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THE STRESSES IN A FLAT CIRCULAR RING
WITH A THROUGH-CUT

BY

YU-TSAI CHEN

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Thesis Director's Signature:

Approved: J. C. Wilhoit Jr., May 24, 1963

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This thesis treats a new approach to the problem of a complete ring, with one cut, supported on one face of the cut and subjected to a non-self-equilibrating load. A standard method of approach makes use of the Fourier Integral. Professor J. N. Goodier of Stanford University suggested a multiple ring approach to Professor Wilhoit in 1953. Although Professor Wilhoit made a preliminary study, no comparison of the two methods was attempted. A study is made of the stresses in the flat circular ring with a through-cut subjected to an arbitrary load. It is shown by the solution of an example that certain problems can be treated by the multiple-ring method with greater ease than by the Fourier-integral method.
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INTRODUCTION

A Fourier-integral solution for the stresses in flat circular bars with concentrated radial loads (5)* were published by W. C. Nelson and C. J. Ancker (7, 8). Solutions for the stresses in complete rings were published. Solutions of some special cases for circular rings (6, 9, 12) were given in detail. This thesis presents a multiple-ring solution (6) for the plane stress problem of a circular ring or a curved bar bounded by two concentric circles and loaded by arbitrary loads on the circular boundaries. The general procedure of the Fourier-integral method is also described. An example is included to show how superposition may be used to obtain the stresses for a loading condition.

In the solutions for Fig. 1-1 the location of the wall at the built-in end of the circular beam is not specified. The wall is a fictitious one in the sense that the solution does not hold in the immediate vicinity of the wall, and the wall is shown only in order to indicate which end of the beam is supported. By Saint-Venant's principle, if a system of forces acting on a small portion of the surface of an elastic body is replaced by another statically equivalent system of forces acting on the same portion of the surface, this redistribution of loading produces substantial changes in the stresses only in the immediate neighborhood of the loading, and the stresses are essentially the same in the parts of the

*Numbers denote items in the Bibliography on page
body which are at large distances in comparison with the
linear dimensions of the surface on which the forces are changed.

In this thesis, the thicknesses of the bodies perpendicular to the plane of the paper are always assumed to be unity so that any load P should be considered as force per unit thickness.
CHAPTER I

THE MULTIPLE-RING METHOD

The ring of unit thickness shown in Figure 1-1 is complete except for a very narrow through-cut on the left. The upper face of the cut is supported, the lower one is free, and an arbitrary loading is indicated around both the inside and the outside boundaries. This loading may have resultant forces in x and y directions and is consequently balanced by the opposite forces on the supported face of the cut. A bending moment also may result and is also balanced by the bending moment on the supported end. Since there are stresses on this face, and there are none on the free lower end, there is a discontinuity of stress in passing from one to the other, and a corresponding prescribed discontinuity of resultant force and moment.

This problem of the cut ring whose surfaces r=a and r=b are loaded by a non-self-equilibrating load can be solved by a multiple-ring solution with two or more sheets. It is possible to consider that the physical ring is located on the first sheet
of the surface which rejoins itself after transversing \( m \) times around. The unbalanced load then may be thought of as being balanced on the \( m \) loops of the ring, which actually will be disregarded in the final solution of the problem.

Any of the four loading terms \((\mathbf{r})_a\), \((\mathbf{r})_b\), \((\mathbf{e})_a\), and \((\mathbf{e})_b\) is a function of \( e \), say \( F(e) \), and can be expressed as the sum of an odd and an even function by

\[
F(e) = \frac{1}{2} \left[ F(e) + F(-e) \right] + \frac{1}{2} \left[ F(e) - F(-e) \right].
\]

Assume that the loading terms are defined in the interval \(-\pi \leq \theta \leq \pi\). The functions are expanded over the two loops, that is the interval \(-3\pi \leq \theta \leq \pi\); over the three loops, that is the interval \(-3\pi \leq \theta \leq 3\pi\); over the four loops, that is the interval \(-5\pi \leq \theta \leq 3\pi\), etc. to the \( m \) loops on which the forces are balanced, giving:

\[
(\mathbf{r})_b = a_0 + a_m \cos e + a'_m \sin e + \sum_{n=1}^{\infty} \left( a_n \cos \frac{ne}{m} + a'_n \sin \frac{ne}{m} \right),
\]

\[
(\mathbf{r})_a = b_0 + b_m \cos e + b'_m \sin e + \sum_{n=1}^{\infty} \left( b_n \cos \frac{ne}{m} + b'_n \sin \frac{ne}{m} \right),
\]

\[
(\mathbf{e})_b = c_0 + c_m \cos e + c'_m \sin e + \sum_{n=1}^{\infty} \left( c_n \cos \frac{ne}{m} + c'_n \sin \frac{ne}{m} \right),
\]

\[
(\mathbf{e})_a = d_0 + d_m \cos e + d'_m \sin e + \sum_{n=1}^{\infty} \left( d_n \cos \frac{ne}{m} + d'_n \sin \frac{ne}{m} \right).
\]

(1-1)

The four loading terms may be expanded to the \( m \) loops in any desired manner which maintains equilibrium. There are an infinite number of ways the resultant force and moment on the loop \(-\pi \leq \theta \leq \pi\) can be balanced on the \( m \) loops, the total forces taken over both boundaries parallel to \( e = 0 \) and \( e = \frac{\pi}{2} \) must be zero, that is \( \sum F_x = \sum F_y = 0 \), and the moments of all edge tractions about the origin must vanish. Therefore,
we have
\[ \int_d^{d + 2m\pi} [(\hat{r})_b \cos \theta - (\hat{e})_b \sin \theta] \, b \, d\theta \]

\[ = \int_d^{d + 2m\pi} [(\hat{r})_a \cos \theta - (\hat{e})_a \sin \theta] \, a \, d\theta, \]

\[ = \int_d^{d - 2m\pi} [(\hat{r})_b \sin \theta - (\hat{e})_b \cos \theta] \, b \, d\theta \]

\[ = \int_d^{d - 2m\pi} [(\hat{r})_a \sin \theta - (\hat{e})_a \cos \theta] \, a \, d\theta, \]

and
\[ \int_d^{d + 2m\pi} (\hat{e})_b \, b^2 \, d\theta = \int_d^{d + 2m\pi} (\hat{e})_a \, a^2 \, d\theta. \]

in which \( d = -m\pi \) if \( m \) is an odd number, or \( d = -(m + 1)\pi \) if \( m \) is an even number.

(1-2)

Substituting Equations (1-1) into Equations (1-2), it is easy to find the relations of equilibrium on \( m \) loops.

\[ (a_m - c'_m) \, b = (b_m - d'_m) \, a, \]

\[ (a'_m + c_m) \, b = (b'_m + d_m) \, a, \]

\[ c_o \, b^2 = d_o \, a^2. \]

(1-3)

In this chapter, the solutions of the plane-stress problems considered will be derived from Airy's stress function \( \phi \). This stress function must satisfy the biharmonic equation

\[ \nabla^2 \nabla^2 \phi = 0 \]

(1-4)
where $\nabla^2$ is the Laplace operator which, in polar coordinates $(r, \theta)$ is
\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.
\]
The stresses are to be determined from the stress function by the relations
\[
\sigma_r = \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2},
\]
\[
\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2},
\]
\[
\tau_r\theta = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right).
\]
(1-5)

Solving the biharmonic equation, we find the general solution of the two-dimensional problem in polar coordinates $(4, 16)$. The stress function $\phi$ is
\[
\phi = A_o r^2 + B_o \log r + C_o \frac{\theta}{m} + D_o r^2 \cos \theta + E_o r \log r
\]
\[+ (A_m r^3 + B_m r^{-1}) \cos \theta + (D_m r \log r \cos \theta - C_m r \theta \sin \theta)
\]
\[+ (A'_m r^3 + B'_m r^{-1}) \sin \theta + (D'_m r \log r \sin \theta - C'_m r \theta \cos \theta)
\]
\[+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( A_n r^{m+2} + B_n r^{-n} + C_n r^{-m} + D_n r^{-m+2} \right) \cos \frac{n\theta}{m}
\]
\[+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( A'_n r^{m+2} + B'_n r^{-n} + C'_n r^{-m} + D'_n r^{-m+2} \right) \sin \frac{n\theta}{m}
\]
(1-6)

where $m$ and $n$ are integers to anticipate our problem.

For a complete multiple ring, the displacements must be single-valued, and to satisfy this condition we have
\[D_o = o, \ E_o = o, \ D_m = \frac{1-\nu}{2} C_m, \ D'_m = \frac{1-\nu}{2} C'_m.
\]
(1-7)
The derivation of relations (1-7) is almost the same as that described in Timoshenko and Goodier's Theory of Elasticity (4). Therefore, we have the stress function $\phi$ of the complete multiple ring of $m$ loops.

$$
\phi = A_o r^2 + B_o \log r + C_o \frac{\theta}{m} \\
+ (A_m r^3 + B_m r^{-1}) \cos \theta + C_m \left( \frac{1-\nu}{2} r \log r \sin \theta - r \theta \sin \theta \right) \\
+ (A'_m r^3 + B'_m r^{-1}) \sin \theta + C'_m \left( \frac{1-\nu}{2} r \log r \cos \theta - r \theta \cos \theta \right) \\
+ \sum_{n=1}^{\infty} \left( A_n r^{n+2} + B_n r^{-n} + C_n r^{n} + D_n r^{-n+2} \right) \cos \frac{n\theta}{m} \\
+ \sum_{n=1}^{\infty} \left( A'_n r^{n+2} + B'_n r^{-n} + C'_n r^{n} + D'_n r^{-n+2} \right) \sin \frac{n\theta}{m}.
$$

(1-8)

Using the Equations (1-5), we have

$$
\hat{r} = 2A_o + B_o \frac{\theta}{r^2} + (2A_m r - 2B_m r^{-3} - \frac{3+\nu}{2} C_m) \cos \theta \\
+ (2A'_m - 2B'_m r^{-3} - \frac{3+\nu}{2} C'_m) \sin \theta
$$

$$
- \sum_{n=1}^{\infty} \left[ \frac{(n+1)(n-2)}{n} A_n r^{n+2} + \frac{n}{m} \left( \frac{n}{m} + 1 \right) B_n r^{-n-2} \\
+ \frac{\nu}{m} \left( \frac{n}{m} - 1 \right) C_n r^{-n-2} + \left( \frac{n}{m} - 1 \right) \left( \frac{n}{m} + 2 \right) D_n r^{-n-2} \right] \cos \frac{n\theta}{m} \\
- \sum_{n=1}^{\infty} \left[ \frac{(n+1)(n-2)}{n} A'_n r^{n+2} + \frac{n}{m} \left( \frac{n}{m} - 1 \right) B'_n r^{-n-2} \\
+ \frac{\nu}{m} \left( \frac{n}{m} - 1 \right) C'_n r^{-n-2} + \left( \frac{n}{m} - 1 \right) \left( \frac{n}{m} + 2 \right) D'_n r^{-n-2} \right] \sin \frac{n\theta}{m},
$$
\[
\hat{e} = 2A_o - B_o r^{-2} + (6A_m r + 2B_m r^{-3} + \frac{1-v}{2\Gamma} C_m) \cos e
\]
\[+ (6A'_m r + 2B'_m r^{-3} + \frac{1-v}{2\Gamma} C'_m) \sin e
\]
\[+ \sum_{n=m}^{\infty} \left\{ \left( \frac{n}{m} + 1 \right) \frac{n}{m} A_n r^{\frac{n}{m}} + \frac{n}{m} \left( \frac{n}{m} + 1 \right) B_n r^{-\frac{n}{m}-2} \right\} \cos \frac{n\theta}{m}
\]
\[+ \sum_{n=m}^{\infty} \left\{ \left( \frac{n}{m} + 1 \right) \frac{n}{m} C_n r^{\frac{n}{m}-2} + \left( \frac{n}{m} - 1 \right) \frac{n}{m} A_n r^{-\frac{n}{m}} \right\} \sin \frac{n\theta}{m}
\]
\[\hat{e} = \frac{c_o r^{-2}}{m} + (2A_m r^{-2}B_m r^{-3} + \frac{1-v}{2\Gamma} C_m) \sin e
\]
\[+ (-2A'_m r + 2B'_m r^{-3} - \frac{1-v}{2\Gamma} C'_m) \cos e
\]
\[+ \sum_{n=m}^{\infty} \left\{ \frac{n}{m} \left( \frac{n}{m} + 1 \right) A_n r^{\frac{n}{m}} - \frac{n}{m} \left( \frac{n}{m} + 1 \right) B_n r^{-\frac{n}{m}-2} \right\} \cos \frac{n\theta}{m}
\]
\[+ \sum_{n=m}^{\infty} \left\{ \frac{n}{m} \left( \frac{n}{m} + 1 \right) C_n r^{\frac{n}{m}-2} - \frac{n}{m} \left( \frac{n}{m} - 1 \right) A_n r^{-\frac{n}{m}} \right\} \sin \frac{n\theta}{m}
\]  

\[\text{(1-9)}\]

Using the above expressions for \(\hat{r}\) and \(\hat{e}\) at \(r=a\) and \(r=b\) and comparing with Equations (1-1), we have the following cases:

For \(n=0\), that is for the terms independent of \(e\), there are four equations for the three unknown constants \(A_o\), \(B_o\), and \(C_o\), but, by virtue of the last expression in Equations (1-3) they are consistent.
From equations \(2A_o + \frac{B_o}{b^2} = a_o, \quad 2A_o + \frac{B_o}{a^2} = b_o\)

we have

\[
A_o = \frac{1}{2} \frac{a_o - b_o q^2}{1 - q^2}, \quad B_o = -\frac{a^2 q^2}{1 - q^2} (a_o - b_o)
\]

where \(q = \frac{a}{b}\) is an abbreviation which we shall use systematically from now on.

Further \(C_o = mc_o b^2 = md_o a^2\)

(1-11)

For \(n=m\), that is for the terms involving \(\sin \phi\) and \(\cos \phi\), there will be eight equations for the six unknowns \(A_m, B_m, C_m, A'_m, B'_m,\) and \(C'_m\); but by virtue of the first two expressions of Equations (1-3), these equations are consistent, and we have

\[
2A_m b - 2B_m b^{-3} - \frac{3 + \nu}{2} \frac{C_m}{b} = a_m
\]

\[
2A_m b - 2B_m b^{-3} + \frac{1 - \nu}{2} \frac{C_m}{b} = c'_m
\]

\[
2A'_m b - 2B'_m b^{-3} - \frac{3 + \nu}{2} \frac{C'_m}{b} = a'_m
\]

\[
2A'_m b - 2B'_m b^{-3} + \frac{1 - \nu}{2} \frac{C'_m}{b} = -c_m
\]

\[
2A_m a - 2B_m a^{-3} - \frac{3 + \nu}{2} \frac{C_m}{a} = b_m
\]

\[
2A_m a - 2B_m a^{-3} + \frac{1 - \nu}{2} \frac{C_m}{a} = d'_m
\]

\[
2A'_m a - 2B'_m a^{-3} - \frac{3 + \nu}{2} \frac{C'_m}{a} = b'_m
\]

\[
2A'_m a - 2B'_m a^{-3} + \frac{1 - \nu}{2} \frac{C'_m}{a} = -d_m
\]
On subtraction in pairs, the solution takes this form:

\[ A_m = \frac{(1-V)(a_m-b_m q^3) + (3+V)(c_m-d_m q^3)}{8b (1-q^4)} \]

\[ A'_m = \frac{(1-V)(a'_m-b'_m q^3) - (3+V)(c'_m-d'_m q^3)}{8b (1-q^4)} \]

\[ B_m = b^3 \frac{(1-V)(a_m q^4 - b_m q^5) + (3+V)(c_m q^4 - d_m q^5)}{8 (1-q^4)} \]

\[ B'_m = b^3 \frac{(1-V)(a'_m q^4 - b'_m q^5) - (3+V)(c'_m q^4 - d'_m q^5)}{8 (1-q^4)} \]

\[ C_m = \frac{b}{2} (c_m - a_m) = \frac{a}{2} (d_m - b_m) \]

\[ C'_m = -\frac{b}{2} (a_m' + c_m) = -\frac{a}{2} (b_m' + d_m) \]

\[ \text{(1-12)} \]

For \( n \neq m \), equating the coefficients of \( \cos \frac{n\Theta}{m} \) and \( \sin \frac{n\Theta}{m} \) will give four equations for the unknown constants.

\[ A_n, B_n, C_n, \text{ and } D_n \]

and four equations for the unknown constants

\[ A'_n, B'_n, C'_n, \text{ and } D'_n \].

\[ \left( \frac{n}{m} + 1 \right) \left( \frac{n}{m} - 2 \right) A_n b \frac{n}{m} + \frac{n}{m} \left( \frac{n}{m} + 1 \right) B_n b^{-\frac{n}{m}-2} \]

\[ + \frac{n}{m} \left( \frac{n}{m} - 1 \right) C_n b^{-\frac{n}{m}-2} + \left( \frac{n}{m} - 1 \right) \left( \frac{n}{m} + 2 \right) D_n b^{-\frac{n}{m}} = -a \]

\[ \frac{n}{m} \left( \frac{n}{m} + 1 \right) A_n b \frac{n}{m} = \frac{n}{m} \left( \frac{n}{m} + 1 \right) B_n b^{-\frac{n}{m}-2} \]

\[ + \frac{n}{m} \left( \frac{n}{m} - 1 \right) C_n b^{-\frac{n}{m}-2} - \frac{n}{m} \left( \frac{n}{m} - 1 \right) D_n b^{-\frac{n}{m}} = c_n' \]

\[ \left( \frac{n}{m} + 1 \right) \left( \frac{n}{m} - 2 \right) A_n a \frac{n}{m} + \frac{n}{m} \left( \frac{n}{m} + 1 \right) B_n a^{-\frac{n}{m}-2} \]

\[ + \frac{n}{m} \left( \frac{n}{m} - 1 \right) C_n a^{-\frac{n}{m}-2} + \left( \frac{n}{m} - 1 \right) \left( \frac{n}{m} + 2 \right) D_n a^{-\frac{n}{m}} = -b_n \]
\[
\frac{n}{m} \left( \frac{n}{m} + 1 \right) A_n a^{\frac{n}{m}} - \frac{n}{m} \left( \frac{n}{m} + 1 \right) B_n a^{\frac{n-2}{m}} + \frac{n}{m} \left( \frac{n}{m} - 1 \right) C_n a^{\frac{n-2}{m}} - \frac{n}{m} \left( \frac{n}{m} - 1 \right) D_n a^{\frac{n}{m}} = d_n
\]

Similarly we have four equations in which \( A_n, B_n, C_n, D_n \) are replaced by \( A'_n, B'_n, C'_n, D'_n \), respectively, and \( a_n, b_n, c_n, d_n \) by \( a'_n, b'_n, c'_n, d'_n \), respectively. Thus, it is possible to solve all the unknown constants \( A_n, A_n, \) etc. in terms of \( a_n, a_n, \) etc. to give the desired loading over the \( m \) loops of the ring. This solution is generally not very difficult to work out in a given problem; therefore, it will not be given here.

Now we see that the problem of the multiple ring subjected to an arbitrary load on surfaces of first loop \( r=a \) and \( r=b \) is completely determinate. It is clear that the solution for a complete ring \((3, 4)\), whose surfaces \( r=a \) and \( r=b \) are loaded by a self-equilibrating load, is the special case of a multiple ring solution when \( m=1 \).

Now let us consider the problem of a cut ring or a portion of a ring subjected to an arbitrary edge load. First, we consider the portion of the ring as part of a multiple ring. The arbitrary loading is extended over the multiple ring of \( m \) loops so that equilibrium is preserved. The multiple ring is solved as above. A cut is now made in the multiple ring corresponding to the desired free end of the portion under consideration. The resultant force and moment on this face are removed by the use of the Volterra dislocations of orders 1, 2, and 6, which will merely be tabulated here for the purposes of this thesis since these are well-known classical solutions \((1)\).
Figure 1–2 indicates the Volterra dislocation of order 1, for the discontinuity in the radial displacement \( U_r \) at \( \theta = \pm \pi \). The stress function, stresses, and displacements are given below:

\[
\sigma_r = C \left[ \frac{r^3}{2(a^2+b^2)} - \frac{a^2 b^2}{2(a^2+b^2)} \right] \frac{l}{r} \sin \phi
\]

\[
\sigma_r = C \left[ \frac{r}{a^2+b^2} + \frac{a^2 b^2}{a^2+b^2} \right] \frac{l}{r^3} - \frac{l}{r^3} \sin \phi
\]

\[
\sigma_\theta = C \left[ \frac{\theta^2}{a^2+b^2} - \frac{a^2 b^2}{a^2+b^2} \right] \frac{l}{r^3} \cos \theta
\]

\[
U_r = \frac{C}{E} \left\{ 2 \tan \phi - \sin \phi \left[ (1-V) \log r - \frac{(1-3 V) r^2}{2 (a^2+b^2)} \right] + \frac{(1+V) a^2 b^2}{2 (a^2+b^2)} \frac{l}{r^2} - (1 + V) \right\}
\]

\[
U_\theta = \frac{C}{E} \left\{ -2 \sin \phi + \cos \phi \left[ -(1-V) \log r - \frac{(5+V) r^2}{2 (a^2+b^2)} \right] + \frac{(1+V) a^2 b^2}{2 (a^2+b^2)} \frac{l}{r^2} \right\}
\]

(1-14)
Figure 1-3 indicates the Volterra dislocation of order 2, for the discontinuity in the tangential displacement $U_e$, at $\phi = \pm \pi$. The stress function, stresses, and displacements are given below:

$$c = \frac{F_y (a^2 + b^2)}{a^2 - b^2 + (a^2 + b^2) \log \frac{b}{a}}$$

$$\phi_2 = c \left[- \frac{R^3}{2 (a^2 + b^2)} + \frac{a^2 b^2}{2 (a^2 + b^2)} \frac{l}{R} + r \log r \right] \cos \phi$$

$$\tilde{\sigma} = c \left[- \frac{r}{a^2 + b^2} - \frac{a^2 b^2}{a^2 + b^2} \frac{l}{r^3} + \frac{l}{r} \right] \cos \phi$$

$$\tilde{\tau} = c \left[- \frac{3r}{a^2 + b^2} + \frac{a^2 b^2}{a^2 + b^2} \frac{l}{r^3} + \frac{l}{r} \right] \sin \theta$$

$$U_r = \frac{c}{E} \left\{ 2 \sin \phi - \cos \phi \left[ (1 - \nu) \log r - (1 + \nu) \frac{(1 + \nu)a^2 b^2}{2(a^2 + b^2)} \frac{l}{r^3} - \frac{(1 - 3\nu)R^2}{2 (a^2 + b^2)} \right] \right\}$$

$$U_e = \frac{c}{E} \left\{ 2 \cos \phi - \sin \phi \left[ (1 - \nu) \log r \right. \right.$$

$$\left. - \frac{(1 + \nu)a^2 b^2}{2 (a^2 + b^2)} \frac{l}{r^3} - \frac{(5 + \nu)R^2}{2 (a^2 + b^2)} \right\}$$

(1-15)
Figure 1-4 indicates the Volterra dislocation of order 12, for the discontinuity in tangential displacement proportional to the radius, at $\theta = \pm \Pi$. The stress function, stresses, and displacements are given below:

$$
C = \frac{2(a^2 - b^2) M}{(a^2 - b^2)^2 - 4a^2b^2 (\log \frac{b}{a})^2}
$$

$$
D = \frac{a^2 - b^2 + 2a^2 / \log a - 2b^2 / \log b}{2(a^2 - b^2)}
$$

$$
\phi_3 = C \left[ D + r^2 \log r 
- \frac{2a^2 b^2}{a^2 - b^2} (\log \frac{b}{a}) \log r 
+ (1 + 2 \log r) \right]
$$

$$
\hat{w} = C \left[ D + \frac{2a^2 b^2}{a^2 - b^2} (\log \frac{b}{a}) \frac{1}{r^2} + (3 + 2 \log r) \right]
$$

$$
\hat{e} = \phi
$$

$$
\hat{u}_r = C \left[ (1 - \nu) Dr - (1 + \nu) r + \frac{2(1+\nu) a^2 b^2}{a^2 - b^2} (\log \frac{b}{a}) \frac{1}{r}
+ 2 (1 - \nu) r \log r \right]
$$

$$
\hat{u}_\theta = \frac{4C \rho \theta}{E}
$$

(1-16)
CHAPTER II

THE FOURIER-INTEGRAL METHOD

Every function is the sum of an odd function and an even function. Therefore, we can write every function in the form

\[ F(e) = G(e) + H(e) \]

where \( G(e) = \frac{1}{2} F(e) + F(-e) \) is an even function

and \( H(e) = \frac{1}{2} [F(e) - F(-e)] \) is an odd function.

For the present, it is sufficient to note that the formula for expressing a function as a Fourier integral (15) gives us

\[ F(e) = \frac{2}{\pi} \int_{0}^{\infty} \cos n\pi \ln \int_{0}^{\infty} G(\lambda) \cos n\lambda \, d\lambda \\
+ \frac{2}{\pi} \int_{0}^{\infty} \sin n\pi \ln \int_{0}^{\infty} H(\lambda) \sin n\lambda \, d\lambda \]

It is valid for \(-\infty < \theta < \infty\)

Let us consider the ring problem of Figure 1-1, Chapter I, subjected to arbitrary edge force on its surfaces \( r=a \) and \( r=b \), fixed at one end and free at another. This is a non-self-equilibrating load problem, which can be considered the
combination of Figure 2-1 and Figure 2-2. The solution of Figure 2-2 is the combination of Volterra dislocations of orders 1, 2, and 6, which have been tabulated in Chapter I.

In Figure 2-1 the edge loading generally has the resultant moment $2M$ and resultant forces $2F_x$ and $2F_y$ applied at the center of the ring. They are balanced by the end moment $2M$ and end forces $2F_x$ and $2F_y$ also applied at the center of the ring. The stresses produced in the ring by the edge forces can be solved by the Fourier integral. For solving such a problem, we still use Airy's stress function $\phi$ which must satisfy the biharmonic equation (1-4), $\nabla^2 \nabla^2 \phi = 0$, and the boundary conditions.

The boundary conditions of Figure 2-1 written in the Fourier integral forms are

\[
(r^b_r)_b = F_1(e) = G_1(e) + H_1(e)
\]
\[
= \frac{2}{\pi} \int_0^\infty \cos n\theta \, dn \int_0^\infty G_1(\lambda) \cos n\lambda \, d\lambda
\]
\[
+ \frac{2}{\pi} \int_0^\infty \sin n\theta \, dn \int_0^\infty H_1(\lambda) \sin n\lambda \, d\lambda
\]

\[
(r^b_r)_a = F_2(e) = G_2(e) + H_2(e)
\]
\[
= \frac{2}{\pi} \int_0^\infty \cos n\theta \, dn \int_0^\infty G_2(\lambda) \cos n\lambda \, d\lambda
\]
\[
+ \frac{2}{\pi} \int_0^\infty \sin n\theta \, dn \int_0^\infty H_2(\lambda) \sin n\lambda \, d\lambda
\]

\[
(r^b_\theta)_b = F_3(e) = G_3(e) + H_3(e)
\]
\[
= \frac{2}{\pi} \int_0^\infty \cos n\theta \, dn \int_0^\infty G_3(\lambda) \cos n\lambda \, d\lambda
\]
\[
+ \frac{2}{\pi} \int_0^\infty \sin n\theta \, dn \int_0^\infty H_3(\lambda) \sin n\lambda \, d\lambda
\]
(\tilde{\sigma})_a = F_4(\epsilon) = G_4(\epsilon) + H_4(\epsilon)

= \frac{2}{\pi} \int_0^\infty \cos n \epsilon \, dn \int_0^\infty G_4(\lambda) \cos \nu \lambda \, d\lambda

+ \frac{2}{\pi} \int_0^\infty \sin n \epsilon \, dn \int_0^\infty H_4(\lambda) \sin \nu \lambda \, d\lambda

(2-1)

The stress function \( \phi \) each multiplied by \( \Delta n \), will still satisfy equation (1-4).

\[ \phi = \sum_{n=0,\Delta n,2\Delta n}^\infty \phi_n \, \Delta n \] (2-2)

The same is true of the stress function

\[ \phi = \int_0^\infty \phi_n \, dn \] (2-3)

which is the limit of the foregoing infinite series of stress functions as \( \Delta n \) approaches zero.

Where \( \phi_n = (A_n r^{n+2} + B_n r^{-n} + C_n r^n + D_n r^{-n+2}) \cos \nu \epsilon \)

\[ + (A_n' r^{n+2} + B_n' r^{-n} + C_n' r^n + D_n' r^{-n+2}) \sin \nu \epsilon \] (2-4)

with two exceptions in which

if \( n=0 \), \( \phi_o = A_0 r^2 + B_0 \log r + C_0 \epsilon + D_0 r \log r \)

and if \( n=1 \), \( \phi = (A_1 r^3 + B_1 r^{-1} + C_1 \log r) \cos \epsilon + D_1 r \epsilon \sin \epsilon \)

\[ + (A_1' r^3 + B_1' r^{-1} + C_1' \log r) \sin \epsilon + D_1' r \epsilon \cos \epsilon \]

Boundary conditions (2-1) remain to be satisfied. Application of Equations (1-5) gives

\[ \tilde{\tau} = \int_0^\infty \left( \frac{1}{r} \frac{\partial \phi_n}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi_n}{\partial \theta^2} \right) \, dn \]

and \( \tilde{\epsilon} = -\int_0^\infty \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \phi_n}{\partial \theta} \right) \, dn \). (2-5)
Substitution of \( r=a \) and \( r=b \) in Equations (2-4) and application of boundary conditions (2-1) lead to the results

\[
\left( \frac{1}{r} \frac{\partial \phi_n}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi_n}{\partial \theta^2} \right)_{r=b} = \frac{2}{\pi} \cos n\theta \int_0^\infty G_1(\lambda) \cos n\lambda d\lambda + \frac{2}{\pi} \sin n\theta \int_0^\infty H_1(\lambda) \sin n\lambda d\lambda
\]

\[
\left( \frac{1}{r} \frac{\partial \phi_n}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi_n}{\partial \theta^2} \right)_{r=a} = \frac{2}{\pi} \cos n\theta \int_0^\infty G_2(\lambda) \cos n\lambda d\lambda + \frac{2}{\pi} \sin n\theta \int_0^\infty H_2(\lambda) \sin n\lambda d\lambda
\]

\[
-\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \phi_n}{\partial \theta} \right)_{r=b} = \frac{2}{\pi} \cos n\theta \int_0^\infty G_3(\lambda) \cos n\lambda d\lambda + \frac{2}{\pi} \sin n\theta \int_0^\infty H_3(\lambda) \sin n\lambda d\lambda
\]

\[
-\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \phi_n}{\partial \theta} \right)_{r=a} = \frac{2}{\pi} \cos n\theta \int_0^\infty G_4(\lambda) \cos n\lambda d\lambda + \frac{2}{\pi} \sin n\theta \int_0^\infty H_4(\lambda) \sin n\lambda d\lambda
\]

(2-6)

Equating the coefficients of \( \sin n\theta \) and \( \cos n\theta \) respectively, will give four equations for the unknown constants \( A_n, B_n, C_n \), and \( D_n \) and four equations for the unknown constants \( A'_n, B'_n, C'_n \), and \( D'_n \)
\[(n + 1)(n - 2)A_n b^n + n(n + 1) B_n b^{n-2} + n(n - 1) C_n b^{n-2}\]
\[+ (n - 1)(n + 2) D_n b^n = -\frac{2}{\tau} \int_0^\infty G_1(\lambda) \cos n\lambda \, d\lambda,
\]
\[(n + 1)(n - 2) A_n a^n + n(n + 1) B_n a^{n-2} + n(n - 1) C_n a^{n-2}\]
\[+ (n - 1)(n + 2) D_n a^n = -\frac{2}{\tau} \int_0^\infty G_2(\lambda) \cos n\lambda \, d\lambda,
\]
\[n(n + 1) A_n b^n - n(n + 1) B_n b^{n-2} + n(n - 1) C_n b^{n-2} - n(n - 1) D_n b^n\]
\[= -\frac{2}{\tau} \int_0^\infty H_3(\lambda) \sin n\lambda \, d\lambda,
\]
\[n(n + 1) A_n a^n - n(n + 1) B_n a^{n-2} + n(n - 1) C_n a^{n-2} - n(n - 1) D_n a^n\]
\[= -\frac{2}{\tau} \int_0^\infty H_4(\lambda) \sin n\lambda \, d\lambda,
\]
\[(n + 1)(n - 2) A'_n b^n + n(n + 1) B'_n b^{n-2} + n(n - 1) C'_n b^{n-2}\]
\[+ (n - 1)(n + 2) D'_n b^n = -\frac{2}{\tau} \int_0^\infty H_1(\lambda) \sin n\lambda \, d\lambda,
\]
\[(n + 1)(n - 2) A'_n a^n + n(n + 1) B'_n a^{n-2} + n(n - 1) C'_n a^{n-2}\]
\[+ (n - 1)(n + 2) D'_n a^n = -\frac{2}{\tau} \int_0^\infty H_2(\lambda) \sin n\lambda \, d\lambda,
\]
\[n(n + 1) A'_n b^n - n(n + 1) B'_n b^{n-2} + n(n - 1) C'_n b^{n-2} - n(n - 1) D'_n b^n\]
\[= -\frac{2}{\tau} \int_0^\infty G_3(\lambda) \cos n\lambda \, d\lambda,
\]
\[n(n + 1) A'_n a^n - n(n + 1) B'_n a^{n-2} + n(n - 1) C'_n a^{n-2} - n(n - 1) D'_n a^n\]
\[= -\frac{2}{\tau} \int_0^\infty G_4(\lambda) \cos n\lambda \, d\lambda.
\]
\[(2-7)\]
G (e) and H (e) are known functions of e. Therefore, we can integrate these equations without much difficulty. Then we can solve eight equations simultaneously for eight unknown constants A, A_n, etc. This will not be done here, since it is not very difficult to work out in a given problem. After determining the eight unknown constants, Equation (2-3) can be integrated completely. Substituting stress function φ into Equations (1-5), the stresses produced by edge loads are solved. The solutions of φ_0 and φ_1 can be found easily in Timoshenko and Goodier's Theory of Elasticity (4).

By proper superposition of Volterra dislocations of orders 1, 2, and 6, the end moment M and the end forces F_x and F_y applied at the lower face of the cut can be removed. Thus the problem is solved completely.
CHAPTER III
EXAMPLE

A. Use of the Multiple Ring

In order to illustrate how the methods described in the foregoing chapters should be applied, the cut ring shown in Figure 3-1 is taken as a simple example.

Figure 3-1 shows a complete ring of unit thickness with a cut at $\phi = \pm \pi$, and subjected to a non-self-equilibrating normal load of cosine distribution on the edge $r=a$. It is evident that the stresses at the cut must be discontinuous to satisfy the equilibrium.

Solving the stresses in the cut ring by the multiple ring method, we can directly apply the formulas derived in Chapter I.

First, let us expand the loading function over two loops so that statical equilibrium is preserved. It is clear that equations (1-1) becomes the form

$$(rr)_p = 0$$

$$(rr)_a = -\frac{3P}{2a} \cos \frac{\theta}{2}$$

$$(re)_p = 0$$

$$(re)_a = 0$$

$$\text{(3-1)}$$
From which, it is obvious that all the coefficients of the right terms in equations (1-1) are zeros, except

\[ b_1 = -\frac{3p}{2a} \]  \hspace{1cm} (3-2)

The equilibrium conditions (1-3) are satisfied.

Solving Equations (1-10), (1-11), (1-12), (1-13), all the unknown constants \( A_0, B_0 \) etc. are zeros, except \( A_1, B_1, C_1 \), and \( D_1 \) which we can solve by applying Equations (1-13) where we take \( m=2 \) and \( n=1 \), in this example. Therefore, we have

\[ \begin{align*}
-\frac{3}{4}A_1b^{\frac{5}{2}} + \frac{3}{4}B_1b^{-\frac{2}{2}} - \frac{3}{4}C_1b^{-\frac{3}{2}} - \frac{3}{4}D_1b^{-\frac{1}{2}} &= 0 \\
\frac{3}{4}A_1b^{-\frac{1}{2}} - \frac{3}{4}B_1b^{-\frac{2}{2}} - \frac{4}{4}C_1b^{-\frac{3}{2}} + \frac{4}{4}D_1b^{-\frac{1}{2}} &= 0 \\
-\frac{3}{4}A_1a^{\frac{5}{2}} + \frac{3}{4}B_1a^{-\frac{2}{2}} - \frac{3}{4}C_1a^{-\frac{3}{2}} - \frac{3}{4}D_1a^{-\frac{1}{2}} &= \frac{3p}{2a} \\
\frac{3}{4}A_1a^{-\frac{1}{2}} - \frac{3}{4}B_1a^{-\frac{2}{2}} - \frac{4}{4}C_1a^{-\frac{3}{2}} + \frac{4}{4}D_1a^{-\frac{1}{2}} &= 0
\end{align*} \]  \hspace{1cm} (3-3)

Solving above equations simultaneously, we have

\[ A_1 = -\frac{pa^{\frac{1}{2}}(a + 3b)}{(b-a)^3} \]
\[ B_1 = -\frac{pa^{\frac{3}{2}}b^2(3a + b)}{(b-a)^3} \]
\[ C_1 = -\frac{3pa^{\frac{1}{2}}(2a^2 + ab + b^2)b}{(b-a)^3} \]
\[ D_1 = \frac{3pa^{\frac{1}{2}}(a^2 + ab + 2b^2)}{(b-a)^3} \]  \hspace{1cm} (3-4)
Thus, the solution of multiple ring can be given below:

\[ \phi = (A_1 r^2 + B_1 r^{-\frac{1}{2}} + C_1 r^{\frac{1}{2}} + D_1 r^{-\frac{1}{2}}) \cos \frac{\theta}{2} \]

\[ \tilde{r} = \frac{1}{4} (9 \ A_1 r^2 - 3B_1 r^{-\frac{5}{2}} + C_1 r^{\frac{3}{2}} + 5D_1 r^{-\frac{1}{2}}) \cos \frac{\theta}{2} \]

\[ \tilde{e} = \frac{1}{4} (15 \ A_1 r^2 + 3B_1 r^{-\frac{5}{2}} - C_1 r^{\frac{3}{2}} + 3D_1 r^{-\frac{1}{2}}) \cos \frac{\theta}{2} \]

\[ \tilde{e} = \frac{1}{4} (3A_1 r^2 - 3B_1 r^{-\frac{5}{2}} - C_1 r^{\frac{3}{2}} + D_1 r^{-\frac{1}{2}}) \sin \frac{\theta}{2} \]

\[ U_r = \frac{\cos \frac{\theta}{2}}{4E} \left\{ 2(3-5\nu) \ A_1 r^2 + 2(1 + \nu) \ B_1 r^{-\frac{3}{2}} - 2(1 + \nu) \ C_1 r^{-\frac{1}{2}} + 2(5 - 3 \nu) \ D_1 r^{-\frac{1}{2}} \right\} \]

\[ U_e = \frac{\sin \frac{\theta}{2}}{4E} \left\{ (9 + \nu) \ A_1 r^2 + (1 + \nu) \ B_1 r^{-\frac{3}{2}} + (1 + \nu) \ C_1 r^{-\frac{1}{2}} - (7 - \nu) \ D_1 r^{-\frac{1}{2}} \right\} \]

(3-5)

Now, a cut is made at \( \theta = \pm \pi \). The resultant force and moment on this cut face can be calculated.

\[ M = - \int_a^b (\tilde{r} \tilde{e})_{\theta = -\pi} \ r \ dr = 0 \]

\[ F_x = - \int_a^b (r \tilde{e})_{\theta = -\pi} \ dr = P \]

\[ F_y = - \int_a^b (\tilde{e} \tilde{e})_{\theta = -\pi} \ dr = 0 \]

(3-6)

Let \( F_x \) be \(-P\) in Equations (1-14). Adding Equations (1-14) to Equations (3-5) respectively with some suitable calculation for the upper end conditions, finally we have the solution of cut ring shown in Figure 3-1.
\[
c = \frac{-P(a^2 + b^2)}{a^2 - b^2 + (a^2 + b^2) \log \frac{b}{a}}
\]
\[
\phi = (A, r^\frac{3}{2} + B, r^{-\frac{1}{2}} + C, r^{-\frac{3}{2}} + D, r^{-\frac{3}{2}}) \cos \frac{\Theta}{2}
\]
\[
+ C \left[ \frac{r^3}{2(a^2 + b^2)} - \frac{a^2 b^2}{2(a^2 + b^2)} \frac{1}{r} \right] r \log r \sin \theta
\]
\[
\hat{r} = \frac{1}{4} (9 A, r^2 - 3 B, r^\frac{3}{2} + C, r^{-\frac{3}{2}} + 5 D, r^{-\frac{3}{2}}) \cos \frac{\Theta}{2}
\]
\[
+ C \left( \frac{r}{a^2 + b^2} - \frac{a^2 b^2}{a^2 + b^2} \frac{1}{r^3} - \frac{1}{r} \right) \sin \theta
\]
\[
\hat{e} = \frac{1}{4} (15 A, r^\frac{3}{2} + 3 B, r^{-\frac{3}{2}} - C, r^{-\frac{3}{2}} + 3 D, r^{-\frac{3}{2}}) \cos \frac{\Theta}{2}
\]
\[
+ C \left( \frac{3r}{a^2 + b^2} - \frac{a^2 b^2}{a^2 + b^2} \frac{1}{r^3} - \frac{1}{r} \right) \sin \theta
\]
\[
\hat{e} = \frac{1}{4} (3 A, r^\frac{3}{2} - 3 B, r^{-\frac{3}{2}} - C, r^{-\frac{3}{2}} + D, r^{-\frac{3}{2}}) \sin \frac{\Theta}{2}
\]
\[
+ C \left( \frac{-r}{a^2 + b^2} - \frac{a^2 b^2}{a^2 + b^2} \frac{1}{r^3} - \frac{1}{r} \right) \cos \theta
\]
\[
U_r = \frac{\cos \frac{\Theta}{2}}{4 E} \left\{ 2(3 - 3 V) A, r^\frac{3}{2} + 2(1 + V) B, r^{-\frac{3}{2}} - 2(1 + V) C, r^{-\frac{3}{2}} \right\}
\]
\[
+ 2(5 - 3 V) D, r^\frac{1}{2} \right\} + \frac{C}{E} \left\{ -2 \pi \cos \theta + 2 \theta \cos \theta \right\}
\]
\[
- \sin \theta \left[ (1 - V) \log r - \frac{(1 - 3 V) r^3}{2(a^2 + b^2)} - \frac{(1 + V) a^2 b^2}{2(a^2 + b^2) r^2} - (1 + V) \right]
\]
\[
- K \sin \theta
\]
\[
U_e = \frac{5 \sin \frac{\Theta}{2}}{4 E} \left\{ (9 + V) A, r^\frac{3}{2} + (1 + V) B, r^{-\frac{3}{2}} \right\}
\]
\[
+ (1 + V) C, r^{-\frac{3}{2}} - (7 - V) D, r^\frac{1}{2} \right\} + \frac{C}{E} \left\{ 2(\pi - \Theta) \sin \theta \right\}
\]
\[
+ \cos \theta \left[ -(1 - V) \log r - \frac{(5 + V) r^3}{2(a^2 + b^2)} - \frac{(1 + V) a^2 b^2}{2(a^2 + b^2) r^2} \right]
\]
\[
+ K \cos \theta + H r
\]

(3-7)
Where the constants $A_1$, $B_1$, $C_1$, and $D_1$ are shown in Equations (3-4). $L$, $H$, $K$ are derived from the end condition of rigidly fixed end. That is

$$H = \frac{1}{BE} \left\{ - 3 (9 + \nu) A_1 R_0^{3/2} + 5 (1 + \nu) R_0^{-3/2} + 3 (1 + \nu) C_1 R_0^{-3/2} + (7 - \nu) D_1 R_0^{-3/2} \right\}$$

$$K = \frac{1}{BE} \left\{ - (9 + \nu) A_1 R_0^{3/2} + 5 (1 + \nu) R_0^{-3/2} + 3 (1 + \nu) C_1 R_0^{-3/2} \right\}$$

$$+ \frac{C}{E} \left\{ \log R_0 + \frac{(5 + \nu) R_0^2}{2(a^2 + b^2)} - \frac{(1 + \nu) a^2 b^2}{2(a^2 + b^2) R_0^2} \right\}$$

where $R_0 = \frac{1}{2}(a + b)$

B. Use of the Fourier Integral

The stresses due to the edge force of the inner boundary of the same problem described as in Figure 3-1 also can be solved by the Fourier Integral. The boundary conditions of this specific problem, written in the form of the Fourier Integral, are

$$(\hat{r}r)_b = 0$$

$$(\hat{r}r)_a = F_2(e) = G_2(e) = \frac{2}{\pi} \int_0^\infty \cos n\theta \, dn \int_0^\infty G_2(\lambda) \cos n\lambda \, d\lambda$$

$$(\hat{r}e)_b = 0$$

$$(\hat{r}e)_a = 0$$

(3-8)
By applying Equations (2-7), there are four simultaneous equations

\[(n+1)(n-2)A_n b^n + n(n+1)B_n b^{-n-2} + n(n-1)C_n b^{-n} + (n-1)(n+2)D_n b^{-n} = 0\]

\[(n+1)(n-2)A_n a^n + n(n+1)B_n a^{-n-2} + n(n-1)C_n a^{-n} + (n-1)(n+2)D_n a^{-n} = S\]

\[n(n+1)A_n b^n - n(n+1)B_n b^{-n-2} + n(n-1)C_n b^{-n} - n(n-1)D_n b^{-n} = 0\]

\[n(n+1)A_n a^n - n(n+1)B_n a^{-n-2} + n(n-1)C_n a^{-n} - n(n-1)D_n a^{-n} = 0.\]

where

\[S = -\frac{2}{\pi} \int_0^\infty G_2(\lambda) \cos n\lambda d\lambda\]

Solving Equations (3-9) simultaneously, we have

\[A_n = F(n)(n-1)(n a^{-n+2} b^2 + a^n b^{-2n} - n a^{-n} a^{-n})\]

\[B_n = F(n)(n-1)(a^{-n+2} b^{2n} + n a^n b^2 - n a^{n+2} - a^{n+2})\]

\[C_n = F(n)(n+1)(-n a^{-n+2} + a^{n+2} + n a^n b^2 - a^{n+2} b^{-2n})\]

\[D_n = F(n)(n+1)(-n a^n + a^n + n a^{n+2} b^{-2} - a^{-n} b^{2n})\]
Where

\[ F(n) = \frac{\int_{\alpha}^{\infty} G_2(\alpha) \cos n \alpha d\alpha}{\Pi (n^2 - 1) \left( (a^n b^m - a^n b^{-m})^2 - n^2 (a^2 b - a b^{-1})^2 \right)} \]

Substituting Equations (3-10) into Equations (2-3) and rearranging them, we find the stress function \( \phi \) for Figure 3-2.

\[
\phi = \frac{-a}{2\pi} \left\{ \int_{0}^{a} g(r, n) \cos n \theta \, dr \int_{0}^{\alpha} G_2(\alpha) \cos n \alpha d\alpha \right. \\
- \left. \int_{0}^{\alpha} j(r, n) \cos n \theta \, dr \int_{0}^{\alpha} G_2(\alpha) \cos n \alpha d\alpha \right\}
\]

(3-11)

Where

\[
g(r, n) = \frac{n}{b} \left( b^2 - r^2 \right) \left( a^n r^n + a^n r^{-n} \right) + \frac{n}{a} \left( r^2 - a^2 \right) \left( b^n r^{-n} + b^{-n} r^n \right) \\
+ b^{-1} \left( b^2 + r^2 \right) \left( a^n r^n - a^n r^{-n} \right) + a^{-1} \left( r^2 + a^2 \right) \left( b^n r^{-n} - b^{-n} r^n \right)
\]

and

\[
j(r, n) = \frac{n}{b} \left( b^2 - r^2 \right) \left( a^n r^n + a^n r^{-n} \right) - \frac{n}{a} \left( r^2 - a^2 \right) \left( b^n r^{-n} + b^{-n} r^n \right) \\
+ b^{-1} \left( b^2 + r^2 \right) \left( a^n r^n - a^n r^{-n} \right) - a^{-1} \left( r^2 + a^2 \right) \left( b^n r^{-n} - b^{-n} r^n \right).
\]
In the specific problem of Figure 3-2, we have

\[ G_2(\theta) = 0 \quad \text{For } -\infty < \theta < -\pi \]
\[ G_2(\theta) = -\frac{3P}{2a} \cos \frac{\theta}{2} \quad \text{For } -\pi < \theta < \pi \]
\[ G_2(\theta) = 0 \quad \text{For } \pi < \theta < \infty \]

therefore

\[ \int_0^\infty G_2(\lambda) \cos n\lambda d\lambda = \int_0^\infty -\frac{3P}{2a} \cos \frac{\lambda}{2} \cos n\lambda d\lambda = -\frac{3P \cos n\pi}{a(1 - 4n^2)} \]

Substituting above result into Equation (3-11), we have

\[ \phi = \frac{3P}{2\pi} \left\{ \int_0^\infty \frac{g(r, n) \cos n\theta \cos n\pi d\lambda}{(1 - 4n^2)(n^2 - 1)[(a^n b^n - a^n b^{-n}) + n(a^n b - a^{-n} b)]} \right\} - \frac{3P}{2\pi} \left\{ \int_0^\infty \frac{h(r, n) \cos n\theta \cos n\pi d\lambda}{(1 - 4n^2)(n^2 - 1)[(a^n b^n - a^n b^{-n})^2 + n^2(a^n b - a^{-n} b)]} \right\} \]

\[ \phi = \frac{3P}{2\pi} \int_0^\infty \frac{h(r, n) \cos n\theta \cos n\pi d\lambda}{(1 - 4n^2)(n^2 - 1)[(a^n b^n - a^n b^{-n})^2 + n^2(a^n b - a^{-n} b)^2]} \]

(3-12)

Where

\[ h(r, n) = (a^n b - a^n b^{-n}) \left[ \frac{n}{a} (r^2 - a^2) (b^n r^{-n} + b^{-n} r^n) \right] \]
\[ + a^{-1} (r^2 + a^2) (b^n r^{-n} - b^{-n} r^n) \]
\[ - n (a^{-1} b - a^{-1} b^n) \left[ \frac{n}{b} (b^2 - r^2) (a^n r^n + a^n r^{-n}) \right] \]
\[ + b^{-1} (b^2 + r^2) (a^n r^{-n} - a^n r^n) \]

Now, the following new symbols are introduced:

\[ S = \frac{\theta}{\log (b/a)}, \quad t = \pi / \log (b/a), \quad z = n \log \frac{b}{a}, \]
\[ A = \log (r/a) / \log \frac{b}{a}, \quad B = \log \left( \frac{b}{r} \right) / \log \left( \frac{b}{a} \right), \]
Now, the stress function \( \phi \) for Figure 3-2 can be written in the form

\[
\phi = -P \frac{3P}{\pi \log \frac{b}{a}} \int_{0}^{\infty} H(z) \cos s \theta \cos t \theta \, dz
\]

where

\[
H(z) = \sinh z \left[ (r^2-a^2) \cosh b \frac{z}{a} \right]
\]

\[
- \frac{a^{-1}(a^2+r^2) \sinh b \frac{z}{a}}{z}
\]

\[
+ \frac{x z \left[ (b^2-r^2) \cosh A \frac{z}{a} \right]}{b \log \frac{b}{a}}
\]

\[
- \frac{b^{-1}(b^2+r^2) \sinh A \frac{z}{a}}{z}
\]

Let \( F_x \) be -P in Equations (1-14). Adding the stress function \( \phi \) in Equations (1-14) to Equation (3-14), the stress function \( \phi \) for Figure 3-1 can be found.

\[
\phi = C \left[ \frac{r^3}{2(a^2+b^2)} - \frac{a^2b^2}{2(a^2+b^2)} \frac{1}{r} - r \log r \right] \sin \theta
\]

\[
- \frac{3P}{\pi \log \frac{b}{a}} \int_{0}^{\infty} H(z) \cos s \theta \cos t \theta \, dz
\]

\[
\frac{1}{(1-4z^2/(\log \frac{b}{a})^2) \left[ z^2/(\log \frac{b}{a})^2 - 1 \right] \left[ \sinh^2 z - x^2 \right]}
\]

where

\[
C = -P \frac{(a^2+b^2)}{\left[ a^2 - b^2 + (a^2+b^2) \log \frac{b}{a} \right]}
\]

The corresponding stresses can be obtained from the stress function of Equation (3-15) by applying Equations (1-3).
results are given in the following form because the expressions
for \( \hat{r}r + \hat{e}e \) and \( \hat{r}r - \hat{e}e \) are shorter than those for \( \hat{r}r \) and \( \hat{e}e \).

\[
\hat{r}r + \hat{e}e = 2P \left[ r^{-1}(a^2+b^2)-2r \right] \sin \theta / \left[ a^2-b^2+(a^2+b^2) \log \frac{1}{\lambda} \right]
+ \frac{12\pi}{\pi \log \frac{1}{\lambda}} \left\{ \frac{x}{b} \int_0^\infty \frac{z \sinh Az \cos Sz \cos tz dz}{[1-4z^2/(\log \frac{1}{\lambda})^2] \left[ \sinh^2 z - x^2 z^2 \right]} \right\}
- \frac{1}{\lambda} \left\{ \frac{x}{b} \int_0^\infty \frac{\sinh z \sinh Bz \cos Sz \cos tz dz}{[1-4z^2/(\log \frac{1}{\lambda})^2] \left[ \sinh^2 z - x^2 z^2 \right]} \right\}. \tag{3-16}
\]

\[
\hat{r}r - \hat{e}e = 2P \left[ r^{-1}(a^2+b^2)-2r \right] \sin \theta / \left[ a^2-b^2+(a^2+b^2) \log \frac{1}{\lambda} \right]
+ \frac{6\pi}{\pi \log \frac{1}{\lambda}} \left\{ \frac{x}{b} \int_0^\infty \frac{z^2 \cosh Az \cos Sz \cos tz dz}{[1-4z^2/(\log \frac{1}{\lambda})^2] \left[ \sinh^2 z - x^2 z^2 \right]} \right\}
- (a^{-1} - a^{-2}) \left\{ \frac{x}{b} \int_0^\infty \frac{\sinh z \cosh Bz \cos Sz \cos tz dz}{[1-4z^2/(\log \frac{1}{\lambda})^2] \left[ \sinh^2 z - x^2 z^2 \right]} \right\}. \tag{3-17}
\]

\[
\hat{e}e = P \left[ r^{-1}(a^2+b^2)-2r^{-1} + a^2 a^{-2} r^{-3} \right] \cos \theta / \left[ a^2-b^2+(a^2+b^2) \log \frac{1}{\lambda} \right]
- \frac{3\pi}{\pi \log \frac{1}{\lambda}} \left\{ \frac{x}{b} \int_0^\infty \frac{z^2 \sinh Az \sin Sz \cos tz dz}{[1-4z^2/(\log \frac{1}{\lambda})^2] \left[ \sinh^2 z - x^2 z^2 \right]} \right\}
+ (a^{-1} - a^{-2}) \left\{ \frac{x}{b} \int_0^\infty \frac{\sinh z \sinh Bz \sin Sz \cos tz dz}{[1-4z^2/(\log \frac{1}{\lambda})^2] \left[ \sinh^2 z - x^2 z^2 \right]} \right\}. \tag{3-18}
\]
The first integral in Equation (3-17) can be obtained by taking \( \frac{\partial}{\partial A} \) of the first integral in Equation (3-16); and the first integral in Equation (3-18) is obtained by taking \(-\frac{\partial}{\partial S}\) of the first integral in Equation (3-16). Similarly, the second integrals in Equations (3-17) and (3-18) can be obtained by taking \( \frac{\partial}{\partial B} \) and \(-\frac{\partial}{\partial S} \) of the second integral of Equation (3-16), respectively. It will be noted that A and B are both positive numbers between zero and unity.

After solving these mathematical equations, the stresses in the ring shown as Figure 3-1 are obtained. A numerical solution will be given in the next section.

C. Numerical Solution

Let \( a = 4.0 \) and \( b = 5.0 \) in the previous example. The solution of tangential stress along the circular line \( r = 4.2 \), and along the section \( \theta = 90^\circ \) calculated by both multiple-ring method and Fourier-integral method will be evaluated by the Rice University computer. Equations (16) and (17) are integrated from \( z = 0 \) to \( z = 0.22 \), and then from \( z = 0.226712 \) to \( z = 6 \); because in this numerical example, there is a singular point at \( z = 0.223356 \). Thus the singular point can be avoided (13). Part of the truncation error is caused in this singular point. This truncation error may be reduced by a better computer program.

After \( z = 6 \), the ordinates are extremely small; therefore, the integration from \( z = 6 \) to \( z = \infty \) are neglected. This causes the main part of the truncation errors, and its maximum value
is approximately equal to 1.05 at $\phi = 0^\circ$. Another part of the truncation errors is the inaccuracy arising from central differences. These three parts of truncation errors cause the differences between the values of stress solved by the multiple-ring method and the Fourier-integral method. The values of differences are within 1.1 as shown in Tables I and II. The values of tangential stress $\theta$ at the points $r = 4.2, \phi = -180^\circ$ and $r = 4.2, \phi = 180^\circ$ apparently should be zero, because of their zero bending moment. The numerical solutions of Fourier-integral method give the answers with the value 0.75238. These discrepancies as well as all the other discrepancies are caused by the three parts of the truncation errors. The round-off error is disregarded in this example.

The maximum stress of this example is at the point $r = 4.0, \phi = 60^\circ$ with a value 78.48715.

The approximate solution by thin straight beam theory of "Strength of Materials" is given by slide rule in Tables I and II for comparison.

In Tables I and II, the values of tangential stress are written in the non-dimensional form $\theta = \frac{P}{b - a}$. 
CONCLUSIONS

The following conclusions have been made as a result of the work described in this thesis:

I. It is feasible to solve all the circular ring problems of elasticity by the multiple-ring method. The solution of complete ring is a special case of multiple ring. Any error here would be due to replacing an infinite series with a finite series.

II. The problems formulated by Fourier-integral method in Chapter II can be solved by numerical means. Attention must be paid to any singular points and accuracy must be considered. The accuracy is dependent on computer program.

III. Certain case of the ring problems with boundary conditions of finite terms of trigonometric functions can be solved by multiple-ring method much easier than by Fourier-integral solution.
TABLE I  COMPARISON OF THE VALUES OF TANGENTIAL STRESSES ALONG THE CIRCULAR LINE WITH A CONSTANT RADIUS $r = 4.2$ FOR LOADING SHOWN IN FIGURE 3 - 1

<table>
<thead>
<tr>
<th>Angle</th>
<th>$\sigma$ Computed by Multiple-ring Method</th>
<th>$\sigma$ Computed by Fourier-Integral Solution</th>
<th>$\sigma$ Computed by the Approximation thin Straight Beam of &quot;Strength of Materials&quot;</th>
</tr>
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<tr>
<td>-180</td>
<td>0.00000</td>
<td>0.75238</td>
<td>0.00</td>
</tr>
<tr>
<td>-150</td>
<td>0.30218</td>
<td>0.50024</td>
<td>0.32</td>
</tr>
<tr>
<td>-120</td>
<td>2.30377</td>
<td>2.48460</td>
<td>2.23</td>
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<tr>
<td>-90</td>
<td>7.12715</td>
<td>6.98029</td>
<td>6.78</td>
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<tr>
<td>-60</td>
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<tr>
<td>-30</td>
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<td>23.62638</td>
<td>23.30</td>
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<tr>
<td>0</td>
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</tr>
<tr>
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<td>41.56195</td>
<td>41.41473</td>
<td>39.10</td>
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<td>0.00</td>
</tr>
<tr>
<td>Radius</td>
<td>(\pi \text{ Computed by Multiple-ring Method}</td>
<td>(\pi \text{ Computed by Fourier-Integral Solution}</td>
<td>(\pi \text{ Computed by the Approximation of thin Straight Beam of &quot;Strength of Materials&quot;}</td>
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<td>--------------------------</td>
<td>--------------------------------------------------</td>
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<td>-65.20</td>
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</table>
FIG. I COMPARISON OF THE VALUES OF TANGENTIAL STRESSES ALONG THE CIRCULAR LINE WITH A CONSTANT RADIUS $r = 4.2$ FOR LOADING SHOWN IN FIGURE 3-1
FIG. II COMPARISON OF THE VALUES OF TANGENTIAL STRESSES ALONG THE CIRCULAR LINE WITH A CONSTANT RADIUS $r = 4.2$ FOR LOADING SHOWN IN FIGURE 3-1

By Approximation of Thin Straight Beam Theory of "Strength of Materials"

By Multiple-Ring Method And Fourier-Integral Method
BIBLIOGRAPHY


