RICE UNIVERSITY

AN EXPERIMENTAL STUDY OF THE PROPAGATION OF FLEXURAL WAVES IN AN ELASTIC BEAM OF CIRCULAR CROSS SECTION

by

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ABSTRACT

An experimental study of the propagation of flexural waves in an elastic beam of circular cross section subjected to an approximate step function bending moment is given. The test beam was a low carbon steel bar 2 inches in diameter by 30 feet long and was suspended in a vertical position by a pin located near the upper end. The step moment was applied at the upper pinned end of the bar by an arrangement of two high pressure, nitrogen operated cylinders. The strains were measured with strain gages located at eight stations along the bar.

The experimental results are compared with results obtained from a solution of the Timoshenko theory's differential equations by E. E. Zajac and W. Flugge. The experimental results are correlated with the predictions of the theoretical solution.
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NOTATION

$x, z$ ........ cartesian coordinates

t ........... time

$a$ ........... radius of bar

$A$ ........... cross sectional area

$A_s$ ........ effective shear area

$C_b$ ........ bar velocity

$C_m$ ........ propagation velocity of a discontinuity in bending

$C_v$ ........ propagation velocity of a discontinuity in shear

$E$ ........... Young's modulus

$G$ ........... shear modulus

$I_e$ ........ elastic moment of inertia

$I_i$ ........ inertial moment of inertia

$M$ ........... bending moment

$M_o$ ........ moment input

$R$ ........... radius of curvature

$V$ ........... shear force

$W$ ........... deflection in z-direction

$\lambda$ ........ wave length

$\nu$ ........ Possion's ratio

$\xi$ ........ dimensionless space coordinate

$\rho$ ........ density of beam

$\tau$ ........ dimensionless time

$\varphi$ ........ angular rotation
INTRODUCTION

The Bernoulli-Euler beam vibration equation shows that the propagation velocity of flexural waves depends upon wave length and also that very short wave lengths, such as those associated with impact loading, tend toward infinite velocities. A modification of the elementary theory to account for rotatory inertia by Lord Rayleigh, 1894, and for shear distortion by Timoshenko, 1921, resulted in a more physically acceptable equation of motion. W. Flugge\textsuperscript{1} demonstrated in 1942 that the Timoshenko theory predicted that a suddenly applied disturbance would propagate at a finite velocity.

In recent years a number of solutions based on the Timoshenko beam theory have been accomplished.\textsuperscript{2-8} For the most part the analytical solutions were obtained by use of integral transform techniques and contour integration which led to results in terms of real definite integrals. Numerical evaluation of the definite integrals obtained in this way was quite involved and has required the use of high speed computers. Numerical methods of solution such as utilization of finite difference equations have also been used but they too involve considerable computer work.

W. Flugge and E. E. Zajac\textsuperscript{9} presented a solution of the Timoshenko theory's differential equations in 1959.

\textsuperscript{*Numbers refer to references of the bibliography.}
consisting in part of methods which do not appear to have been applied previously to this particular problem. The solution consisted of the use of a shifting theorem from Laplace transform theory together with the expansion of the resulting transform in powers of $1/s$ and term by term inversion. From this was obtained the power series solution useful for computing response near the origin of the $(x, t)$ plane and a series of rapidly convergent Bessel functions yielding the solution at the leading wave front. An asymptotic solution was obtained for large times which indicates how near the point of impact the integral from the contour integration technique can be evaluated. Also the classical stationary phase method of Kelvin was applied to obtain an asymptotic solution in time over the beam. Although none of the methods yield a complete solution individually the composite gives an almost complete solution over the entire beam. A considerable advantage is offered by this solution over previous solutions since numerical evaluations can be carried out on a desk calculator.

This thesis is an attempt to determine the validity of the solution of Flugge and Zajac by comparing experimental and analytical results for the specimen problem, Figure 1,** of a simply supported, semi-infinite, elastic, circular beam subjected to a step function bending moment at the supported end.

**Figures are given in sequence beginning on page 19.
THE TIMOSHENKO BEAM

This section contains a brief summary of the well known Timoshenko beam equations.

The assumption is made that initially plane cross sections remain plane during deformation but, unlike the elementary Bernoulli-Euler theory, that plane cross sections do not necessarily remain normal to the neutral axis. The removal of the latter restriction eliminates the requirement of purely translatory motion in the z-direction and permits a rotary motion of the beam elements.

Consider the differential element of Figure 2 which shows the forces and deflections associated with a loaded element. From the condition of equilibrium and a summation of forces in the z-direction:

\[-V \cos \psi + (V + \Delta V) \cos (\psi + \Delta \psi) - \rho A \Delta x \frac{\partial^2 w}{\partial z^2} = 0\]

By rearranging the equation, considering small values of \( \psi \) and letting \( \Delta x \to 0 \), the first fundamental equation is obtained:

\[ \frac{\partial v}{\partial x} = \rho A \frac{\partial^2 w}{\partial z^2} \] (1)

Summing moments about the element and equating them to the product of the mass moment of inertia of the element and its angular acceleration:

\[ M - (M + \Delta M) + V \cos \psi \Delta x = \rho I_z \Delta x \frac{\partial^2 \psi}{\partial z^2} \]
Dividing through by $\Delta x$ and letting $\Delta x \to 0$, a second fundamental equation is obtained:

$$\frac{2M}{\Delta x} = V - \rho l_0 \frac{\partial^2 W}{\partial x^2}$$

(2)

A third relation is obtained from a consideration of Hooke's law:

$$M = \frac{-EI_0}{R} = -EI_0 \frac{\partial W}{\partial x}$$

(3)

By considering the total deflection of the center of gravity of the element shown in Figure 2 is made up of contributions due to rotation and shear the following relation can be deduced:

$$\frac{\partial W}{\partial x} = x' + \beta$$

$$\beta = \frac{\partial W}{\partial x} - x'$$

Where the angle change $\beta$ is assumed to be caused by a shear force $V$ acting on an effective shear area, $A_s$, or:

$$V = GA_s \left[ \frac{\partial W}{\partial x} - x' \right]$$

(4)

By rearranging equations (1), (2), (3) and (4) the following four equations can be obtained:

$$\frac{\partial W}{\partial x} = \frac{\partial^2 W}{\partial x^2} - \frac{\rho A}{G A_s} \frac{\partial^2 W}{\partial x^2}$$

(5)
\[
\frac{\partial^3 W}{\partial x \partial t^2} = \frac{\partial^4 W}{\partial x^2 \partial t^2} - \frac{\rho A}{GA_s} \frac{\partial^4 W}{\partial x^2 \partial t^2} 
\] (6)

\[
\frac{\partial^3 W}{\partial x^3} = \frac{\partial^4 W}{\partial x^4} - \frac{\rho A}{GA_s} \frac{\partial^4 W}{\partial x^2 \partial t^2} 
\] (7)

\[
EI_e \frac{\partial^3 W}{\partial x^3} + GA_s \left( \frac{\partial^2 W}{\partial x^2} - \frac{\partial W}{\partial x} \right) - \rho I_e \frac{\partial^3 W}{\partial x \partial t^2} = 0 
\] (8)

Substitution of equations (5), (6) and (7) into (8) yields a single fourth order equation in terms of the deflection \( W \):

\[
\frac{\partial^4 W}{\partial x^4} - \left[ \frac{1}{C_m^2} + \frac{1}{C_v^2} \right] \frac{\partial^2 W}{\partial x^2} - \frac{1}{C_m^2 C_v^2} \frac{\partial^2 W}{\partial x \partial t^2} + \frac{\rho A}{EI_e} \frac{\partial^3 W}{\partial x \partial t^2} = 0 
\] (9)

\[C_m = \sqrt{\frac{E}{\rho}} \quad \text{(For A Homogeneous Beam)}\]

\[C_v = \sqrt{\frac{GA_s}{\rho A}}\]

\[K = \frac{\rho A}{EI_e}\]

The bending wave velocity, \( C_m \), represents the velocity at which a discontinuity in \( M \), \( \frac{\partial M}{\partial x} \) or \( \frac{\partial M}{\partial t} \) will propagate. The shear wave velocity, \( C_v \), represents the velocity at which a discontinuity in \( V \), \( \frac{\partial V}{\partial x} \) or \( \frac{\partial V}{\partial t} \) will propagate.
Introducing the dimensionless variables $\xi$ and $\tau$, the equation of motion becomes:

$$\frac{\partial^2 W}{\partial \xi^2} - \left[ 1 + \frac{C^2}{C_v^2} \right] \frac{\partial^2 W}{\partial \xi^2 \partial \tau^2} + \frac{C^2}{C_v^2} \frac{\partial^2 W}{\partial \tau^2} + \frac{\partial^2 W}{\partial \tau^2}$$

(10)

$$\xi = x \sqrt{\frac{A}{I}}$$

$$\tau = c_m x \sqrt{\frac{A}{I}}$$
EXPERIMENTAL PROCEDURE

A low carbon, hot rolled steel beam, 360 inches long by 2 inches in diameter was used to simulate the semi-infinite test specimen. The beam was suspended in a vertical position by a 1/2 inch diameter pin located 2-1/4 inches from the upper end of the beam. The lower end of the beam was loosely pinned to prevent excessive rotation of the beam after application of the bending moment. The fastening at the lower end did not affect the test results since the duration of each test was so short as to prohibit the inclusion of reflections. The lower pinned end did affect the distribution of bending moment at static equilibrium in that the bending moment decreased continuously from a maximum, $M_0$, at the upper pin to zero at the lower pin whereas in a semi-infinite beam the bending moment would decrease to zero only at infinity.

Sudden application of the bending moment at the upper end of the beam was accomplished by an arrangement of two high pressure nitrogen operated cylinders. The arrangement consisted of a beam loading cylinder and a restraining cylinder attached to the beam at a distance of 1.5 inches above the upper pin. The cylinders acted in opposing directions and were mounted such that the restraining cylinder was incapable of causing the beam to deflect from its free hanging vertical position. The arrangement of the test beam and cylinders is shown in Figure 3.
The purpose of the restraining cylinder was to obtain desirable preloading of the beam loading cylinder without a resultant beam deflection. A tensile specimen was used to link the restraining cylinder to the beam. Upon failure of the tensile specimen the preloaded beam loading cylinder quickly applied and maintained a predetermined load. The value of bending moment used in these tests was adjusted so as to cause a maximum dynamic fiber stress of approximately 28000 PSI at the upper pinned end of the beam. This condition of maximum stress was obtained by establishing a regulated pressure of 1010 PSIG in the beam loading cylinder prior to release by the restraining cylinder. The procedure used in applying the simulated step function bending moment was:

1. Attach the restraining cylinder to the beam by means of a tensile specimen designed to fail at approximately 8000 pounds.

2. Adjust the pressure in the restraining cylinder to exert a static pull of 7500 pounds.

3. Regulate the pressure in the beam loading cylinder to obtain a preload of 7000 pounds.

4. Increase the pressure to the restraining cylinder until failure of the tensile specimen occurred which consequently released the beam from its constrained vertical position.
Oscilloscope recordings made from strain gages located at the upper pinned end show that maximum fiber stress increased from zero to 15600 PSI, the equilibrium value, in approximately 66 microseconds. The rate of load application was equal to about 109 million pounds per second.

The nitrogen supply for the loading cylinders is shown schematically in Figure 4. The use of an accumulator between the beam loading cylinder and nitrogen supply cylinder was necessary to prevent an appreciable pressure drop during initial beam deflection. The pressure in the accumulator was regulated with a 500 CFH regulator located on the nitrogen bottle and was adjusted in accordance with a 0-1500 PSI test gage. The adjustment of pressure in the restraining cylinder involved the use of a needle valve, a 0-3000 PSI pressure gage and a length of 1/8 inch stainless steel tubing. The pressure gage was used in preloading the restraining cylinder and in determining the load at which failure of the tensile specimen occurred.

The tensile specimens used in these tests were made of a high carbon steel, Carpenter #11 special. The specimens were austenitized at 1525°F, oil quenched from 1525°F and tempered at 450°F. After heat treatment the specimen hardness was approximately 60 Rockwell C. The specimens failed at approximately 8000 pounds tensile
load without appreciable elongation. The brittle specimens were used in order to obtain a rapid release by the restraining cylinder.

Sixteen strain gages to indicate maximum fiber stress were cemented in pairs diametrically opposite each other at eight stations along the beam. Location of the stations on the beam is shown in Figure 5. Twelve additional strain gages were located along the neutral axis in pairs and diametrically opposite at stations 2 through 7. The gages located along the neutral axis are labeled "shear gages" and those at 90° from the neutral axis, "bending gages". The alignment of the strain gages is shown in Figure 6. The gages were Tatnall metal film, type C6-121 and were attached with Tatnall B-3 cement.

Individual pairs of strain gages were connected to an Ellis bridge amplifier, model BAM-1, to obtain amplification of the resulting strain signals. The gages were connected so as to eliminate symmetric strains in all tests. The strain signals were displayed with a type 545A Tektronics oscilloscope and photographed with a Hewlett-Packard oscilloscope camera for a permanent record. The oscilloscope beam was triggered externally by the tensile specimen used to link the restraining cylinder to the test bar. A schematic view of the instrumentation setup is shown in Figure 7.
RESULTS

A constant moment input, \( M_0 \), was used in all tests and a minimum of two tests were made with each pair of gages in order to check uniformity.

The first series of tests was made with the pairs of bending gages in which maximum fiber stress at each of the eight stations was recorded as a function of time. These results are shown in Figure 8. The horizontal time scale was 200 microseconds per division in each test. The vertical strain scale was varied from \( \pm 666 \) microinches per inch full scale at station one to \( \pm 159 \) microinches per inch full scale at station eight. The vertical strain calibration was varied in order to utilize maximum strain sensitivity at each station. Because of this the appropriate vertical calibration is listed beside each photograph.

The wave fronts in the photographs of Figure 8 are shown to be propagating at a constant velocity of 10700 feet per second. This velocity was determined experimentally by connecting pairs of bending gages from two stations in parallel permitting a determination of the difference in arrival times of the wave front at each of the two stations. Knowing the distance separating the pairs of gages and the time required for the wave front to traverse the distance, the velocity of propagation was calculated. Velocity determinations using station three as a base point are shown in Figure 9.
Further examination of the photographs in Figure 8 shows that at a velocity of approximately 9500 feet per second the form of the strain oscillation changes noticeably. The change in form, represented by a velocity line labeled 9500 FPS, occurs at a time corresponding to the component of shear distortion.

According to Kolsky, exact theory predicts that the group velocity of flexural waves of the first mode will have a maximum at a ratio of radius of bar to wave length, $a/\lambda$, equal to about 0.3. This implies that wave lengths that are approximately three times the radius of the bar should precede the components of other wave lengths and should therefore appear at the head of the pulse and should propagate at a velocity of approximately 0.63 times the bar velocity, $C_b$. The bending vibrations shown in Figure 8 are due mainly to contributions of the first transverse mode and correlation is obtained with Kolsky's prediction as pertains to velocity and wave length.

The oscilloscope beam intensity required to resolve the large amplitude bending vibrations, shown in Figure 8, was too high to clearly show the high frequency, low amplitude vibrations due to the higher modes. By reducing the beam intensity to a level which barely resolved the first mode contribution and increasing the strain sensitivity to a maximum permitted by the instrumentation the presence of the higher modes was defined. Recordings,
shown in Figure 10, made with pairs of bending gages at stations 3, 4, and 5 demonstrate this effect. Loss of resolution over portions of the trace was a definite disadvantage but tests with the shear gages located on the neutral axis of the bar alleviated the difficulties.

The strain gages located along the neutral axis and inclined 45° to that axis indicated an average value of shear stress over the area covered by the gage and were therefore not used to determine the specific magnitude of the strains present. But since the shear strain variations were in phase with the outer fiber strain variations the shear recordings were used to determine the component wave velocities.

Test results obtained from individual pairs of shear gages located at stations 2 through 7 are shown in Figure 11. The leading wave front in each photograph is positioned to correspond with a velocity line of 16950 feet per second. The wave front velocity was determined experimentally in the same manner used to determine the velocity of the wave front associated with the first mode vibrations. Velocity determinations using station two as the base point are shown in Figure 12. Velocity lines corresponding to the arrival of the shear wave and first mode bending wave front are also shown on Figures 11 and 12.

The solution of Flugge and Zajac predicts that a rapid fluctuation develops immediately behind a flexural
wave front as it progresses along the bar. At sufficiently large values of $\tau$, the wave front assumes the form of a jump directly followed by a steep sloping off of the amplitude and frequency of the oscillations. Consider the special case where the state of strain along some longitudinal reference line on the test bar is frozen instantaneously at $\tau$ equal to 500. At this instant the leading wave front has traveled a distance of $\xi$ equal to 500. Examining the condition of strain in the bar subjected to this "frozen" condition an observer could see, according to the solution of Flugge and Zajac:

1. At values of $\xi$ greater than 500 the bar exists in an undisturbed state.
2. At a value of $\xi$ equal to 500 the wave front stands in the form of a jump.
3. Immediately behind the jump the wave form slopes off steeply in amplitude and frequency.
4. Following the steeply sloped portion the wave form assumes a gradual decrease in amplitude and frequency. At a wave length corresponding to approximately $3a$ the contribution from the first mode appears although this arrival does not disrupt the form of the wave.
5. Somewhat behind the front of the first mode contribution the leading portion of the
shear wave appears, resulting in a momentary increase in frequency.

6. Behind the shear wave front the wave form increases in amplitude and decreases in frequency until, at $\xi$ equal to zero, the strain assumes a value corresponding to the step input of bending moment, $M_0$. The greater values of strain disturbance appear between $\xi$ equal to zero and the value of $\xi$ at the shear wave front.

Similar examinations of the strain state at smaller values of $\tau$ show how the flexural wave gradually progresses, due to dispersion, into the form described at $\tau = 500$.

In general the solution of Flugge and Zajac was supported by the experimental results. The overall manner in which the disturbance propagated along the beam as well as the propagational velocities of the leading disturbance, the wave front of the first mode contribution and the shear wave front conformed to the predictions. One difference between the experimental results and theoretical solution involved the failure to record a jump at the leading wave front at large times. Since the jump consists of very high frequency oscillations, frequencies which approach infinite values, its lack of appearance in the oscilloscope traces can be due largely to two factors. The first factor concerns the approximate step input of bending moment at the
upper pinned end of the beam. Full coverage of the frequency spectrum is obtained only by applying a true step input. It is likely that the approximate step input used in these tests did not adequately cover the frequency range necessary for the development of a jump. A second factor is related to the frequency response of the Ellis bridge amplifier used in these tests. The instrument effectively acts as an attenuator for frequencies above a certain level thus the existence of a jump could be hidden due to the characteristics of the instrumentation.

Another difference between the experimental and analytical results was an out of phase condition at the juncture of the first mode and higher mode bending vibrations and at the juncture of the bending and shear vibrations. The theoretical solution predicted a smooth merging between the wave components.
CONCLUSIONS

1. The experimentally determined velocities correlate closely with the Timoshenko theory's predictions concerning: arrival of the initial disturbance traveling at the bar velocity, $C_b$; arrival of the first mode wave group traveling at a velocity of approximately $0.63 C_b$; and arrival of the shear wave front. In accordance with Timoshenko theory no disturbance was observed corresponding to a velocity greater than the bar velocity.

2. In accordance with the prediction of Flugge and Zajac, the greater part of the bending disturbance propagated at a velocity below the shear velocity.

3. Within the limits of the experimental technique the solution of the Timoshenko beam obtained by Flugge and Zajac was shown to provide an adequate representation of the propagational characteristics of flexural waves.
BIBLIOGRAPHY


SIMPLY SUPPORTED SEMI-INFINITE BEAM

STEP FUNCTION BENDING MOMENT

FIGURE 1

SPECIMEN PROBLEM SOLVED BY W. FLUGGE AND E.E. ZAJAC
FIGURE 2
FREE BODY DIAGRAM OF A DIFFERENTIAL ELEMENT OF THE TIMOSHENKO BEAM
FIGURE 3
BEAM SUSPENSION AND CYLINDER ATTACHMENT

1 - RESTRAINING CYLINDER
2 - TENSILE SPECIMEN
3 - TEST BEAM 2" DIA. x 30' LG.
4 - 1" DIA. PIN SUPPORTING BEAM
5 - BEAM LOADING CYLINDER
FIGURE 4
BEAM LOADING EQUIPMENT ARRANGEMENT
FIGURE 5
LOCATION OF STRAIN GAGE STATIONS ON TEST BAR
ARRANGEMENT OF GAGES AT A STATION FOR DETECTION OF BENDING AND SHEAR WAVES.
FIGURE 7
INSTRUMENTATION SETUP
VARIATION OF OUTER FIBER STRAIN AS A FUNCTION OF TIME
RECORDED WITH BENDING GAGES
\[ x = \text{distance from upper pin support} \]

horizontal scale: 200 microsec. per div.

STA. 3
\[ x = 6 \text{ ft.} \]

STAS. 3 & 4
\[ x = 6 \text{ & 9 ft.} \]

STAS. 3 & 5
\[ x = 6 \text{ & 12 ft.} \]

STAS. 3 & 6
\[ x = 6 \text{ & 15 ft.} \]

STAS. 3 & 7
\[ x = 6 \text{ & 18 ft.} \]

10700 ft. per sec.

0 500 1000 1500 2000 2500

TIME — microseconds —

FIGURE 9
WAVE FRONT VELOCITY
RECORDED WITH BENDING GAGES
$x =$ distance from upper pin support
horizontal scale: 200 microsec. per div.

STA. 3
\[ x = 6 \text{ ft.} \]

STA. 4
\[ x = 9 \text{ ft.} \]

STA. 5
\[ x = 12 \text{ ft.} \]

16950 ft. per sec. 10700 ft. per sec

0 500 1000 1500 2000 2500
TIME — microseconds —

**FIGURE 10**
OUTER FIBER STRAIN VARIATION
AND HIGH STRAIN SENSITIVITY
RECORDED WITH BENDING GAGES
x = distance from upper pin support
horizontal scale: 200 microsec. per div.

FIGURE 11
VARIATION OF STRAIN ALONG NEUTRAL AXIS
RECORDED WITH SHEAR GAGES
$x = \text{distance from upper pin support}$

horizontal scale: 200 microsec. per div.

FIGURE 12

WAVE COMPONENT VELOCITIES
RECORDED WITH SHEAR GAGES