A CALCULATION OF PRESSURE RECOVERY
BEHIND A NORMAL SHOCK IN
HYPERSONIC FLOW

by

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ABSTRACT

An analytical model is devised for investigating the flow immediately behind the normal shock which stands in the throat of the conical shock diffuser of a hypersonic wind tunnel.

A method is developed for calculating the flow properties, specifically the static and total pressures, for the case of large heat losses to the wall. An example is worked out to illustrate the method and some typical results.
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INTRODUCTION

Several considerations in the design of a hypersonic wind tunnel require a knowledge of the static and total pressures at various sections between the test section and the exhaust tank.

First, steady conditions must be achieved and maintained in the test section for as long as possible, and this time (the "running time") must be known. In order to achieve steady conditions in the test section at all, the flow must be choked at a point downstream from the test section so that the test section will be isolated from the changing conditions in the exhaust tank.

Since the ratio of such a choking area to the test section area depends on the ratio between the total pressures at these two sections, the behavior of the total pressures must be known to insure choking at the proper section. In order to calculate the running time, the "critical back pressure" (the maximum exhaust tank pressure for choked flow) must be known. This is because the running time is the length of time taken for the back pressure to increase (as mass flows into the closed tank) from its initial value (which is fixed by vacuum limitations) to the critical back pressure.

Second, it is desirable to maximize the test section Mach Number and area. This implies the test section should be minimized, since the test section mass flow density and temperature are fixed by other considerations. To what extent this pressure minimization is possible depends on the ratio of the total pressure in the test section to that in the exhaust tank.

Third, for comparative purposes, the diffuser efficiency should be calculated. This factor also depends on the ratio of the total pressure
in the test section to that in the exhaust tank. This ratio is called the "pressure recovery" of the diffuser.
Axially Symmetric
Not to Scale
**NOTATION**

A  cross-sectional area

c  local acoustic velocity

c_f  skin friction coefficient, \( \frac{\tau_w}{2\rho A} \)  (Fanning factor)

c_p  constant pressure specific heat

c_v  constant volume specific heat

°C  degrees Centigrade

D  diameter of circular cross-section or hydraulic diameter of annular cross-section, \( 4A/WP = D_i - D_o \)

D_i  inside diameter of annular cross-section

D_o  outside diameter of annular cross-section

f  pipe friction factor, \( 4\pi n/\rho A^2 \)  (d'Arcy factor)

°F  degrees Fahrenheit

G  mass flow density, \( \frac{\omega}{A} \)

g  standard acceleration of gravity, 32.2 ft/sec^2

\( g_c \)  dimensional constant (unit conversion factor), 32.2 \( \frac{lb \cdot ft}{lb \cdot sec^2} \)

h  enthalpy/unit mass

H  convective heat transfer coefficient

k  specific heat ratio, \( C_p/C_v \)

°K  degrees Kelvin

\( l \)  length, along duct axis, or a particular region, \( x_2 - x_1 \)

M  local Mach Number, \( \frac{\nu}{c} \)

Nu  Nusselt Number, \( HD/\lambda \)

P  Pressure

WP  wetted perimeter

Pr  Prandtl Number, \( \frac{\nu C_p}{\lambda} \)
q  heat added/unit mass
r  recovery factor, \(T_{aw}-T/To-T\)
R  gas constant, 386 lb\(_f\)-ft/lb \(\cdot \)°R for helium
°R degrees Rankine
Re  Reynolds Number,
S  Stanton Number
s  entropy/unit mass
t  time
t\(_o\)  running time of test
T  temperature
T\(_w\)  wall temperature
T\(_{aw}\)  adiabatic wall temperature
u  internal energy/unit mass
V  volume of exhaust tank
w  external work performed/unit mass
x  displacement along duct axis
z  non-dimensional displacement, \(x/l\)
\(\alpha\)  ratio of total temperature to static temperature, \(T_o/T\)
\(\varrho\)  \(l + kM^2\)
\(\gamma\)  \(l - M^2\)
\(\delta\)  diameter ratio in an annular cross-section, \(D_1/D_o\)
\(\lambda\)  thermal conductivity
\(\mu\)  dynamic viscosity
\(\phi\)  semi-cone angle of rear of diffuser plug
\(\Omega\)  vorticity factor, \(S/S^\circ = f/f^\circ\)
\(\omega\)  mass flow rate
\(\rho\)  mass density
\( \theta \) non-dimensional temperature, \( T/T_w \)

\( \tau_w \) shearing stress at wall

\( \nu \) flow velocity

( ) \(_o\) total (stagnation) conditions

( ) \(_m\) mean value, at a given cross-section

( ) \(_x\) conditions just upstream of a shock wave

( ) \(_y\) conditions just downstream of a shock wave

( )* nominal value

( )* state at which the Mach Number is unity

( )' first approximation, in an iterative analysis

(\( \bar{\cdot} \)) mean value, in a given region

\( d(\ ) \) exact differential

\( \delta(\ ) \) inexact differential

\( \#\) defined as

\( \equiv\) identically equal

\( \propto\) proportional

\( \sim\) in the order of

\( \approx\) approximately equal
THE PROBLEM

The problem under consideration is purely theoretical. However, it is made to correspond as closely as possible to the related physical problem pertaining to the hypersonic wind tunnel of the Rice Institute Mechanical Engineering Department. This correlation is affected by three means. First, a similar configuration is adopted. Second, the analysis employs the proper numerical values where dictated by other considerations. And third, use is made of results of preliminary observations made on the wind tunnel itself.

Before the present problem is described in detail, the important general features of the wind tunnel will be outlined. The wind tunnel in question is a blowdown-type tunnel, but running times are in the order of several seconds. The entire system can be considered to consist of four regions. First, at stagnation pressure the gas is transformed into a plasma by passing an electric arc through it. Second, the gas is accelerated to hypersonic velocity by a specially designed converging-diverging nozzle, [Ref. 2]. Third, the gas is passed into the test section as a free jet. Fourth, the flow is passed through a conical shock diffuser, is decelerated, cooled, and exhausted into a large capacity vacuum tank.

The present problem is concerned with the behavior of the flow properties, specifically the static and total pressures, from the section just behind the normal shock (which stands in the diffuser throat) to the exhaust tank. In order to accomplish this, the behavior of the static and total temperature, the Mach Number, the mass flow density, and the Reynolds Number, must also be determined.

In the annular passage behind the normal shock [(D) to (C)], the flow is cooled to around the ambient temperature as it is decelerated.
In the enlarging passage, [(C) to (S)], the flow is smoothed and further decelerated. In the choking region [(S) to (V)], the flow is first accelerated to sonic velocity (i.e., becomes "choked") and then decelerated and passed into a constant area duct at (V). This choking section corresponds physically to a valve which opens just as the test begins. The valve constriction (*) is used as the choking section because in the physical case, (1) it has a fixed area, and (2) it has the smallest area of any section between the test section and the exhaust tank. From the valve outlet (V), the flow is passed into a constant area duct, through the duct exit (E), and into the exhaust tank (B).
GENERAL ASSUMPTIONS

In this problem, simplifying assumptions must be made in order to achieve an analytical solution. It was felt that a tabulation of these general assumptions, with a discussion of their validity, would increase the utility of this analysis. The assumptions for which no discussion is offered must be considered to define the special case for which a solution is presented.

1. The working fluid is assumed to satisfy the conditions required by simple kinetic theory namely [Ref's 5, 7, and 11]:
   
   (a) When in equilibrium, the gas molecules are all alike,
   (b) Newton's Laws of Motion govern their movement,
   (c) The molecules behave as perfectly rigid, elastic spheres,
   (d) The diameter of the spheres is very small compared to the mean distance between them, so that the space occupied by the molecules is negligible,
   (e) No appreciable attractive or repulsive forces are exerted by the molecules on each other or on their container,
   (f) All energy other than translational energy is negligible;
   i.e., no internal motions:
      (i) no rotational energy of molecule
      (ii) no vibration between atoms in molecule
      (iii) no electronic re-arrangement.
   
   Obviously, for a monatomic gas, (i) and (ii) are perfectly satisfied.

   It is interesting to compare the simple kinetic theory assumption to the ideal gas assumption, the latter requiring that: (1) the Perfect Gas Law holds, and (2) the internal energy is a function of temperature
only. From the results of simple kinetic theory [Eq's 1 (a,b)] it can be seen that the ideal gas assumption is a necessary but not a sufficient condition for the simple kinetic theory assumption.

Since the wind tunnel to which this analysis applies uses helium, an inert, monatomic gas, the conditions of simple kinetic theory are met as well as practically possible, provided the temperature is properly limited, (i.e., to the range of temperature in which negligible ionization occurs). According to Fowler [Ref. 7], "The energy step associated with the change from the normal state to the nearest excited state occurs at 18,000°R for nearly all atoms capable of existing in the free state." Therefore, as long as the analysis is limited to temperature below 18,000°R, the simple kinetic theory assumption should yield valid results.

The simplicity and accuracy of the Perfect Gas Law makes it the best choice (in this problem) for the equation of state, regardless of whether the other results of simple kinetic theory are employed. However, since the other results of the theory are employed, the errors thereby incurred will be more consistent.

2. It is assumed throughout the problem that the one-dimensional approximation for flow analysis is valid. According to Shapiro [Ref. 15], the conditions for this approximation are:

(a) The percentage change of cross-sectional area, with respect to displacement along duct axis, must be small (or zero). We can always design the configuration such that this condition is satisfied.

(b) The radius of curvature of the duct axis must be small compared to the duct diameter. Since the duct axis is straight in this problem, this condition is satisfied perfectly.

(c) The shapes of the velocity and temperature profiles must remain
approximately unchanged from section to section along the duct axis.

3. The flow is assumed to choke instantaneously upon starting; therefore, steady conditions apply throughout the problem in all regions upstream from the choking section [Illus. (*)]. As mentioned before, since the wind tunnel is of the blowdown type with a running time in the order of several seconds, the time elapsed in establishing choked flow is exceedingly small compared to the length of a test run.

4. It is assumed that a conical shock diffuser can be designed such that the starting shock can be swallowed, and the normal shock will stand just behind the diffuser throat. It is also assumed that the conical shocks are sufficiently weak for the entropy increases across them to be negligible. This, then, permits the assumption that the flow is isentropic from the test section (T) to the section immediately in front of the normal shock (U).

5. For an infinite upstream Mach Number, it is assumed that in spite of imperfect gas effects encountered in a strong shock, normal shock relations are valid for calculating both the downstream Mach Number and the ratio of diffuser throat area to test section area. This assumption should be permissible since, according to normal shock relations [Eq.'s2(o,p,q,r)] both the downstream Mach Number and this particular area ratio approach finite limits as the upstream Mach Number becomes infinite.

6. The temperature immediately behind the normal shock [section (D)] is assumed to be 18,000°R. This is based on the work of Martinez [Ref. 12], which indicates that the temperature at any measurable distance behind a shock wave will not exceed a certain characteristic temperature which depends on the gas. For helium, this characteristic temperature is
18,000°R (see General Assumptions, part 1).

7. It is assumed that the hydraulic diameter can be substituted for the circular diameter in the appropriate equations where applied in non-circular regions throughout the problems. [Ref.'s 4 and 13].

8. It is assumed that heat can be transferred away from the duct wall sufficiently fast to maintain a constant wall temperature. Since the entire system is at the ambient (room) temperature at the beginning of a test run, the wall temperature is taken to be constant at 80°F (540°R) throughout the problem. This assumption is based on two factors. First, this permits an important simplification of the heat transfer relationships. Second, this assumption would be least likely to hold in a region where the heat transfer rate is high, i.e., just behind the shock. However, the effects of the large fluid-to-wall temperature differential which produces the high heat transfer rate in such a region will tend to dominate any effects of a change in wall temperature.

9. Heat transfer by radiation is assumed to be negligible in all regions behind the normal shock, thus defining a special case. Including radiation heat transfer would increase the rate at which heat is extracted from the flow. This would, in turn, reduce the length of duct necessary to cool the flow to around the ambient temperature, thereby reducing friction losses. Both the increase in the heat transfer rate and the decrease in friction losses tend to retard the decrease in total pressure [Eq. 2(k)], thus increasing the efficiency of the diffuser. Therefore, neglecting radiation is a conservative assumption.

10. The flow in the region just behind the normal shock [(D) to (C)] is assumed to be fully developed turbulent flow in the continuum region, due to shock induced turbulence. This is based on the work of Martinez
[Ref. 12] and Beckmann [Ref. 3], and will be discussed more completely where appropriate in The Analysis.

11. The arithmetic mean film temperature is used for the evaluation of all transport properties. This is based on a formula recently presented by Eckert [Ref. 6], which has been modified for use in this analysis (App. 5).

12. The working fluid is assumed to have a constant molecular weight. This also defines a special case.
THE ANALYSIS

It was felt that the utility, continuity, and clarity of the analysis would be greatly improved by relegating to the Appendix the development of many supporting elements. Therefore the equations used in this section have been derived (and/or modified) and cataloged in the Appendix. In addition, certain simplifications and approximations have been presented in the Appendix, along with any secondary developments that would impede the reader's progress.

In order to obtain a solution to this problem, many "specific" assumptions (those applying only in a specific region) must be made, in addition to the general assumptions already tabulated. Thus a general solution was out of the question. However, it was felt that the working out of a typical example would be easier to follow and would offer a "feeling" for the numbers involved. The opportunities for modification or extrapolation of the analysis and its results at least partially make up for this loss of generality.

Therefore, such an example is worked out, using a region-by-region analysis. The specific assumptions will be discussed as they are employed. Frequent reference is made to the illustration, to the equation catalog in the Appendix, and to the list of symbols.

It should be noted that this is a steady flow analysis, resulting in a constant mass flow rate ($\omega$). Actually, this is only true upstream from the choking section (*). However, steady conditions are also assumed downstream from the choking section since, in this problem, the exhaust tank (or back) pressure is considered to be constant at the "critical back pressure" (the pressure in the tank at which the flow velocity in the choking section is no longer sonic).
**Determination of Test Section Quantities (T)**

Although the problem does not include the normal shock or any region upstream from it, several of the given quantities are test section values, and therefore, all test section values must be calculated in order to proceed to the actual beginning section: i.e., that section immediately behind the normal shock (D).

The given values in this problem are:

\[ M_T = 40 \text{ (with a special exception)} \]
\[ \omega = 1 \text{ gm/sec} \]
\[ D_T = D_{o_D} = D_{o_C} = D_s = D_v = D_E = 3 \text{ inches} \]
\[ T_T = 430^\circ R \]
\[ T_D = 18,000^\circ R \]
\[ T_w = 80^\circ F = 540^\circ R \]
\[ R = 386 \text{ ft-lb}^2/\text{°R-lb} \text{ m} = \text{constant} \]

The area determines the mass flow density:

\[ A_T = \frac{T}{4} D_T^2 = \frac{T}{4} (2.250)^2 = 0.0490 \text{ ft}^2 \]

thus,

\[ G_T = \frac{\omega}{A_T} = \frac{1}{454} \cdot \frac{1}{0.0490} = 0.450 \frac{\text{lb}}{\text{sec-ft}^2} \]

The static pressure can now be found from Eq. 3(c):

\[ \frac{P_T}{R} = \frac{G_T \sqrt{F_T}}{A_T} = \sqrt{\frac{386}{454 \cdot 32.2}} \cdot \frac{0.0450 \sqrt{430}}{40} \]

\[ P_T = 0.0625 \text{ psf} \]

The ratio of total temperature to static temperature can be found from Eq. 2(c):

\[ \alpha_T = \frac{T_T}{T_R} = 1 + \frac{M_T^2}{3} = 1 + \frac{40^2}{3} = 5.34 \]

which leads to

\[ T_R = \alpha_T T_T = 5.34 \cdot 430 = 230,000^\circ R \]

From Eq. 2(h), the total pressure is determined

\[ P^\circ_T = P_T \alpha^{5/2} = 0.0625 \cdot (534)^{5/2} = 412,000 \text{ psf} \text{a}. \]
Therefore, the test section values are:

\[ D_T = 0.250 \text{ ft} \]
\[ A_T = 0.0490 \text{ ft}^2 \]
\[ G_T = 0.0450 \frac{\text{lb}}{\text{sec-ft}^2} \]
\[ M_T = 40 \]
\[ \alpha_T = 534 \]
\[ T_T = 430^\circ\text{R} \]
\[ T_{0T} = 230,000^\circ\text{R} \]
\[ P_T = 0.0625 \text{ psfa} \]
\[ P_{0T} = 412,000 \text{ psfa}. \]
In order that the configuration won't be a limiting factor in designing for a maximum test section Mach Number, the Mach Number immediately behind the shock and the ratio of diffuser throat area to test section area are calculated on the basis of an infinite test section Mach Number. Therefore $M_D^2 = \frac{1}{5}$ from Eq. 2(q), and $\frac{A_D}{A_T} = 0.700$ from Eq. 2(r).

Using Eq.'s 7(d) and 7(b), the annular diameter ratio is

$$\delta = \left(\frac{D_D}{D_T}\right) = \sqrt{1-\frac{A_T}{A_D}} = \sqrt{1-0.700} = 0.542,$$

and the hydraulic diameter is

$$D_D = D_T (1 - \delta) = 0.250 (1 - 0.548) = 0.1130 \text{ ft}.$$

Also, $A_D = \left(\frac{A_D}{A_T}\right) A_T = 0.700 (0.0490) = 0.0343 \text{ ft}^2$,

leading to

$$G_D = \frac{\omega}{A_D} = \frac{1}{454} \frac{1}{0.0343} = 0.0643 \frac{1 \text{ lb}}{\text{sec-ft}^2}.$$

From Eq. 2(c)

$$\alpha_p \leq \frac{T_D}{T_p} = 1 + \frac{M_D^2}{2} = 1 + \left(\frac{1}{5}\right)\left(\frac{1}{5}\right) = \frac{\kappa}{5} = 1.0667;$$

and the assumption that $T_D = 18,000^\circ\text{R}$ leads to

$$T_p = \alpha_p T_D = 1.067 (18,000) = 19,200^\circ\text{R}.$$

Eq. 3(c) gives the static pressure:

$$P_D = \frac{\sqrt{R} C_p \sqrt{T_p}}{M_D} = \frac{2.68 (0.0643) \sqrt{18,000}}{\sqrt{0.200}}$$

$$P_D = 51.3 \text{ psf}.$$

Eq. 2(h) then gives the total pressure:

$$P_{p0} = P_p \alpha_p^{\frac{5}{2}} = 51.3 \left(1.0667\right)^{\frac{5}{2}} = 59.3 \text{ psf}.$$

The non-dimensional temperatures are

$$\theta_D \leq \frac{T_D}{T_w} = 18,000/540 = 33.4$$

and $$\theta_{p0} \leq \frac{T_{p0}}{T_w} = 19,200/540 = 35.6.$$
Finally, Eq. 8(a) (plus App. 6) gives the Reynolds's Number:

\[ Re_D = \frac{G_D D_D}{B \sqrt{\frac{\mu}{\rho}} \sqrt{\theta_p + 1}} = \frac{0.0643 (0.1130)}{7.68 \cdot 10^{-7} \sqrt{1540^2}} \sqrt{33.4 + 1} \]

\[ Re_D = 98.0. \]

Thus the values at (D) are:

\[ \frac{A_D}{A_T} = 0.700 \]
\[ \delta_D = 0.548 \]
\[ D_D = 0.1130 \text{ ft.} \]
\[ A_D = 0.0343 \text{ ft}^2 \]
\[ G_D = 0.0643 \text{ lb m} \]
\[ M_D = \sqrt{M_D^2} = \sqrt{1/5} = 0.447 \]
\[ \alpha_D = 16/15 = 1.0667 \]
\[ T_D = 18,000^\circ R \]
\[ T_{oD} = 19,200^\circ R \]
\[ Q_D = 33.4 \]
\[ Q_{oD} = 35.6 \]
\[ P_D = 51.3 \text{ psfa} \]
\[ P_{oD} = 59.3 \text{ psfa} \]
\[ Re_{D} = 98.0. \]
Flow in the Cooling Region: Determination of Quantities at \( T_c \):

This is the most crucial region in the analysis, since the fluid, having just passed through a strong normal shock, must be cooled from an extremely high temperature (~18,000°R) to around room temperature in a few diameters length (~5).

Although the Reynolds Numbers are sufficiently low (<400) to indicate laminar flow in this region, the normal shock induces considerable turbulence, so that the flow exhibits turbulent characteristics for a significant distance downstream [Ref.'s 3 and 12].

This shock induced turbulence is accounted for by introducing a "vorticity factor" \( \Omega \) which is taken to be 20, by an assumed analogy with Beckmann's problem [Ref. 3]. This vorticity factor multiplies both the friction factor and the convective heat transfer coefficient (and therefore the Nusselt and Stanton Numbers).

Throughout this region, the cross-section is annular and the area is constant. (In the physical application, the diffuser throat area \( A_D \) is made slightly less than the cross-sectional area in the cooling region downstream in order to insure that the normal shock remains in the diffuser throat).

It is immediately obvious that

\[
A_c = A_D = 0.0343 \text{ ft}^2
\]

and

\[
\dot{s}_c = \dot{s}_D = 0.548
\]

and therefore, \( G_c = G_D = 0.0643 \frac{\text{lbm}}{\text{sec-ft}^2} \)

Now, selecting a value for the static temperature at \( (C) \), and non-dimensionalizing by dividing through by the wall temperature which is assumed constant,
\[ \Theta_c = \frac{T_e}{T_w} = 1.1, \text{ (implying that } T_e = 594 \text{°R)} \]

from which a mean Reynolds Number can be calculated [Eq. 8(b)]:

\[ \text{Re}_{Dc} = \sqrt{\frac{\Theta_e + 1}{\Theta_e + 1}} \ln \frac{\Theta_e + 1}{\Theta_e + 1} \text{ Re}_{Dc} \]

\[ \text{Re}_{Dc} = \sqrt{\frac{3.4 + 1}{3.4 + 1}} \ln \frac{22.4 + 1}{4.1 + 1} \text{ (Eq. 8.0)} \]

\[ \text{Re}_{Dc} = 182.3. \]

Using this mean Reynolds Number and the von Karman Modification of the Prandtl Analogy [Eq. 9(b)] to evaluate the mean Stanton Number:

\[ \overline{S}_{Dc} = 0 \overline{S}_{Dc} = \frac{0.0396 \left( \frac{\text{Re}_{Dc}}{1.823} \right)^{1/4}}{1 - 0.659 \left( \frac{\text{Re}_{Dc}}{1.823} \right)^{1/4}} = \frac{(20) 0.0396 (182.3)^{1/4}}{1 - 0.659 (182.3)^{1/4}} \]

\[ \overline{S}_{Dc} = 0.328. \]

Using this mean Reynolds Number and the von Karman-Nikuradse Formula [Eq. 10(c)] to evaluate the mean friction factor,

\[ \sqrt{f_{Dc}} = \frac{1.15}{\ln \left( \frac{\text{Re}_{Dc}}{2.51} \right)} = \frac{1.15}{\ln (182.3 (0.355))} = 0.354, \]

therefore

\[ f_{Dc} = \sqrt{f_{Dc}} = 20 \left( \sqrt{f_{Dc}} \right)^2 = 20 (0.354)^2 = 2.51. \]

Since this is an iterative process, a rough plot of the von Karman-Nikuradse Formula [Eq. 10(b)] gives a sufficiently good first approximation (in this example, \( \text{Re}_{Dc} = 182.3 \) implies \( \sqrt{f_{Dc}} = 0.355 \)) that very few repititions are required for convergence.

Now \( \sqrt{f_{Dc}} = \left( \frac{\text{Re}}{4 \overline{S}_{Dc}} \right) \) can be calculated,

\[ \left( \frac{\sqrt{f_{Dc}}}{4 \overline{S}_{Dc}} \right) = \left( \frac{2.51}{4 (0.328)} \right) = 1.913. \]

It should be noted that the assumption of Reynolds Analogy (Pr=1), rather
than the von Karman Modification of the Prandtl Analogy would have resulted in

\[
f = \frac{4C}{4S} = \frac{C}{S} = 2.
\]

Now, drawing heavily on Section 12 of the Appendix, and taking as a rough approximation \( \alpha_c' \approx (\gamma_s')_c = 1 \), the ratio of total pressures can be found, [Eq. 12(d)]:

\[
1 - \left( \frac{P_{oc}'}{P_o} \right)^2 = \frac{k_0}{2} \left( \frac{\alpha_c'}{\alpha_o} \right)^2 \left[ \frac{1}{1 + \frac{\alpha_c'}{\alpha_o}} \right] \left[ \left( \frac{f}{4.5} \right)^2 - 1 \right] \left( \theta_o - \theta_o \right) + \frac{f}{4.5} \ln \frac{\theta_o}{\theta_c} + 1 \}
\]

\[
\left( \frac{P_{oc}'}{P_o} \right)^2 = 1 - \left( \frac{15}{15} \right) \left( \frac{15}{150} \right) \left[ \left( \frac{1}{15} \right)^2 \right] \left( \frac{15}{150} \right) + \frac{1}{150} \ln \frac{35.6}{11} - 1
\]

\[
\left( \frac{P_{oc}'}{P_o} \right)^2 = 0.645
\]

Eq. 12(e) now gives

\[
M_c^2' = \left( \frac{\alpha_c'}{\alpha_o} \right)^2 \theta_o \frac{\theta_o}{\theta_o} \left( \frac{P_{oc}'}{P_o} \right)^2 M_o^2
\]

\[
M_c^2' = \left( \frac{15}{150} \right)^2 \frac{15}{35.6} \left( \frac{15}{0.645} \right)^2 \left( \frac{15}{150} \right)
\]

\[
M_c^2' = 0.00740
\]

leading to [Eq. 2(c)]:

\[
\alpha_c'' = 1 + \frac{M_c^2'}{3} = 1 + \frac{0.00740}{3} = 1.002467
\]

Now correcting,

\[
M_c^2'' = \left( \frac{\alpha_c''}{\alpha_o} \right)^2 \frac{\alpha_c'' \theta_c}{\theta_o} \left( \frac{P_{oc}'}{P_o} \right)^2 M_o^2
\]
\[ M_c^{\prime\prime} = \left( \frac{\alpha_c^\prime}{\alpha_c^\prime} \right)^5 M_c^{\prime} = \left( \frac{1.002467}{1} \right)^5 0.00740 \]

\[ M_c^{\prime\prime} = 0.00749, \]

which gives:

\[ \alpha_c^\prime = \frac{1 + M_c^{\prime\prime}}{3} = \frac{1 + 0.00749}{3} = 1.002497, \]

\[ \Theta_c^{\prime\prime} = \Theta_c \alpha_c^{\prime\prime} = 1.1 \times (1.0025) = 1.10275. \]

One further correction can now be made by using these values in Eq. 12(d) giving:

\[ \left( \frac{P_{oc}}{P_{ob}} \right)^2 = 1 - \frac{1}{2}\frac{M_c^2}{2} \left[ 1 + \left( \frac{\alpha_c^{\prime\prime}}{\alpha_c^b} \right) \left( \frac{f}{45\theta_c^b} - 1 \right) (\Theta_c^{\prime\prime} - \Theta_c^b) + \frac{f}{45\theta_c^b} \ln \frac{\Theta_c^{\prime\prime} - 1}{\Theta_c^b - 1} \right] \]

\[ \left( \frac{P_{oc}}{P_{ob}} \right)^2 = 1 - \frac{(0.00749)}{2(35.6)} \left[ 1 + \left( \frac{1.0025}{1.0667} \right)^4 \right] \left( 1.913 - 1 \right) (35.6 - 1.1) + 1.913 \ln \frac{35.6 - 1}{1.10275 - 1} \]

\[ \left( \frac{P_{oc}}{P_{ob}} \right)^2 = 0.641, \]

leading to:

\[ \frac{P_{oc}}{P_{ob}} = 0.801. \]

Eq. 12(d) now gives:

\[ M_c^2 = \left( \frac{P_{oc}}{P_{ob}} \right)^2 \left( \frac{P_{oc}}{P_{ob}} \right)^2 \left( \frac{\alpha_c^{\prime\prime}}{\alpha_c^b} \right)^5 M_c^{\prime\prime} \]

\[ M_c^2 = 0.641 \left( \frac{1}{0.645} \right) \left( \frac{1.002467}{1.002467} \right)^5 0.00749 \]

\[ M_c^2 = 0.00746, \]

leading to:

\[ \alpha_c = 1 + \frac{M_c^2}{3} = 1 + \frac{0.00746}{3} = 1.00249 \]

and finally:

\[ \Theta_c^{\prime\prime} = \alpha_c \Theta_c = 1.00249 \times (1.1) = 1.1027. \]
The length of the passage necessary to cool the flow to the specified static temperature ($Q_c = 1.1$) can now be found using Eq. 11(c):

$$\frac{L_{bc}}{D_c} = \frac{L_{bc}}{D_c} \frac{D_p}{D_T} = \frac{L}{D_T} = \frac{1}{4} \frac{L}{D_T} \ln \frac{\Theta_{bc} - 1}{\Theta_{bc} - 1} = \frac{1}{4} \frac{L}{D_T} \ln \frac{35.6 - 1}{10.27 - 1}$$

$$\left(\frac{L}{D_T} \right)_{bc} = 4.43.$$ 

Writing the result in a different manner [Eq. 7(b)],

$$\frac{L_{bc}}{D_T} = \frac{L_{bc}}{D_c} \frac{D_p}{D_T} = \frac{L}{D_T} (1 - \delta) = 4.43 (1 - 0.548)$$

$$\left(\frac{L}{D_T} \right)_{bc} = 2.00.$$ 

Now the total temperature and total pressure can be found,

$$T_{Tc} = \Theta_{Tc} T_w = 1.1027(540) = 596 \, ^\circ R,$$

and,

$$P_{Tc} = (\frac{P_{Tc}}{P_D}) P_D = (0.801) 59.3 = 47.5 \, psf.$$ 

The static pressure is given by Eq. 2(h):

$$P_c = P_{Tc} \frac{r^2}{52} = 47.5 (1.00249)^{-\frac{52}{52}} = 47.2 \, psf.$$ 

The Reynolds Number is given by Eq. 8(b):

$$\text{Re}_c = \sqrt{\frac{Q_D + 1}{Q_c + 1}} \quad \text{Re}_D = \sqrt{\frac{33.4 + 1}{1.1 + 1}} = 98.0 = 397$$

Therefore, the values at (C) are:

- $\delta_c = 0.548$
- $A_c = 0.0343 \, \text{ft}^2$
- $G_c = 0.0643 \, \text{lb} \, \text{m} \, \text{sec}^{-1} \, \text{ft}^2$
- $M_c = \sqrt{M_c^2} = \sqrt{0.00746} = 0.0864$
- $C_c = 1.00249$
- $(L_{bc}/D_T) = 2.00$
- $T_c = 594 \, ^\circ R$
To_c = 596°R
Q_c = 1.1
Qo_c = 1.1027
P_c = 47.2 psfa
Po_c = 47.5 psfa
Re_c = 397.
Flow in the Smoothing Region: Determination of Quantities at (S)-

In this region, the cross-section is annular with the outer annular diameter constant and equal to the test section diameter. The inner annular diameter decreases linearly with displacement along the duct axis and finally becomes zero at section (S).

This increase in area tends to smooth and stabilize the flow. Although the turbulence induced by the shock is decreased, in the absence of better knowledge the vorticity factor is assumed constant and equal to 20 as in the cooling region. In this example, the value of the semi-vertex angle (φ) at the rear of the diffuser plug is taken to be 5°. Since the outer annular area as (S) is the same as the test section area, and since the inner annular diameter at (S) is zero,

\[ \frac{A_g}{A_T} = 0.0490 \text{ ft}^2, \]

and

\[ G_s = G_T = 0.0450 \frac{lb}{sec-ft^2}. \]

A first approximation for the mean Reynolds Number results from assuming the flow to be isothermal in this region. Therefore, Eq. 8(b) gives:

\[ \text{Re}_{cs} = \frac{(1 + \frac{1}{2}) \text{Re}_c}{1 + \frac{1}{2}(\phi_c + \phi_s)} = \frac{1 + \phi_c}{1 + \frac{1}{2}\phi_c} \cdot \text{Re}_c = \frac{1 + 0.548}{1 + 0.274} = 397 \]

\[ \text{Re}_{cs} = 481. \]

Eq. 9(b) then gives the approximate mean Stanton Number:

\[ \overline{\text{St}}'_{cs} = 0.0396 \left( \frac{\text{Re}_{cs}}{1 + 0.274} \right)^{-0.4} \]

\[ \overline{\text{St}}'_{cs} = (20) 0.0396 (481)^{-0.4} \]

\[ \overline{\text{St}}'_{cs} = 0.244. \]
Now the approximate total temperature can be found from Eq. 11(d):

\[
\Theta_{os}' = 1 + (\Theta_{os} - 1) \left( \frac{1}{1 - \delta_c} \right) \alpha_c \Theta_{os}^2
\]

\[
\Theta_{os}' = 1 + (1.1027 - 1) \left( \frac{1}{1 - 0.548} \right) 2(\cos 5^\circ) 0.244
\]

\[
\Theta_{os}' = 1.001213.
\]

Therefore,

\[
\frac{\Theta_{os}'}{\Theta_{os}} = \frac{1.001213}{1.1027} = 0.907.
\]

For low Mach Numbers \( M_c^2 = M_c^2 = 0.0864 \) the difference between the total temperature and the static temperature is sufficiently small \( (T_o - T_e \approx 2^\circ R) \), that negligible error is introduced by calculating the mean Reynolds Number in the following manner, [Eq. 8(b)]

\[
\begin{align*}
\overline{Re}_{cs} &= \frac{2 \left[ \Theta_{os} + 1 \right]}{\left[ \Theta_{os} + 1 \right] + \left[ \Theta_{os} + 1 \right]} \overline{Re}_{cs}' \\
&\approx \frac{2 \sqrt{\Theta_{os} + 1}}{\left[ \Theta_{os} + 1 \right] + \left[ \Theta_{os} + 1 \right]} \overline{Re}_{cs}'
\end{align*}
\]

\[
\overline{Re}_{cs} \approx 487.
\]

This value is so near the first approximate value that the mean Stanton Number, and therefore the total temperature \( (\Theta_{os}) \), would be unchanged by an iteration using this corrected value. The mean friction factor is given by Eq. 10(c) using \( f_{cs}' = 0.0823 \) as a first approximation:

\[
\sqrt{f_{cs}'} = \frac{115}{2.51} \ln \left[ \frac{487 (0.0823)}{1.15} \right] = 0.268,
\]

leading to:

\[
f_{cs} = \frac{f_{cs}^2}{2} = 20 (0.268)^2 = 1.660.
\]

Now a first approximation for \( M_c^2 \) can be made by assuming that:

\[
\frac{M_c^2}{A_e} = \frac{M_c^2}{A_e} = \frac{A_c}{A_e} \frac{\Theta_{os}'}{\Theta_{os}} \frac{\overline{Re}_{cs}}{\overline{Re}_{cs}} = 1.
\]
Therefore
\[ M_s^2' = \left( \frac{A_{1'}}{A_T} \right) M_c^2 = (0.700)^2 0.00746 = 0.00366. \]

Now an approximate mean Mach Number is:
\[ \frac{M_{c_s}^2'}{M_{c_s}^2} = \frac{1}{2} (M_c^2 + M_s^2') = \frac{1}{2} (0.00746 + 0.00366) \]
\[ M_{c_s}^2' = 0.00556, \]
which gives
\[ \frac{\alpha_{c_s}^2}{\alpha_c^2} = \frac{1}{2} (\alpha_c + \alpha_s') = 1 + \frac{1}{2} \frac{M_{c_s}^2'}{M_{c_s}^2} = 1 + \frac{0.00556}{3} = 1.00185, \]
\[ \beta_{c_s}^2 = \frac{1}{2} (\beta_c + \beta_s') = 1 + \frac{1}{2} \frac{M_{c_s}^2'}{M_{c_s}^2} = 1 + \frac{1}{2} \frac{0.00556}{3} = 1.0026, \]
\[ \gamma_{c_s}^2 = \frac{1}{2} (\gamma_c + \gamma_s') = 1 - \frac{M_{c_s}^2'}{M_{c_s}^2} = 1 - 0.00556 = 0.9944. \]

\[ \frac{\alpha_s}{\alpha_c} \left( \frac{P_{sc}}{P_c} \right)^{\frac{2}{3}} = 1 \]

implies that:
\[ \left[ \frac{\alpha_s}{\alpha_c} \left( \frac{P_{sc}}{P_c} \right)^{\frac{2}{3}} \right] = 1 \]

Therefore a second approximation for \( M_s^2 \) can be found from Eq. 13(e)
\[ \ln \left\{ \frac{M_s^2''}{M_s^2} \right\} \left( \frac{A_T}{A_b} \right)^{\frac{2}{3} \frac{\gamma}{\gamma-1}} \left( \frac{\Theta_{oc}}{\Theta_{oc}} \right)^{\frac{\gamma}{\gamma-1}} \left[ \frac{T_{sc}}{T_{cs}} \left( \frac{P_{sc}}{P_c} \right) \right]^2 \]

\[ \ln \frac{M_s^2''}{M_s^2} + \ln \left\{ \left( \frac{1}{0.700} \right)^{0.104} \left( \frac{1}{0.907} \right)^{1.016} \right\} = \frac{\gamma}{2} \left( 1.166 \right) 0.00746 \left( 0.700 \right)^2 \left( 1.066 \right) 1.007 \]

\[ M_s^2'' = 0.471 \]
\[ M_c^2 = 0.471 \]
\[ \frac{M_s^2''}{M_c^2} = (0.471)^2 0.00746 \]
\[ M_s^2'' = 0.00351. \]
Therefore
\[ \alpha_s'' = 1 + \frac{1}{2} M_s'^{2''} = 1 + \frac{1}{2} (0.00351) = 1.00117, \]
and,
\[ \bar{M}_{cs''} = \frac{1}{2} (M_c^2 + M_s'^{2''}) = \frac{1}{2} (0.00746 + 0.00351) = 0.00548, \]
leading to
\[ \hat{\alpha}_{cs} = 1 + \frac{1}{2} M_{cs}^{2''} = 1 + \frac{0.00548}{3} = 1.00183, \]
\[ \hat{c}_{cs} = 1 + \frac{1}{2} M_{cs}^{2''} = 1 + \frac{1}{2} (0.00548) = 1.00914, \]
\[ \hat{e}_{cs} = 1 - M_{cs}^{2''} = 1 - 0.00548 = 0.99452, \]
all of which are practically the same as for the first approximation. Now Eq. 13(e) can be solved by taking the following arithmetic mean value
\[ \left( \frac{\alpha}{\alpha_c} \right)^4 \frac{\theta_o}{\theta_{o_c}} \left( \frac{P_o}{P_o} \right)^2 = \frac{1}{2} \left\{ 1 + \left( \frac{\alpha_c}{\alpha} \right)^4 \frac{\theta_{o_c}}{\theta_o} \left( \frac{P_o}{P_o} \right)^2 \right\} \]
\[ \left( \frac{\alpha}{\alpha_c} \right)^4 \frac{\theta_o}{\theta_{o_c}} \left( \frac{P_o}{P_o} \right)^2 = \frac{1}{2} \left\{ 1 + \frac{M_{cs}^{2''}}{M_c^2} \left( \frac{A_T}{A_D} \right)^2 \right\} \]
\[ \left( \frac{\alpha}{\alpha_c} \right)^4 \frac{\theta_o}{\theta_{o_c}} \left( \frac{P_o}{P_o} \right)^2 = \frac{1}{2} \left\{ 1 + \frac{0.00351}{0.00746} \left( \frac{1}{0.700} \right)^2 \right\} \]
\[ \left( \frac{\alpha}{\alpha_c} \right)^4 \frac{\theta_o}{\theta_{o_c}} \left( \frac{P_o}{P_o} \right)^2 = 0.980. \]
Making this substitution in Eq. 13(e), the Mach Number at (S) can be found:
\[ \ln \frac{M_c^2}{M_s^2} + \ln \left\{ \left( \frac{A_T}{A_D} \right)^2 \left( \frac{\theta_o}{\theta_{o_c}} \right) \right\} = \frac{k T_o}{2 (\tan \phi)} \left( \frac{M_c^2}{M_s^2} \right)^2 (1.066) \frac{\alpha}{\alpha_c} \frac{\theta_{o_c}}{\theta_o} \left( \frac{P_o}{P_o} \right)^2 \]
\[ \ln \frac{M_s^2}{M_c^2} = \ln \left( \frac{(0.700)^{1.066}}{(0.907)^{2.07}} \right) = \frac{2.07(1.066)(0.00746)^2}{2 (\tan 5^\circ)} \ln (0.700) \ln (1.007) 0.980 \]
\[ M_s^2 = \left( \frac{M_c^2}{M_s^2} \right) M_c^2 = (0.470)(0.00746) = 0.00350, \]
which is the same as the second approximation.
Therefore
\[ \alpha_s'' \equiv \alpha'' = 1.00117. \]
The total pressure at (S) can now be found using Eq. 3(c):

\[
\frac{P_S}{P_c} = \frac{\rho_0}{\rho} \left(\frac{\alpha_s}{\alpha_c}\right)^n \frac{\Theta_0}{\Theta_c}
\]

\[
\left(\frac{P_S}{P_c}\right)^2 = \frac{1}{0.470} \left(0.700\right)^n \left(\frac{1.00127}{1.00247}\right)^n \approx 0.907
\]

\[
\frac{P_S}{P_c} = \sqrt{\left(\frac{P_S}{P_c}\right)^2} = \sqrt{0.938} = 0.968
\]

\[
P_{SC} = (P_S/P_c) P_c = (0.968) 475 = 460 \text{ psf}.
\]

Now Eq. 2(c) gives:

\[
\rho_t = \frac{\rho_0}{\rho} \frac{\Theta_0}{\Theta_c} = 1.001213 = 1.00004.
\]

Therefore,

\[
T_s = \rho_t T_c = 1000(540) = 540 \text{ °R},
\]

\[
T_{SC} = \rho_t T_c = (T_s \alpha_s) T_w = [0.1] 1001 \times 540
\]

\[
T_{SC} = 540.5 \text{ °R}.
\]

The static pressure is given by Eq. 2(h):

\[
P_s = P_{SC} \alpha_s \frac{\rho_c}{\rho_c} = 460 \left(\frac{1.00127}{1.00247}\right)^n = 457 \text{ psf}.
\]

The Reynolds Number is [Eq. 8(b)]:

\[
Re_s = \frac{\rho_c + l}{\rho_t + l} \frac{1 + \delta_s}{1 + \delta_s} Re_c
\]

\[
Re_s = \sqrt{\frac{11}{1+1}} \frac{1 + 0.548}{1} 397
\]

\[
Re_s = 635.
\]

Therefore, the values at (S) are:

\[
\delta_s = 0
\]

\[
A_s = A_t = 0.0490 \text{ ft}^2
\]

\[
G_s = G_t = 0.0450 \frac{1b_m}{	ext{sec-ft}^2}
\]

\[
\frac{\ell_s}{D_s} = \frac{\ell_s}{D_t} = \frac{\ell_c}{2 \tan \varphi} = \frac{0.548}{2 \tan 5^\circ} = 3.15
\]
\[ M_s = \sqrt{M_s^2} = \sqrt{0.00351} = 0.0592 \]

\[ \alpha_s = 1.00117 \]

\[ T_s = 540^\circ R \]

\[ T_{to_s} = 540.5^\circ R \]

\[ \theta_s = 1 \]

\[ \theta_{to_s} = 1.0012 \]

\[ P_s = 45.8 \text{ psfa} \]

\[ P_{o_s} = 46.0 \text{ psfa} \]

\[ R_{e_s} = 635^\circ \]
Flow in the Choking Region: (1) Determination of Quantities at (*):

As previously mentioned, the flow must be choked at some section between the test section and the exhaust tank, otherwise the pressure increase that occurs in the tank as the mass flow enters will prevent maintenance of steady test section conditions during a test run. This problem will correspond more closely to the physical problem if the flow is choked in a constricted section between section (s) and the exhaust pipe; since, in the corresponding physical problem a valve (having a constricted cross-sectional area) is located in this region. Therefore, once the flow has been smoothed, [at section (S)], it will be accelerated to sonic velocity at the choking section (*) and then decelerated to the valve outlet section (V), [which has the same area at the test section (T), smoothing section (S), and the exhaust pipe]. Actually, at any time between the start of a test run and the instant at which pressure in the exhaust tank reaches the "critical back pressure", the flow will be choked, but will continue to accelerate (thus becoming supersonic) upon leaving the choking section and then shock somewhere downstream in order to reach the existing instantaneous back pressure. However, this can be thought of as the means by which the flow adjusts to the pressure boundary condition in the exhaust tank, and since the calculations in this problem are for the limiting case anyway, (in order to determine the length of a test run among other things), the problem is analyzed as if choking was on the verge of breaking down. Therefore, subsonic flow is assumed to exist at all points downstream from the choking section (*).

In the region between section (S) and the choking section (*) the flow is assumed to be isentropic; adiabatic because the difference between the total temperature of the flow and the wall temperature is
initially so small \((T_{o_s} - T_w < 1^o R)\) and because the region is quite short; reversible, since the design of a converging passage for subsonic flow is not too crucial, provided the area changes continuously, (a condition met in this problem). In the region between the choking section (*) (the valve constriction) and the valve outlet (V), the most conservative assumption is made: that the pressure remains constant (i.e., that the increase in kinetic energy when the flow was accelerated to choking is entirely dissipated through turbulence and wall friction). Since this region is also quite short, the flow is assumed adiabatic. As mentioned before, the pressure in the exhaust tank is considered constant at the "critical back pressure" throughout the problem, therefore, the flow is considered to be entirely subsonic in this region [i.e., from (*) to (V)].

The isentropic assumption immediately implies that [Eq. 2(m)]:

\[
\begin{align*}
Po_\ast &= Po_s = 46.0 \text{ psfa} \\
and \quad To_\ast &= To_s = 540.5^o F.
\end{align*}
\]

Since the flow is choked (by design), the Mach Number is unity:

\[
M_\ast = M^* = 1.
\]

Therefore, by Eq. 2(c),

\[
\alpha_\ast = \frac{T_o}{T_\ast} = 1 + \frac{M^2}{2} = 1 + \frac{1}{2}(1) = \frac{4}{3},
\]

which gives

\[
T_\ast = T_o / \alpha_\ast = (\frac{2}{3}) 540.5 = 405^o R.
\]

Eq. 2(h) gives the static pressure:

\[
P_\ast = \frac{P_o}{\alpha_\ast} \alpha_\ast^{\gamma/\gamma'} = 46.0 (\frac{4}{3})^{5/4} = 22.4 \text{ psfa}.
\]

The choking area is given by Eq. 3(b), (remembering that the flow is isentropic):

\[
\frac{A_\ast}{A_s} = \left(\frac{A_*}{A_s}\right) = \frac{16}{9} \left(\frac{M^2}{\alpha_\ast^2}\right) = \frac{16 (0.0522)}{9 \left(1.00117\right)^2} = 0.1050
\]
or

\[ A_* = \left( \frac{A_*/A_s}{A_T} \right) A_T = (0.1050) 0.0490 = 0.00515 \text{ ft}^2, \]

which gives

\[ D_* = \left[ \frac{4}{\pi} A_* \right]^{1/2} = \left[ \frac{4}{\pi} (0.00515) \right]^{1/2} = 0.0810 \text{ ft.} (= 0.973 \text{ in.}) \]

Also, the mass flow density is

\[ G_* = \frac{A_*}{A_T} G_s = \left( \frac{1}{0.1050} \right) 0.0450 = 0.428 \frac{\text{lbm}}{\text{sec-ft}^2}. \]

Finally, the Reynolds Number is given by Eq. 8(a):

\[ Re_* = \frac{G_* D_*}{B \cdot 10^{-2} \cdot (\rho/\rho_0) + 1} = \frac{0.428 (0.0810)}{7.68 (10^{-7}) \sqrt[4]{0.973}, \sqrt[4]{0.973} + 1} = 2060. \]

Therefore, the values in the choking section (*) are:

\[ Re_* = 2060 \]

\[ M_* = 1 \]

\[ \alpha_* = 4/3 \]

\[ A_* / A_T = 0.1050 \]

\[ A_* = 0.00515 \]

\[ D_* = 0.0810 \text{ ft.} (= 0.973 \text{ in.}) \]

\[ G_* = 0.428 \frac{\text{lbm}}{\text{sec-ft}^2} \]

\[ T_* = 405^\circ \text{R} \]

\[ T_{0*} = 540^\circ \text{R} \]

\[ P_* = 22.4 \text{ psfsa} \]

\[ P_{0*} = 46.0 \text{ psfsa}. \]

**Flow in the Choking Region:** (2) **Determination of Quantities at the Valve Outlet (V)**

Since the area at (V) is equal to the test section area, the area at (S), and the exhaust pipe area (by design), the mass flow density is known:

\[ A_v = A_T = 0.0490 \text{ ft}^2 \]

\[ G_v = G_T = 0.0450 \frac{\text{lbm}}{\text{sec-ft}^2} \]

The assumption of constant static pressure gives:

\[ P_v = P_* = 22.4 \text{ psfsa}. \]
and the adiabatic assumption gives:

$$T_{0v} = T_{0\ast} = 540°R.$$  

Eq. 3(c) can then be used to find $M_v^i$:

$$M_v \sqrt{\alpha_v} = \frac{G_v}{k_\ast} \frac{\sqrt{R_v}}{P_v} = \frac{0.0450 \times (2.68)}{540} \sqrt{22.4}$$

$$M_v \sqrt{\alpha_v} = 0.1248.$$  

Letting $\alpha_v = 1$ gives $M_v^i = 0.1248$

or $$M_v^{\prime} = 0.01560,$$

leading to

$$\alpha_v'' = 1 + \frac{1}{3} M_v^2 = 1 + \frac{0.01560}{3} = 1.0052.$$  

Correcting $M_v$:

$$M_v^2 (\alpha_v'') = (0.1248)^2 = 0.01560$$

$$M_v = \frac{0.01560}{\alpha_v''} = 0.01560 = 0.01550.$$  

Therefore,

$$M_v = \sqrt{0.01550} = 0.1245$$

and $$\alpha_v = 1 + \frac{1}{3} M_v^2 = 1 + \frac{0.01550}{3} = 1.00517.$$  

The static temperature can be found [Eq. 2(c)]:

$$T_v = T_{0v} / \alpha_v = 540 / 1.005 = 537°R.$$  

The total pressure is given by Eq. 2(h):

$$P_v = P_v \alpha_v^{\frac{5}{2}} = 22.4 \times (1.00517)^{\frac{5}{2}} = 22.7 \text{ psf}.$$  

Finally, the Reynolds Number is [Eq. 8(b)]:

$$Re_v = \sqrt{\frac{\alpha_v + 1}{\alpha_v + 1}} Re_s = \sqrt{1 + \frac{1}{537/540 + 1}} 635 = 635.$$  

Therefore, the values at (V) are:

$$M_v = \sqrt{M_v^2} = \sqrt{0.01550} = 0.1245,$$

$$\alpha_v = 1.00517$$

$$A_v = 0.0490 \text{ ft}^2$$

$$G_v = 0.0450 \text{ lb}_m / \text{sec-ft}^2$$

$$T_v = 537°R$$

$$T_{0v} = 540°R$$

$$P_v = 22.4 \text{ psf}.$$
Flow in the Exhaust Pipe: Determination of Quantities in the Exhaust Pipe Exit (E)

In this region, isothermal flow is assumed in a constant area, circular duct of 100 diameters length. The isothermal assumption is made to facilitate the analysis, and is based on the fact that for Mach Numbers below \( \sqrt{\frac{1}{\sqrt{\gamma}} } \left( = \sqrt{\frac{3}{5}} = 0.774 \right) \), friction tends to increase the static temperature while heat subtraction tends to decrease it [Eq. 2(1)]. Therefore, as the temperature increases due to friction, a temperature differential is established causing heat to be transferred to the wall (i.e., \( \mathcal{J} T < 0 \)): This, in turn, lowers the temperature.

The offsetting effects of friction and heat transfer thus suppress errors in the isothermal assumption. It is further assumed that the flow is now laminar, since the Reynolds Numbers is constant [by Eq. 8(a), \( dA = dT = 0 \)] and well within the laminar region (\( Re = Re_v = 635 \)), and since this region is sufficiently far removed from the normal shock in the diffuser throat that the shock induced turbulence has probably disappeared. Since the flow is assumed laminar, the friction factor can be found by the familiar relationship [Eq. 10(d)],

\[
\frac{f_v}{Re_v} = \frac{64}{Re_v} = \frac{64}{635} = 0.1008.
\]

Eq. 14(d) can now be used to find the Mach Number in the pipe exit.

For a first approximation, neglect the logarithmic term:

\[
\frac{1}{M_e^2} = \frac{1}{M_v^2} - 1 \cdot f_v \left( \frac{K_v}{D} \right) = \frac{1}{0.01550} - \frac{5}{3} \cdot (0.1008) \cdot 100 = 51.1.
\]
Correcting $M_E^2$:

$$\frac{1}{M_E^2} = \frac{1}{M_{E'}^2} + \frac{k}{2} \ln \frac{M_{E'}^2}{M_E^2} = 511.1 + \frac{50}{2} \ln \frac{0.01550}{511.1}$$

$$\frac{1}{M_E^2} = 50.7$$

or,

$$M_E = \sqrt[3]{50.7} = \sqrt[3]{0.01970} = 0.1403$$

Therefore, the total temperature can be found [Eq. 2(c)], remembering the isothermal assumption:

$$\frac{T_{OE}}{T_{OV}} = \frac{T_{OE}}{T_{OE} - \frac{A_e}{\rho_e}} \approx E = 1 + \frac{M_E^2}{3} = 1 + \frac{0.01970}{3} = 1.00657$$

which gives

$$T_{OE} = T_{OV} E = 537 (1.00657) = 541^\circ R.$$  

For isothermal flow in a constant area duct, Eq. 3(c) gives the static pressure:

$$P_{OE} = \frac{M_E}{M_{OE}} P_v = \frac{0.1245}{0.1403} 22.4 = 19.87 \text{ psfa}$$

The total pressure can now be found from Eq. 2(h):

$$P_{OE} = P_{OE} E = 19.87 (1.00657)^{5/2} = 20.2 \text{ psfa}.$$  

Therefore, the values at the pipe exit (E) are:

$$M_E = \sqrt{M_{E}^2} = \sqrt{0.01970} = 0.1403$$

$$E = 1.00657$$

$$A_E = A_T = 0.0490 \text{ ft}^2$$

$$C_E = C_T = 0.0450 \text{ lb}_m/\text{sec-ft}^2$$

$$T_{OE} = 537 \ ^\circ R \ (= 77^\circ F)$$

$$T_{OE} = 541^\circ R$$

$$P_{OE} = 19.87 \text{ psfa}.$$  

$$P_{OE} = 20.2 \text{ psfa}.$$  

$$Re_E = 635.$$
Flow into the Exhaust Tank: Determination of Quantities at (B) -

In the physical problem, the exhaust tank is a spherical shell (about 12 feet in diameter) which is pumped out to a high vacuum before each test run. In this portion of the analysis: (1) the initial pressure in the tank is neglected, being very small compared to the critical back pressure; (2) the tank is assumed to be perfectly insulated; and (3) the conservative assumption is made: that none of the kinetic energy of the flow in the pipe exit is recovered as static pressure as the flow decelerates to zero velocity in the tank. Therefore the critical back pressure \( P_B = P_{0B} \) is equal to the exit static pressure:

\[ P_{0B} = P_B = P_E = 19.87 \text{ psfa}. \]

The adiabatic assumption gives the temperature in the tank:

\[ T_B = T_{0B} = T_{0E} = 541^\circ R (= 81^\circ F). \]

The running time \( t_o \) can now be calculated as follows: from the equation of state [Eq. 1(a)] (assuming the initial tank pressure to be zero),

\[ P_B = \frac{\omega t_o}{V} R T_B \]

or,

\[ t_o = \frac{P_B}{\omega R T_B} \frac{V}{19.87} \left( \frac{12}{5454} \right)^3 = 39.0 \text{ sec}. \]

The overall total pressure ratio is:

\[ \frac{P_{0R}}{P_{0p}} = \frac{P_B}{P_{0p}} = \frac{19.87}{59.3} = 0.335 = 33.5\%. \]
DISCUSSION

In this problem, (specifically, in the cooling region), the von Karman Modification of the Prandtl Analogy is used to relate the Stanton Number (heat transfer coefficient) to the Reynolds Number, and the von Karman-Nikuradse Formula is employed to find the friction factor as a function of the Reynolds Number. This leads to the result that, in the absence of body forces, external drag forces, or mass injection (or rejection), the total pressure cannot increase. This result is obvious in the case of heat addition, but is surprising in this problem since Eq. 2(k) suggests that for a sufficiently large heat loss rate such as that encountered in the region immediately behind the normal shock, the heat transfer effect on total pressure might overcome the frictional effect.

However, a comparison of Eq. 12(a) and Eq. 2(m) shows that for heat losses ($dT_0 < 0$), the total pressure can increase only if:

$$\frac{f}{4S} \frac{T_0}{T_0 - T_w} < \frac{1}{L}$$

or

$$\frac{f}{4S} < \frac{T_0 - T_w}{T_0 - T_*}$$

where

$$T_* > T_w$$

For a Prandtl Number of $2/3$, and for the range of Reynolds Numbers encountered in this problem ($90 < Re < 650$). Moreover, Eq. 12(c) shows that as the difference between the total temperature and the wall temperature becomes very small, the total pressure begins decreasing drastically. Therefore, no optimum value for the total temperature exists.
APPENDIX

1. Simple Kinetic Theory-

The conditions of simple kinetic theory are tabulated in the General Assumptions. The results of simple kinetic theory are:

(a) \( P = \rho RT \)
(b) \( u = \frac{3}{2} RT \)
(c) \( h = u + \frac{p}{\rho} = \frac{5}{2} RT \)
(d) \( c_v = \frac{3}{2} R \)
(e) \( c_p = \frac{5}{2} R \)
(f) \( k = \frac{c_p}{c_v} = \frac{5}{3} \)
(g) \( \mu = B/\sqrt{T}, \text{ B constant} \)
(h) \( \lambda = \frac{5}{2} c_v \mu \)
(i) \( Pr = \frac{\mu c_p}{\lambda} = \frac{\mu c_p}{5/2 c_v \mu} = \frac{2}{5} k = 2/3 \)

2. One-Dimensional Flow Equations-

These flow equations are derived for the special case of one-dimensional, steady, continuum flow of a constant molecular weight gas which satisfies the conditions of the Perfect Gas Law. Also, no consideration is given to: external work, chemical reaction, phase changes, or mass injection. Since the development of these equations closely follows that of Shapiro [Ref. 15], only the final form of each equation will be presented in most cases. Where a second form appears, the equation has been evaluated for: \( k = 5/3 \).

(a) Equation of State: \( P = \rho RT \)
(b) Continuity Equation: \( \omega = \rho A \omega = \text{constant} \)
(c) Energy Equation (First law):
\[
dq - dw = dh_o = dh + d\left(\frac{\omega^2}{2}\right)\]
where
\[ h_0 = h + \left( \frac{\nu^2}{2} \right) = u + \frac{P}{\rho} + \left( \frac{\nu^2}{2} \right) \]
leading to
\[ \alpha \triangleq \frac{T_0}{T} = 1 + \frac{k-1}{2} M^2 = 1 + \frac{M^2}{3} \]
(d) Second Law:
\[ ds \triangleq \frac{dT}{T}_{\text{rev}} \geq \frac{dT}{T}_{\text{actual}} \]
(e) Combination of First and Second Laws:
\[ T \frac{ds}{c_p} = d\ln T - \frac{R}{c_p} \frac{dP}{P} = d\ln T - \frac{k-1}{k} \frac{dP}{P} \]
(f) Polytropic Equation (Equation of state for isentropic processes):
\[ ds = 0 = \frac{dT}{T} - \frac{k-1}{k} \frac{dP}{P} \]
\[ \frac{dT}{T} = \frac{k-1}{k} \frac{dP}{P} \]
\[ \frac{P}{k T} = \text{constant} \]
leading to \[ \frac{P}{\rho k} = \text{constant} \] and \[ \frac{T}{\rho k-1} = \text{constant} \] .
(g) Local Acoustic Velocity (Velocity of propagation of an infinitesimal disturbance in an Ideal Gas):
\[ c^2 = \frac{\left( \frac{dP}{d\rho} \right)}{\left( \frac{d\rho}{d\rho} \right)} = \frac{\left( \text{Const.} \rho^k \right)}{\rho} = \text{Const.} \frac{\rho}{k} \]
\[ c^2 = k \frac{\rho}{\rho} = k \frac{T}{T} . \]
(h) Dimensionless Pressure Ratio:
\[ \frac{P_0}{P} = \left( \frac{P_0}{T} \right)^{k-1} = \alpha \sqrt{k-1} = \alpha \frac{5}{2} \]
Momentum Equation:
\[ PA + P dA - (P + dP)(A + dA) - T_w (wP) dx = w(\nu d\nu) - wN \]
\[ \frac{dP}{P} = -\frac{k M^2}{2} \left( \frac{d(\nu^2)}{\nu^2} + f \frac{dx}{D} \right) \]
(1) \[ \frac{dP}{P} = -\frac{k M^2}{2} \left( \frac{dT}{T} + \frac{dM^2}{M^2} + f \frac{dx}{D} \right) \]
Influence Coefficients:
(3) \[ \frac{dM^2}{M^2} = \frac{\alpha}{\gamma} \left\{ -2 \frac{dA}{A} + \beta \frac{dT_0}{T_0} + k M^2 f \frac{dx}{D} \right\} \]
where
\[ \alpha \triangleq \frac{T_0}{T} = 1 + \frac{k-1}{2} M^2 = 1 + \frac{M^2}{3} , \]
\[ \beta \triangleq \frac{1 + k M^2}{1 + \frac{M^2}{3}} \]
\[ \gamma \triangleq 1 - M^2 . \]
Alternate Expressions for Entropy Differential:

\[
\frac{d s}{c_p} = \frac{d T_0}{T_0} - \frac{k-1}{k} \frac{d P_0}{P_0}
\]

\[
\frac{d s}{c_p} = \frac{d T}{T} + \frac{k-1}{2} M^2 f \frac{d x}{D}.
\]

Normal Shock Relations:

\[
M_y^2 = \frac{(k-1) M_x^2 + 2}{2k M_x^2 - (k-1)} = \frac{M_x^2 + 3}{5 M_x^2 - 1}
\]

\[
\left(\frac{P_{yX}}{P_{oX}}\right)_{M_x} = \left[\frac{(k+1) M_x^2}{(k-1) M_x^2 + 2}\right]^{\frac{k}{k-1}} \left[\frac{k+1}{2k M_x^2 - (k-1)}\right]^{\frac{1}{k-1}}
\]

\[
\left(\frac{P_{oY}}{P_{oX}}\right)_{M_x} = \left[\frac{4 M_x^2}{5 M_x^2 + 3}\right]^{\frac{3}{2}} \left[\frac{4}{5 M_x^2 - 1}\right]^{\frac{3}{2}}.
\]

Normal Shock Relations for an Infinite Mach Number:

\[
(M_y^2)_{M_u} = \frac{M_x^2}{5 M_x^2} = \frac{1}{5}
\]

\[
\left(\frac{A_p}{A_{T}}\right)_{A u} = \left(\frac{A^*}{A}\right)_{A u} = \frac{A^*_{M_u}}{A_{M_u}} \left(\frac{A^*}{A}\right)_{A u} = \left(\frac{P_{oX}}{P_{oY}}\right)_{M_u} \left(\frac{A^*}{A}\right)_{A u}
\]

\[
\left(\frac{A_p}{A_{T}}\right)_{M_u} = \left[\frac{\left(M_u^2 + 2\right)^{\frac{3}{2}}}{4 M_u^2}\right] \left(\frac{5 M_u^2 - 1}{4}\right)^{\frac{3}{2}} \frac{16 M_u}{9} \alpha_u^2
\]

\[
\left(\frac{A_p}{A_{T}}\right)_{M_u} = \left[\frac{\left(M_u^2\right)^{\frac{3}{2}}}{4 M_u^2}\right] \left(\frac{5 M_u^2}{4}\right)^{\frac{3}{2}} \frac{16 M_u}{M_u^4}
\]

\[
\left(\frac{A_p}{A_{T}}\right)_{M_u} = 0.700.
\]

3. Alternate Expressions of Continuity Equation--

\[
\omega = \frac{P A}{R T} = \frac{P A M}{R T} = \sqrt{k g R T} = \sqrt{k g} \frac{P A M}{R T}
\]

\[
\left(\frac{A^*}{A}\right) = \omega \left(\frac{A^*}{A}\right) \left(\frac{P_o}{P^*}\right) \left(\frac{M^*}{M}\right) = \frac{\alpha^*^2 M}{M^*} \alpha^2 = \frac{16 M^2}{9} \alpha^2
\]
4. Adiabatic Wall (Recovery) Temperature--

The adiabatic wall temperature \( T_{aw} \) is defined as the temperature at the wall of a perfectly insulated pipe in which fluid is flowing under steady conditions.

In the analysis, the total temperature will be used in place of the adiabatic wall temperature wherever low Mach Numbers occur, specifically in the cooling region immediately behind the shock [(D) to (C)]. The error incurred in such a substitution is tolerable, as shown below.

(a) By definition
\[
\rho \sqrt{\frac{R}{K}} = \rho \frac{PAM}{\sqrt{T}} = \rho \frac{PAM}{\sqrt{\gamma T}} = \rho \frac{PAM}{\sqrt{\gamma T_0}} = \rho \frac{PAM}{\sqrt{\gamma T_0}} = \sqrt{\frac{\gamma}{1}} \frac{PAM}{\sqrt{T_0}}
\]

(b) Therefore
\[
\frac{T_{aw}}{T} = \frac{T_{aw}}{T} = \frac{T_{aw}}{T} - 1.
\]

(c) Error:
\[
\text{Error} = \frac{T_{aw}}{T} - \left( \frac{T_{aw}}{T} \right) = \frac{T_{aw}}{T} \left( \frac{1 + \frac{k - 1}{2} M^2}{1 + \frac{k - 1}{2} M^2} \right) - \frac{T_{aw}}{T} \left( \frac{1 + \frac{k - 1}{2} M^2}{1 + \frac{k - 1}{2} M^2} \right)
\]

\[
\text{Error} < \left( \frac{1 - r}{1 - r} \right) \frac{k - 1}{2} M^2 = \frac{1 - r}{3} M^2
\]

From App. 5,
\[
r = \left( \frac{\rho}{\rho_0} \right)^{1/2} = \left( \frac{3}{5} \right)^{1/2} = 0.874
\]

and for \( M^2 < M_0^2 = 1/5 \),

(c) Error:
\[
\text{Error} < \left( 1 - 0.874 \right) \left( 1/5 \right) = 0.040 \%
\]

5. Transport Property Evaluation Temperature--

Eckert's empirical formula for the temperature at which transport properties should be evaluated [Ref. 6] gives

(a) \[
T_E = \frac{1}{2} (T + T_w) + 0.22 r \frac{k - 1}{2} M^2 T = T_m + 0.22 r \frac{k - 1}{2} M^2 T
\]

where \( r \), the recovery factor, \( \frac{T_{aw} - T}{T_{aw} - T} \).

However, at low Mach Numbers, the error involed in using the arithmetic mean film temperature \( T_m \) in place of Eckert's suggested temperature
(T_e) is quite small. For example, assuming \( M^2 < \frac{1}{5} \),
as shown in The Analysis, the Mach Number immediately behind the normal
shock \( M_D \) equals \( \sqrt{\frac{1}{5}} \), after which the flow decelerates to section
(3), and taking \( r = (P_r)^{\frac{1}{3}} = (\frac{2}{3})^{\frac{1}{3}} = 0.874 \)
for fully developed turbulent flow [Ref. 4].

Therefore:

\[
\text{Error} = \frac{T_e - T_m}{T_e} = \frac{0.22 \times 10^{-7} \frac{k^2}{M^3} T}{T_e} < \frac{0.22(0.874)(\frac{1}{3})(\frac{1}{3})T}{0.5T}
\]

(b) Error < 2.56%.

6. Viscosity-

From the results of simple kinetic theory.

(a) \( \mu = B \sqrt{T_m} \).

As shown in the preceding paragraph, \( T_m \) is the arithmetic mean tempera-
ture, \( \frac{1}{2} (T + T_w) \).

The constant \( B \) was evaluated by taking the viscosity at the highest tem-
perature for helium (\( T = 1000^\circ C = 2291^\circ R \)) in the Project SQUID Technical
Report 37 [Ref. 10].

The constant was thus determined to be:

(b) \( B = 7.68 \times 10^{-7} \) \( \frac{\text{lbm}}{\text{sec} \cdot \text{ft}^2} \).

This value checked out very closely at the lower temperature listed
in the same report.

7. Annular Relations--

(a) \( \delta = \frac{D_i}{D_o} \),

(b) \( D_{annulus} = \frac{4 \pi A}{W_P} = D_o - D_i = D_o (1 - \delta) \),

(c) \( A_{annulus} = A_0 - A_1 = A_0 (1 - \delta^2) \)

(d) \( \delta = \sqrt{1 - \frac{A_{annulus}}{A_o}} \).
8. Reynolds Number—

(a) \[ \text{Re} = \frac{\rho x D}{\mu} = \frac{G D}{B T_w} = \frac{G D}{B T_w/2} \frac{1}{\Theta + l} \]

In regions where the cross-section is either circular or annular, and where either \( D_0 \) (if annular) or \( D \) (if circular) is equal to the test section diameter \( (D_T) \) [see Illus.] another form can be used. Such a formula (derived below) can be used in regions of circular cross-section by letting equal zero. Now in an annular region having an outside annular diameter the same as the test section diameter,

\[ G_T A_r = \omega_T = \omega = GA = GA_r(1 - \delta^2) = GA_T (1 - \delta^2) \]

and

\[ D = D_0 (1 - \delta) = D_T (1 - \delta). \]

Therefore,

\[ \text{Re} = \left[ \frac{G_T A_r}{A_T (1 - \delta^2)} \right] \frac{D_T (1 - \delta)}{B T_w/2} \frac{1}{\Theta + l} \]

(b) \[ \text{Re} = \frac{G_T D_T}{B T_w/2} \frac{1}{1 + \delta} \frac{1}{\Theta + l} \]

(c) \[ \text{Re} = \frac{4 \omega}{\pi D_T B T_w/2} \frac{1}{1 + \delta} \frac{1}{\Theta + l}. \]

9. von Karman Modification of Prandtl Analogy—

For turbulent forced convection inside tubes, taking into account the effect of a buffer zone between the laminar sub-layer and the turbulent region, von Karman presents the following equation relating the Nusselt, Prandtl, and Reynolds Numbers [Ref. 8],

(a) \[ N_u^0 \triangleq \frac{H^o D}{\lambda} = \frac{0.0396 \left( R_e \right)^{3/4}}{1 + (R_e)^{-1/4} \left( (Pr - 1) + \ln \left[ 1 + \frac{1}{2} (Pr - 1) \right] \right)} \]

This can be expressed in terms of the Stanton Number as:

\[ S^0 \triangleq \frac{H^o c_p}{\text{Pr} \text{Re}} = \frac{N_u}{\text{Pr} \text{Re}} = \frac{0.0396 \left( R_e \right)^{-1/4}}{1 + (R_e)^{-1/4} \left( (Pr - 1) + \ln \left[ 1 + \frac{1}{2} (Pr - 1) \right] \right)} \]

which becomes, upon substitution of \( \text{Pr} = 2/3 \),

(b) \[ S^0 = \frac{0.0396 \left( R_e \right)^{-1/4}}{1 - 0.659 \left( R_e \right)^{-1/4}} \]

The meaning of the subscript \( (o) \) will be explained where appropriate in The Analysis.
10. Friction Factor---

I. Turbulent Flow--

von Karman-Nikuradse Formula

This formula is given in Ref. 15 [from Ref.'s 9 and 14] as:

\[ \frac{1}{f^{\infty}} = -0.8 + 2 \log_{10} \left( \frac{\text{Re}}{f^{\infty}} \right) \]

It becomes more useful after it has been re-arranged as shown:

\[ \ln \left( e^{1/f^{\infty}} \right) = \ln \left( e^{-0.8} \right) + \frac{1.15}{1.15} \ln \left( \frac{\text{Re}}{f^{\infty}} \right) \]

(b) \[ \text{Re} = \frac{2.51 \cdot e^{1/f^{\infty}}}{f^{\infty}} \]

(c) \[ f^{\infty} = \frac{1.15}{\ln \left( \frac{\text{Re}}{2.51} f^{\infty} \right)} \]

The last equation will be used to determine the friction factor (by iteration) when the Reynolds Number is known and will be employed even for the low Reynolds Numbers encountered in this problem [Ref. 1].

II. Laminar flow:

In the case of fully developed, isothermal, laminar flow, the parabolic velocity profile leads to an exact analytical solution for the friction factor in terms of the Reynolds Number [Ref. 4].

The resulting relationship is:

(d) \[ f = \frac{64}{\text{Re}} \]

11. Heat Balance--

From the First Law,

\[ \dot{q} - \dot{q}_{w} = dh + d \left( \frac{\omega}{2} \right) = dh_{o} = c_{p} \, dT_{o} \]

\[ \omega \, \dot{q} = \omega \, c_{p} \, dT_{o} = P \, A \, \pi \, c_{p} \, dT_{o} = H (wp) \, dk \left( T_{w} - T_{aw} \right) \]

\[ dT_{o} = \frac{4 \cdot H \cdot \frac{(wp)}{4 \pi \cdot c_{p}}} {4 \pi \cdot c_{p}} \left( T_{w} - T_{aw} \right) \, dk \]

From App. 4, \( T_{o} \approx T_{aw} \), and assuming the wall temperature( \( T_{w} \)) to be constant,
(a) \[ dT_0 = -4 \leq (T_0 - T_w) \frac{dx}{D} \]

(b) \[ \frac{dT_0}{T_0 - T_w} = \frac{d\Theta_0}{\Theta_0 - 1} = \frac{d(\Theta_0 - 1)}{\Theta_0 - 1} = -4 \leq \frac{dx}{D} \]

From 11(b), assuming the circular diameter or the outside annular diameter is equal to the test section diameter, constant area implies that \( \dot{\gamma} \) is constant. For this case,

(c) \[ \ln \frac{\Theta_0 - 1}{\Theta_o, - 1} = -4 \frac{\Sigma_2}{\delta} \left( \frac{l_{2x}}{D_T} \right) \left( \frac{1}{1 - \dot{\gamma}} \right) \]

For the case of an annular passage where the outside annular diameter is constant and equal to the test section diameter, and where

\[ \dot{\gamma} = \delta_1 \left( 1 - \frac{\gamma}{\gamma_2} \right) = \delta_1 \left( 1 - \gamma \right), \quad \text{Eq. 11(b) becomes:} \]

\[ \frac{d(\Theta_0 - 1)}{\Theta_0 - 1} = -4 \frac{\Sigma_2}{\delta} \left( \frac{l_{2x}}{D_T} \right) \frac{d\left[ 1 - \delta_1 \left( 1 - \gamma \right) \right]}{1 - \delta_1 \left( 1 - \gamma \right)} \]

Since this case is met only in the smoothing region [(C) to (S)] of The Analysis, the subscript will be altered now: integrating,

\[ \ln \frac{\Theta_o - 1}{\Theta_o, - 1} = -4 \frac{\Sigma_2}{\delta} \left( \frac{l_{2x}}{D_s} \right) \ln \left( \frac{1}{1 - \delta_s} \right) \]

Now, from the configuration [Illus.]

\[ \frac{1}{\delta_s} \left( \frac{l_{cs}}{D_s} \right) = \frac{\cot \phi}{2}, \quad \text{where} \ \phi \ \text{is the semi-cone angle of the "plug".} \]

(d) Therefore

\[ \ln \frac{\Theta_o - 1}{\Theta_o, - 1} = -2 \frac{\Sigma_2}{\delta_s} \left( \cot \phi \right) \ln \left( \frac{1}{1 - \delta_s} \right) \]

or,

\[ \frac{\Theta_o - 1}{\Theta_o, - 1} = \left( \frac{1}{1 - \delta_s} \right)^{-2} \frac{\Sigma_2}{\delta_s} \left( \cot \phi \right) \]

12. Approximate Method for Low Mach Numbers and No Area Change-

From Eq. 2(k),

\[ \frac{dP_o}{P_o} = -k \frac{M^2}{2} \left( \frac{dT_0}{T_0} + \frac{f}{D} \frac{dx}{D} \right) \]

Eliminating \( \frac{dx}{D} \) with the aid of Eq. 11(a) and converting to the non-dimensional total temperature:

(a) \[ \frac{dP_o}{P_o} = -k \frac{M^2}{2} \left[ 1 - \frac{f}{45} \frac{\Theta_o}{\Theta_o - 1} \right] \frac{d\Theta_o}{\Theta_o} \]
Now, using Eq. 3(c) to eliminate \( M^2 \) and non-dimensionalize \( P_0 \),

\[
-2 \int \left( \frac{P_2}{P_{01}} \right) d \left( \frac{P_0}{P_{01}} \right) = k M_i \int \frac{\theta_0^2}{\theta_1^2} \left( \frac{\alpha}{\alpha_1} \right)^\gamma \left\{ \frac{f}{4S} \left( \frac{\theta_0}{\theta_1} - 1 \right) \right\} d \theta_0
\]

and therefore,

\[
(b) \quad 1 - \left( \frac{P_{02}}{P_{01}} \right)^2 = \frac{k M_i^2}{\theta_0^2 \alpha_1^\gamma} \left\{ \frac{\theta_0^2}{\theta_1^2} \left[ f \frac{\theta_0}{4S} \theta_1^2 \theta_0 \left( \frac{\theta_0}{\theta_1} - 1 \right) - 1 \right] d \theta_0 \right\}
\]

Now,

\[
\alpha^\gamma = \left( 1 + \frac{M^2}{2} \right)^\gamma = \left( 1 + \frac{M^2}{2} \right) + \ldots
\]

which suggests that for low Mach Numbers the error in \( \Delta P_0 \) will be small if we assume \( \alpha^\gamma \) to be constant at some mean value. Also since \( S \) varies approximately with the \( \frac{1}{2} \)th power of the Reynolds Number and since \( f \frac{\theta_0}{4S} \) varies only slightly with the Reynolds Number, it is assumed that \( f \) and \( S \) are constant at suitably chosen mean values. With these assumptions, the right side of Eq. 12(a) can now be integrated as shown below:

\[
1 - \left( \frac{P_{02}}{P_{01}} \right)^2 = \frac{k M_i^2}{\theta_0^2 \alpha_1^\gamma} \left\{ \frac{\theta_0^2}{\theta_1^2} \left[ f \frac{\theta_0}{4S} \theta_1^2 \theta_0 \left( \frac{\theta_0}{\theta_1} - 1 \right) - 1 \right] d \theta_0 \right\}
\]

Following Shapiro [Ref. 15], the arithmetic mean for \( \alpha^\gamma \) is used in evaluating the equation, giving

\[
(d) \quad 1 - \left( \frac{P_{02}}{P_{01}} \right)^2 = \frac{k M_i^2}{\theta_0^2 \alpha_1^\gamma} \left\{ \frac{\theta_0^2}{\theta_1^2} \left[ \int f \frac{\theta_0}{4S} \theta_1^2 \theta_0 \left( \frac{\theta_0}{\theta_1} - 1 \right) d \theta_0 \right] \right\}
\]

The assignment of mean values to \( f \) and \( S \) will be taken up where appropriate in the Analysis. Once \( Q_2 \) has been selected, \( \alpha^\gamma_2 \) has been guessed and \( \frac{f}{4S} \) has been evaluated, \( P_0 \) can be calculated from the last equation. From this result, Eq. 3(c) can be used to determine \( M_2^2 \):

\[
(e) \quad M_2^2 = \left( \frac{\alpha_2}{\alpha_1} \right)^\gamma \left( \frac{Q_2}{Q_1} \right) \left( \frac{P_{02}}{P_{01}} \right) M_1^2
\]
Now $\alpha^2_2$ can be corrected and the calculations carried through again, until convergence is obtained.

13. Approximate Method for Low Mach Numbers--

From Eq. 2(j),
\[
\frac{dM^2}{M^2} = \frac{\alpha}{\gamma} \left( -2 \frac{dA}{A} + \beta \frac{d\Theta_0}{\Theta_0} + k \frac{M^2 f}{D} \right)
\]
where:
\[
\alpha = \frac{T_c}{\gamma} = 1 + \frac{7}{2} M^2
\]
\[
\beta = 1 - \frac{7}{2} M^2
\]
\[
\gamma = 1 - M^2
\]

Eliminating $M^2$ from the friction term with the aid of Eq. 3(c), and applying the equation to the smoothing region [(C) to (S)]:
(a)\[
\frac{\gamma}{\alpha} \frac{dM^2}{M^2} = -2 \frac{dA}{A} + \beta \frac{d\Theta_0}{\Theta_0} + k \frac{M^2 f}{D} \frac{(P_{0c})}{(P_0)} \left( \frac{A_c}{A} \right) \left( \frac{c}{c_c} \right)^4 \frac{dx}{D}
\]

From the configuration that has been adopted,
\[
A^2 = A_0^2 (1 - s^2)^2 = A_c^2 (1 - s)^2
\]
\[
P = D_0 (1 - s) = D_c (1 - s)
\]
Assuming a linear relationship,
\[
\delta = \delta_c (1 - \frac{y}{A_c}) = \delta_c (1 - z)
\]
and from App. 11
\[
cot \phi = \frac{\delta}{\delta_c} \left( \frac{D_0}{D_c} \right)
\]

Using these facts, the last equations can now be re-written:
(b)\[
\int \frac{dM^2}{M^2} = -2 \int \frac{dA}{A} + \beta \int \frac{d\Theta_0}{\Theta_0} + k \frac{M^2 f}{2 (\tan \phi)} \left( \frac{A_c}{A} \right)^2 \left( \frac{c}{c_c} \right)^4 \frac{dx}{D}
\]
\[
\times \left[ \frac{(\alpha_c)^4 \Theta_0}{\Theta_0} \left( \frac{P_{0c}}{P_0} \right)^2 \right] \int_{0}^{\delta_c} \frac{ds}{(1 - s^2)^2 (1 - s)}
\]

where mean values are indicated by bars.

This reduces to:
(c)\[
\int \left\{ \frac{M^2}{M_c^2} \left( \frac{A_c}{A_c} \right)^2 \frac{\alpha}{\gamma} \frac{d\delta}{\delta} \right\} = k \frac{M^2 f}{2 (\tan \phi)} \left( \frac{A_c}{A_c} \right)^2 \frac{\alpha}{\gamma} \frac{\delta_c}{\delta}
\]
\[
\times \left[ \frac{(\alpha_c)^4 \Theta_0}{\Theta_0} \left( \frac{P_{0c}}{P_0} \right)^2 \right] \int_{0}^{\delta_c} \frac{ds}{(1 - s^2)^2 (1 - s)}
\]

The remaining integral can be evaluated as follows:
\[
\int_{0}^{\delta_c} \frac{ds}{(1 - s^2)^2 (1 - s)} = \int_{0}^{\delta_c} \frac{ds}{(1 - s)^2 (1 + s)^2}
\]
\[
\int_0^{\delta_c} \frac{d\delta}{(1-\delta)^2} = \int_0^{\delta_c} \left( \frac{1}{4(1-\delta)^2} + \frac{1}{8(1-\delta)^2} + \frac{3}{16 \ln \frac{1+\delta_e}{1-\delta}} \right) d\delta
\]

\[
\int_0^{\delta_c} \frac{d\delta}{(1-\delta)^2 (1-\epsilon)} = \left\{ \frac{1}{16 \ln \frac{1+\delta_e}{1-\delta}} \right\}
\]

In this problem, \( \delta_c = \delta_D = \sqrt{1 - \left( \frac{A_y}{A_T} \right) \frac{M_T}{M_{c,T}} = \sqrt{1 - 0.700} \)

leading to: \( \delta_c = 0.548 \).

Therefore:

\[
\int_0^{0.548} \frac{d\delta}{(1-\delta)^2 (1-\epsilon)} = \left\{ \frac{1}{4} + \frac{1}{8(1-0.54)}^2 + \frac{1}{8(1-0.54)} + \frac{3}{16 \ln \frac{1+0.54}{1-0.54}} \right\}
\]

Substituting back into the last equation:

\[
(e) \ln \left\{ \frac{M_c^2}{M_T^2} \left( \frac{A_T}{A_y} \right)^{\frac{3}{2}} \frac{\partial P}{\partial \delta} \left( \frac{\partial \epsilon}{\partial \delta} \right)^{\frac{3}{2}} \right\} = \frac{k T_{\epsilon 2} M_c^4}{2 (\ln \delta)} A_T \left( \frac{\partial \epsilon}{\partial \delta} \right)^{\frac{3}{2}} \ln \left( \frac{1.066}{\frac{\epsilon}{\epsilon_0}} \frac{P_0}{(P_0)^2} \right).
\]

14. Isothermal Flow in a Constant Area Duct-

from Eq. 2(i),

(a) \( \frac{dP}{P} + \frac{k M^2}{2} \left( \frac{dT}{T} + \frac{dM^2}{M^2} + f \frac{dx}{D} \right) = 0 \),

where \( dT = 0 \) because the flow is isothermal. Due to this fact, plus

the fact that the area is constant, Eq. 3(c) takes this form

(b) \( G \frac{\frac{RT}{\xi g_c}}{\xi} = \text{constant} = P M \)

so that the logarithmic differentiation yields

(c) \( \frac{dP}{P} = -\frac{1}{2} \frac{dM^2}{M^2} \).
Substituting into Eq. 14(a),
\[
(1 - \frac{1}{k M^2}) \frac{d M^2}{M^2} + f \frac{d x}{D} = 0
\]
which leads to:
\[
\int_0^x f \left( \frac{D}{D} \right) d x = \int \frac{M^2}{K} \left( \frac{k M^2}{k M^2} - 1 \right) d M^2
\]
Since the duct area is constant and the flow is isothermal, the Reynolds Number will be constant [Eq. 7(a)]. Assuming that the flow is laminar, the friction factor is related to the Reynolds Number by the familiar expression [Eq. 10(d)]
\[
f = \frac{64}{
\]
and therefore the friction factor will be constant. Performing the integration:
\[
\int f \left( \frac{D}{D} \right) = \frac{1}{k} \left( \frac{1}{M_1^2} - \frac{1}{M_2^2} \right) + \ln \frac{M_1^2}{M_2^2}
\]
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