A STUDY OF THE QUASI-FRICTIONAL FORCES
RESULTING ON SERRATED CLAMPING ELEMENTS
by
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ABSTRACT

It is intuitive that a clamping element will grip better if the surface is provided with teeth which penetrate into the surface of the clamped object. This paper is concerned with the problem of determining a quantitative comparison of the gripping ability of various penetrator shapes. It is presumed that the effects noted for a single penetrator will be applicable to an array of penetrators as on a serrated clamping element.

Mathematical considerations show that a vertical penetrator load is held in equilibrium by the vertical components of the yield pressure which act on the penetrator surfaces. This is mathematically equivalent to the force given by the yield pressure acting over the area of the indentation projected on a plane normal to the direction of the load. This paper endeavors to show that the same concept will apply to forces applied transversely. Therefore, the maximum possible transverse force which can be applied to a penetrator is theoretically equal to the product of the yield pressure of the penetrated material and the area of the penetrator face projected on a plane normal to the transverse force.

Plasticity considerations show that the yield pressure for a given material varies according to the penetrator shape. Strain hardening is also a factor in most materials.

The influence of conventional friction is also shown
to vary with the penetrator shape. In some cases vectorial addition of the frictional effect is required, but special cases permit algebraic addition. It is shown analytically that a friction reversal occurs at the surface of the penetrator as the transverse force reaches its maximum value. The result of the friction reversal is to lower the yield pressure against the penetrator face.

Experiments show that the foregoing theories are justified for the cases tested.
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TABLE OF NOTATION

A  Cross sectional area of ploughed out track
a  Meyer Constant
\( a_1, a_2 \)  Horizontal force components
\( b_1, b_2 \)  Vertical force components
C  A constant
\( c_1, c_2 \)  Forces normal to wedge face
\( d \)  Width of scratch (Meyer scratch hardness equation)
\( d, d' \)  Depth of penetration Eq. (1)
d  Impression diameter
\( d_b \)  Impression diameter from 10 mm. ball - mm.
D  Arbitrary ball diameter - in.
F  Transverse force
\( F_c \)  Cutting force (referring to metal cutting; comparable to \( F \))
\( F_t \)  Thrust force (referring to metal cutting; comparable to \( L \))
\( F_f \)  Frictional force along wedge face.
L  Axial load on penetrator
n  Meyer strain-hardening exponent
P  Load in Kg. (Meyer scratch hardness equation)
\( P \)  Ploughing force (Bowden, Tabor, and Moore)
\( p \)  Normal yield pressure
\( P_m \)  Mean pressure (where pressure distribution is uncertain)
\( P_r \)  Meyer scratch hardness number
R  Resultant force
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INTRODUCTION

An effect closely associated with static friction is that obtained by a surface containing a number of wedge-like elements which engage another more ductile surface by means of plastic penetration. Gripping surfaces of this type are commonly called serrated surfaces. We should expect that resistance to relative motion between the two engaged surfaces would be greater than the corresponding case of two relatively smooth surfaces under the same normal load.

The principle of the serrated clamping member to increase resistance to slippage is used in many tools. In most cases, designers of equipment using serrated clamping members will assume frictional coefficients established for non-penetrating type surfaces. This practice is probably justified on the basis that the clamping device will continue to give satisfactory service even after the penetrators have worn flat. Yet, it is apparent that certain advantages would be gained if more exact relationships between normal loads and apparent frictional forces were known.

If such relationships were known for serrated surfaces containing penetrators of various shapes and sizes, the designer would have a more rational basis upon which to design the device required to produce or sustain the normal load. Furthermore, he could choose a penetrator
shape which would fulfill the clamping requirements, yet whose indentation would not seriously damage the clamped member. We should expect such relationships to involve factors other than those present in the case of ordinary friction.

It is the purpose of this paper to discuss the factors involved in the analytical approach to this problem. A further object is to present an experimental method of obtaining the type of information required, to tabulate some typical relationships obtained therefrom, and to compare some of these relationships with those obtained by analytical methods.

The first step in the study of the penetrating type of clamping element might well be the study of the manner in which single penetrators of various geometrical shapes behave under the influence of axial and transverse loads in a plastic medium. The axial load would be regarded as corresponding to the normal load and the transverse load would correspond to the apparent frictional force. In our study we assume that the total normal and apparent frictional loads acting on a serrated gripping surface would be equally divided among the individual penetrators.

Considering the case of an individual penetrator, we recognize that the loading is similar to that of the stylus in a scratch hardness test. Meyer, in his well
known work on hardness testing (1) defines scratch hardness as the specific pressure between the bearing surface of a cone point and the material, determined by the relationship of the vertical load on the cone point to the bearing surface area projected on a horizontal plane, or:

\[ p_r = \frac{2P}{\pi d^2/4} \]

where \( p_r \) = scratch hardness number in kg./mm\(^2\).

\( P \) = load in kg.

\( d \) = width in millimeters of a scratch produced by a cone pointed stylus.

In a scratch hardness test it is understood that the entire load must be supported by the half of the point surface which lies in the direction of movement. This fact is accounted for in the numerator of the above equation.

Meyer also refers to \( p_r \) as the specific compressive resistance of the material. This might also be regarded as the pressure required to cause plastic flow under the conditions imposed. It is apparent that the transverse components of this same pressure would constitute the force required to propagate the scratch if the effect of coulomb friction is neglected.

This concept was discussed in a paper by Bowden,
Tabor and Moore\(^{(2)}\) in which it was suggested that the total force of friction might be expressed as:

\[ F = S + P \]

in which \( S \) is the effect of friction in the usual sense and \( P \) is the force required to displace the softer metal from the path of the slider. This is expressed as:

\[ P = A'p \]

where \( A' \) is the cross-sectional area of the track ploughed out by the slider, and \( p \) is the pressure required to cause plastic flow of the softer metal.

Experiments by the authors show that these relationships are very nearly true for the cases tested. The forces involved were small since the very soft metal indium was used as the plane surface and the depth of penetration was of a low order of magnitude. In each case cited, the slider produced a track whose cross section was a circular segment. A sphere, a cylinder of substantial length, and a cylinder of very minute length (a spade) were tested, the latter being used to eliminate the area over which the usual friction might act. The experimental results consisted of curves of the transverse force plotted as a function of track width. No data was given showing the maximum transverse force possible as a function of the axial load applied.
FIG. 1 STRESS-STRAIN RELATIONSHIPS SHOWING CONCEPT OF IDEALLY PLASTIC MATERIAL
According to the theory of plasticity, the pressure required by a penetrator to produce plastic flow of an ideally plastic material (see Fig. 1) depends not only on the yield characteristics of the material, but also on the geometrical shape of the penetrator. Prandtl (3) and Honsky (4), who were forerunners in the field of plasticity, determined the stress distribution caused by penetration of various two-dimensional indenters. These solutions did not consider the deformation of the surface due to the displacement of the metal. Later work by Hill, Lee, and Tupper (5) did consider in detail the effect of the displaced metal in the case of a two-dimensional, perfectly lubricated wedge under the influence of an axial load.

Results of wedge indentation experiments by Dugdale (6) were published in 1953. The object in this case was to calculate, by means of his experimental data, a value of shearing yield stress which he compared with a value he obtained by other experimental means not associated with indentation. The effect of friction and strain hardening were considered in his calculations.

Although knowledge of the plastic behavior of the material is of extreme importance in understanding what occurs in the application of an axial and a transverse load, a statical relation between the two forces may easily be derived if the penetrator is straight sided and sharp pointed. This approach is used by Merchant (7) in
the study of metal cutting. The relation between the cutting force and the thrust force is quite analogous to the type of relation we seek. This analogy will be dealt with at length in the following pages. It will be shown that the force relations are independent of the material or of strain-hardening effects if the penetrator is straight sided and sharp edged. Since geometric similarity of strain is maintained at any stage of penetration, the average strain hardness would be constant.

Penetration by a spherical indenter would involve an increasing value of mean pressure required to produce plastic flow as the penetration progressed. This is due to the increasing average strain of the metal surrounding the indentation which results in increased strain hardening. This effect has been analyzed by Meyer as a major part of his aforementioned work on hardness testing, and will be discussed later in connection with our investigation of spherical tipped penetrators.

Concerning the present paper, it is of interest to note the case of the "Drill Pipe or Tubing Slip" used extensively in oil field operations. The "Slip" is a tool which is used to grip and support the string of pipe which hangs vertically in the well. The weight of this string may be in excess of 100 tons. A typical type of slip construction is shown schematically in Fig. 2. In operation, the pipe, upon contacting the slip segments, drags the segments deeper into the conical
FIG. 2 SCHEMATIC DRILL PIPE SLIP

FIG. 3 (A) INITIAL STAGES OF PENETRATION UNDER AXIAL AND TRANSVERSE LOADING

FIG. 3 (B)
bowl causing the penetrators on the clamping surface of the slip segments to be forced deeply into the surface of the pipe. The downward motion continues until the penetrator engagement is sufficient to completely support the weight of the pipe. This might be considered as a self-energized type of tool.

It may be shown by analysis that a coefficient of friction greater than that obtainable by non-penetrating surfaces is required for the self-energization of the slip.

The indentations produced on the surfaces of oil well tubular goods by the use of penetrating slip elements have been recognized as potential points of failure. This is especially true in the case of drill pipe, where the pipe is subjected to combined tension, torsion, and bending stresses. Obviously, sharp pointed or sharp wedge-like penetrators are the worst offenders in their ability to produce points of high stress concentration, yet these shapes are most commonly used. Although round nosed penetrators may not grip as well as the sharp pointed ones, we have little information available regarding their gripping ability.

This paper is presented in the hope that it may provide some information which may be useful in further study relating to problems of this nature.
Mode of Penetration

We consider first the simplest case, that of a two-dimensional wedge having straight sides and a sharp loading edge. We assume that the width is great enough compared to the anticipated depth of penetration to render any end effects negligible.

We must realize that a symmetrical wedge loaded axially transmits equal forces from each of its surfaces to the penetrated medium. If we assume that we have an ideally plastic material (see Fig. 1), then the unit contact pressure at the wedge face will always be the same regardless of how much the load is increased. The area of contact will simply be increased until equilibrium is obtained. Fig. 3(a) indicates a wedge held in equilibrium by the forces $L_1$, $c_1$, and $c_2$, where $L$ is the axial load per unit width and $c_1$ and $c_2$ are the resultants of the normal yield pressure acting over each of the contact areas. If we let:

\[
d = \text{the depth of penetration}
\]
\[
p = \text{normal yield pressure (constant)}
\]

we may say,

\[
c_1 = c_2 = \frac{d}{\cos \alpha} \cdot p
\]

and $a_1$, $a_2$, $b_1$, and $b_2$ are the horizontal and vertical components respectively.
If a small horizontal force $F$ per unit width is applied, the horizontal and vertical components must still bear their original ratio (if perfect lubrication exists). From Fig. 3(b) we see that for equilibrium:

$$a_2' = a_1' + F$$

Since $a_1$ has become smaller, $a_1$ must also have decreased. We therefore conclude that the force $a_1'$ is due to elastic compressive stress over the wedge face. By the same reasoning $a_2$ becomes greater, but since $p$ cannot increase, the contact area must increase. This can occur only if the wedge sinks deeper into the material.

Thus:

$$a_1' = \frac{d'}{\cos \alpha} \cdot \sigma_e, \quad a_2' = \frac{d'}{\cos \alpha} \cdot p$$

where $d'$ = the new depth of penetration

$\sigma_e$ = elastic compressive stress

Then

$$a_1' = a_1 \cos \alpha = d' \sigma_e$$

$$a_2' = d' \sigma_e + F$$

$$a_2' = \frac{a_2'}{\cos \alpha} = \frac{d' \sigma_e + F}{\cos \alpha}$$
equating both expressions for $c_2^i$:

\[
\frac{d^i \sigma \cdot F}{\cos \alpha} = \frac{d^i p}{\cos \alpha}
\]

\[
F = d^i (p - \sigma_e)
\]  
(1)

In order to express $d^i$ in terms of the axial load $L$, we may write:

\[
b_1^i = c_1^i \sin \alpha = \frac{d^i \sigma \cdot \sin \alpha}{\cos \alpha} = d^i \sigma_e \tan \alpha
\]

\[
b_2^i = c_2^i \sin \alpha = \frac{d^i}{\cos \alpha} \cdot p \sin \alpha = d^i p \tan \alpha
\]

but

\[
L = b_1^i + b_2^i = d^i \sigma_e \tan \alpha + d^i p \tan \alpha
\]

\[
= d^i (\sigma_e + p) \tan \alpha
\]

therefore

\[
d^i = \frac{L}{(\sigma_e + p) \tan \alpha}
\]

substituting:

\[
F = \frac{L(p - \sigma_e)}{(\sigma_e + p) \tan \alpha}
\]  
(2)

We find that the maximum $d^i$ and the maximum $F$ both occur when $\sigma_e$ becomes zero. It is at this instant that the entire axial load $L$ is borne by the forward face of the wedge. Since $F$ cannot be increased beyond this point, we define our point of failure as the instant when $\sigma_e$ becomes zero.

Hereafter we will be concerned only with the forces
acting against the forward face of the penetrator, regardless of the shape of the penetrator, since the same reasoning will hold for all two-dimensional cases and is assumed to hold in the three-dimensional cases.

Rewriting equations (1) and (2) we have for failure:

\[ F = pd^* \] (3) \quad or \quad \[ F = \frac{L}{\tan \alpha} \] (4)

We note that the term \( pd^* \) is simply the pressure at which plastic flow will occur multiplied by the area (since unit width is assumed) of the wedge in contact with the material projected on a plane perpendicular to the direction of the applied transverse force \( F \). This is equivalent to the "ploughing term" as given by Bowden, Tabor and Moore\(^{(2)}\), and is independent of the angle of the penetrator.

The expression \( L/\tan \alpha \), however, more useful for the special case of the sharp-nosed penetrator since it is independent of the plastic properties of the material.

**Effect of Friction**

In order to consider the effect of conventional friction we follow the analysis which has been given by M. E. Merchant in his well known study of the mechanics of the metal cutting process\(^{(7)}\). An expression is derived which relates the cutting force \( F_c \) and the thrust force \( F_t \) in terms of the geometry and the coefficient of friction \( \mu \) of the system. This analysis, as
FIG. 4 TOOL WITH POSITIVE RAKE ANGLE

FIG. 5 TOOL WITH NEGATIVE RAKE ANGLE
Merchant points out, is applicable whether the rake angle is positive or negative. The rake angle of a cutting tool is defined as the angle between the face of the tool along which the chip flows, and a line normal to the cutting direction. Positive and negative rake angles are further defined as shown in Figs. 4 and 5.

It will be shown that a wedge shaped penetrator, loaded axially and transversely, is merely a special case of a negative rake tool and conforms to Merchant's analysis with one exception. In the usual metal cutting operation, the tools are rigidly supported so that the thrust force will always bear a definite relation to the cutting force applied. However, as our problem is defined, we wish to know what cutting force must be applied for failure of the metal to occur if the thrust load is held constant. As we shall see, the latter problem involves a reversal of the frictional component along the tool face.

In his paper, Merchant considers the chip as an independent body held in equilibrium by the action of two equal and opposite forces $R$ and $R'$, (see Fig. 6) The force $R$ actually is transmitted to the chip at the shear plane and is the resultant of the shear and the normal forces acting along the shear plane. However, $R$ may also be resolved into vertical and horizontal components $F_t$ and $F_n$, which represent the thrust and the cutting forces respectively which must be opposed
FIG. 6  FORCE SYSTEM ACTING IN CASE OF POSITIVE RAKE TOOL SHOWING CHIP IN EQUILIBRIUM (FROM MERCHANT)
EQUILIBRIUM STATES WITH SAME THRUST FORCE AND REVERSED FRICTIONAL DIRECTION.

FORCE SYSTEM ACTING IN CASE OF NEGATIVE RAKE TOOL (WEDGE) SHOWING PLASTIC ZONE IN TWO EQUILIBRIUM STATES WITH SAME THRUST FORCE AND REVERSED FRICTIONAL DIRECTION.
reactively by the tool.

The corresponding case for a tool with large negative rake is shown in Fig. 7. This corresponds more closely to our problem. However, we will consider as the free body, the metal occupying the zone of plasticity adjacent to the tool face. The shape of this zone is treated at length by Hill, Lee, and Tupper (5), but is not of extreme importance in this phase of our discussion.

Here again, we have a normal force and a frictional force whose resultant opposes the flow of metal along the tool face. The equilibrating forces are transmitted to the plastic zone through the shear and compressive stresses acting along the elastic-plastic boundary, and resolve into the vertical and horizontal components \( F_t \) and \( F_c \) as shown. Since \( R \) and \( R' \) are equal and opposite forces we may represent their components as components of a single vector \( R \) as shown in Fig. 7.

From the figure, it is evident that if metal is being displaced outward along the face of the wedge, that is to say, if the frictional force acts in the direction \( DC \), then the relationship between \( F_c \) and \( F_t \) may be expressed:

\[
F_c = F_t \cot (\phi + \alpha)
\]
However, we have defined our particular problem as one in which \( F^t \) must remain constant, and \( F^c \) is increased until equilibrium no longer exists. If we let the transverse force equal \( F^t_0 \), holding \( F^t \) constant, it is evident that equilibrium will still exist, the absence of the frictional component implying that the metal no longer tries to flow along the tool face. Increasing the transverse force to \( F^t'' \), we find equilibrium conditions still satisfied since the frictional force \( F^t''_0 \) has changed direction.

We may therefore express the magnitude of the maximum possible transverse force \( F^t''_0 \) which may be applied to a wedge while it is under the influence of an axial force \( F^t \) as:

\[
F^t''_0 = F^t \cot(\alpha - \phi) = \frac{F^t}{\tan(\alpha - \phi)}
\]

or

\[
F^t''_0 = F^t \left(1 + \tan \alpha \tan \phi \right) \left(\tan \alpha - \tan \phi \right)
\]  

(5)

but \( \tan \phi = \mu = \) coefficient of friction, therefore:

\[
F^t''_0 = F^t \left(1 + \mu \tan \alpha \right) \left(\tan \alpha - \mu \right)
\]  

(6)

This is the equivalent of equation (4), where \( F = F^t''_0 \), \( L = F^t \), \( \mu = 0 \).

As mentioned previously the plastic flow pressure will depend not only on the penetrated medium but also on the geometrical shape of the penetrator. In the
special case given above, the frictional direction will also substantially affect the flow pressure.

**Plastic Behavior**

We therefore consider at this point the behavior of the plastic region in contact with the tool face under the influence of the friction reversal. In order to find a solution, we will assume that we have an ideally plastic material, that is, one which exhibits a constant yield stress which cannot be exceeded. We will also adopt the Tresca criterion for plastic flow which states that the difference between the two principal stresses remains constant, and the difference is equal to twice the maximum shear stress, or:

\[ (\sigma_1 - \sigma_2) = 2\tau_{\text{yield}}, \text{ constant} \quad (7) \]

We will further assume a condition of plane strain which assumes no strain of the material normal to the plane considered.

It has been shown by Hill, Lee, and Tupper\(^5\) that a solution exists for a wedge penetrating vertically under the foregoing conditions. A feature of the solution is that the surface of the displaced metal remains straight. Also, the solution provides a uniformly distributed pressure on the wedge. As our problem was originally defined, the normal load was to be applied first, thus following the solution of Hill, Lee, and Tupper. However,
as we have previously shown, the subsequent penetration is oblique during the application of the transverse load. Now the metal is displaced only in the direction of the transverse load, and the geometry of the aforementioned solution no longer applies. However, we will assume that our type of penetration also produces a straight lip of displaced metal, assuming that we may determine the angle by graphical methods if necessary.

Thus we consider the plastic region in Fig. 8 bounded by a free surface AB, a surface under an unknown pressure distribution BC, and a plastic-elastic boundary AC of which little is known.

In general, the solution of two-dimensional plasticity problems consists of satisfying the boundary conditions, satisfying the equilibrium equations which are true for any material, and satisfying a third equation, the "condition of plasticity," which replaces the compatibility equation used in elasticity problems. When rectangular coordinates are considered, the equilibrium equations in the absence of body forces are:

\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \]  \hspace{1cm} (8)

\[ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} = 0 \]  \hspace{1cm} (9)
RAISED LIP DUE TO OBLIQUE PENETRATION

INITIAL AXIAL INDENTATION

FIG. 8 TYPE OF DEFORMATION DUE TO TRANSVERSE LOADING

FIG. 9 ZONING OF ASSUMED PLASTIC REGION
The condition of plasticity (see p. 20) is:

\[ (\sigma_v - \sigma_z) = 2T_{\text{YIELD}} = \text{constant} \]  

(7)

If the stresses are not given in the principal directions the condition may be expressed as:

\[ \left( \frac{\sigma_y - \sigma_x}{2} \right)^2 + T_{xy}^2 = T_{\text{YIELD}}^2 = \text{constant} \]  

(10)

In polar coordinates, the equilibrium equations are:

\[ \frac{\partial \sigma_\theta}{\partial \theta} + 2T_{r\theta} + r \frac{\partial T_{re}}{\partial r} = 0 \]  

(11)

\[ (\sigma_r - \sigma_\theta) + r \frac{\partial \sigma_r}{\partial r} + \frac{\partial T_{re}}{\partial \theta} = 0 \]  

(12)

In polar coordinates, the condition of plasticity is expressed as:

\[ \left( \frac{\sigma_\theta - \sigma_r}{2} \right)^2 + T_{r\theta}^2 = T_{\text{YIELD}}^2 = \text{constant} \]  

(13)

To satisfy the boundary conditions of our problem, it is necessary to divide the plastic region into three zones as shown in Fig. 9. The two zones adjacent to the surfaces AB and BC respectively each will be shown to have a different but uniform stress state throughout, while the sector lying between these two zones will be a zone of transition which provides stress continuity between the other two zones.

Considering first zone I, we let the inclined
surface of the raised lip coincide with the x axis and investigate the stress state in an element on the surface AB (Fig. 10). Since AB is a free surface, we know that \( \sigma_y = 0 \) and \( \tau_{xy} = 0 \). Since a principal plane may be defined as one in which no shear stress exists and whose normal stresses are maximum or minimum, we may say that the surface along the x axis is a principal plane and both x and y are principal directions.

It is apparent that the assumption of a uniform stress state throughout zone I satisfies equations (8) and (9).

By equation (10) we have:

\[
\left[ \frac{0 - \sigma_x}{2} \right]^2 + 0 = \tau_{\text{yield}}^2; \quad \sigma_x = -2 \tau_{\text{yield}}
\]

A solution would also exist for \( \sigma_x = +2 \tau_{\text{yield}} \), but we reject it in favor of the former solution, since we intuitively expect that \( \sigma_x \) shall be a compressive stress due to the physical conditions of the problem.

As we know from elementary theory, a plane of maximum shear stress occurs at \( 45^\circ \) to the principal planes. Thus we may represent the planes of maximum shear stress with a series of lines \( \pm 45^\circ \) from the free surface AB. This defines the shape of zone I, although we still do not know the length of AB.
Fig. 10 Stress condition throughout Zone I

$\sigma_r = \sigma_\theta = \sigma_n$; $\sigma_n$ constant with change of $r$, $\sigma_n$ varying with $\theta$

Fig. 11 Assumed stress condition in Zone II

Mohr Stress Circle for Zone I

$\sigma_y = \sigma_x = \sigma_n$

$\tau_{xy} = -\tau_m$
Considering zone III on an independent rectangular coordinate system with BC coincident with the x axis (Fig. 12), we investigate an element at the surface subjected to a normal pressure \( p = -\sigma_y \) and a shear stress \( \mu \tau \), where \( \mu \) is the conventional coefficient of friction.

Although we do not have enough information at this point to determine the stress state by use of a Mohr stress diagram, we can make a schematic construction (see Fig. 12) which incorporates the stress relationships we do know. Since the radius represents the maximum shear stress, a quantity which we might know for a given material, we may draw it to scale. However, the location of the center of the circle is as yet undetermined, and will affect the value of \( p \).

We observe from Fig. 12 that the principal direction is displaced by an angle \( \omega \) from the direction in which \( p \) acts. Therefore we rotate the lines of maximum shear by an angle \( \omega \) since they must lie at an angle of 45° from the principal direction.

To determine the orientation of the planes of maximum shear stress we consider the stresses acting on a triangular element bounded by a surface coincident with the x axis and by two mutually perpendicular planes (Fig. 14). If we call these planes \( x' \) and \( y' \) we may write the following equilibrium equations:
FIG. 12 ASSUMED STRESS CONDITION THROUGHOUT ZONE III

FIG. 13 CHANGE IN SHAPE OF PLASTIC REGION DUE TO FRICTION REVERSAL
FIG. 14  ELEMENT CONSIDERED IN ORIENTATION OF MAXIMUM SHEAR PLANES
\[
\Sigma F_x = 0: \quad \sigma_y \sin \gamma = \sigma_x \sin \gamma - \tau_{xy} \cos \gamma + \tau_{yx} \cos \gamma
\]

\[
\Sigma F_y = 0: \quad \sigma_y \cos \gamma = \sigma_y \cos \gamma - \tau_{xy} \sin \gamma - \tau_{yx} \sin \gamma
\]

where the area upon which \( \sigma_y \) acts is taken as unity.

Solving for \( \sigma_y \) in each equation and equating the results, we have:

\[
\sigma_x - \tau_{xy} \frac{\cos \gamma}{\sin \gamma} + \tau_{xy} \frac{\cos \gamma}{\sin \gamma} = \sigma_y - \tau_{xy} \frac{\sin \gamma}{\cos \gamma} - \tau_{yx} \frac{\sin \gamma}{\cos \gamma}
\]

\[
(\sigma_x - \sigma_y) = \tau_{xy} \left[ \frac{\cos \gamma - \sin \gamma}{\sin \gamma} \right] = -\tau_{xy} \left[ \frac{\cos \gamma + \sin \gamma}{\sin \gamma} \right]
\]

Multiplying through by \( \sin \gamma \cos \gamma \), we have:

\[
\left[ \frac{\sigma_x - \sigma_y}{2} \right] \sin 2\gamma - \tau_{xy} \cos 2\gamma = -\tau_{yx} (14)
\]

If \( \gamma \) is the angle between a plane of maximum shear stress and the surface upon which \( \sigma_y \) acts, then we may say:

\[
\cos 2\gamma = \frac{\tau_{xy}}{\tau_{yx}} (15)
\]

since it can be shown that the normal stresses acting on perpendicular planes of maximum shear are equal.

The first term of equation (14) is thereby eliminated.
For the special case of zone III, \( \tau_{xy} = -\mu p \) and \( \tau_{xy} = \tau_m \) (see Fig. 9), therefore:

\[
\cos 2\gamma = -\frac{\mu p}{\tau_m} ; \quad \gamma > 45^\circ
\]

which indicates counter clockwise rotation of the maximum shear plane system since no friction \( \tau_{xy} = 0 \) would give:

\[
\cos 2\gamma = 0 ; \quad \gamma = 45^\circ
\]

If the frictional force actually acts in the direction shown in Fig. 14 \( \tau_{xy} = \mu p \) then we have:

\[
\cos 2\gamma = \frac{\mu p}{\tau_m} ; \quad \gamma < 45^\circ
\]

As indicated in Fig. 14, we define \( \omega \) as:

\[
\omega = 45^\circ - \gamma
\]

or \( \omega = \frac{\pi}{4} - \gamma \)  \hspace{1cm} (16)

By geometry, we find the angle DBC of Fig. 12 to be \( (\frac{\pi}{4} - \omega) \) for the frictional direction assumed. By further geometrical considerations, we find that the sector angle EBD = \( (\alpha - \beta + \omega) \), where \( \alpha \) is the semi-wedge angle, and \( \beta \) = the angle of inclination of the lip as shown in Fig. 13.

The stress distribution in the plastic sector is expressed in polar coordinates. It has been noted by Prandtl and Hencky that a solution can be found by assuming that the radial and tangential stresses are
equal at any point. It must be further assumed that radial and tangential stresses vary according to the angular position of the element in question, but are constant at any radial position. (see Fig. 11)

We may then say:

\[ \sigma_\theta = \sigma_r = \sigma_n \]

where \( \sigma_n \) represents the normal stress which acts on the plane of maximum shear stress.

Expressing the foregoing assumptions in the equilibrium equations (11) and (12), together with the condition of plasticity (13) we have:

\[ \frac{\partial \sigma_n}{\partial \phi} + 2\tau_m + 0 = 0 \]

\[ 0 + 0 + 0 = 0 \]

It is to be noted that \( \tau_{\rho \phi} = \tau_m \) by equation (12).

Integrating the above equation gives:

\[ \sigma_n = -2\tau_m \phi + C \]  \hspace{1cm} (17)

The constant \( C \) must be determined from the boundary conditions of the plastic sector. Since the stress condition in zone I has already been uniquely defined, and since a stress continuity must exist at the junction between zone I and zone II, it is helpful to fix the boundary BE as \( \phi = 0 \) as shown in Fig. 11.
Then at this boundary:

\[ \sigma_n = \sigma \]

If we consider the stress normal to the surface of an element at this boundary (Fig. 10), we find:

\[ \sigma_n = -\tau_m, \quad \sigma = -\tau_m \]

Thus we may rewrite equation (17):

\[ \sigma_n = -\tau_m (1 + 2\Theta) \quad \text{(18)} \]

In order to establish the normal stress on an element at the boundary between zone II and zone III, we let \( \Theta = (\alpha - \beta + \omega) \), the sector angle which we described earlier. Therefore the normal stress on surface BD is:

\[ \sigma_n = -\tau_m \left[ 1 + 2(\alpha - \beta + \omega) \right] \quad \text{(19)} \]

Since \( \sigma_n \) represents the mean normal stress acting on the plane at which the maximum shear stress occurs, we may represent \( \sigma_n \) as the center of a Mohr's circle as shown in Fig. 12. As in zone I, we may also satisfy the equilibrium equations (8) and (9) in zone III by the assumption of a uniform stress state throughout. Thus we assume that the above equation for \( \sigma_n \) also holds true at the surface BC. Therefore we may express \( p \) (the normal pressure required to produce yielding) at the surface BC as:

\[ -p = -\tau_m \left[ 1 + 2(\alpha - \beta + \omega) \right] - \tau_m \cos 2\omega \quad \text{(20)} \]
This is most readily seen by reference to the Mohr stress circle in Fig. 12.

However, it is also seen from Fig. 12 that the absolute value of \( p \) may also be expressed as:

\[
p = \frac{\tau_m \sin 2\omega}{\mu}
\]

Equating (20) and (21), we have:

\[
1 + 2(\alpha - \beta + \omega) + \cos 2\omega = \frac{\sin 2\omega}{\mu}
\]

By plotting each side of the above equation as a function of \( \omega \) (Fig. 15), the value of \( \omega \) which satisfies the equality is determined by their intersection.

Since \( \beta \) represents the lip angle, we observe from Fig. 8 that it could not be determined, even graphically, until its length were known since the volume of the raised lip should equal the volume of the material displaced below the original surface. However, the length of the lip, according to our assumed plastic region depends also on the angle \( \omega \). Therefore a family of curves for the left side of the equality might be plotted for different values of \( \beta \) to determine corresponding values of \( \omega \).

The correct set of values could then be determined by a graphical check.

However, if we neglect the presence of the raised
GRAPHICAL DETERMINATION OF $\omega$

WHERE

\[ 1 + 2(\alpha - \beta \pm \omega) + \cos 2\omega = \sin \frac{2\omega}{\mu} \]

\[ \frac{\alpha}{\pi/4} \]
\[ \frac{\beta}{0} \]
\[ \frac{\mu}{0.15} \]

\[ f(\omega) \]

\[ 1 + 2(\pi/4 + \omega) + \cos 2\omega \]
\[ 1 + 2(\pi/4 - \omega) + \cos 2\omega \]

$\omega = 0.327$
$\omega = 0.233$

FIG. 15
lip, letting $\beta = 0$, we may still make an interesting qualitative observation. As we have seen, a reversal of the friction component is possible as the transverse force on the penetrator is increased (see Fig. 7). If the friction is reversed, the normal stress on the interface BC required to cause yielding would, according to the foregoing analysis, be substantially reduced. This can best be seen from Fig. 15, in which the upper curve represents the function:

$$1 + 2(\alpha - \beta + \omega) + \cos 2\omega$$

where $\alpha = \pi/4$; $\beta = 0$. The lower curve represents the same function with the sign of $\omega$ reversed in the first term, indicating clockwise rotation of the maximum shear plane system in zone III. Thus two values may be obtained for $p$ from equations (20) or (21), the lower of which indicates the state after the friction reversal.

According to the foregoing theory, the penetrator should sink deeper into the material after the friction reversal since it is under a constant axial load. However, relative motion between the penetrator and the material due to the sinking in would reestablish the original friction direction. Therefore it is difficult to assess the effect of the friction reversal on theoretical grounds.
Spherically Tipped Penetrators

We consider now a sphere in contact with a plastic medium. The contact area projected on a plane normal to the direction of the axial load is described approximately by a half circle as shown in Fig. 16.

The axial load $L$ may be equated to the force due to the mean pressure $p_m$ acting over the described contact area. Thus:

$$L = p_m \cdot \frac{\pi d^2}{8}$$

(22)

where $d$ is the diameter of the half circle under consideration.

If we assume that a relation exists for this case similar to that described by equation (3) on page 13, we must determine the contact area of the spherical tip projected in a plane normal to the direction of the transverse force $F$, and multiply this projected area by the mean pressure $p_m$ in order to calculate the transverse force (friction neglected). This projection would be in the form of a circular segment whose area is described by:

$$A = \frac{R^2}{2} (\psi - \sin \psi)$$

(23)

where $R$ is the radius of the spherical tip, and $\psi$ is the angle subtended by the arc of the described segment (see Fig. 16).
FIG. 16 PROJECTED AREAS AFFECTING AXIAL AND TRANSVERSE FORCES IN CASE OF SPHERICAL TIPPED PENETRATOR

FIG. 17 APPROXIMATE EFFECTIVE WEDGE ANGLE FOR CONSIDERATION OF FRICTION
This angle may be written:

\[ \psi = 2 \sin^{-1} \frac{r}{R} \]

or

\[ \psi = 2 \sin^{-1} \frac{d}{D} \]  \hspace{1cm} (24)

The equation for the transverse force may then be written:

\[ F = p_m \frac{R^2}{2} (\psi - \sin \psi) \]  \hspace{1cm} (25)

We assume at this point that the mean pressure \( p_m \) is distributed in an approximately uniform manner. This is confirmed by Ishlinsky to be reasonably justified in the case of a full diameter impression. Further justification for this assumption is given by Van Iterson in his analysis of the Brinell hardness test.

In order to use the foregoing equation it is necessary to be able to determine \( p_m \) for the material under test.

In our analysis of the straight sided sharp wedge, we found the relationship \( F/L \) to be independent of the material, thus permitting us to disregard the effect of strain hardening. In the case of a spherical penetrator, however, strain hardening must be considered because the deformation angle \( \psi \) becomes greater as the penetration progresses. Therefore we should expect \( p_m \) to increase if the material is of a strain hardening type (see lower portion of Fig. 1). Even if strain hardening was neglected, a satisfactory analytical determination of \( p_m \)
would not be available, since three-dimensional problems of this type have not as yet been solved rigorously.

Experimental determination would yield a different value of $p_m$ for each different load; tabulation of such variation would in effect describe the strain hardening characteristics of the material. Since we have no better way to determine $p_m$, we shall proceed with our analysis in terms of the experimental $p_m$ which includes the effect of strain hardening.

A convenient method for obtaining $p_m$ is by use of the Meyer Analysis. Meyer’s Law refers to the indentations produced in the case of a full diameter impression, but we shall assume that the relationship will be reasonably true in our case. Meyer’s work shows that resistance to penetration is increased according to the relation:

$$ L = ad^n $$

(26)

where $a$ and $n$ are constants which define the characteristics of the material subject to penetration by a ball of fixed diameter. These constants may be determined for a given material by use of a standard Brinell hardness testing instrument.

Taking the logarithm of each side of equation (26), we have:

$$ \log L = \log a + n \log d $$
A graphical determination of \( a \) and \( n \) is obtained in Fig. 18 by plotting the logarithms of the impression diameters as the abscissae and the logarithms of the corresponding loads as the ordinates. Then \( n \) is represented by the slope of the resulting line, and \( a \) is the intercept of the line at \( d = 1.0 \). The exponent \( n \) is a measure of the strain hardenability of the material under test, where \( n = 2 \) represents a perfectly non-strain hardening material. This follows from equation (26) which then describes the reactive load as being proportional to the square of the impression diameter (i.e., the area).

Since \( a \) is the extrapolated load required to produce an indentation of 1 unit diameter, it may be inferred that this load gives an extremely small degree of permanent deformation if the ball is of appreciable diameter (say 10 units). Thus the value \( 4a/\pi \) might justifiably be regarded as the "true" hardness of the undeformed material under test.

Meyer has further advocated the use of the mean pressure obtained by the formula:

\[
 p_m = \frac{4L}{\pi d^2} 
\]  

(27)

as the means of describing penetration hardness. This value is known as the Meyer hardness and is usually given in units of kilograms/sq. millimeter. Since the Meyer hardness for a given material depends on the
FIG. 18 MEYER ANALYSIS DATA

LOAD ON BALL - KILOGRAMS

303 STAINLESS STEEL (ANNEALED)

n = 2.45

1020 ROLLED STEEL

n = 2.10

IMPRESSION DIAMETER - MM. (WITH 10 MM. BALL)
degree of deformation, we observe that the same $p_m$ will result regardless of the ball diameter if geometrically similar indentations result.

That is:

$$\frac{d_1}{D_1} = \frac{d_2}{D_2}$$  \hspace{1cm} (28)

or

$$\frac{L_1}{d_1^2} = \frac{L_2}{d_2^2} ; \quad \frac{L_1}{D_1^2} = \frac{L_2}{D_2^2}$$  \hspace{1cm} (29)

where $D$ is the ball diameter.

We can obtain an expression for $d$ for an arbitrary load $L$ if we know the Meyer constants determined by the usual metric standards.

Let: $L =$ arbitrary load - lbs.

$D =$ arbitrary ball diameter - in.

d = resulting impression diameter - in.

$L_b =$ load on 10 mm ball - Kg.

$D_b =$ 10 mm

d$_b =$ resulting impression diameter - mm.

From: (29):

$$L_b = \frac{(10)^2}{D^2} \times \frac{1}{2.2} \times \frac{1}{2 \text{ lbs}} \times \frac{.0394 \text{ in.}}{(1 \text{ mm})^2}$$

$$L_b = .0703 \frac{L}{D^2}$$

From (26):

$$d_b = \left[ \frac{L_b}{a} \right]^{1/n} = \left[ \frac{.0703L}{aD^2} \right]^{1/n}$$
From equation (22) we see that the diameter $d$ of the half circle impression area would be the same as that for the full circle if the load were doubled. Therefore we modify equation (30) as:

$$d = \frac{D}{10} \left[ \frac{.0703L}{aD^2} \right]^{1/n} \text{ inches (30)}$$

From equation (22) we see that the diameter $d$ of the half circle impression area would be the same as that for the full circle if the load were doubled. Therefore we modify equation (30) as:

$$d = \frac{D}{10} \left[ \frac{.1496L}{aD^2} \right]^{1/n} \text{ (31)}$$

from which the deformation angle $\psi$ in equation (24), and the mean pressure $p_m$ in equation (22) may be calculated.

Finally, employing the foregoing values in equation (25) we determine the theoretical transverse force $F$ (pounds) which (neglecting friction) may be applied to a ball tipped penetrator of diameter $D$ (inches) under the normal load $L$ (pounds) in a material whose Meyer constants are $n$ and $a$ determined by metric loading with a 10-millimeter ball.

In writing equation (31) we imply that the impression diameter resulting from an application of twice the normal load is equal to that resulting from the single normal load sustained only on half of the area of the impression as shown in Fig. 16. This implication is not strictly true. Physically, the following process occurs.

When no transverse force is applied, the normal load
is supported fully all around. As the transverse force is increased, both the vertical and horizontal components are relieved from that side of the ball, and the ball sinks deeper into the material to establish a new equilibrium position. The transverse force may be increased until an impression diameter of such size is created as will support the normal load on half of its area as shown in Fig. 16. Since the ball will no longer sink into the material, no new contact area will be created by additional transverse loading.

It is apparent that the deformations occurring in this process are more severe than those in a standard ball penetration test producing the same diameter impression. This would tend to cause an increase in the yield point of the metal due to greater strain hardening and consequently affect the mean pressure \( p_m \). However, since most of the metal would be displaced toward the direction of movement, a higher lip would occur on that side of the indentation than would normally be found in a vertically loaded test. Plasticity considerations\(^\text{5}\) show that when a higher lip exists, the mean pressure required for yielding becomes less. Consequently, we expect that in some cases these opposing factors may approximately cancel.

An exact analysis of the effect of conventional friction in the case of the ball would involve certain difficulties. However, if we observe that an approximate effective wedge angle exists as shown in Fig. 17,
we may determine the approximate frictional effect by the following reasoning.

Equation (6) may be rewritten:

\[ F''_o = F_t \left[ \frac{1}{\tan \alpha - \mu} + \frac{\mu \tan \alpha}{\tan \alpha - \mu} \right] \]

If \( \tan \alpha \) is large compared to \( \mu \), this may be approximated:

\[ F''_o = F_t \left[ \frac{1}{\tan \alpha} + \mu \right] \]

which is the equivalent of adding the frictional force algebraically, since equation (6) would reduce to:

\[ F''_o = F_t \cdot \frac{1}{\tan \alpha} \]

if \( \mu \) was set equal to zero.

In the case of a sphere, a deformation angle of as much as \( \psi = 90^\circ \) would give an approximate effective wedge angle \( \alpha \) of \( 67.5^\circ \), and \( \tan \alpha \) would be about 2.41. This is relatively large compared to the usual measured values for the conventional coefficient of friction \( \mu \).

Consequently, for deformation angles up to about \( 90^\circ \), algebraic addition of the friction effect is justified.

As previously mentioned on page 4, Bowden, Tabor, and Moore(2) have expressed the foregoing concept in the equation:

\[ F = S + P \]
where $F$ is the total transverse force, $S$ is the effect of friction in the usual sense, and $P$ is the force required to displace the softer metal from the path of the slider. Although the above authors were concerned with very small deformation angles, it is indicated here that the above equation is also justified for cases where deformation is much greater.
EXPERIMENTAL PROCEDURE

Description of Apparatus

To determine experimentally the relationships between axial loads and maximum possible transverse loads on a penetrator resisted by a plastic medium, a special apparatus was designed and built. A cross sectional view of the machine is illustrated in Fig. 19, and an exterior view of the machine is shown in the photograph, Fig. 20.

Two identical penetrators were mounted with a means of applying equal, measurable axial forces to them. This was achieved by forming the penetrators at the head of concentric circular shanks, which in turn mounted in concentrically bored holes extending part way into two identical pistons. The accommodating cylinders, being a straight through bore, permitted the boring and honing to be completed in the two cylinders by a single machining set-up. Thus it may be assumed that the penetrator points possessed near perfect alignment. The force was applied hydraulically by a smaller piston actuated by a handscrew as shown in the drawing, the pressure being transmitted equally to the opposite cylinder by means of a copper tube not visible in the drawing. A pressure gauge installed in one of the cylinder heads served as an indication of the force applied to the pistons. The calibration of the gauges will be described later.

In order to apply a measurable transverse force, a
cylinder, piston, and centering cone were provided on an axis perpendicular to that of the penetrators. At the lower end of the frame, an opposite centering cone was also provided. The latter was lightly spring loaded to facilitate installation of the material specimens into the machine in the proper position for testing. The lower centering cone, being well guided though free to slide axially, also provided lateral support to that end of the specimen. The force along the axis of the specimen (the transverse force) was applied and measured in the same manner as the axial force previously described.

Tool steel was used as the penetrator material. The penetrators were machine finished, highly polished, and heat treated to a hardness of 63-65 Rockwell "C". The penetrating surfaces were repolished after heat treating.

For the penetrator types which required a specific orientation (i.e., non-circular types) a locating means was provided in the penetrator shank socket. A guide frame attached to the pistons (visible in Fig. 20) prevented rotation of the piston within the cylinder.

Calibration of Apparatus

Since the piston packing friction constituted an unknown frictional force, it was deemed inaccurate to base penetrator force determinations on pressure and piston area relationships. For this reason, a
calibration bar was built of special size and capacity to suit the apparatus. The calibration bar (seen in foreground of Fig. 20) was made such that deflections at the open end of the split could be measured with standard micrometers. Provision was made for loading in the center of the bar, thereby causing the deflection at the open end to be greater than that at the point of load application. This feature increased the accuracy of the deflection measurements.

The load-deflection characteristics of the bar were determined on a standard testing machine known to be accurate within .10%. The bar was then placed between centers in the penetration apparatus, and loaded similarly. The pressure gauge readings were plotted against the deflection measurements of the bar, with corresponding forces noted.

Although this method of calibration eliminates the need for extremely accurate pressure measurements, the gauges were checked on a dead weight tester for accuracy and consistency throughout their range. The gauges, having a range to 2000 psi were capable of indicating forces in the order of 8000 pounds due to the size of the pistons. It is estimated that gauge readings were consistently accurate within plus or minus 5 psi. Since the relationship of gauge reading versus force was established by averaging a large number of points (plotting of curves)
it is reasonable to assume that the error would be very slight. Thus the major source of error in force measurement would lie in gauge reading, and would be in the order of plus or minus 20 pounds.

Lubrication of Penetrators

Since the bearing pressure between the penetrators and the penetrated metal was very large (200,000 psi or more) it was necessary to choose a lubricant of very high film strength in order to prevent galling and seizing of the metal to the penetrator. A mixture of molybdenum disulphide in a grease base proved satisfactory for this purpose. The same mixture was used throughout all the tests, including the measurement of the conventional friction coefficients.

Measurement of Conventional Coefficient of Friction

As we have seen previously, a mathematical prediction of the transverse load on a penetrator requires previous knowledge of the usual frictional properties of the contacting materials. Due to the extremely high bearing pressures involved during actual penetration, use of a frictional coefficient obtained at moderate pressures is probably unjustified. Therefore the following technique was followed:

A blunt nosed penetrator was applied under a normal load sufficient to cause plastic deformation. A transverse
load was applied so that a track was formed in the material. The normal load was then reduced sufficiently to avoid further plastic penetration and the penetrator rerun through the same track. During this operation, the transverse force required was observed. The normal force was then increased incrementally and the process repeated until a normal force capable of producing plastic penetration was reached. The relationship between the forces measured just prior to the latter state were assumed to be representative of the friction occurring during actual plastic penetration.

**Determination of Meyer Constants**

As previously mentioned, knowledge of the physical properties of the plastic medium, namely the hardness of the undeformed metal and the strain hardening characteristics, is necessary in the calculation of the performance of certain types of penetrators.

Since the Meyer relationships were discussed in a previous section, the testing procedure will not be repeated here. The methods described by R. H. Meyer in his work on hardness conversion relationships were generally followed with the following exceptions:

Instead of a universal testing machine, a Brinell hardness testing machine was used to produce the indentations. The machine was the "Original" type, made by Aktiebolaget Alpha, Stockholm, Sweden. The impression
diameters were measured with an illuminating type Brinell microscope with magnification of 20 diameters. The estimated accuracy was within plus or minus .02 millimeters. Two impressions were made at each loading to determine an average. Instead of a tungsten carbide ball, the standard 10 mm. steel ball was used. Molybdenum disulphide in a grease base was used as lubrication on the ball although lubrication is usually considered unnecessary.

Application and Measurement of Penetrator Forces

If a penetrator under the influence of a constant axial load is acted upon by a transverse force it will sink deeper into the material. This we have shown by theoretical considerations. However, in the case of the apparatus used, if the penetrators were to sink in under constant load, hydraulic compensation would have been required. Furthermore, the increased friction of the penetrator pistons against their cylinder walls due to the influence of the transverse loading would have made accurate maintenance of a constant axial load difficult. For these reasons, the transverse load was increased in stages as follows:

The desired axial penetrator load was applied. A transverse load was slowly applied until a failure occurred. Failure in this case was defined as the point at which the pressure gauge failed to respond to further
load application. The axial load gauge was then observed. If a decrease of the pressure was indicated, the transverse load was removed and the original axial load restored. The transverse load was again applied until a new point of failure was observed. The process was repeated until the axial load gauge indicated no decrease of pressure. The transverse pressure gauge reading was recorded at this point. This method of load application is presumed to produce approximately the same type of deformation as the ideal case described earlier.

It is to be noted that the load which we refer to as the "axial" load was the load applied by the penetrators at diametrically opposite points on the specimen. The "transverse" load was the load applied along the axis of the specimen. In interpreting the experimental data, only half of the actual transverse load was used since two penetrators were involved.

Lubrication was applied before and after the initial axial load application only.
DISCUSSION OF EXPERIMENTAL RESULTS

Scope of Experimental Work

Basically, only one type of material was used in the tests. A low carbon (.15°/o - .25°/o), cold rolled steel was used, the object being to obtain a material which is relatively low in strain hardenability. It is to be expected that use of such material will result in better correlation with theory.

In several instances, however, other materials were used for specific experiments which will be discussed later.

The material specimens were all approximately 1/2" diameter. It is probable that some error was introduced due to penetration into a curved surface. Comparisons were made between forces measured on the round specimens with forces measured in comparable penetrations in flat specimens. The differences were found to be small.

Penetrators tested were of the following four basic types:

1) Two-dimensional shapes in which the width was uniform (3/16" wide) and the profiles comprised a wedge with 90° included angle, the leading edge radius being variable for this series of tests.

2) Cone shapes of 90° included angle, the tip radius being variable.

3) Spherical shapes of two different radii.
4) Pyramidal shapes comprising four equal sides with 90° included angle, one test of the series using a sharp point, and another using a truncated (flat) point.

Of the first series of penetrators described, the wedge having a leading edge of radius zero corresponds to the theoretical problem of the straight sided wedge.

Of the same series of penetrators, the rounded leading edges of radii 1/8" and 1/4" represent cylinders of these radii with the forces acting normal to the cylindrical axes. The penetration in these cases was not beyond the curved surface of the tip. The penetrators having leading edges of 1/32" and 1/16" represent intermediate cases, since penetration under the lighter loads involved the curved surfaces, but under higher loads involved also the straight sides of the wedge.

The experimental data taken for all of the penetrators of this series is plotted in Fig. 21. The data for the penetrators with rounded loading edges is given only as statistical data. No analytical considerations have been made for the latter.

Comparison of Straight-Sided Sharp Wedge with Theory

The average value for F/L in the case of the sharp wedge (R=0), as shown in Fig. 21, is about 1.22. If equation (6) is evaluated for this case, the value of F/L is 1.35 for a coefficient of friction $\mu$ equal to .15 (measured as described previously for steel).
RELATIONSHIPS BETWEEN AXIAL AND TRANSVERSE FORCES FOR TWO-DIMENSIONAL PENETRATORS

(SHADED AREA INDICATES THEORETICAL UPPER AND LOWER LIMITS FOR R = 0)

MAT'L - 1020 CR STEEL

FIG. 21
This discrepancy might be explained by the fact that failure probably occurs before the reverse component of friction has completely developed to the position of $F''_c$ as indicated in Fig. 7. As we have previously pointed out, the penetrator must sink deeper into the material after the friction reversal. This cannot occur, however, without reestablishing the original friction direction. The net result of this apparent conflict is shown experimentally to be a somewhat lower than maximum theoretical value of $F/L$. The shaded area of Fig. 21 indicates the upper and lower theoretical limits for this case. The upper limit is that value obtained by application of equation (6) and the lower limit is that obtained by the assumption that no relative motion will occur at the interface at the final stage of failure (no friction).

In practice, the maximum transverse force seemed to vary somewhat with the rate of application of the force. Since the loads were applied manually throughout the experiments, the foregoing fact might help to explain the scattering of values along the curves. Sensitivity to the strain rate seemed to be more pronounced in the case of the sharp pointed penetrators.

An interesting feature of the two-dimensional sharp wedge experiment is shown in the photographs (Fig. 22) in which an indented specimen of cold rolled 1040 steel is illustrated.
FIG. 22 (UPPER) INDENTATION BY PURE AXIAL LOADING;
(LOWER) INDENTATION BY COMBINED LOADING X 9
The indentation in the upper photograph was produced by a pure axial load of 2330 pounds, and measurements indicate that the mean pressure normal to the wedge faces required to produce this indentation was about 207,000 pounds per square inch. The equation for the mean pressure $p$ at a wedge face in this case is:

\[-p = -T_m \left[ 1 + 2(\alpha - \beta + \omega) \right] - T_m \cos 2\omega \quad (20)\]

The $+\omega$ indicates friction acting upward along the wedge face in the direction opposing penetration.

By measurement, we find $\beta$ to be about 60° (0.105 rad.). The semi-angle $\alpha$ is 45° or $\pi/4$. We may find $\omega$ in the same manner as shown in Fig. 15, by plotting the function

\[1 + 2 \left( \frac{\pi}{4} - 0.105 + \omega \right) + \cos 2\omega\]

and we find $\omega$ to be about 30° or 17.5°. Therefore, from equation (21) we may write:

\[T_m = \frac{\mu p}{\sin 2\omega}\]

or

\[T_m = \frac{(207,000)(115)}{\sin 350}\]

\[T_m = 54,200 \text{ psi}\]

which is an entirely reasonable value for the shear strength of the strain hardened steel under test.
The indentation in the lower photograph was produced by an axial load of 2330 pounds in combination with a transverse load of 2820 pounds (the point of incipient failure). Measurements of the bearing surface indicate that a mean pressure of 155,000 pounds per square inch was required to produce the deformation. By measurement, we find the lip angle to be about $15^\circ (0.262 \text{ rad.})$.

If we assume this to be a case in which a reversal of the friction has occurred, we plot the function:

$$1 + 2 \left( \frac{\pi}{4} - 0.262 - \omega \right) + \cos 2\omega$$

in which $-\omega$ indicates the friction reversal. In this case $\omega$ is found to be about $0.203$ or $11.6^\circ$.

By equation (16) we find:

$$\tau_m = \frac{(155,000)(.15)}{\sin 23.2^\circ}$$

$$\tau_m = 58,900 \text{ psi}$$

Since the deformation was more severe in this case, (see Fig. 8) we should expect the shear strength to be somewhat higher than in the case of the pure axial indentation.

The foregoing illustration indicates an apparent agreement of the plastic solution with the experimental findings.

**Sharp Pointed Cone**

In the second series of penetrators the sharp
pointed cone was intended to check the theory that a sharp pointed indenter of constant geometrical shape will have an F/L value independent of the nature of the penetrated material. From simple geometrical considerations (not shown here) the value F/L for a sharp cone point should be:

\[ \frac{F}{L} = \frac{2}{\pi} \cot \alpha \]

where \( \alpha \) is the semi-angle of the cone, and the effect of friction is neglected. For a cone having an included angle of 90°, the value is equal to .637.

From Fig. 23, it is seen from the slope of the curve for the sharp pointed cone (R 0) that the experimental value of F/L is about .65, indicating the presence of the conventional friction component. A significant fact is that almost identical values were obtained for two different steels, in which a wide variation of strain hardenability existed.

Repeating the test with aluminum, a close correlation was also found.

The forces measured in the case of similar cones having radii of 1/32" and 1/16" at the tips are plotted in the same figure. The curves indicate rather clearly the quantitative effects of blunting the points of the penetrators in the manner described.
RELATIONSHIPS BETWEEN AXIAL AND TRANSVERSE FORCES FOR CONE TYPE PENETRATORS

![Diagram of relationships between axial and transverse forces for cone type penetrators.](image)

**Fig. 23**
RELATIONSHIPS BETWEEN AXIAL AND TRANSVERSE FORCES FOR SPHERICAL TYPE PENETRATORS

CURVES REPRESENT THEORETICAL VALUES. POINTS INDICATE EXPERIMENTAL RESULTS.

MEYER CONSTANTS

\[
\begin{align*}
\alpha &= 90 \\
n &= 2.10
\end{align*}
\]

(10 MM BALL)

\(R = \frac{1}{8}\)
\(R = \frac{1}{4}\)

\(\mu = 0.15\)

MAT'L: 1020 CR STEEL

MEYER CONSTANTS

\[
\begin{align*}
\alpha &= 75 \\
n &= 2.45
\end{align*}
\]

(10 MM BALL)

\(R = \frac{1}{8}\)
\(R = \frac{1}{4}\)

\(\mu = 0.12\)

MAT'L: 303 STAINLESS

AXIAL LOAD \(L\) (LBS)

TRANSVERSE FORCE \(F\) (LBS)

FIG. 24
Spherical Penetrators

In Fig. 2, two sets of curves are given. The upper portion of the figure indicates the results of spherical penetrators in the cold rolled steel, and the lower portion relates to measurements made in a stainless steel of apparently high strain hardening capacity. The latter experiments were made for the purpose of testing the adequacy of the theory given here for penetrators of this type.

All of the curves plotted in Fig. 2 are theoretical curves based on the physical properties of the materials as determined from the Meyer analyses of Fig. 13. The conventional friction effect based on Coulomb's law was added algebraically in this case since the effective wedge angles (Fig. 17) for this series of tests were great enough to make such algebraic addition approximately valid.

The experimental values are plotted as individual points and show reasonable agreement with the theoretical curves. Since most of the experimental points in the lower set of curves lie slightly above the theoretical line, the possibility is suggested that the theory does not account sufficiently for the additional strain hardening of this mode of deformation over that of a pure axial indentation.

The component of F due to the conventional coefficient of friction μ (by algebraic addition) is indicated additionally in each case. It is apparent that as L is
decreased to such values as will produce only elastic deformations, \( F/L \) approaches \( \mu \).

**Pyramidal Penetrators**

Results of the tests made with penetrators of this type are shown in Fig. 25. An interesting feature of these tests is the effect of the truncated point as compared with the sharp point. As indicated in the figure, the sharp point gives an \( F/L \) relationship approximately independent of load. However, in the case of the blunt tip, \( F/L = \mu \) (presumably during the elastic loading) until a given point is reached (probably the point of incipient plastic penetration) where the curve rises with the same slope as the former curve.
RELATIONSHIPS BETWEEN AXIAL AND TRANSVERSE FORCES FOR PYRAMIDAL TYPE PENETRATORS

FIG. 25
CONCLUSIONS AND GENERAL DISCUSSION

On the basis of the foregoing work, it is indicated that reasonably accurate predictions may be made analytically for the relation between axial and maximum possible transverse forces sustained by a penetrator in a plastic material.

When penetrators have sharp leading edges and straight sides so as to produce indentations of constantly similar geometrical shapes, the foregoing relation will be independent of the load and of the material.

It is indicated both theoretically and experimentally that a reversal of friction occurs at the penetrator surface in contact with the material after the maximum penetration has occurred. It is also indicated that the reversal causes a decrease in the normal yield pressure at the surfaces of contact.

Although the theory indicates that the conventional friction effect should be added vectorially to the plastic displacement effect, algebraic addition is justified when the effective wedge angle is large compared to the conventional coefficient of friction. This is in agreement with the work of Bowden, Tabor, and Moore.\(^2\)

For spherical indentations, the Meyer analysis provides reasonable information regarding the nature of the material under varying degrees of deformation.
It is recognized that, in certain types of loading, the axial load will not be applied fully before the transverse force is applied. In the case of a self-energized gripping element, the axial load will always depend on the transverse load. In general, however, the relationships found in this work will probably apply. The degree of deformation of the penetrated medium will be different in the two cases, but, as we have seen, this will not greatly affect the assumed relationship.

In presuming that our theory will apply to a multiplicity of penetrators as on a serrated surface, we must limit our use of this theory only to cases where the penetrators are spaced such that the plastic zones of each penetrator are not affected by their near neighbors. This limitation leads to another field for investigation. It would be of interest to determine an ideal penetrator shape which, among other factors, would probably be determined by the shape of plastic zone it produced.


Other Related References


Sherwood, Robert S., "The Mechanism of Dry Friction." *Iowa State College Bulletin Engineering Report No. 6 of the Iowa Engineering Experiment Station*