RICE UNIVERSITY

OPTIMIZATION OF CONTROL SYSTEM
WITH TIME LAG

by

Shinya Ochiai

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ABSTRACT

The integral-square value of the error in the output to a general second order linear system disturbed by a unit impulse was calculated. The condition that this integral be a minimum was considered to be an indication of optimum control. The value of the proportional control factor that thus optimizes the system was investigated under the condition that there exist time lags in the control system or in the controlled system. Pade approximations for the exponential function were used in the analysis.
INTRODUCTION

In this thesis, the optimization of a control system containing a pure time delay was studied. The criterion of the optimum control was assumed to be that the integral square error of the output from the controlled system be a minimum. The major contribution of this work is the consideration of errors introduced by an analysis which uses a Padé approximation to reduce a transcendental transfer function to rational form.

Several works which discuss optimum controller setting have extended their discussions to the case in which a controlled system has a time lag. Ziegler and Nichols (Bibliography 11,) developed a method of optimum controller setting based on a 25 percent damping ratio. Takahashi (9) made a study of the optimum setting that makes the integral of the absolute value of the error a minimum. Studies by Hazebrock and van der Waerden (3) were based on the same criterion used in this thesis. They used Parsaval's theorem in cases that the system has no time lag. They did not, however, use the theorem for a system having a time lag.
### NOMENCLATURE*

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Defined</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>system constant, $\frac{\alpha}{L}$</td>
<td>following (2.6)</td>
</tr>
<tr>
<td>$A$</td>
<td>cross sectional area of tank</td>
<td>following (1.3)</td>
</tr>
<tr>
<td>$b$</td>
<td>system parameter, $\frac{\beta}{L}$</td>
<td>following (2.23)</td>
</tr>
<tr>
<td>$c$</td>
<td>system parameter, $\frac{\gamma}{L}$</td>
<td>following (2.23)</td>
</tr>
<tr>
<td>$c_1$</td>
<td>coefficient of polynomial</td>
<td>(1.9)</td>
</tr>
<tr>
<td>$c^*$</td>
<td>semicircle in the complex plane</td>
<td>Fig. 2.1</td>
</tr>
<tr>
<td>$d_1$</td>
<td>coefficient of polynomial</td>
<td>(1.10)</td>
</tr>
<tr>
<td>$D(t)$</td>
<td>disturbance</td>
<td>following (3.4)</td>
</tr>
<tr>
<td>$D(s)$</td>
<td>Laplace transform of $D(t)$</td>
<td></td>
</tr>
<tr>
<td>$e(t)$</td>
<td>voltage in the field circuit</td>
<td>following (3.4)</td>
</tr>
<tr>
<td>$E_{\text{max}}$</td>
<td>maximum value of $e(t)$</td>
<td></td>
</tr>
<tr>
<td>$E(s)$</td>
<td>Laplace transform of $e(t)$</td>
<td></td>
</tr>
<tr>
<td>$f(Z)$</td>
<td>function of $Z$</td>
<td>(2.13)</td>
</tr>
<tr>
<td>$f_C$</td>
<td>friction coefficient</td>
<td>following (3.4)</td>
</tr>
<tr>
<td>$g(Z_1)$</td>
<td>function of $Z_1$</td>
<td>(2.16)</td>
</tr>
<tr>
<td>$h(t)$</td>
<td>liquid level (also for output)</td>
<td>following (1.3)</td>
</tr>
<tr>
<td>$H(s)$</td>
<td>Laplace transform of $h(t)$</td>
<td>following (1.3)</td>
</tr>
<tr>
<td>$h_1(t)$</td>
<td>pressure head of outlet pipe</td>
<td>following (1.3)</td>
</tr>
<tr>
<td>$H_1(s)$</td>
<td>Laplace transform of $h_1(t)$</td>
<td>also for input</td>
</tr>
</tbody>
</table>

* Letters used only in Appendixes are not listed here.
i(t) current in the field circuit following (3.4)
I(s) Laplace transform of i(t)
j \sqrt{-1}
J moment of inertia of servomotor following (3.4)
K proportional control factor (2.1) \sim (2.3), (3.15) following (1.4)
K_a gain factor of amplifier following (3.4)
K_c proportional control factor following (1.3)
K_f constant in the field circuit following (3.4)
K_p gain factor of potentiometer following (3.4)
K_t constant of tachometer following (3.4)
L time lag following (1.3)
L_f inductance of field following (3.4)
P_i pole of \frac{Z}{H(\frac{Z}{L})H(\frac{-Z}{L})} following (2.11)
q_i inlet flow following (1.3)
q_o outlet flow following (1.3)
R resistance of outlet pipe following (1.3)
R_f resistance of the field circuit following (3.4)
s complex variable
T(t) torque of the servomotor following (3.4)
T(s) Laplace transform of T(t)
U_i real part of Z_i
\( V_i \) imaginary part of \( Z_i \)  
\( Z \) complex variable, \( SL \) following (2.10)  
\( Z_i \) \( i \)th root of \( f(Z) \) = 0 following (2.15)  
\( \alpha \) system constant \( (2.1) \sim (2.3) \)  
\( \beta \) " " \( (2.3) \)  
\( \gamma \) " " \( (2.3) \)  
\( \tau_m \) time constant of servomotor following (3.5)  
\( \tau_f \) time constant of field circuit following (3.5)  
\( \theta \) angle in the complex plane Fig. 2.1  
\( \theta(t) \) rotational angle of servomotor following (3.4)  
\( \theta(s) \) Laplace transform of \( \theta(t) \)  
\( \dot{\theta}(t) \) rotational speed of servomotor following (3.4)  
\( \hat{\theta}(s) \) Laplace transform of \( \dot{\theta}(t) \)
METHOD OF APPROACH

The optimization of a control system containing a pure time delay* was analyzed. The following conditions were assumed to exist:

1. The characteristics of the controlled system are expressed by a linear first or second order differential-difference equation.

2. The forcing term in the describing equation is a unit impulse disturbance.

3. The desired value of the output from the system is constant.

4. Only simple proportional control is considered.

In the analysis the following procedure was followed:

1. The integral-square value of the output was calculated by Parseval's theorem.

2. Only the first order Padé approximation for an exponential function:

\[ e^{-x} = \frac{1 - \frac{1}{2}x}{1 + \frac{1}{2}x} \]

was used in the transfer function for the time lag in the study of each system except

* For a case in which the controlled system also contains a time lag see Appendix VII.
the simplest one. For the simplest system that is described in Chapter I, the approximation

\[ e^{-x} = \frac{1 - \frac{1}{2}x + \frac{x^2}{12}}{1 + \frac{1}{2}x + \frac{x^2}{12}} \]

was employed.

3. The results obtained by these approximate methods were checked by means of exact calculations using the residue theorem.
CHAPTER I

STUDY OF THE SIMPLEST SYSTEM

The first system considered in this analysis has the following properties.

1. The level of the liquid in the tank illustrated in Fig. 1.1 is controlled by a proportional controller having a time lag.
2. This device controls the inlet flow by changing the valve position by an amount proportional to the deviation.
3. The cross sectional area of the tank is constant.

![Diagram of the Simplest System](image)

Figure 1.1 The Simplest System

The purpose of this chapter is to show how the procedure described in the previous section was applied to a practical problem. The following steps were executed:
(1) A differential equation for the system was derived.

(2) Taking the Laplace transform of the differential equation, the relation between a disturbance and an output was expressed by a transfer function.

(3) Assuming a unit impulse to be the disturbance, the Padé approximation was used for the exponential function. The output could then be expressed in the form of the ratio of polynomials.

(4) By use of Parseval's theorem the integral square error of the output was calculated, and the values of the proportional factor that makes this integral square error a minimum was obtained.

(5) In order to check the result obtained by this approximate method the differential equation for the system was solved numerically and the integral square error was calculated.

Explanations for these steps are as follows. Here numbers (1), (2) .....(5) indicate that descriptions following them are the specific example of general descriptions given previously.

(1) Equations for this system are:

\[
A \frac{dh(t)}{dt} = qi(t) - qo(t)
\]

(1.1)
\[ q_0(t) = \frac{1}{R} (h(t) - h_i(t)) \quad (1.2) \]

\[ q_i(t) = -K_c h(t-L) \quad (1.3) \]

where

- \( h \) = deviation of the liquid level from the desired level ft
- \( h_i \) = pressure at the outlet ft
- \( q_i \) = inlet flow rate \( ft^3/sec \)
- \( q_o \) = outlet flow rate \( ft^3/sec \)
- \( A \) = cross sectional area of the tank \( ft^2 \)
- \( R \) = resistance of the pipe \( sec/ft^2 \)
- \( K_c \) = proportional control factor \( ft^2/sec \)
- \( L \) = time lag sec

The terms \( h_i, q_i, \) and \( q_o \) represent deviations from steady state values. From equations (1.1), (1.2) and (1.3) the equation

\[
\frac{dh(t)}{dt} + Kh(t-L) = \frac{1}{AR} (h_i(t) - h(t))
\]

was obtained, where

\[ K = \frac{K_c}{A} \]

In many cases, the effect of the deviation of the liquid level on the change of the outlet flow rate is negligible compared with the forcing term \( h_i \). Equation (1.4) is then written as follows.
\[ \frac{dh(t)}{dt} + Kh(t-L) = \frac{1}{AR} h_1(t) \]  

(1.4')

(2) The Laplace transform of equation (1.4') is,

\[ (s + Ke^{-Ls})H(s) - \frac{1}{AR} H_1(s) \]

or

\[ \frac{H(s)}{H_1(s)} = \frac{1}{AR(s + Ke^{-Ls})} \]  

(1.5)

where \( H(s) \) and \( H_1(s) \) are the Laplace transforms of \( h(t) \) and \( h_1(t) \) respectively.

(3) The object of our discussion is to find the value of \( K \) that makes the integral square value of the time function \( h(t) \)

\[ I = \int_{-\infty}^{t_0} h^2(t) \, dt \]  

(1.6)

a minimum for a given time lag. Here the unit impulse disturbance \( H_1(s) = 1 \) was applied. The ideal output is zero, and the condition that the integral square value of the error be a minimum is assumed to be the condition of optimum control. Then the value of \( K \) afforded by the above method should indicate the optimum controller setting when the unit impulse disturbance occurs. Since the system is assumed to be linear, this is the optimum setting for any impulse input.
(4) By Parseval's theorem, equation (1.6) was expressed in the following form,

\[ I = \frac{1}{2\pi j} \int_{-j \infty}^{j \infty} H(s) H(-s) \, ds \]  

(1.7)

(Bibliography 5, page 44) In case \( H(s) \) is expressed in the form

\[ H(s) = \frac{c(s)}{d(s)} \]  

(1.8)

where

\[ c(s) = c_{n-1} s^{n-1} + c_{n-2} s^{n-2} + \ldots + c_0 \]  

(1.9)

\[ d(s) = d_n s^n + d_{n-1} s^{n-1} + \ldots + d_0 \]  

(1.10)

and \( n \) is a positive integer, then the value of \( I \) is described in terms of \( c_n \) and \( d_n \). That is,

\[ I_1 = \frac{c_0^2}{2d_0 d_1} \]

\[ I_2 = \frac{c_1^2 d_0 + c_0^2 d_2}{2d_0 d_1 d_2} \]

\[ I_3 = \frac{c_2^2 d_0 d_1 + (c_1^2 - 2c_0 c_2) d_0 d_3 + c_0^2 d_2 d_3}{2d_0 d_3 (-d_0 d_3 + d_1 d_2)} \]

\[ \vdots \]

\[ \vdots \]

Here \( d(s) \) has zeros in the left half-plane only.
In our case

\[ H(s) = \frac{1}{AR \left(s + Ke^{-sL}\right)} \quad (1.11) \]

In order to obtain \( H(s) \) in the form of (1.8), \( e^{-sL} \) should be expanded into a polynomial.

For this purpose the Padé approximation for \( e^{-x} \) furnishes a particular simple algebraic function. If the following form is chosen,

\[ e^{-x} = \frac{1 - \frac{1}{2}x + \frac{x^2}{12}}{1 + \frac{1}{2}x + \frac{x^2}{12}} \quad (1.12) \]

\( H(s) \) becomes

\[ H(s) = \frac{1}{AR} \cdot \frac{1 + \frac{L}{2}s + \frac{L^2}{12}s^2}{K + (1 - \frac{KL}{2})s + \frac{1}{2}(1 + \frac{KL}{6})Ls^2 + \frac{L^2}{12}s^3} \quad (1.13) \]

Comparing terms in (1.8), (1.9) and (1.10) to those in (1.13) shows that

\[ c_0 = 1, \quad c_1 = \frac{L}{2}, \quad c_2 = \frac{L^2}{12} \]

\[ d_0 = K, \quad d_1 = 1 - \frac{KL}{2}, \quad d_2 = \frac{1}{2}(1 + \frac{KL}{6})L \]

\[ d_3 = \frac{L^2}{12} \]

Thus

\[ I_3 = \frac{1}{AR} \cdot \frac{12 + 6KL - K^2L^2}{2K \left(12 - 6KL - K^2L^2\right)} \quad (1.14) \]
I₃ assumes the minimum value \(1.531\frac{L}{AR}\) when \(KL = 0.74\).

(5) Since this result was obtained by the approximation for \(e^{-Ls}\), its accuracy should be checked. For this purpose equation (1.4') was solved and (1.6) was calculated numerically by digital computer.

(see Appendix V) In equation (1.6)

\[
I = \int_{-\infty}^{+\infty} h^2(t) \, dt = \int_0^{+\infty} h^2(t) \, dt
\]

since

\[h(t) = 0 \quad \text{for } t < 0.\]

The upper limit, \(\infty\), of the integral was replaced by \(t_f\) where \(h^2(t)\) was negligibly small for all values of \(t \geq t_f\). This method indicated that \(I_3\) takes a minimum value \(1.439\frac{L}{AR}\) when \(KL = 0.74\).

So it was seen that the calculations obtained by the Padé approximation for an exponential function gave a very accurate value of the \(K\) that makes \(I_3\) a minimum. \(I_3\) calculated by the approximate method was about 6.3% in error. Therefore it is concluded that for the system described by the equation

\[
\frac{dh(t)}{dt} + Kh(t-L) = \frac{1}{AR} h_1(t), \quad (1.4')
\]
the optimum proportional controller setting is given by

$$ K = \frac{0.74}{L} $$

for a unit impulse disturbance. As expected, different orders and forms of Padé approximations for the exponential function give different errors. (These are summarized in Appendix II.)
CHAPTER II
STUDY OF MORE COMPLICATED SYSTEMS

In Chapter I, the simplest case in which the relationship between input $H_1(s)$ and output $H(s)$ is expressed by

$$\frac{H(s)}{H_1(s)} = \frac{1}{AR} \frac{1}{s + Ke^{-SL}}$$

was studied. In this chapter the more complicated systems as follows were investigated.

$$\frac{H(s)}{H_1(s)} = \frac{1}{1 + \alpha s + Ke^{-SL}} \quad (2.1)$$

$$\frac{H(s)}{H_1(s)} = \frac{1}{(\alpha s + 1) s + Ke^{-SL}} \quad (2.2)$$

$$\frac{H(s)}{H_1(s)} = \frac{\gamma s + 1}{\alpha s^2 + \beta s + 1 + Ke^{-SL}} \quad (2.3)$$

where $\alpha$, $\beta$ and $\gamma$ are system constants. However instead of solving the differential equation numerically (the method described in (5) in Chapter I), the residue theorem was used for the calculation of the integral square error as a check on the accuracy of the approximation.

Case 2.1 $\frac{H(s)}{H_1(s)} = \frac{1}{1 + \alpha s + Ke^{-SL}}$ \quad (2.1)

Here again numbers (2) ..... (5) indicate that descriptions following them are the specific example of general descriptions given previously (page 10).
(2) The transfer function is described above.

(3) The Padé approximation for the exponential

\[ e^{-x} = \frac{1 - \frac{x}{2}}{1 + \frac{x}{2}} \quad (2.4) \]

was adopted. As in the previous case,

\[ H_1(s) = 1 \]

\[ H(s) = \frac{1 + \frac{L}{2} s}{(1 + K) + (\alpha + \frac{L}{2} - \frac{KL}{2}) s + \frac{\alpha L}{2} s^2} \quad (2.5) \]

So that,

\[ I_{2L} = \frac{1}{2a} \cdot \frac{\frac{1}{2}(1 + K) + a}{(1 + K) \left\{ \frac{1}{2}(1 - K) + a \right\}} \quad (2.6) \]

where

\[ \frac{\alpha}{L} = a, \]

and

\[ \frac{dI_2}{dk} = 0 \]

for

\[ K = -(1 + 2a) + 2 \sqrt{a (1 + 2a)} \quad (2.7) \]
The term $\frac{dI_2}{dk}$ changes its sign from minus to plus at this value of $K$ for which $I_2 \geq 0$. This means $I_2$ becomes a minimum at this value of $K$. When $a = 1$,

$$K = 0.464$$  \hspace{1cm} (2.8)

and

$$I_2 \min = 0.465$$  \hspace{1cm} (2.9)

(5) In order to examine the accuracy of this result,

$$I = \int_0^\infty h^2(t) \, dt$$

should be calculated. The above integral was evaluated by means of the residue theorem.

Residues were calculated in the following way.

In equation (1.7),

$$I = \frac{1}{2\pi i} \int_{-j\infty}^{+j\infty} H(s) H(-s) \, ds$$

$$= \lim_{R \to \infty} \frac{1}{2\pi i Lj} \int_0^{-2\pi} H(\frac{Z}{L}) H(-\frac{Z}{L}) dZ$$

$$- \lim_{R \to \infty} \frac{1}{2\pi i Lj} \int_{\theta = \frac{\pi}{2}}^{\theta = \frac{\pi}{2} + \pi} H(\frac{Z}{L}) H(-\frac{Z}{L}) dZ$$  \hspace{1cm} (2.10)

where

$$s_L = Z$$

and $c_R$ is a semi-circle in the right-half plane with center at the origin and diameter $2R$. (Fig 2·1)
Fig. 2.1 Z plane

Since

$$H(s) = \frac{1}{1 + \alpha s + Ke^{-sL}}$$

then,

$$H(s) = H\left(\frac{Z}{L}\right) = \frac{1}{1 + aZ + Ke^{-Z}}$$

where

$$a = \frac{\alpha}{L}$$

In equation (2.10)

$$\lim_{R \to \infty} \frac{1}{2\pi L} \int_{\theta}^{\theta} \frac{\theta - \frac{\pi}{2}}{H\left(\frac{Z}{L}\right)H\left(-\frac{Z}{L}\right)}d\theta$$

$$\theta = \frac{\pi}{2}$$
Consequently,

\[
I = \lim_{R \to \infty} \frac{1}{2\pi LR} \int_{C_R} \frac{dz}{(1 + aZ + Ke^Z)(1 - aZ + Ke^Z)}
\]

\[= 0\]  
(see Appendix III)

where \(P_i\) is the \(i\)th pole of \(H(Z)H(-Z)\), and \(\text{Res} (P_i)\) is the residue at \(Z = P_i\). Thus, \(P_i\) is a root of

\[f (-Z) \equiv 1 - aZ + Ke^Z = 0 \quad (2.13')\]

since

\[f (Z) \equiv 1 + aZ + Ke^Z \quad (2.13)\]

has zeros in the left half-plane only.

Now equation (1.7) is expressed as

\[
I = \frac{1}{2\pi LR} \int_{C_R} \frac{dz}{f(Z)f(-Z)}
\]

\[\quad (2.14)\]

therefore,
Let $Z_i$ be a root of $f(Z) = 0$. Then $-Z_i$ is the root of $f(-Z) = 0$. Since $f(Z)$ has zeros in the left-half plane only, residue should be calculated only for the zeros of $f(-Z)$. This fact is shown in Fig. 2.2.

![Z plane diagram](image)

Then, in equation (2,15)

$$\text{Res}(P_i) = \left( \frac{1}{\frac{d}{dZ} \{f(Z) f(-Z)\}} \right)_{Z = Z_i}$$

(2,15')

* $\left( \frac{d}{dZ} \{f(Z) f(-Z)\} \right)_{Z = P_i} \neq 0$ is required
Since
\[ f_\cdot (Z_i) = 0, \]
\[ \left( \frac{d \{ f(Z) f(-Z) \}}{dZ} \right)_Z = -Z_i \]
\[ + f(-Z_i) \left( \frac{d f(-Z)}{dZ} \right)_Z = -Z_i \]
\[ = f(-Z_i) \left( \frac{d f(-Z)}{dZ} \right)_Z = -Z_i \]
\[ = (1 - a Z_i + Ke^{Z_i}) (-a + Ke^{-Z_i}) \]
\[ \equiv g(Z_i) \quad (2.16) \]

For the purpose of obtaining the sum of the residues, the property that the sum of two conjugate complex numbers is real should be made use of.

The term \( f(-Z) \) has complex conjugate poles. Let \( P_{i+1} = P_i \) where \( i \) is an odd integer. From equation (2.12)
\[ IL = - \{ \text{Res} (P_{i+1}) + \text{Res} (P_2) \} - \{ \text{Res} (P_3) + \text{Res} (P_4) \} + \ldots \]
\[ = -\{\text{Res}(P_1) + \text{Res}(P_{1})\} - \{\text{Res} (P_3) + \text{Res} (P_3)\} + \ldots \]
\[ (2.17) \]

This is an infinite series; however, it is expected that in absolute value \( \text{Res} (P_1) \) and \( \text{Res} (P_{1}) \) are very large compared with the rest of the terms.

As an example Table 2.1 shows this fact in case
a = 1.

Table 2.1 Sum of the Conjugate Residues

<table>
<thead>
<tr>
<th>i</th>
<th>Z_i</th>
<th>Res (-Z_i) + Res (-Z_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1.15 ± 1.39 j</td>
<td>-0.43109</td>
</tr>
<tr>
<td>3</td>
<td>-2.85 ± 7.42 j</td>
<td>0.02659</td>
</tr>
<tr>
<td>5</td>
<td>-3.40 ± 13.6 j</td>
<td>0.00937</td>
</tr>
<tr>
<td>7</td>
<td>-3.70 ± 19.5 j</td>
<td>0.00481</td>
</tr>
<tr>
<td>9</td>
<td>-4.05 ± 26.5 j</td>
<td>-0.00265</td>
</tr>
</tbody>
</table>

Formulas for calculation of residues are shown as follows:

\[
\text{Res (} P_i \text{)} + \text{Res (} P_{i+1} \text{)} = \text{Res (} -Z_i \text{)} + \text{Res (} -Z_i \text{)}
\]

\[
= \frac{2 \text{Real}\{g(Z_i)\}}{|g(Z_i)|^2}
\]

\[
= \frac{2(AC - BD)}{(AC - BD)^2 + (BC + AD)^2}
\]

(2.18)

where

\[A = 1 - aU_i - \frac{k^2(1 + aU_i)}{(1 + aU_i)^2 + (aV_i)^2}\]

\[B = -aV_i + \frac{k^2V_i}{(1 + aU_i)^2 + (aV_i)^2}\]

\[C = -(1 + a + aU_i)\]

\[D = -aV_i\]

and

\[Z_i = U_i + jV_i\]

Obtain \(U_i\) and \(V_i\) from
\[ f \left( Z^i \right) = 1 + aZ^i + Ke^{-Z^i} = 0, \]
(see Formula 3 in Appendix IV) and calculate by
(2.17) for \( K = 0.464 \), \( a = 1 \) as in previous example

equation (2.8)

\[ (IL)_{\text{min}} = 0.475. \]

Comparison of this value with (2.9),

\[ (I_{2L})_{\text{min}} = 0.465 \]

shows that the use of Padé approximation resulted
in about 2% error in the magnitude of \((I_{2L})_{\text{min}}\).
Though the Padé approximation (2.4) is less
accurate than the one which was used in the
previous case (1.12), the error caused in this
case (2.1) is smaller. This means that in this
case the controlled system has a self-regulating
tendency, that is in terms of equation (1.4),
the term \( h(t) \) is not negligibly small.

\[
\text{Case 2.2} \quad \frac{H(s)}{H_1(s)} = \frac{1}{(1 + \chi s) s + Ke^{-sL}}
\]

(2.2)

Here again numbers (2) .... (5) indicate that
descriptions following them are the specific ex-
ample of general descriptions given previously
(page 10).

(2) To the transfer function (2.2),
(3) unit step function \( H_1(s) = 1 \) was applied.

The Padé approximation for the exponential function,
\[
e^{-x} = \frac{1 - \frac{1}{2} x}{1 + \frac{1}{2} x}
\]
was used.

So that,
\[
H(s) = \frac{1 + \frac{L}{2} s}{K + (1 - \frac{KL}{2}) s + (\alpha + \frac{L}{2}) s^2 + \alpha \frac{L}{2} s^3}
\]

(4) By Parseval's theorem
\[
\frac{I_3}{L} = \frac{(a + \frac{1}{2}) + \frac{1}{4} KL}{2 KL \left( (a + \frac{1}{2}) - (a + \frac{1}{4}) KL \right)}
\]
was obtained. \( I_3 \) takes a minimum value for
\[
KL = 2(2a + 1) \left[ -1 + \sqrt{\frac{2(2a + 1)}{4a + 1}} \right]
\]
where \( a = \frac{\alpha}{L} \);
\[
\left( \frac{I_3}{L} \right)_{\text{min}} = 1.83 \quad (2.20)
\]
for \( KL = 0.572 \) \quad (2.21)
in case \( a = 1 \)

(5) Residues are derived in the following way
I = \int_{-\infty}^{\infty} h^2(t) \, dt

= \frac{1}{2\pi} \int_{-j\infty}^{j\infty} H(s) H(-s) \, ds

= \frac{1}{2\pi} \int_{-j\infty}^{j\infty} \left( \frac{L}{(1 - aZ) Z + Ke^{-Z}} \right) \, \frac{dz}{L}

= \frac{1}{2\pi} (-2\pi j \sum_{i=1}^{\infty} \text{Res} \left( P_i \right))

(For calculations of residue see Formula 1 in Appendix IV)

where \( a = \frac{\alpha}{L} \)

\[ Z = sL \]

and \( P_i \) is the pole of

\[ \frac{1}{\{ (1 + aZ) Z + Ke^{-Z} \} \{ (1 - aZ) Z + Ke^{-Z} \}} \]

For the value of \( K \) in (2.21), residues were calculated, which gave

\[ \left( \frac{I}{L} \right)_{\text{min}} = 1.87 \]  \hspace{1cm} (2.22)

for

\[ KL = 0.572 \]  \hspace{1cm} (2.21)

The error in \( \left( \frac{I}{L} \right)_{\text{min}} \) is 2\%.
Case 2.3

\[ \frac{H(s)}{H_1(s)} = \frac{\gamma s + 1}{\alpha s^2 + \beta s + 1 + Ke^{-sL}} \]  

(2.3)

In the same manner as in the previous case;

(2) \[ \frac{H(s)}{H_1(s)} = \frac{\gamma s + 1}{\alpha s^2 + \beta s + 1 + Ke^{-sL}} \]  

(2.3)

(3) \[ H_1(s) = 1 \]

\[ e^{-x} = \frac{1 - \frac{1}{x}}{2} \]

\[ 1 + \frac{1}{x} \]

\[ H(s) = \frac{1 + (\frac{\gamma + \frac{L}{2}}{s}) + \frac{L}{2} s^2}{(K + 1) + (\beta + \frac{L-KL}{2}) s + (\alpha + \frac{KL}{2}) s^2 + \frac{KL}{2} s^3} \]

(4) By Parsaval's theorem,

\[ I_3L = \frac{c^2}{4} (K + 1)(b + \frac{1-K}{2}) + (c^2 + \frac{1}{4})(K + 1) \frac{a}{2} + (a + b) \frac{a}{2} \]

\[ (K + 1)a \left[ - (K+1) \frac{a}{2} + (b + \frac{1-K}{2})(a + b) \frac{a}{2} \right] \]

(2.23)

where

\[ a = \frac{\alpha}{L^2} \]

\[ b = \frac{\beta}{L} \]

\[ c = \frac{\gamma}{L} \]
In case

\[ a = b = 1 \]
\[ c = 0.1 \]
\[ K = 0.10 \] (2.24)

\[ (I_{3L})_{\text{min}} = 0.501 \] (2.25)

were obtained.

(5) By the exact calculation using the residue theorem,

(see Formula 2 in Appendix IV), for the value of

K in (2.24)

\[ K = 0.10 \] (2.24)

\[ (I_{3L})_{\text{min}} = 0.503 \]

The error in \((I_{3L})\) is 0.2%.
CHAPTER III

EXAMPLE

In order to illustrate how the method described in the previous chapters should be applied, the speed control of the induction servomotor was taken as an example. The schematic diagram of this system is shown in Fig. 3-1.

Fig. 3.1 Schematic diagram of the system

The following things should be noticed.

1. The speed of the servomotor is detected by the tachometer. This signal is transported through the potentiometer to the magnetic amplifier having a time lag.
2. The output torque from the servomotor is considered to be proportional to the current in the field circuit, where the amount of current is varied depending on the input voltage.

3. The spring coefficient of the motor is assumed to be negligible.

4. The value of the proportional control factor is changed continuously by the potentiometer.

The values of system parameters $a$, $b$, $c$, and $K$ were obtained as follows:

Equations for this system are,

\[ e(t) = R_f i_f(t) + L_f \frac{di_f(t)}{dt} \]  
(3.1)

\[ T(t) = K_f i(t) \]  
(3.2)

\[ T(t) - D(t) = J \frac{d^2 \theta(t)}{dt} + f_c \frac{d \theta(t)}{dt} \]  
(3.3)

\[ e(t) = -K_t K_p K_a \dot{\theta}(t-L) \]  
(3.4)

where

- $R_f$ = resistance in the field circuit  \( \Omega \)
- $L_f$ = inductance in the field circuit  \( H \)
- $K_f$ = constant in the field circuit  \( \text{lb} \cdot \text{ft.}/\text{amp} \)
- $e$ = voltage in the field circuit  \( \text{volt} \)
- $i$ = current in the field circuit  \( \text{amp} \)
- $T$ = torque of the servomotor  \( \text{lb} \cdot \text{ft.} \)
\[ J = \text{moment of inertia of the servomotor} \quad \text{lb. ft. sec.}^2 \]

\[ f_c = \text{friction coefficient of the servomotor} \quad \text{lb. ft. sec.} \]

\[ \theta = \text{output angle of the servomotor} \quad \text{rad.} \]

\[ \dot{\theta} = \text{output speed of the servomotor} \quad \text{rad./sec.} \]

\[ D(t) = \text{disturbance by the change of the torque from the load} \quad \text{lb. ft.} \]

\[ K_t = \text{constant of the tachometer} \quad \text{ft./rad./sec.} \]

\[ K_a = \text{gain factor of the amplifier} \quad \text{volt/volt} \]

\[ K_p = \text{gain factor of the potentiometer} \quad \text{volt/ft.} \]

The Laplace transformation of equations (3.1) through (3.4) yields after rearranging terms.

\[
\frac{\dot{\theta}(s)}{D(s)} = \frac{1}{f_c} \frac{C_f s + 1}{C_m C_f s^2 + (C_m + C_f) s + 1 + K e^{-L s}}
\]

\[ (3.5) \]

where

\[ C_m = \frac{J}{f_c} \]

\[ C_f = \frac{L_f}{R_f} \]

\[ K = \frac{K_f K_p K_t K_a}{f_c R_f} \]

and \( \dot{\theta}(s) \) and \( D(s) \) are the Laplace transforms of \( \dot{\theta}(t) \) and \( D(t) \). Equation (3.5) has the same form of the transfer function as case 2.3 in Chapter II. The
The block diagram of the system is shown in Fig. 3.2.

\[
\frac{\dot{\theta}(Z)}{D(Z)} = \frac{1}{f_c} \cdot \frac{cZ + 1}{aZ^2 + bZ + 1 + Ke^{-Z}}
\]

where

\[
a = \left(\frac{C_m}{L}\right)\left(\frac{C_f}{L}\right)
\]

\[
b = \frac{C_m}{L} + \frac{C_f}{L}
\]

The time lag between each block is considered to be small compared with that of the amplifier.

Equation (3.5) is expressed as

Fig. 3.2 Block diagram of the servomotor
Numerical values for $a$, $b$, $c$, and $K$ were calculated for a 60 cycle induction servomotor\(^{(1)}\) a servo-amplifier\(^{(2)}\) and a servomotor tachometer\(^{(3)}\).

The following data were available in the catalogues.

- **frequency** $f = 60$ cycle
- **maximum voltage** $E_{\text{max}} = 115$ volt
- **rotor inertia** $J = 4.0$ g-cm\(^2\)
- **time constant** $\tau_m = 0.00584$ sec.
- **resistance in the field** $R_f = 803$ Ω
- **inductance in the field** $2\pi f L = 790$ Ω
- **motor characteristics** Fig. 3.3.

\(^{(1)}\) Motor Type R160-2, Kearfott, Bibliography 4.
\(^{(2)}\) Type A3300-01, Kearfott, Bibliography 4.
\(^{(3)}\) Type M102, Servomechanism Inc. Bibliography 8.
From these data the following values are obtained:

\[
L_f = \frac{790}{2 \pi (60')} = 2.10 \text{ henry}
\]

\[
\tau_f = \frac{R_f}{L_f} = 2.61 \times 10^{-3} \text{ sec}
\]

\[
f_c = \frac{J}{\tau_m} = 680 \text{ g.cm}^2/\text{sec.}
\]

\[K_f\] was obtained as follows:

\[T(t) = K_f i(t)\] (3.2)

Taking the Laplace transform

\[T(s) = K_f I(s)\] (3.7)
where \( T(s) \) and \( I(s) \) are the Laplace transforms of \( T(t) \) and \( i(t) \) respectively. Since

\[
\frac{I(s)}{E(s)} = \frac{1}{Lfs + Rf} \quad (3.8)
\]

\[
\frac{\dot{\Theta}(s)}{T(s)} = \frac{1}{Js + f_c} \quad (3.9)
\]

then

from equations (3.7), (3.8) and (3.9)

\[
\frac{\dot{\Theta}(s)}{E(s)} = \frac{K_f}{(Lfs + Rf)(Js + f_c)}
\]

If a step input is applied, the final value of \( \dot{\Theta}(t) \) is given by,

\[
\lim_{t \to \infty} \dot{\Theta}(t) = \lim_{s \to 0} \frac{K_fs}{(Lfs + Rf)(Js + f_c)} \cdot 0.70 \frac{E_{max}}{s}
\]

\[
= 0.70 \frac{K_f E_{max}}{R_f f_c}
\]

Here it is assumed that 70 percent rated voltage is used for operation in the steady state.

From Fig. 3.3 \( \dot{\Theta}(t) \) corresponding to 70 percent rated voltage is 3300 r.p.m.
Then,

\[
0.70 \frac{K_f E_{\text{max}}}{R_{fC}} = 3300 \text{ r.p.m.}
\]

or

\[
\frac{K_f}{R_{fC}} = \frac{3300 \text{ r.p.m.}}{0.70 \times 115 \text{ volt}} = 41.0 \text{ r.p.m./volt}
\]  \hspace{1cm} (3.10)

For the amplifier

\[
K_a = 80 \text{ volt/volt}
\]  \hspace{1cm} (3.11)

and for the tachometer,

\[
K_t = 6 \text{ volt/1000r.p.m.} = 6 \times 10^{-3} \text{ volt/r.p.m.}
\]  \hspace{1cm} (3.12)

The time lag \( L \) in the magnetic amplifier is one half of the period of a cycle of the applied voltage.

\[
L = \frac{1}{2f} = \frac{1}{120} \text{ sec.}
\]  \hspace{1cm} (3.13)

Then from (3.6') and (3.13)

\[
a = \frac{C_m}{L} \frac{C_f}{L} = 0.22
\]

\[
b = \frac{C_m}{L} \frac{C_f}{L} = 1.0
\]  \hspace{1cm} (3.14)

\[
c = \frac{C_f}{L} = 0.31
\]

and from (3.10), (3.11) and (3.12)

\[
K = \frac{K_f K_t K_a K_p}{R_f R_{fC}}
\]

\[
= 19.7 K_p
\]  \hspace{1cm} (3.15)
Substituting (3.14) to (2.23), $I_3L$ is expressed in term of $K$. By numerical calculation it was found that $I_3L$ takes a minimum value when

$$K = 0.13$$  \hspace{1cm} (3.16)

From (3.15) and (3.16)

$$19.7K_p = 0.13$$

or

$$K_p = 0.064$$  \hspace{1cm} (3.17)

Thus (3.17) gives the value of the gain factor of the potentiometer which yields an optimum control system subjected to isolated impulse disturbances.
CONCLUSIONS

For the systems whose transfer functions were expressed by

\[
\frac{1}{Z + Ke^{-Z}}
\]

\[
\frac{1}{aZ + 1 + Ke^{-Z}}
\]

\[
\frac{1}{Z(aZ + 1) + Ke^{-Z}}
\]

optimum values of K were expressed in terms of a system parameter a.

For the system

\[
\frac{cZ + 1}{aZ^2 + bZ + 1 + Ke^{-Z}}
\]

it was impossible to express the optimum value of K in terms of a, b, and c. In this case K should be calculated numerically for different values of each of the parameters a, b, and c.

Padé approximation for the exponential function shown in Table A1.1 in Appendix I gave accurate results for both \(I_{\text{min}}\) and the optimum value of K. It was shown that the accuracy remained good for every possible combination of values of a, b, and c in a range from 0.1 to 100 for each parameter.
ACKNOWLEDGEMENTS

The author wishes to acknowledge the aid of Mr. H. M. Bourland on the application of this theory to the servomotor. The author's acknowledgements are due also to Messrs. C. Kaplan and J. C. Smithson for their correcting mistakes in English usage. The author's most sincere gratitude goes to Dr. Sam. H. Davis whose guidance was invaluable.
BIBLIOGRAPHY


# APPENDIX I

## TABULATION OF RESULTS

<table>
<thead>
<tr>
<th>Chapter or case</th>
<th>transfer function</th>
<th>Pade Approximation</th>
<th>optimum setting</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{Z + Ke^{-Z}} )</td>
<td>( e^{-x} = \frac{1 - \frac{x}{2} + \frac{x^2}{12}}{1 + \frac{x}{2} + \frac{x^2}{12}} )</td>
<td>KL = 0.74</td>
</tr>
<tr>
<td>2.1</td>
<td>( \frac{1}{aZ + 1 + Ke^{-Z}} )</td>
<td>( e^{-x} = \frac{1 - \frac{x}{2}}{1 + \frac{x}{2}} )</td>
<td>K = 2\sqrt{\frac{a(2a+1)}{(2a+1)}}</td>
</tr>
<tr>
<td>2.2</td>
<td>( \frac{1}{(aZ + 1)Z + Ke^{-Z}} )</td>
<td>&quot;</td>
<td>KL = 2(2a+1) \left[ \frac{\sqrt{2(a+1)}}{4a+1} \right] -1</td>
</tr>
<tr>
<td>2.3</td>
<td>( \frac{aZ + 1}{aZ^2 + bZ + 1 + Ke^{-Z}} )</td>
<td>&quot;</td>
<td>Not obtainable in the form of K = K(a,b,c)</td>
</tr>
</tbody>
</table>

Table A1.1 Table of Results
Table A1.1  Table of Results (continued)

(1) If the error of $I_{\text{min}}$ is small, the error of $K_{\text{optimum}}$ is small. This is shown in Table A.2 in Appendix II. Another example of this fact is shown in this system

$$\frac{cz + 1}{az^2 + bz + 1 + Ke^{-Z}}$$

in Table A1.2

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$a = 10$</th>
<th>$b = 1$</th>
<th>$c = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>0</td>
<td>0.4995</td>
<td>0.4986</td>
</tr>
<tr>
<td>0.01</td>
<td>0.4995</td>
<td>0.5005</td>
<td></td>
</tr>
</tbody>
</table>

Table A1.2  Optimum Value of $K$

* Integral square error for the particular value of the parameters shown in the last column.
By the calculation using Padé approximation

\[ K = 0.01 \]

was obtained as an optimum value. As is shown in Table A1.2, this was proved to be correct.
APPENDIX II

FORMS OF PADÉ APPROXIMATIONS

There are various kinds of forms in Padé approximations for the exponential function $e^{-x}$. Errors caused by use of these are presented here.

For analysis of the system

$$\frac{H(s)}{H_1(s)} = \frac{1}{AK(s + Ke^{-sL})} \quad (1.5)$$

the following two types of the Padé approximation were used

(A2.1)

$$e^{-x} = 1 - x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!}$$

(A2.2)

$$e^{-x} = \frac{1 - \frac{1}{2}x + \frac{x^2}{12}}{1 + \frac{1}{2}x + \frac{x^2}{12}}$$

The error of the integral-square value of $h(t)$ caused by these approximations are shown in Table A2.1.
### Table A2.1 Error caused by Padé approximations

<table>
<thead>
<tr>
<th>Padé approximations</th>
<th>error of KL</th>
<th>error of $I_{\text{min}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{-x} = 1 - x$</td>
<td>34 %</td>
<td>34 %</td>
</tr>
<tr>
<td>$e^{-x} = 1 - x + \frac{x^2}{2}$</td>
<td>34</td>
<td>34</td>
</tr>
<tr>
<td>$e^{-x} = 1 - x + \frac{x^2}{2} + \frac{x^3}{3!}$</td>
<td>48</td>
<td>74</td>
</tr>
<tr>
<td>$e^{-x} = 1 - x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}$</td>
<td>8</td>
<td>7.2</td>
</tr>
<tr>
<td>$e^{-x} = \frac{1 - \frac{x^3}{2}}{1 + x}$</td>
<td>12</td>
<td>7.0</td>
</tr>
<tr>
<td>$e^{-x} = \frac{1 - \frac{1}{2}x + \frac{x^2}{12}}{1 + \frac{1}{2}x + \frac{x^2}{12}}$</td>
<td>0</td>
<td>6.5</td>
</tr>
</tbody>
</table>
APPENDIX III
A PROOF OF AN INTEGRATION

A proof of an equation (2.11)

\[
\lim_{R \to \infty} \frac{1}{2\pi i} \int_{\gamma_0} H(\frac{z}{L})H(-\frac{z}{L})dz = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{\gamma_0} \frac{dz}{(1 + aZ + Ke^Z)(1 - aZ + Ke^Z)} = 0
\]

(A3.1)

is presented here.

Instead of calculating an integration of \(H(\frac{z}{L})H(-\frac{z}{L})\) along a semi-circle \(C_{R_0}\) in Fig. A3.1, an integration along a path \(C\) in Fig. A3.2 was calculated.

In Fig. A3.2 a rectangle enclosed by the y axis and the path \(C\) contains a finite number of poles of \(H(\frac{z}{L})H(-\frac{z}{L})\). When the interger \(N\) approaches infinity,

\[
\lim_{N \to \infty} L = \lim_{N \to \infty} \frac{1}{2\pi i} \int_{\gamma_0} H(\frac{z}{L})H(-\frac{z}{L})dz = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{\gamma_0} H(\frac{z}{L})H(-\frac{z}{L})dz
\]

(A3.2)
Let \( I_c \) be

\[
I_c = I_{c_1} + I_{c_2} + I_{c_3}
\]

where

\[
I_{c_1} = \frac{1}{2\pi L_j} \int_{c_1} H\left(\frac{Z}{L}\right)H\left(-\frac{Z}{L}\right) dZ
\]

\[
I_{c_2} = \frac{1}{2\pi L_j} \int_{c_2} H\left(\frac{Z}{L}\right)H\left(-\frac{Z}{L}\right) dZ
\]

\[
I_{c_3} = \frac{1}{2\pi L_j} \int_{c_3} H\left(\frac{Z}{L}\right)H\left(-\frac{Z}{L}\right) dZ
\]

In the path \( C_1 \),

\[
y = (2N + 1)\pi
\]

as it is shown in Fig. A3.2.

So that,

\[
H\left(\frac{Z}{L}\right) = \frac{1}{1 + aZ + Ke^{-Z}}
\]

\[
= \frac{1}{(1 + ax - Ke^{-x}) + j \{a(2N + 1)\pi\}}
\]

and

\[
\left| H\left(\frac{Z}{L}\right) \right| \leq \frac{1}{a(2N + 1)\pi}
\]

Similarly,

\[
\left| H\left(-\frac{Z}{L}\right) \right| \leq \frac{1}{a(2N + 1)\pi}
\]

The absolute value of \( I_{c_1} \) is,

\[
|I_{c_1}| = \left| \frac{1}{2\pi L_j} \int_{0}^{(2N+1)\pi} H\left(\frac{Z}{L}\right)H\left(-\frac{Z}{L}\right) dZ \right|
\]

\[
\leq \left| \frac{1}{2\pi L_j} \int_{0}^{(2N+1)\pi} \left| H\left(\frac{Z}{L}\right) \right| \left| H\left(-\frac{Z}{L}\right) \right| dZ \right|
\]

\[
\leq \frac{1}{2\pi L} \int_{0}^{(2N+1)\pi} \left\{ \frac{1}{a(2N + 1)\pi} \right\}^2 dx
\]

\[
= \frac{1}{2\pi L} \cdot \frac{1}{a^2(2N + 1)\pi}
\]
In the path $C_3$

$$y = -(2N + 1)\pi$$  \hspace{1cm} (A3.5)

In the same way as the previous case,

$$|H\left(\frac{Z}{L}\right)| \leq \frac{1}{a(2N + 1)\pi}$$

$$|H\left(-\frac{Z}{L}\right)| \leq \frac{1}{a(2N + 1)\pi}$$

was derived.

Then the absolute value of $I_{C_3}$ is,

$$|I_{C_3}| = \left| \frac{1}{2\pi L} \int_{(2N+1)\pi}^{0} H\left(\frac{Z}{L}\right)H\left(-\frac{Z}{L}\right) dZ \right|$$

$$\leq \frac{1}{2\pi L} \int_{(2N+1)\pi}^{0} \left|H\left(\frac{Z}{L}\right)\right| \left|H\left(-\frac{Z}{L}\right)\right| d\xi$$

$$\leq \frac{1}{2\pi L} \frac{1}{a^2(2N + 1)\pi}$$  \hspace{1cm} (A3.6)

In the path $C_2$

$$x = (2N + 1)\pi$$  \hspace{1cm} (A3.7)

So that

$$H\left(\frac{Z}{L}\right) = \frac{1}{\left(1 + a(2N + 1)\pi + Ke^{-(2N+1)\pi}\cos y\right) + j\left(-ay + Ke^{-(2N+1)\pi}\sin y\right)}$$

$$\left|H\left(\frac{Z}{L}\right)\right| \leq \frac{1}{a(2N + 1)\pi}$$

only if $N$ is sufficiently large that $|Ke^{-(2N+1)\pi}| \leq 1$.

Now

$$H\left(-\frac{Z}{L}\right) = \frac{1}{\left(1 - a(2N + 1)\pi + Ke^{2(2N+1)\pi}\cos y\right) + j\left(-ay + Ke^{2(2N+1)\pi}\sin y\right)}$$

$$\left|H\left(-\frac{Z}{L}\right)\right| = \frac{1}{e^{(2N+1)\pi} \sqrt{K^2 + S_N}}$$

where,

$$S_N = \frac{1}{e^{2(2N+1)\pi}} \left[\left\{a(2N + 1)\pi - 1\right\}^2 + a^2y^2\right]$$

$$- \frac{2K}{e^{(2N+1)\pi}} \left[\left\{a(2N + 1)\pi - 1\right\}\cos y + ay\sin y\right]$$
Then,
\[
|\mathcal{S}_N| < \frac{1}{e^{2(2N+1)\pi}} \left[ \frac{a(2N+1)\pi - 1}{e^{2(2N+1)\pi}} + \{a(2N+1)\pi\}^2 \right] + \frac{2\sqrt{2}Ka(2N+1)\pi}{e^{(2N+1)\pi}}
\]
(A3.8)

if only \( N \) is sufficiently large that \( a(2N+1)\pi \geq 1 \).

Here the fact,\[-(2N+1)\pi \leq y \leq (2N+1)\pi\]
in the path \( C_2 \) was used. It is obvious that if \( N \) approaches infinity
\[
\lim_{N \to \infty} |\mathcal{S}_N| = 0.
\]
(A3.9)

So that,
\[
|I_{C_2}| = \left| \frac{1}{2\pi i} \int_{-\gamma(2N+1)\pi}^{\gamma(2N+1)\pi} \frac{\mathcal{H}(\frac{z}{\lambda})\mathcal{H}(\frac{-z}{\lambda})dz}{a(2N+1)\pi e^{2(2N+1)\pi} \sqrt{x^2 + \delta_N}} \right|
\]
\[
\leq \frac{1}{2\pi i} \int_{-\gamma(2N+1)\pi}^{\gamma(2N+1)\pi} \frac{1}{a(2N+1)\pi e^{2(2N+1)\pi} \sqrt{x^2 + \delta_N}} dy
\]
\[
= \frac{1}{2\pi i} \frac{2}{ae^{2(2N+1)\pi} \sqrt{x^2 + \delta_N}}
\]
(A3.10)

Since
\[
I_C = I_{C_1} + I_{C_2} + I_{C_3},
\]
the absolute value of \( I_C \) is
\[
|I_C| \leq |I_{C_1}| + |I_{C_2}| + |I_{C_3}|.
\]
From (A3.4), (A3.6) and (A3.10)
\[
|I_C| \leq \frac{1}{\pi L} \left\{ \frac{1}{a^2(2N+1)\pi} + \frac{1}{ae^{2(2N+1)\pi} \sqrt{x^2 + \delta_N}} \right\}
\]

Refering to (A3.9) yields
\[
\lim_{N \to \infty} |I_C| = 0.
\]
(A3.11)

(A3.11) and (A3.2) shows that (A3.1) is proved.
APPENDIX IV
CALCULATIONS BY THE RESIDUE THEOREM

Formulas for the calculations by the residue theorem are presented here.

Formula 1.  \[
\text{Residue of } \frac{H(Z)}{H_1(Z)} = \frac{1}{Z(1 + aZ) + Ke^{-Z}}
\]

The method described in (5) at case 2.1 in Chapter II was repeated.

\[
f(Z) = (1 + aZ)Z + Ke^{-Z} \quad \text{(A4.1)}
\]

\[
f(-Z_i)\left(\frac{df(-Z)}{dz}\right)_{Z=-z} = (-Z_i + aZ_i^2 + Ke^{-Z_i})(-1 - 2aZ_i + Ke^{-Z_i})
\]

\[
= g(Z_i) \quad \text{(A4.2)}
\]

The way of obtaining \(Z_i\), the root of \(f(Z) = 0\) is described in Formula 3 in Appendix IV.

\[
\text{Res } (-Z_i) = \frac{1}{g(Z_i)}
\]

\[
\text{Res } (-Z_i) + \text{Res } (-Z_i) = \frac{2 \text{ Real}\{g(Z_i)\}}{|g(Z_i)^2|}
\]

\[
= \frac{2(AC - BD)}{(AC - BD)^2 + (BC + AD)^2}
\]

where \(Z = U_i + jV_i\)
A = -U_i + a(U_i^2 - V_i^2) - \frac{K^2(1 + aU_i)U_i}{S} - aV_i^2

B = -1 + 2aU_i + \frac{K^2(1 + 2aU_i)}{S}

C = -1 - (2a + 1)U_i - a(U_i^2 - V_i^2)

D = -(2a + 1 + 2aU_i)V_i

S = (U_i^2 + V_i^2)\{ (1 + aU_i)^2 + (aV_i)^2 \}

Formula 2.

Residue of \( \frac{H(Z)}{L} \) = \frac{cz + 1}{az^2 + bZ + 1 + Ke^{-Z}}

By the same method as the previous case,

\[ f(Z) = az^2 + bZ + 1 + Ke^{-Z} \]  \hspace{1cm} (A4.3)

\[ f(-Z_i) \left( \frac{df(-Z_i)}{dZ} \right)_{Z=-Z_i} \]

\[ = (az_i^2 - bZ_i + 1 + Ke^{Z_i})(-2aZ_i - b + Ke^{-Z_i}) \]  \hspace{1cm} (A4.4)

\[ \text{Res}(-Z_i) = \frac{(cz_i + 1)(-cz_i + 1)}{(az_i^2 - bZ_i + 1 + Ke^{Z_i})(-2aZ_i - b + Ke^{-Z_i})} \]  \hspace{1cm} (A4.5)

Let

\[ g_1(Z_i) = az_i^2 - bZ_i + 1 + Ke^{Z_i} = A + Bj \]

\[ g_2(Z_i) = -2aZ_i - b + Ke^{-Z_i} = C + Dj \]

\[ g_3(Z_i) = (cz_i + 1)(-cz_i + 1) = E + Fj \]

Then,

\[ \text{Res}(-Z_i) + \text{Res}(-Z_i) \]

\[ = \frac{2 \text{Real} \{ g_3(Z_i) \overline{g_1(Z_i)} g_2(Z_i) \}}{|g_1(Z_i) g_2(Z_i)|^2} \]

\[ = \frac{2 \{ E(AC - BD) + F(BC + AD) \}}{(AC - BD)^2 + (BC + AD)^2} \]  \hspace{1cm} (A4.6)
Formula 3. Root of \( aZ^2 + bZ + c + Ke^Z = 0 \)

\[
f(Z) = aZ^2 + bZ + c + Ke^Z = 0 \tag{A4.7}
\]

is solved in the following way.

\[
Z = U + jV \tag{A4.8}
\]

Then

\[
a(U + jV)^2 + b(U + jV) + c + Ke^{-(U + jV)} = 0
\]

or

\[
bU + a(U^2 - V^2) + c = -Ke^{-U}\cos V \tag{A4.9}
\]

\[
v(b + 2aU) = Ke^{-U}\sin V \tag{A4.10}
\]

From (A4.9) and (A4.10)

\[
a^2(V^2)^2 + GV^2 + H = 0
\]

where

\[
G = (b^2 - 2ac) + 2abU + 2a^2U^2
\]

\[
H = (c + bU + aU^2) - K^2e^{-2U}
\]

Then

\[
V = \pm \frac{1}{a} \sqrt{\frac{1}{2} \left[ -G + \sqrt{G^2 - 4a^2H} \right]} \tag{A4.11}
\]

By using a computer, the value of \( V \) corresponding to \( U \) was calculated by this formula and \( U \) and \( V \) thus obtained were substituted into equations (A4.9) and (A4.10). \( U \) and \( V \) that satisfies both (A4.9) and (A4.10) are the solution of (A4.7).
APPENDIX V

NUMERICAL SOLUTION OF A DIFFERENTIAL EQUATION

Numerical solution of the differential equation (1.4)

\[ \frac{dh(t)}{dt} = -K(t) + \frac{1}{AR} h_1(t) \]  \hspace{1cm} (A5.1)

and calculation of

\[ \int_0^\infty h^2(t) dt \]

is presented here.

As \( \frac{1}{AR} h_1(t) \) is a unit impulse function and

\( h(t) = 0 \)

for

\( t < 0 \)

equation (A5.1) is equivalent to the following equations:

\[ \frac{dh(t)}{dt} = -K(t-L) \]

\( h(0) = 1 \)

\( h(t) = 0 \) for \( t < 0 \)  \hspace{1cm} (A5.2)

The Runge-Kutta method for the numerical solution for

the differential equation

\[ \frac{dy}{dx} = f(x) \]  \hspace{1cm} (A5.3)

was employed. Let \( \Delta x \) denote the interval between equidistant values of \( x \). Then if the arbitrary value of \( x \) is \( x_n \), the next increment in \( y \) is computed from the formula

\[ \Delta y \equiv \frac{\Delta x}{6} \left[ f(x_n) + 4f(x_n + \frac{\Delta x}{2}) + f(x_n + \Delta x) \right] \]  \hspace{1cm} (A5.4)
If $\Delta x$ is sufficiently small

$$\Delta y = \frac{\Delta x}{2} \left[ f(x_n) + f(x_n + \Delta x) \right]$$

(A5.5)

then

$$x_{n+1} = x_n + \Delta x$$

$$y_{n+1} = y_n + \Delta y$$

(A5.2) was solved by replacing $x$ with $t$, $y$ with $h(t)$ and $f(x)$ with $h(t-L)$ in equation (A5.5). That is,

$$\Delta h = \frac{\Delta t}{2} \left[ h(t_n - L) + h(t_n + \Delta t - L) \right]$$

$$t_{n+1} = t_n + \Delta t$$

$$h_{n+1} = h_n + \Delta h$$

(A5.6)

The value of

$$\int_0^\infty h^2(t)dt$$

was obtained by calculating

$$\sum_{t=0}^{t_f} \left\{ h_n(t) \right\}^2 (\Delta t)$$

where $\left\{ h_n(t) \right\}^2$ was sufficiently small for every value of $t \geq t_f$. 
APPENDIX VI

ERRORS CAUSED BY THE PÄDE APPROXIMATION

How errors caused by the Padé approximation for the exponential varies for the values of the system parameters is discussed here. Whether the parameters $a$, $b$ and $c$ can take the values shown in Fig. A6.1 or not is also discussed. Here $a$, $b$ and $c$ are the parameters in the transfer function,

$$
\frac{H(\frac{Z}{L})}{H_1(\frac{Z}{L})} = \frac{cz + 1}{az^2 + bz + 1 + Ke^{-Z}}
$$

(A6.1)

![Fig. A6.1 Values of system parameters](image)

Errors caused by using the Padé approximation for the exponential function are shown in Table A6.1.
Table A6.1 Table of errors

Several comments are mentioned about the following cases.

Case 2, 3 and 4

The combination of the values of a, b and c corresponding to cases 2, 3 and 4 is not physically possible for the actual system. The reason for this is as follows.

The example in chapter III shows that there are following relationships between the parameters a, b and c.

---

The combination of the values of a, b and c indicated is not physically possible.
\[ a = \left( \frac{C_m}{L} \right) \left( \frac{C_f}{L} \right) \]
\[ b = \frac{C_m}{L} + \frac{C_f}{L} \]
\[ c = \frac{C_f}{L} \quad (*) \quad (A6.2) \]

This relationship between \( a \) and \( b \) is seen in most of the systems whose characteristics are expressed by differential equations of the second order. (For example, Bibliography 9, page 97.) That is, \( a \) is the multiple of two time constants and \( b \) is the sum of them. Thus case 2 and 4 are not physically realizable. Likewise case 3 is unobtainable, because \( c \leq b \).

**Case 5**

In this case equation (A6.1) is expressed by the equation,
\[ H(\frac{Z}{L}) = \frac{0.1Z + 1}{0.1Z^2 + 100Z + 1 + Ke^{-Z}} \]
which may be approximated as
\[ \frac{H(\frac{Z}{L})}{H_1(\frac{Z}{L})} = \frac{1}{100(Z + \frac{K}{100}e^{-Z})} \]
This is the same form as the simplest case in chapter I.

**Case 1, 6, 7 and 8**

Using the residue theorem errors shown in Table A6.1 were obtained.

---

* The value of \( c \) varies depending on from where the disturbance comes into the system.
Errors caused by Pade approximations were shown in Fig. A6.2 when \( b = 1 \) and \( a \) and \( c \) take different values.

<table>
<thead>
<tr>
<th>( c )</th>
<th>1 %</th>
<th>0.01 %</th>
<th>0.1 %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>6</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>0.01</td>
<td></td>
<td>2</td>
<td>0.4</td>
</tr>
</tbody>
</table>

\[ \begin{array}{cccc}
0.1 & 1 & 10 & a \\
\end{array} \]

Fig. A6.2 Errors and system parameters \((b = 1)\)
APPENDIX VII

CONTROLLED SYSTEM WITH TIME LAG

A discussion is presented here in the case that there is a time lag in the controlled system.

The block diagram for a controlled system with a time lag is shown in Fig. A7.1.

\[
\begin{align*}
E_i(s) & \rightarrow G(s)e^{-sL_1} \rightarrow E_o(s) \\
& \bigg\downarrow \quad \bigg\uparrow \\
& \bigg\uparrow \quad \bigg\downarrow \\
& H(s)e^{-sL_2}
\end{align*}
\]

Fig. A7.1  Controlled system with time lag

In Fig. A7.1

\[
\begin{align*}
E_i(s) & = \text{input to the system} \\
E_o(s) & = \text{output from the system} \\
G(s) & = \text{controlled system} \\
H(s) & = \text{control system} \\
L_1 & = \text{time lag in a controlled system} \\
L_2 & = \text{time lag in a control system}
\end{align*}
\]

(A7.1)
Fig. A7.1 is redrawn as Fig. A7.2

\[ E_i(s) e^{-sL_1} + E_i'(s) - G(s) = E_o(s) \]

\[ H(s) e^{-s(L_1 + L_2)} \]

Fig. 7.2 Block diagram equivalent to Fig. A7.1

The integral-square value of \( E_o(s) \) corresponding to \( E_i(s) \) and \( E_i'(s) \) is the same for either case. By this reason, when a time lag exists in the controlled system, the same result may be obtained by treating \( L_1 + L_2 \) as a time lag in the control system.