2} + 1 \right)
\end{align*}
\] (57)
The solution $\gamma(\xi)$ is

$$\gamma(\xi) = E e^{P \xi} + F e^{Q \xi} + G e^{R \xi} + H e^{P \xi}$$

(58)

where $E$, $F$, $G$, and $H$ are complex constants. Denoting

$$E = E_R + i E_I$$
$$F = F_R + i F_I$$
$$G = G_R + i G_I$$
$$H = H_R + i H_I$$

(59)

where $E_R$, $E_I$, $F_R$, $F_I$, $G_R$, $G_I$, $H_R$, $H_I$ are all real constants, the solution

(49) becomes

$$\eta(\xi, \tau) = \left[ e^{a \xi} \left( E_R \cos b \xi + E_I \sin b \xi \right) + e^{-a \xi} \left( F_R \cos b \xi + F_I \sin b \xi \right) + e^{n \xi} \left( (G_R \cos n \xi - G_I \sin n \xi) \right) + e^{-n \xi} \left( (H_R \cos n \xi + H_I \sin n \xi) \right) \right] \cos \omega \tau$$

$$+ \left[ e^{a \xi} \left( E_R \sin b \xi - E_I \cos b \xi \right) + e^{-a \xi} \left( -F_R \sin b \xi - F_I \cos b \xi \right) + e^{n \xi} \left( -G_R \sin n \xi - G_I \cos n \xi \right) + e^{-n \xi} \left( H_R \sin n \xi - H_I \cos n \xi \right) \right] \sin \omega \tau$$

(60)

and the derivatives are
\[ \frac{\partial}{\partial \varphi} = \left[ E_R e^{a\varphi} (a \cos \varphi - b \sin \varphi) + E_I e^{a\varphi} (a \sin \varphi + b \cos \varphi) ight. \\
+ F_R e^{-a\varphi} (-a \cos \varphi - b \sin \varphi) + F_I e^{-a\varphi} (a \sin \varphi - b \cos \varphi) \\
+ G_R e^{a\varphi} (u \cos \varphi - v \sin \varphi) + G_I e^{a\varphi} (-u \sin \varphi - v \cos \varphi) \\
\left. + H_R e^{-a\varphi} (-u \sin \varphi + v \cos \varphi) + H_I e^{-a\varphi} (u \cos \varphi + v \sin \varphi) \right] \cos \omega \tau \\
+ \left[ E_R e^{a\varphi} (a \cos \varphi + b \sin \varphi) + E_I e^{a\varphi} (-a \cos \varphi + b \sin \varphi) \\
+ F_R e^{-a\varphi} (a \cos \varphi - b \sin \varphi) + F_I e^{-a\varphi} (-a \cos \varphi + b \sin \varphi) \\
+ G_R e^{a\varphi} (-u \cos \varphi + v \sin \varphi) + G_I e^{a\varphi} (u \cos \varphi + v \sin \varphi) \\
\left. + H_R e^{-a\varphi} (-u \cos \varphi - v \sin \varphi) + H_I e^{-a\varphi} (u \cos \varphi + v \sin \varphi) \right] \sin \omega \tau \] (61)

\[ \frac{\partial^2}{\partial \varphi^2} = \left[ E_R e^{a\varphi} \{ (a^2 - b^2) \cos \varphi - 2ab \sin \varphi \} \\
+ E_I e^{a\varphi} \{ (a^2 - b^2) \sin \varphi + 2ab \cos \varphi \} \\
+ F_R e^{-a\varphi} \{ (a^2 - b^2) \cos \varphi + 2ab \sin \varphi \} \\
+ F_I e^{-a\varphi} \{ (a^2 - b^2) \sin \varphi - 2ab \cos \varphi \} \\
+ G_R e^{a\varphi} \{ (u^2 - v^2) \cos \varphi - 2uv \sin \varphi \} \\
+ G_I e^{a\varphi} \{ (u^2 - v^2) \sin \varphi - 2uv \cos \varphi \} \\
\left. + H_R e^{-a\varphi} \{ (u^2 - v^2) \cos \varphi + 2uv \sin \varphi \} \\
+ H_I e^{-a\varphi} \{ (u^2 - v^2) \sin \varphi + 2uv \cos \varphi \} \right] \cos \omega \tau \] (62)

\[ \frac{\partial^3}{\partial \varphi^3} = \left[ E_R e^{a\varphi} \{ (a^3 - 3ab^2) \cos \varphi + (-3a^2b + b^3) \sin \varphi \} \\
+ E_I e^{a\varphi} \{ (3a^2b - b^3) \cos \varphi + (a^3 - 3ab^2) \sin \varphi \} \\
+ F_R e^{-a\varphi} \{ (-a^3 + 3ab^2) \cos \varphi + (-3a^2b + b^3) \sin \varphi \} \right] \sin \omega \tau \] (CONTINUED)
If the dynamic effects are negligible, equation (46) becomes

\[ \frac{\partial^4 \eta}{\partial \xi^4} - K_3 \frac{\partial^2 \eta}{\partial \xi^2} = 0 \]  

whose solution is

\[ \eta = \left[ A + B \xi + C e^{\sqrt{K_3} \xi} + D e^{-\sqrt{K_3} \xi} \right] \cos \omega_0 t 
+ \left[ E + F \xi + G e^{\sqrt{K_3} \xi} + H e^{-\sqrt{K_3} \xi} \right] \sin \omega_0 t \]  

where \((A, B, ..., H)\) are all real constants.

(3) Solution for Beam at Bottom Under Variable Tension

Using the dimensionless variables

\[
\eta = \frac{4}{l} \quad \xi = \left( \frac{T_0 - \rho'' y^2}{\rho g l} \right) \\
\tau = \sqrt{\frac{r g}{\rho l}} t
\]

(66)
the equation of motion (25) becomes

$$\frac{d^4 U}{d \theta^4} - K_0 \frac{d^2}{d \tau^2} \left( \frac{d \theta}{d \tau} \right) + K_2 \frac{d \theta}{d \tau} + K_3 \frac{d^2 \theta}{d \tau^2} = 0$$

(67)

where

$$K_0 = \frac{\rho g l^3}{EI}$$

$$K_2 = \frac{\rho g l^5}{EI} \sqrt{\frac{\rho g}{l}}$$

A harmonic solution to Equation (67) is obtained by assuming

$$\eta(\theta, \tau) = \mathcal{R} \left[ Z(\theta) e^{i \omega \tau} \right]$$

(69)

with

$$\omega = f \sqrt{\frac{\rho g}{l}}$$

$$\omega \tau = ft$$

(70)

This yields

$$\frac{d^4 Z}{d \theta^4} - K_0 \frac{d^2}{d \theta^2} \left( \frac{d Z}{d \theta} \right) - \gamma Z = 0$$

(71)

where

$$\gamma = (K_0 \omega^2) - i(K_2 \omega)$$

(72)

Equation (71) can be solved by the Method of Froebinus setting

$$Z(\theta) = \sum_{n=0}^{\infty} Q_n \theta^{n + q}$$

(73)

where $Q_n$ and $q$ are constants. Substituting the assumed solution (73) into the differential equation (71) and equating the coefficient of each power of $\theta$ to zero shows that there are four different values of $q$

6 Reference 3, pp. 405-409.
which may be used:

\[ q_0 = 0 \quad q_2 = 2 \]
\[ q_1 = 1 \quad q_3 = 3 \]  

(74)

so that there are four linearly independent solutions of the form (73) one for each value of \( q \).

The first four constants of the series (73) are

\[
\begin{align*}
Q_0 &= (\text{arbitrary}) \quad Q_2 = 0 \\
Q_1 &= 0 \quad Q_3 = \frac{K_0 q_0 q}{(q+1)(q+2)(q+3)}
\end{align*}
\]  

(75)

The recursion formula for the remaining constants is

\[
Q_{n+4} = \frac{K_0 (n+q+1)^2 Q_{n+1} + Y Q_n}{(n+q+1)(n+q+2)(n+q+3)(n+q+4)}
\]  

(76)

where \( n = 0, 1, 2, 3, \ldots \). The solution (73) is

\[
Z(\phi) = A \left[ 1 + \frac{Y}{4!} \phi^4 + \frac{4K_0 \phi^7}{7!} + \frac{K_0^2 \phi^8}{8!} + \ldots \right] \\
+ B \left[ \phi + \frac{K_0}{4!} \phi^4 + \frac{K_0^2}{5!} \phi^5 + \frac{4K_0^2 \phi^7}{7!} + \ldots \right] \\
+ C \left[ \phi^2 + \frac{4K_0 \phi^5}{5!} + \frac{2Y \phi^6}{6!} + \frac{20K_0^2 \phi^8}{8!} + \ldots \right] \\
+ D \left[ \phi^3 + \frac{10K_0 \phi^6}{6!} + \frac{3! Y \phi^7}{7!} + \frac{108K_0^2 \phi^9}{9!} + \ldots \right]
\]  

(77)

where \( A, B, C, \) and \( D \) are complex constants. Denoting

\[
\begin{align*}
A &= A_R + i A_I \\
B &= B_R + i B_I \\
C &= C_R + i C_I \\
D &= D_R + i D_I
\end{align*}
\]  

(78)

where \( A_R, A_I, B_R, B_I, C_R, C_I, D_R, \) and \( D_I \) are real constants, and
\[ Y_R = \Re (Y) = K_0 \bar{\omega}^2 \]
\[ Y_I = \Im (Y) = -K_2 \bar{\omega} \]  
(79)

the solution (69) becomes

\[
\eta = \left[ A_R \left( 1 + \frac{K_0}{8i} S^4 + \frac{4iK_0}{7i} S^7 + \frac{K_0^2 - K_2^2}{\bar{r}} S^8 + \cdots \right) \\
- A_I \left( \frac{K_0}{8i} S^4 + \frac{4iK_0}{7i} S^7 + \frac{K_0^2 - K_2^2}{\bar{r}} S^8 + \cdots \right) \\
+ B_R \left( S^4 + \frac{K_0}{8i} S^7 + \frac{4iK_0}{7i} S^8 + \cdots \right) - B_I \left( \frac{K_0}{8i} S^5 + \cdots \right) \\
+ C_R \left( S^2 + \frac{2iK_0}{3i} S^6 + \frac{2iK_0}{3i} S^7 + \cdots \right) - C_I \left( \frac{2iK_0}{3i} S^6 + \cdots \right) \\
+ D_R \left( S^3 + \frac{18K_0}{4i} S^6 + \frac{31iK_0}{4i} S^7 + \frac{108K_0^2}{4i} S^8 + \cdots \right) - D_I \left( \frac{31iK_0}{4i} S^7 + \cdots \right) \cos \bar{\omega} \zeta \\
- \frac{31iK_0}{4i} S^7 + \cdots \right] \cos \bar{\omega} \zeta 
\]

(80)

The derivatives are

\[
\frac{d \eta}{d \bar{\omega}} = \left[ A_R \left( \frac{K_0}{3i} S^3 + \frac{4iK_0}{6i} S^6 + \frac{K_0^2 - K_2^2}{\bar{r}} S^7 + \cdots \right) \\
- A_I \left( \frac{K_0}{3i} S^3 + \frac{4iK_0}{6i} S^6 + \frac{K_0^2 - K_2^2}{\bar{r}} S^7 + \cdots \right) \\
+ B_R \left( 1 + \frac{K_0}{3i} S^3 + \frac{18K_0}{3i} S^6 + \frac{4iK_0}{6i} S^7 + \cdots \right) - B_I \left( \frac{K_0}{3i} S^4 + \cdots \right) \\
+ C_R \left( 2S^2 + \frac{2iK_0}{4i} S^5 + \frac{2iK_0}{4i} S^6 + \frac{20K_0^2}{4i} S^7 + \cdots \right) - C_I \left( \frac{2iK_0}{4i} S^5 + \cdots \right) \\
+ D_R \left( 3S^2 + \frac{18K_0}{5i} S^5 + \frac{31iK_0}{5i} S^6 + \frac{108K_0^2}{5i} S^7 + \cdots \right) - D_I \left( \frac{31iK_0}{5i} S^5 + \cdots \right) \cos \bar{\omega} \zeta \\
- \frac{31iK_0}{5i} S^5 + \cdots \right] \sin \bar{\omega} \zeta 
\]

(Continued)
\[
\begin{align*}
\frac{\partial^2}{\partial \phi^2} & \left[ A_R \left( \frac{R}{2} \frac{\partial^2}{\partial \phi^2} + \frac{4}{5} \frac{\partial^2}{\partial \phi^2} + \frac{2}{6} \frac{\partial^2}{\partial \phi^2} \right) + A_I \left( \frac{R}{2} \frac{\partial^2}{\partial \phi^2} + \frac{4}{5} \frac{\partial^2}{\partial \phi^2} + \frac{2}{6} \frac{\partial^2}{\partial \phi^2} \right) + B_R \left( \frac{R}{2} \frac{\partial^2}{\partial \phi^2} + \frac{4}{5} \frac{\partial^2}{\partial \phi^2} + \frac{2}{6} \frac{\partial^2}{\partial \phi^2} \right) + C_R \left( \frac{R}{2} \frac{\partial^2}{\partial \phi^2} + \frac{4}{5} \frac{\partial^2}{\partial \phi^2} + \frac{2}{6} \frac{\partial^2}{\partial \phi^2} \right) + D_R \left( \frac{R}{2} \frac{\partial^2}{\partial \phi^2} + \frac{4}{5} \frac{\partial^2}{\partial \phi^2} + \frac{2}{6} \frac{\partial^2}{\partial \phi^2} \right) \right] \sin \omega t \\
& - \left[ A_R \left( \frac{R}{2} \frac{\partial^2}{\partial \phi^2} + \frac{4}{5} \frac{\partial^2}{\partial \phi^2} + \frac{2}{6} \frac{\partial^2}{\partial \phi^2} \right) + A_I \left( \frac{R}{2} \frac{\partial^2}{\partial \phi^2} + \frac{4}{5} \frac{\partial^2}{\partial \phi^2} + \frac{2}{6} \frac{\partial^2}{\partial \phi^2} \right) + B_R \left( \frac{R}{2} \frac{\partial^2}{\partial \phi^2} + \frac{4}{5} \frac{\partial^2}{\partial \phi^2} + \frac{2}{6} \frac{\partial^2}{\partial \phi^2} \right) + C_R \left( \frac{R}{2} \frac{\partial^2}{\partial \phi^2} + \frac{4}{5} \frac{\partial^2}{\partial \phi^2} + \frac{2}{6} \frac{\partial^2}{\partial \phi^2} \right) + D_R \left( \frac{R}{2} \frac{\partial^2}{\partial \phi^2} + \frac{4}{5} \frac{\partial^2}{\partial \phi^2} + \frac{2}{6} \frac{\partial^2}{\partial \phi^2} \right) \right] \cos \omega t
\end{align*}
\]

(81)
If dynamic effects are negligible, the solution becomes

\[
\eta = \left[ A (1 + \cdots) + B \left( \varepsilon^2 + \frac{K_0}{4!} \varepsilon^4 + \frac{4K_0^2}{7!} \varepsilon^7 + \cdots \right) + C \left( \varepsilon^2 + \frac{4K_0}{5!} \varepsilon^5 + \frac{20 K_0^2}{8!} \varepsilon^8 + \cdots \right) + D \left( \varepsilon^3 + \frac{18 K_0}{6!} \varepsilon^6 + \frac{108 K_0^2}{9!} \varepsilon^9 + \cdots \right) \right] \cos \omega t
\]

\[
+ \left[ E (1 + \cdots) + F \left( \varepsilon^2 + \frac{K_0}{4!} \varepsilon^4 + \frac{4K_0^2}{7!} \varepsilon^7 + \cdots \right) + G \left( \varepsilon^2 + \frac{4K_0}{5!} \varepsilon^5 + \frac{20 K_0^2}{8!} \varepsilon^8 + \cdots \right) + H \left( \varepsilon^3 + \frac{18 K_0}{6!} \varepsilon^6 + \frac{108 K_0^2}{9!} \varepsilon^9 + \cdots \right) \right] \sin \omega t
\]

(84)

where \(A, B, C, D, E, F, G, H\) are all real constants.
III. BOUNDARY CONDITIONS

The drill string whose motion is to be numerically evaluated is that used by the National Science Foundation in the preliminary drilling for "Project Mohole", carried out in 1961 in deep water off the coast of Guadalupe Island, Mexico. The details of the drill string are given in Reference 2. The drilling conditions to be used here correspond to those encountered in hole #2 at the Guadalupe site: the position of the drill string corresponds to penetration of 566 feet. The situation is indicated schematically in Figure 2.
There are four different sections of drill pipe used above the ocean floor, as indicated in Figure 2; the changes in cross section occur at the junctions (1)-(2), (2)-(3), and (3)-(4).

Since quantities such as \( T, T_0, \rho, D, \ell \), etc. depend upon the section considered, the value of such a quantity in the \( i \)th section is indicated by a subscript \( i \) denoting the section involved, such as \( T_i, \ell_i \), etc. where \( i = 1, 2, 3, 4 \).

The deflection curve of the drill string is a composite function as indicated in Figure 2. The function used for each section is determined according to its anticipated physical behavior as indicated. The functions are coupled via the boundary conditions imposed at each junction.

Neglecting the ship's roll, the slope at the top of section (1) is zero, while the displacement away from the hole is the ship's displacement \( y_0(t) \). The displacement and shearing force across a horizontal plane at the junction (1)-(2) must be continuous; the bending moment at the bottom of section (1) must vanish, since a cable is assumed perfectly flexible. The slope at junction (1)-(2) can be made as nearly continuous as desired by adjusting the length of the section considered to behave as a beam, as explained in the Appendix.

The displacement and shearing force across a horizontal plane at the junction (2)-(3) must be continuous.

At junction (3)-(4), the displacement and shearing force across a horizontal plane must be continuous; the bending moment at the top of section (4) must vanish. The slope can be made as nearly continuous as desired; see Appendix.

The slope and displacement at the bottom of section (4) are both taken to be zero.
The horizontal shear $V_i$ in each beam is

$$V_i = -EI_i \frac{d^3 y}{dx_i^3} + (T_{0i} - \rho_i g \xi_i) \frac{dy}{dx_i}$$  \hspace{1cm} (85)$$

while the horizontal shear $V_i$ in each cable is

$$V_i = (T_{0i} - \rho_i g \xi_i) \frac{dy}{dx_i}$$  \hspace{1cm} (86)$$

Since the vertical tension $T$ must be continuous at every junction

$$T(\xi_i = l_i) = T(\xi_{i+1} = 0)$$  \hspace{1cm} (87)$$
or

$$(T_{0} - \rho_i g l_i) = (T_{0})_{i+1}$$  \hspace{1cm} (88)$$

The boundary conditions become

$$y(\xi_1 = 0) = y_0 \sin ft$$

$$\frac{dy}{dx_1}(\xi_1 = 0) = 0$$

$$y(\xi_1 = l_1) = y(\xi_2 = 0)$$

$$-EI_1 \frac{d^3 y}{dx_1^3} (\xi_1 = l_1) + T_{02} \frac{dy}{dx_1} (\xi_1 = l_1) = T_{02} \frac{dy}{dx_2} (\xi_2 = 0)$$

$$\frac{dy}{dx_1^2} (\xi_1 = l_1) = 0$$

$$y(\xi_2 = l_2) = y(\xi_3 = 0)$$

$$\frac{dy}{dx_2} (\xi_2 = l_2) = \frac{dy}{dx_3} (\xi_3 = 0)$$

$$y(\xi_3 = l_3) = y(\xi_4 = 0)$$

$$T_{04} \frac{dy}{dx_3} (\xi_3 = l_3) = T_{04} \frac{dy}{dx_4} (\xi_4 = 0) - EI_4 \frac{d^3 y}{dx_4^3} (\xi_4 = 0)$$

$$\frac{dy}{dx_4^2} (\xi_4 = 0) = 0$$

$$y(\xi_4 = l_4) = 0$$

$$\frac{dy}{dx_4} (\xi_4 = l_4) = 0$$
Note that at junction (2-3), continuity of shear implies continuity of slope.

When the dimensionless variables (27) and (45) are applied, the boundary conditions become, respectively:

\[ \eta_1 (\xi_1 = \xi_1) = \eta_0 \sin \omega t \]
\[ \frac{\partial \eta_1}{\partial \xi_1} (\xi_1 = \xi_1) = 0 \]
\[ \eta_2 (\xi_2 = \xi_2) = \eta_3 (\xi_3 = \xi_3) \]
\[ \frac{\partial \eta_2}{\partial \xi_2} (\xi_2 = \xi_2) = \frac{\partial \eta_3}{\partial \xi_3} (\xi_3 = \xi_3) \]
\[ \eta_3 (\xi_3 = \xi_3) = \eta_4 (\xi_4 = \xi_4) \]
\[ \frac{\partial \eta_3}{\partial \xi_3} (\xi_3 = \xi_3) = \frac{\partial \eta_4}{\partial \xi_4} (\xi_4 = \xi_4) \]
\[ \eta_4 (\xi_4 = \xi_4) = 0 \]
\[ \frac{\partial \eta_4}{\partial \xi_4} (\xi_4 = \xi_4) = 0 \]
This set of twelve equations uniquely determines all the unknown constants of integration in \( (\eta_1, \eta_2, \eta_3, \eta_4) \), where

\[
\eta_i = \frac{\xi_i}{\xi_i} \\
\xi_0 = \xi_i (\xi_i = 0) \\
\xi_0 = \xi_i (\xi_i = A_i) \\
\eta_i = \frac{\xi_i}{\xi_i}
\]  

(91)

The functions \( \eta_1 \) and \( \eta_4 \) are given by equations (60) or (65); \( \eta_2 \) and \( \eta_3 \) are given by equations (39) or (42).

The physical parameters in the dynamic equations (39) and (60) must be numerically evaluated to determine whether the static or dynamic solution is to be used in each case.

The drill string considered is in water 11,672 feet deep and the hole is 566 feet deep, which requires the drill string to be 12,238 feet long over-all. The drilling fluid involved is sea water, corresponding to practice at the Guadalupe site.

The axial tension at the hole entry, \( T_h \), must be calculated first, since axial tension is affected by the buoyant force which is distributed over the surface of the drill collars, bumper subs, and drill pipe in a complicated fashion. The buoyant force is not a uniformly distributed load which acts to reduce the weight (per unit length) of the drill string; it is the resultant of the distribution of hydrostatic pressure over the wetted surface of the member considered. The buoyant force in this case, with the irregularly-shaped members in the hole, acts essentially on the horizontal surfaces of the members in the hole. This result may be demonstrated by considering a perfectly uniform
hollow cylinder immersed in a vertical position, as in Figure 3. \( T_o \) represents the tension at the surface and \( W \) the weight of the cylinder in air. The hydrostatic pressure \( p \) acts normal to the surface at every point and is given by

\[
p(X) = gage \ pressure = \rho \cdot g \cdot X
\]  

(92)

\[
F_B = p(L) \cdot \frac{\pi}{4} \left( d^2 - \mathcal{O}^2 \right)
\]

Figure 3

(The hydrostatic pressure inside the drill pipe is the same as exists on the outside, since sea water is serving as the drilling fluid).

Since the distribution of forces on the vertical surfaces is radially symmetric, there is no buoyant force component exerted on the sides of the drill pipe. The only buoyant force is exerted at the bottom of the cylinder and is

\[
F_B = p(L) \cdot \frac{\pi}{4} \left( d^2 - \mathcal{O}^2 \right) = \rho \cdot g \cdot L \cdot \frac{\pi}{4} \left( d^2 - \mathcal{O}^2 \right)
\]  

(93)

where \( d \) = outer diameter of cylinder

\( \mathcal{O} \) = inner diameter of cylinder

\( L \) = length of cylinder

Thus the buoyant force equals the weight of water displaced by the cylinder, but it is a localized force distributed over the bottom of the cylinder.
This method can be applied to the problem considered, since the slope of the drill pipe is assumed small (so that there is little contribution to the buoyant force from the pressures on the side of the drill pipe) and the irregularly shaped members are all in the hole. Consider a free-body diagram (Figure 4) enclosing the entire contents of the hole, both the water and drilling equipment. Let the diameter D of the hole be slightly larger than the bit. The hole is assumed vertical.

\[ T_h = \text{axial tension at hole entry} \]
\[ A_1 = \text{cross section area of hole minus cross-section area of pipe at hole entry} \]
\[ A_2 = \text{cross-section area of hole} \]
\[ p_1 = \text{hydrostatic pressure at top of hole} \]
\[ p_2 = \text{hydrostatic pressure at bottom of hole} \]
\( P_B = \) bit pressure

\( W = \) weight of drilling equipment in hole plus weight of water in hole.

The hydrostatic pressures on the side of the hole cancel each other due to radial symmetry.

Given the weight of drilling equipment in the hole and its density, the volume occupied by the drilling equipment may be computed. Subtracting this volume from the volume of the hole yields the volume of water in the hole, and hence the weight of water in the hole; so that \( W \) is known. The pressures \( p_1 \) and \( p_2 \) are the hydrostatic pressures existing at the top and bottom of the hole. \( P_B \) is assumed known. A force balance as indicated in Figure 4 yields \( T_h \).

If the external diameter of the drill string above the hole is constant (as in this example), all the buoyant force is exerted in the hole. If the outer diameter of the drill string above the hole is not constant, additional buoyant forces act wherever the outer diameter changes.

For the drill string studied here, the axial tension at any position equals \( T_h \) plus the weight of drill pipe above the hole, up to the point considered.

Let the displacement impressed by the ship be (in feet)

\[ y(x, t) = 300 \sin ft \]  

and let the maximum velocity attained by 0.2 knots, in accordance with data on the allowed operating area (ship position relative to the hole) and the drilling ship. This yields

\[ y_0 = 300 \text{ feet} \]  

\[ f = 1.13 \times 10^{-3} \text{ sec}^{-1} \]
Let the bit pressure $P_B$ be 17,000 lbs. and let the flow rate of the drilling fluid (sea water) be 300 gal/min., as used at the Guadalupe site. Let the weight of drilling equipment in the hole be 52,200 lbs. with weight density 489 lbs./ft$^3$ as also quoted in Reference 2.

These parameters, together with the drill string dimensions and weights quoted in Reference 2, yield the data of Table 1.

<table>
<thead>
<tr>
<th>SECTION</th>
<th>$l$ (ft.)</th>
<th>$d$ (in.)</th>
<th>$D$ (in.)</th>
<th>$I$ (in.$^4$)</th>
<th>$f$ (lb.-sec.$^2$/ft.$^4$)</th>
<th>$f'$ (lb.-sec.$^2$/ft.$^4$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>132</td>
<td>4.50</td>
<td>3.64</td>
<td>11.50</td>
<td>0.446</td>
<td>0.803</td>
</tr>
<tr>
<td>2</td>
<td>3500</td>
<td>4.50</td>
<td>3.83</td>
<td>11.60</td>
<td>0.497</td>
<td>0.714</td>
</tr>
<tr>
<td>3</td>
<td>7980</td>
<td>4.50</td>
<td>3.96</td>
<td>11.50</td>
<td>0.528</td>
<td>0.635</td>
</tr>
<tr>
<td>4</td>
<td>60</td>
<td>4.50</td>
<td>3.83</td>
<td>9.57</td>
<td>0.497</td>
<td>0.714</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$T_o$ (lbs.)</th>
<th>$T_h$ (lbs.)</th>
<th>$f''v^2$ (lbs.)</th>
<th>$k_o$</th>
<th>$k_2$</th>
<th>$k_3$</th>
<th>$\alpha_R$</th>
<th>$\alpha_I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>193,600</td>
<td>12</td>
<td>0.000</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>190,800</td>
<td>11</td>
<td>3.05</td>
<td>1.0727</td>
<td>0.0727</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>128,300</td>
<td>11</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>8,700</td>
<td>7,600</td>
<td></td>
<td>1.92</td>
<td>0.425</td>
<td>14.68</td>
<td></td>
</tr>
</tbody>
</table>

The data show that the tension in the top and in the bottom beam is quite nearly constant, justifying the treatment of these members as constant-tension beams. The axial tension at the hole, $T_h = 7600$ lbs., is quite different from the weight of equipment (52,200 lbs.) in the hole.

Considering first the beams under constant tension, sections (1) and (4), the dynamic effects are shown by equations (56) and (57)
to be negligible in this example. Consequently the static solution (65) is to be used.

The dynamic effects will be treated for sections (2) and (3). Hence the dynamic solution (39) is to be employed for these sections.

The boundary conditions will now be expressed in terms of the solutions (39) and (65). In order to simplify the notation for the solutions \( \eta_2 \) and \( \eta_3 \), let

\[
\eta_i = \left[ M_{Ri} J_{OR} (\alpha_i \sqrt{\beta_i}) - M_{Ii} J_{OI} (\alpha_i \sqrt{\beta_i}) \right]
+ N_{Ri} Y_{OR} (\alpha_i \sqrt{\beta_i}) - N_{Ii} Y_{OI} (\alpha_i \sqrt{\beta_i}) \right] \cos \omega t
+ \left[ M_{Ii} J_{OR} (\alpha_i \sqrt{\beta_i}) - M_{Ri} J_{OI} (\alpha_i \sqrt{\beta_i}) \right]
- N_{Ii} Y_{OR} (\alpha_i \sqrt{\beta_i}) - N_{Ri} Y_{OI} (\alpha_i \sqrt{\beta_i}) \right] \sin \omega t
\]

(96)

where \( i = 2, 3 \).

For the solutions \( \eta_1 \) and \( \eta_4 \), let

\[
\eta_i = \left[ A_i + B_i \xi + C_i e^{\sqrt{\mu_i} \xi t} + D_i e^{-\sqrt{\mu_i} \xi t} \right] \cos \omega t
+ \left[ E_i + F_i \xi + G_i e^{\sqrt{\mu_i} \xi t} + H_i e^{-\sqrt{\mu_i} \xi t} \right] \sin \omega t
\]

(98)

where \( i = 1, 4 \).
The set of boundary conditions (90) becomes

\[ A_1 + B_1 e^{\sqrt{K_3} \phi_0} + C_1 e^{-\sqrt{K_3} \phi_0} + D_1 e^{-\sqrt{K_3} \phi_0} = 0 \]  
(99a)

\[ E_1 + F_1 e^{\sqrt{K_3} \phi_0} + G_1 e^{-\sqrt{K_3} \phi_0} + H_1 e^{-\sqrt{K_3} \phi_0} = 0 \]  
(99b)

\[ B_1 + \sqrt{K_3} (C_1 e^{\sqrt{K_3} \phi_0} - D_1 e^{-\sqrt{K_3} \phi_0}) = 0 \]  
(99c)

\[ F_1 + \sqrt{K_3} (C_1 e^{\sqrt{K_3} \phi_0} - H_1 e^{-\sqrt{K_3} \phi_0}) = 0 \]  
(99d)

\[ \ell_1 \left[ A_1 + B_1 e^{\sqrt{K_3} \phi_1} + C_1 e^{-\sqrt{K_3} \phi_1} + D_1 e^{-\sqrt{K_3} \phi_1} \right] 
= \ell_2 \left[ M_{R_2} J_{0 \alpha R} (\alpha_2 e^{\sqrt{K_2} \phi_0}) - M_{I_2} J_{0 \alpha I} (\alpha_2 e^{\sqrt{K_2} \phi_0}) + N_{R_2} Y_{0 \alpha R} (\alpha_2 e^{\sqrt{K_2} \phi_0}) - N_{I_2} Y_{0 \alpha I} (\alpha_2 e^{\sqrt{K_2} \phi_0}) \right] \]  
(99e)

\[ \ell_1 \left[ E_1 + F_1 e^{\sqrt{K_3} \phi_1} + G_1 e^{-\sqrt{K_3} \phi_1} + H_1 e^{-\sqrt{K_3} \phi_1} \right] 
= \ell_2 \left[ -M_{I_2} J_{0 \alpha R} (\alpha_2 e^{\sqrt{K_2} \phi_0}) - M_{R_2} (\alpha_2 e^{\sqrt{K_2} \phi_0}) - N_{I_2} Y_{0 \alpha I} (\alpha_2 e^{\sqrt{K_2} \phi_0}) - N_{R_2} Y_{0 \alpha R} (\alpha_2 e^{\sqrt{K_2} \phi_0}) \right] \]  
(99f)

\[ -\frac{E_1}{I_2} K_3 \left[ C_1 e^{\sqrt{K_3} \phi_1} + D_1 e^{-\sqrt{K_3} \phi_1} \right] + T_2 \left[ B_1 + \sqrt{K_3} (C_1 e^{\sqrt{K_3} \phi_1} - D_1 e^{-\sqrt{K_3} \phi_1}) \right] \]  
(99g)

\[ = -T_2 \cdot \frac{1}{2 \sqrt{K_2}} \left[ (M_{R_2} \alpha_2 R_2 - M_{I_2} \alpha_2 I_2) J_{0 \alpha I} (\alpha_2 e^{\sqrt{K_2} \phi_0}) + (M_{I_2} \alpha_2 R_2 + M_{R_2} \alpha_2 I_2) J_{0 \alpha R} (\alpha_2 e^{\sqrt{K_2} \phi_0}) \right. 
- (N_{R_2} \alpha_2 R_2 - N_{I_2} \alpha_2 I_2) Y_{0 \alpha I} (\alpha_2 e^{\sqrt{K_2} \phi_0}) + (N_{I_2} \alpha_2 R_2 + N_{R_2} \alpha_2 I_2) Y_{0 \alpha R} (\alpha_2 e^{\sqrt{K_2} \phi_0}) \]  
(99h)

\[ -\frac{E_1}{I_2} K_3 \left[ G_1 e^{\sqrt{K_3} \phi_1} + H_1 e^{-\sqrt{K_3} \phi_1} \right] + T_2 \left[ F_1 + \sqrt{K_3} (G_1 e^{\sqrt{K_3} \phi_1} - H_1 e^{-\sqrt{K_3} \phi_1}) \right] \]  
(99i)

\[ C_1 e^{\sqrt{K_3} \phi_1} + D_1 e^{-\sqrt{K_3} \phi_1} = 0 \]  
(99j)

\[ G_1 e^{\sqrt{K_3} \phi_1} + H_1 e^{-\sqrt{K_3} \phi_1} = 0 \]  
(99k)
\[ l_2 \left[ M_{R2} J_{OR} (\alpha_1 \sqrt{P_2}) - M_{I2} J_{OI} (\alpha_1 \sqrt{P_2}) + N_{R2} Y_{OR} (\alpha_1 \sqrt{P_2}) - N_{I2} Y_{OI} (\alpha_1 \sqrt{P_2}) \right] \]
\[ = l_3 \left[ M_{R3} J_{OR} (\alpha_3 \sqrt{P_2}) - M_{I3} J_{OI} (\alpha_3 \sqrt{P_2}) + N_{R3} Y_{OR} (\alpha_3 \sqrt{P_2}) - N_{I3} Y_{OI} (\alpha_3 \sqrt{P_2}) \right] \]

\[ l_2 \left[ - M_{I2} J_{OR} (\alpha_1 \sqrt{P_2}) - M_{R2} J_{OI} (\alpha_1 \sqrt{P_2}) - N_{I2} Y_{OR} (\alpha_1 \sqrt{P_2}) - N_{R2} Y_{OI} (\alpha_1 \sqrt{P_2}) \right] \]
\[ = l_3 \left[ - M_{I3} J_{OR} (\alpha_3 \sqrt{P_2}) - M_{R3} J_{OI} (\alpha_3 \sqrt{P_2}) - N_{I3} Y_{OR} (\alpha_3 \sqrt{P_2}) - N_{R3} Y_{OI} (\alpha_3 \sqrt{P_2}) \right] \]

\[ \frac{1}{2K_{P2}} \left[ -(M_{R2} \alpha_1 - M_{I2} \alpha_{I2}) J_{IR} (\alpha_1 \sqrt{P_2}) + (M_{I2} \alpha_{R2} + M_{R2} \alpha_{I2}) J_{II} (\alpha_1 \sqrt{P_2}) \right. \]
\[ \left. - (N_{R2} \alpha_{R2} - N_{I2} \alpha_{I2}) Y_{IR} (\alpha_1 \sqrt{P_2}) + (N_{I2} \alpha_{R2} + N_{R2} \alpha_{I2}) Y_{II} (\alpha_1 \sqrt{P_2}) \right] \]
\[ = \frac{1}{2K_{P3}} \left[ -(M_{R3} \alpha_{R3} - M_{I3} \alpha_{I3}) J_{IR} (\alpha_3 \sqrt{P_2}) + (M_{I3} \alpha_{R3} + M_{R3} \alpha_{I3}) J_{II} (\alpha_3 \sqrt{P_2}) \right. \]
\[ \left. - (N_{R3} \alpha_{R3} - N_{I3} \alpha_{I3}) Y_{IR} (\alpha_3 \sqrt{P_2}) + (N_{I3} \alpha_{R3} + N_{R3} \alpha_{I3}) Y_{II} (\alpha_3 \sqrt{P_2}) \right] \]

\[ \frac{1}{2K_{P2}} \left[ -(M_{R2} \alpha_1 - M_{I2} \alpha_{I2}) J_{IR} (\alpha_1 \sqrt{P_2}) + (M_{I2} \alpha_{R2} + M_{R2} \alpha_{I2}) J_{II} (\alpha_1 \sqrt{P_2}) \right. \]
\[ \left. + (N_{R2} \alpha_{R2} - N_{I2} \alpha_{I2}) Y_{IR} (\alpha_1 \sqrt{P_2}) + (N_{I2} \alpha_{R2} + N_{R2} \alpha_{I2}) Y_{II} (\alpha_1 \sqrt{P_2}) \right] \]
\[ = \frac{1}{2K_{P3}} \left[ -(M_{R3} \alpha_{R3} - M_{I3} \alpha_{I3}) J_{IR} (\alpha_3 \sqrt{P_2}) + (M_{I3} \alpha_{R3} + M_{R3} \alpha_{I3}) J_{II} (\alpha_3 \sqrt{P_2}) \right. \]
\[ \left. + (N_{R3} \alpha_{R3} - N_{I3} \alpha_{I3}) Y_{IR} (\alpha_3 \sqrt{P_2}) + (N_{I3} \alpha_{R3} + N_{R3} \alpha_{I3}) Y_{II} (\alpha_3 \sqrt{P_2}) \right] \]

\[ l_3 \left[ M_{R3} J_{OR} (\alpha_3 \sqrt{P_2}) - M_{I3} J_{OI} (\alpha_3 \sqrt{P_2}) + N_{R3} Y_{OR} (\alpha_3 \sqrt{P_2}) - N_{I3} Y_{OI} (\alpha_3 \sqrt{P_2}) \right] \]
\[ = l_4 \left[ A_4 + B_4 e^{\frac{F_4}{K_4}} + C_4 e^{\frac{F_4}{K_4}} e^{\frac{F_4}{K_4}} + D_4 e^{-\frac{F_4}{K_4}} e^{\frac{F_4}{K_4}} \right] \]

\[ l_3 \left[ - M_{I3} J_{OR} (\alpha_3 \sqrt{P_2}) - M_{R3} J_{OI} (\alpha_3 \sqrt{P_2}) - N_{I3} Y_{OR} (\alpha_3 \sqrt{P_2}) - N_{R3} Y_{OI} (\alpha_3 \sqrt{P_2}) \right] \]
\[ = l_4 \left[ E_4 + F_4 e^{\frac{F_4}{K_4}} + G_4 e^{\frac{F_4}{K_4}} e^{\frac{F_4}{K_4}} + H_4 e^{-\frac{F_4}{K_4}} e^{\frac{F_4}{K_4}} \right] \]

\[ - T_{Q4} \frac{1}{2K_{P3}} \left[ -(M_{R3} \alpha_{R3} - M_{I3} \alpha_{I3}) J_{IR} (\alpha_3 \sqrt{P_2}) + (M_{I3} \alpha_{R3} + M_{R3} \alpha_{I3}) J_{II} (\alpha_3 \sqrt{P_2}) \right. \]
\[ \left. - (N_{R3} \alpha_{R3} - N_{I3} \alpha_{I3}) Y_{IR} (\alpha_3 \sqrt{P_2}) + (N_{I3} \alpha_{R3} + N_{R3} \alpha_{I3}) Y_{II} (\alpha_3 \sqrt{P_2}) \right] \]

\[ - T_{Q4} \frac{1}{2K_{P3}} \left[ -(M_{R3} \alpha_{R3} - M_{I3} \alpha_{I3}) J_{IR} (\alpha_3 \sqrt{P_2}) + (M_{I3} \alpha_{R3} + M_{R3} \alpha_{I3}) J_{II} (\alpha_3 \sqrt{P_2}) \right. \]
\[ + (N_{R3} \alpha_{R3} - N_{I3} \alpha_{I3}) Y_{IR} (\alpha_3 \sqrt{P_2}) + (N_{I3} \alpha_{R3} + N_{R3} \alpha_{I3}) Y_{II} (\alpha_3 \sqrt{P_2}) \right] \]

\[ - T_{Q4} \frac{1}{2K_{P3}} \left[ -(M_{R3} \alpha_{R3} - M_{I3} \alpha_{I3}) J_{IR} (\alpha_3 \sqrt{P_2}) + (M_{I3} \alpha_{R3} + M_{R3} \alpha_{I3}) J_{II} (\alpha_3 \sqrt{P_2}) \right. \]
\[ + (N_{R3} \alpha_{R3} - N_{I3} \alpha_{I3}) Y_{IR} (\alpha_3 \sqrt{P_2}) + (N_{I3} \alpha_{R3} + N_{R3} \alpha_{I3}) Y_{II} (\alpha_3 \sqrt{P_2}) \right] \]
\[
\begin{align*}
\text{(99a)} & \quad c_4 e^{\sqrt{\text{K}_{34}} \xi_{34}^4} + D_4 e^{-\sqrt{\text{K}_{34}} \xi_{34}^4} = 0 \\
\text{(99b)} & \quad G_4 e^{\sqrt{\text{K}_{34}} \xi_{34}^4} + H_4 e^{-\sqrt{\text{K}_{34}} \xi_{34}^4} = 0 \\
\text{(99c)} & \quad A_4 + B_4 \xi_{34}^4 + C_4 e^{\sqrt{\text{K}_{34}} \xi_{34}^4} + D_4 e^{-\sqrt{\text{K}_{34}} \xi_{34}^4} = 0 \\
\text{(99d)} & \quad E_4 + F_4 \xi_{34}^4 + G_4 e^{\sqrt{\text{K}_{34}} \xi_{34}^4} + H_4 e^{-\sqrt{\text{K}_{34}} \xi_{34}^4} = 0 \\
\text{(99e)} & \quad B_4 + \sqrt{\text{K}_{34}} (C_4 e^{\sqrt{\text{K}_{34}} \xi_{34}^4} - D_4 e^{-\sqrt{\text{K}_{34}} \xi_{34}^4}) = 0 \\
\text{(99f)} & \quad F_4 + \sqrt{\text{K}_{34}} (G_4 e^{\sqrt{\text{K}_{34}} \xi_{34}^4} - H_4 e^{-\sqrt{\text{K}_{34}} \xi_{34}^4}) = 0 
\end{align*}
\]
IV. RESULTS

For the numerical example considered, the integration constants are

\[
\begin{align*}
A_1 &= 3.069 \times 10^{-5} \\
B_1 &= -1.148 \times 10^{-3} \\
C_1 &= 1.004 \times 10^{-37} \\
D_1 &= -3.069 \times 10^{-5} \\
E_1 &= 2.2727 \\
F_1 &= -7.789 \times 10^{-3} \\
G_1 &= 6.814 \times 10^{-37} \\
H_1 &= -2.082 \times 10^{-4} \\
M_{R2} &= 3.246 \times 10^{-2} \\
M_{I2} &= -1.239 \times 10^{-1} \\
N_{R2} &= -1.959 \times 10^{-3} \\
N_{I2} &= -7.293 \times 10^{-2} \\
M_{R3} &= 1.729 \times 10^{-2} \\
M_{I3} &= -5.426 \times 10^{-2} \\
N_{R3} &= -1.377 \times 10^{-3} \\
N_{I3} &= -3.838 \times 10^{-2} \\
A_4 &= -4.942 \times 10^{-3} \\
B_4 &= 6.686 \times 10^{-3} \\
C_4 &= -3.788 \times 10^{-5} \\
D_4 &= 3.788 \times 10^{-5} \\
E_4 &= 1.241 \times 10^{-1} \\
F_4 &= -1.679 \times 10^{-1} \\
G_4 &= 0.9514 \times 10^{-3} \\
H_4 &= -0.9514 \times 10^{-3}
\end{align*}
\]

(1.) Bending Stresses

Let the bending stresses* associated with the in-phase \((\sin \phi t)\) and lagging \((\cos \phi t)\) deflections be denoted \(\sigma_{\sin}\) and \(\sigma_{\cos}\), respectively. Note that each stress varies harmonically with time, \(\sigma_{\sin}\) as \(\sin \phi t\) and \(\sigma_{\cos}\) as \(\cos \phi t\).

The maximum stresses in the top beam are

\[
\begin{align*}
[(\sigma_{\cos})_{\text{TOP BEAM}}]_{\text{MAX.}} &= 1,830 \text{ psi} \\
[(\sigma_{\sin})_{\text{TOP BEAM}}]_{\text{MAX.}} &= 12,390 \text{ psi}
\end{align*}
\]

Both maxima occur at the ocean surface.

* computed at the outer diameter
The maximum total bending stress at the top is given by

\[
\left[ (\sigma_{\text{TOTAL}})_{\text{TOP BEAM}} \right]_{\text{MAX.}} = \sqrt{\left[ (\sigma_{\cos})_{\text{TOP BEAM}} \right]_{\text{MAX.}}^2 + \left[ (\sigma_{\sin})_{\text{TOP BEAM}} \right]_{\text{MAX.}}^2}
\]

\[= 12,500 \text{ psi} \]

The distribution of bending stresses in the top beam is plotted in Figures 5 and 6.
Figure 6.
The maximum bending stresses in the bottom beam are

\[
\begin{align*}
[(\sigma_{\cos})_{\text{bottom beam}}]_{\text{max.}} &= -2,400 \text{ psi} \\
[(\sigma_{\sin})_{\text{bottom beam}}]_{\text{max.}} &= 60,240 \text{ psi}
\end{align*}
\]

Both maxima occur at the ocean floor, where the bottom beam is assumed to be built-in. The maximum total bending stress, at the ocean floor, is

\[
[(\sigma_{\text{total}})_{\text{bottom beam}}]_{\text{max.}} = 60,300 \text{ psi}
\]

The distribution of bending stresses in the bottom beam is plotted in Figures 7 and 8.

\[
\sigma_{\cos} \text{ vs. distance for Bottom Beam}
\]

\[
\sigma_{\cos} = -2400 \text{ psi} @ \text{OCEAN FLOOR}
\]

Figure 7.
$\sigma_{\text{sin}}$ vs. distance for Bottom Beam

$\sigma_{\text{sin}} = 60,240 \text{ psi} @ \text{Ocean Floor}$

$\sigma_{\text{sin}}$, psi

$x_4$, feet

Figure 8.
(2.) Deflections.

The in-phase deflection $y_{\sin}$ and the lagging deflection $y_{\cos}$ of the entire drill string are plotted in Figures 9 and 10, respectively. The in-phase and lagging deflections of the two beam sections are plotted in Figures 11 through 14.
Lagging Deflection Mode Shape for Entire Drill String

Figure 10.

Lagging Deflection Mode Shape for Top Beam Section

Figure 11.
In-Phase Deflection Mode Shape for Top Beam

Elevation above bottom of beam, feet

Ocean Surface

Figure 12.

In-Phase Deflection Mode Shape for Bottom Beam

Elevation above ocean floor, feet

\((y_{\sin})_{\text{max.}} = 7.54 \text{ feet}\)

Figure 13.
Discontinuities in Slope at Beam-Cable Junctions.

For the drill string considered, the slopes of the mode shapes $y_{\cos}$ and $y_{\sin}$ are

<table>
<thead>
<tr>
<th>Location</th>
<th>$\left( \frac{\partial y}{\partial x} \right)_{\cos}$</th>
<th>$\left( \frac{\partial y}{\partial x} \right)_{\sin}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bottom of Top Beam</td>
<td>$-1.148 \times 10^{-3}$</td>
<td>$-7.789 \times 10^{-3}$</td>
</tr>
<tr>
<td>Top of Upper Cable</td>
<td>$-1.148 \times 10^{-3}$</td>
<td>$-7.789 \times 10^{-3}$</td>
</tr>
<tr>
<td>Top of Bottom Beam</td>
<td>$6.668 \times 10^{-3}$</td>
<td>$-1.675 \times 10^{-1}$</td>
</tr>
<tr>
<td>Bottom of Lower Cable</td>
<td>$6.397 \times 10^{-3}$</td>
<td>$-1.606 \times 10^{-1}$</td>
</tr>
</tbody>
</table>
The slopes at the upper beam-cable junction are continuous, to four significant figures. The discontinuities in the slopes at the lower beam-cable junction are each only $4\%$, referred to the cable's value.

These results show that nearly exact continuity of slope can be obtained by adjusting the lengths of the sections considered as beams, as described in the Appendix.

(4.) Coriolis Acceleration of Drilling Fluid

A measure of the effect of the Coriolis acceleration of the drilling fluid, ignored in this analysis, may be obtained from the equation of motion (19) by comparing the term $2q\nu\frac{\partial^2 y}{\partial x^2}$ (Coriolis inertia force per unit length) with the term $\frac{\rho g}{\pi} \frac{\partial y}{\partial t}$ (drag force per unit length). The values obtained from the example studied are

<table>
<thead>
<tr>
<th>Station</th>
<th>$\frac{2q\nu\left(\frac{\partial^2 y}{\partial x^2}\right)\cos}{(\text{lbs./ft.})}$</th>
<th>$\frac{\rho \frac{\partial \nu}{\partial t} \left(\frac{\partial y}{\partial t}\right)\cos}{(\text{lbs./ft.})}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$0.02404 \times 10^{-3}$</td>
<td>$0.00353 \times 10^{-3}$</td>
</tr>
<tr>
<td>B</td>
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<td>$0.01145 \times 10^{-3}$</td>
</tr>
<tr>
<td>C</td>
<td>$0.50499 \times 10^{-3}$</td>
<td>$0.45896 \times 10^{-3}$</td>
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</tbody>
</table>

<table>
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<th>Station</th>
<th>$\frac{2q\nu\left(\frac{\partial^2 y}{\partial x^2}\right)\sin}{(\text{lbs./ft.})}$</th>
<th>$\frac{\rho \frac{\partial \nu}{\partial t} \left(\frac{\partial y}{\partial t}\right)\sin}{(\text{lbs./ft.})}$</th>
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<tr>
<td>A</td>
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<td>C</td>
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<td>$0.03290 \times 10^{-3}$</td>
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Stations A, B, and C refer to the top of the 3500' cable, the bottom of the 3500' cable, and the bottom of the 7980' cable, respectively.
Station C approximates the maximum slopes in the drill string, as illustrated by Figures 9 and 10. Hence, the maximum Coriolis inertia load is represented by station C.

The results show that the Coriolis inertia loading only approaches the magnitude of the drag-force loading where the drag-force loading is at a low value, near the bottom of the drill string.
CONCLUSIONS

The bending stress distribution for the drill string studied compares favorably with the bending stresses observed during the Mohole test drilling in 1961 (Reference 2). The values observed for the drill string studied were of the order of 25,000 psi at the surface, under unspecified conditions of off-hole displacement. This measurement serves as a reasonable check on the value 12,390 psi obtained in this study. Further, the Mohole group observed that the bending stress near the surface dies out rapidly with depth, decreasing to 5,000 psi at a depth of 20 feet. A similar rapid decline in bending stress near the surface is predicted by the analysis of this paper, as shown in Figures 5 and 6.

The bending stresses at the bottom of the drill string are higher than actually experienced. The observed values of bending stress are necessarily lower, due to the inability of the ocean floor to hold the slope of the drill pipe fixed. Instead, the slope of the drill pipe may be inclined as much as 5 degrees toward the ship (Reference 2).

The buoyant force is shown to have a considerable influence on the problem. For the drill string studied, the axial tension at the floor was 7,600 lbs. compared to the weight of equipment in the hole of 52,200 lbs. This reduction in axial tension affects the boundary conditions and hence the entire solution.

The drag force exerted by the water has an appreciable effect on the drill string, inducing bending stresses of 1,800 and 2,400 psi at the ocean surface and floor, respectively. These stresses are the values of $\sigma_{\text{cos}}$, which does not exist in the absence of drag forces.
APPENDIX

(1.) Adjustment of length of lower beam section to obtain near-continuity of slope at lower beam-cable junction.

Consider a static deflection of the drill string. The deflection \( \eta \) will be given by (65) as

\[ \eta(x) = \frac{y_4}{l_4} = A + B + C \sinh \left( \sqrt{K_4} x \right) \]

The boundary conditions (89) are assumed to apply so that the moment in the beam at the junction \( (x_4 = 0) \) must vanish, whence

\[ C = -D \]

so that the bending moment \( M \) is given by

\[ M = -EI \frac{d^2 \eta}{dx^2} = -\frac{2EI}{x} K_4 C \sinh \left( \sqrt{K_4} \frac{x_4}{l_4} \right) \]

Since the beam is in static equilibrium,

\[ \left[ -EI \frac{d^3 \eta}{dx^3} + T_0 \frac{d\eta}{dx} \right]_{x=0} = \left[ -EI \frac{d^3 \eta}{dx^3} \right]_{x=l_4} \]

where the boundary condition \( \frac{d\eta}{dx} (l_4) = 0 \) has been applied. At the beam-cable junction, continuity of shear requires

\[ \left[ -EI \frac{d^3 \eta}{dx^3} + T_0 \frac{d\eta}{dx} \right]_{x_3 = l_3} = T_0 \left( \frac{d\eta}{dx} \right)_{x_3 = l_3} \]

If the two slopes \( \frac{d\eta}{dx} \) and \( \frac{d\eta}{dx_3} \) at the beam-cable junction are approximately equal, then

\[ \left( \frac{d\eta}{dx_3} \right)_{x_3 = l_3} = (1 + \varepsilon) \left( \frac{d\eta}{dx} \right)_{x_4 = 0} \]
where $\varepsilon \ll 1$. Equations (102) and (103) then require

$$ \frac{1}{\varepsilon} + 1 = \cosh \sqrt{K_3} $$

as the relation between slope discontinuity and $K_3$.

When the results of this static analysis were applied to the motion of the drill string, the allowed slope discontinuity was set at $6\%$. The dynamic effects acted to reduce the discontinuity to $4\%$.

(2.) Adjustment of length of upper beam section to obtain near-continuity of slope at upper beam-cable junction.

Considering a static deflection of the drill string, the deflection $\gamma_i$ is given by (65) as

$$ \gamma_i (x_i) = \frac{y_i}{x_i} = \lambda_i + B_i \bar{x}_i + C_i e^{\sqrt{K_3} \bar{x}_i} + D_i e^{-\sqrt{K_3} \bar{x}_i} $$

The boundary condition $M(x_1 = \ell_1) = 0$ gives

$$ \sigma_i e^{\sqrt{K_3} \bar{x}_i} + H_i e^{-\sqrt{K_3} \bar{x}_i} = 0 $$

If the slopes are to be approximately equal, then

$$ \left( \frac{d^2 y_2}{dx_2^2} \right)_{x_2 = 0} = (1 + \delta) \left( \frac{d^2 y_1}{dx_1^2} \right)_{x_1 = \ell_1} $$

where $\delta \ll 1$. Equilibrium of the beam and continuity of shear at the junction, together with (108) and (109), require

$$ \cosh \sqrt{K_3} = 1 + \frac{i}{\ell} $$
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