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BLUNT-NOSED MISSILE SHAPES OF MINIMUM BALLISTIC FACTOR

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Abstract

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The class of slender, axisymmetric missiles of given length, base radius, and specific weight is considered under the assumptions that the pressure distribution is modified Newtonian and the surface-averaged skin-friction coefficient is constant. The problem of minimizing the ballistic factor is formulated and solved via the calculus of variations for the case where the nose radius of the missile has a specified value, not necessarily zero.

ACKNOWLEDGMENT

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1. Introduction

Under the assumptions that the density is an exponential function of the altitude, the drag coefficient is constant, the trajectory is almost rectilinear, and the acceleration of gravity is constant, the reentry time of a ballistic missile can be shown to be a monotonically increasing function of the ballistic factor (Ref. 1)

$$K = (\rho_o Hg/2 \sin \theta) C_D S/W \tag{1}$$

where ρ_o is the density of air at sea level, H the scale-height factor of the exponential atmosphere, g the acceleration of gravity, θ the modulus of the inclination of the flight path at the beginning of reentry, C_D the drag coefficient of the vehicle, S the cross-sectional area of the missile, and W the weight of the missile. Therefore, for given initial values of the altitude, velocity, and path inclination, the reentry time is a minimum if the missile is designed so that the ballistic factor is a minimum.

With the introduction of the definitions

$$C_D = D/qS, \qquad W = \gamma V \tag{2}$$

where q denotes the free-stream dynamic pressure, D the drag, γ the average specific weight, and V the volume, the ballistic factor can be rewritten in the form

$$K = (\rho_o Hg/2\gamma \sin \theta)D/qV \tag{3}$$

Hence, for a given value of γ, the problem of minimizing the ballistic factor is identical to that of minimizing the drag per unit free-stream dynamic pressure and volume.

For the class of missiles whose length, base radius, and specific weight are given, the minimization of (3) was the object of papers by Berman (Ref. 2), Fink (Ref. 3), and Miele and Huang (Ref. 4). Berman considered only power-law bodies, that is, bodies described by the relationship

$$y = x^n \tag{4}$$

where y denotes the ordinate-semithickness ratio, x the abscissa-length ratio, and n the power-law exponent; he used some experimental data on the drag and determined that the optimum power-law exponent is $n = 0.62$. Fink neglected the skin-friction drag and employed the calculus of variations to find the optimum shape within the class of continuous curves y(x). Miele and Huang accounted for the skin-friction drag and employed the calculus of variations to find the optimum shape within the class of continuous curves y(x) satisfying the inequality constraint

$$\dot{y} \geq 0 \tag{5}$$

($\dot{y}$ denotes the derivative dy/dx) which expresses the limit of validity of the Newtonian pressure law.

The solutions of Refs. 2 through 4 have a zero radius of curvature at the nose and, therefore, experience severe local aerodynamic heating during reentry (see, for instance, Ref. 1). For this reason, the problem of minimizing the ballistic factor is reinvestigated here for blunt-nosed bodies. The following hypotheses are employed: (a) the body is axisymmetric; (b) the body is longitudinally slender; (c) the angle of attack is zero; (d) the pressure coefficient is modified Newtonian; (e) the surface-averaged skin-friction coefficient is constant; and (f) the base pressure coefficient is zero. With these hypotheses, the missile shape of minimum ballistic factor for given length, base radius, nose radius, and specific weight is determined via the calculus of variations within the class of continuous curves satisfying Ineq. (5).

2. Formulation of the Problem

Consider an axisymmetric, blunt-nosed slender body in hypersonic Newtonian flow (Fig. 1), and let ℓ denote the length, t/2 the base radius, $\beta t/2$ the nose radius, m a factor modifying the Newtonian pressure distribution, and C_f the surface-averaged skin-friction coefficient. Let

$$\tau = t/\ell \tag{6}$$

denote the thickness ratio and

$$\sigma = 4C_f/m \tag{7}$$

denote the friction parameter. Let $x = X/\ell$ denote a dimensionless coordinate in the flow direction and $y = 2Y/t$ a dimensionless coordinate normal to x. With these definitions, the drag and volume may be written as

$$D = (\pi mqt^4/4\ell^2)\, I_1, \qquad V = (\pi t^2 \ell/4)\, I_2 \tag{8}$$

where I_1 and I_2 denote the following dimensionless integrals:

$$I_1 = \int_0^1 y(\dot{y}^3 + \sigma/\tau^3)dx + 2\beta^2/\tau^2 \tag{9}$$

$$I_2 = \int_0^1 y^2 dx \tag{10}$$

which are subjected to the boundary conditions

$$x_i = 0, \quad y_i = \beta, \quad x_f = y_f = 1 \tag{11}$$

where the subscripts i and f denote the initial and final points, respectively.

After Eqs. (3) and (8) are combined, the ballistic factor may be rewritten in the form

$$K = (\rho_0 Hgmt^2/2\gamma\ell^3 \sin\theta)I \tag{12}$$

where I denotes the quality coefficient

$$I = I_1/I_2 \tag{13}$$

Therefore, the problem of minimizing the ballistic factor for given length, base radius, nose radius, and specific weight is identical to that of minimizing the functional (13) subjected to the definitions (9) and (10), the end conditions (11), and the inequality constraint (5). The thickness ratio τ, the friction parameter σ, and the dimensionless initial radius β are given a priori.

3. Necessary Conditions

The inequality constraint (5) can be replaced by the equality constraint

$$\dot{y} - z^2 = 0 \tag{14}$$

where z, a real variable, is a function of x to be determined. With the introduction of this new variable, the minimal problem can be restated as follows: "Find the functions y(x) and z(x), consistent with the differential constraint (14) and the end conditions (11), which yield a minimum value of the functional (13)."

In accordance with the theory presented in Ref. 5, the necessary conditions for extremizing I are identical to those for extremizing the new functional

$$\tilde{I} = \int_0^1 F(y, \dot{y}, z, \lambda, \mu)dx \tag{15}$$

where F denotes the fundamental function

$$F = y(\dot{y}^3 + \sigma/\tau^3) - \lambda y^2 - \mu(\dot{y} - z^2) \tag{16}$$

In Eq. (16), λ is a positive, constant Lagrange multiplier defined by

$$\lambda = I_1/I_2 \tag{17}$$

and $\mu = \mu(x)$ is a variable Lagrange multiplier. It is obvious from the definition (17) that the multiplier λ is identical with the minimum value of the functional (13).

3.1. Euler Equations. From calculus of variations (see, for instance, Ref. 6), it is known that the extremal arc must satisfy the Euler equations

$$F_y - dF_{\dot{y}}/dx = 0, \qquad F_z = 0 \tag{18}$$

which, because of Eq. (16), may be written explicitly as

$$6y\dot{y}\ddot{y} + 2\dot{y}^3 + 2\lambda y - \sigma/\tau^3 - \dot{\mu} = 0, \qquad \mu z = 0 \tag{19}$$

The second Euler equation admits the solutions

$$\mu = 0 \quad \text{or} \quad z = 0 \tag{20}$$

the first of which is called a regular shape and the second of which is called a zero-slope shape. The extremal arc may be composed of one or both of these shapes.

3.2. First Integral. Since the fundamental function does not contain the independent variable x, Eqs. (18) admit the first integral

$$\dot{y}F_{\dot{y}} - F = C \tag{21}$$

which, because of Eqs. (16) and (19-2), is given explicitly by

$$2y\dot{y}^3 - (\sigma/\tau^3)y + \lambda y^2 = C \tag{22}$$

where C is a constant.

3.3. Corner Conditions. If the extremal arc is composed of more than one subarc, then the corner conditions

$$\Delta(F - \dot{y}F_{\dot{y}}) = 0, \qquad \Delta F_{\dot{y}} = 0 \tag{23}$$

must be satisfied at the junction of any two subarcs. Here, $\Delta(\ldots)$ denotes the difference between quantities evaluated immediately after and before the corner. Several conclusions may be reached by examining the corner conditions. First, Eq. (21) may be combined with Eq. (23-1) to give

$$\Delta C = 0 \tag{24}$$

which means that the integration constant C has the same value for every subarc. Then, combining Eqs. (23) and (16) yields the relationships

$$\Delta(y\dot{y}^3) = 0, \qquad \Delta(3y\dot{y}^2 - \mu) = 0 \tag{25}$$

which admit the solutions

$$y = 0, \quad \mu = 0, \quad \Delta\dot{y} \neq 0 \tag{26}$$

and

$$y \neq 0, \quad \mu = 0, \quad \Delta\dot{y} = 0 \tag{27}$$

These solutions imply that (a) a discontinuity in slope can occur only on the axis of symmetry and (b) regardless of whether there is a discontinuity in slope, the relation $\mu = 0$ holds on both sides of the corner.

3.4. <u>Weierstrass Condition.</u> In addition to the first variation conditions, the extremal arc must satisfy the Weierstrass condition

$$E = \Delta F - F_{\dot{y}} \Delta\dot{y} \geq 0 \tag{28}$$

(E is defined as the <u>excess function</u>) for every set of variations $\Delta\dot{y}$ and Δz consistent with the constraint (14). The quantities ΔF, $\Delta\dot{y}$, and Δz are defined by

$$\begin{aligned} &\Delta F = F(y, \dot{y}_*, z_*, \lambda, \mu) - F(y, \dot{y}, z, \lambda, \mu) \\ &\Delta\dot{y} = \dot{y}_* - \dot{y}, \qquad\qquad \Delta z = z_* - z \end{aligned} \tag{29}$$

where the unstarred symbols refer to the extremal arc and the starred symbols to the comparison arc. On account of Eqs. (14), (16), (19-2), and (29), the excess function (28) can be written as

$$E = y(\dot{y}_* - \dot{y})^2(\dot{y}_* + 2\dot{y}) + \mu\dot{y}_* \geq 0 \tag{30}$$

For the regular shape (20-1), the positiveness of E is ensured as long as both $\dot{y}$ and $\dot{y}_*$ satisfy the constraint (14). For the zero-slope shape (20-2), the positiveness of E is ensured providing

$$\mu \geq 0 \tag{31}$$

everywhere.

4. Totality of Solutions

If the first integral is integrated over the interval (0, 1) of the independent variable x, the following result is derived:

$$C = 2\int_0^1 y\dot{y}^3 dx - (\sigma/\tau^3)\int_0^1 y\,dx + \lambda\int_0^1 y^2 dx \qquad (32)$$

Substitution of Eqs. (9), (10), and (17) into Eq. (32) gives

$$C = 3\int_0^1 y\dot{y}^3 dx + 2\beta^2/\tau^2 \qquad (33)$$

As a consequence of Eq. (33), the end conditions (11), and the constraint (14), we see that

$$C > 0 \qquad (34)$$

Next, consider the zero-slope shape, for which

$$\dot{y} = 0 \qquad (35)$$

so that

$$y = \text{Const} \qquad (36)$$

For this subarc, the Euler equation (19-1) and the first integral (22) reduce to

$$\dot{\mu} = 2\lambda y - \sigma/\tau^3, \qquad C = \lambda y^2 - (\sigma/\tau^3)y \qquad (37)$$

which can be combined to yield

$$\dot{\mu} = \lambda y + C/y \tag{38}$$

Upon integrating Eq. (38) and applying the corner condition (27), we obtain the relation

$$\mu = (\lambda y + C/y)(x - x_o) \tag{39}$$

where the subscript o denotes the corner point. Since $\lambda > 0$ and $C > 0$, Eq. (39) is compatible with the Weierstrass condition (31) only if $x \geq x_o$, that is, providing the zero-slope shape follows the regular shape. Therefore, the only possible zero-slope shape is the subarc $y = 1$. In summary, the totality of solutions includes two classes: (I) one-subarc solutions composed of a regular shape $\mu = 0$ and (II) two-subarc solutions composed of a regular shape $\mu = 0$ followed by a zero-slope shape $y = 1$.

5. Extremal Arc

From the previous discussion, the first integral (22), and the initial conditions (11), it follows that the extremal arc is described by the relationships

$$x = \int_{\beta}^{y} A\,dy, \qquad 0 \le x \le x_o$$
$$y = 1, \qquad x_o \le x \le 1 \tag{40}$$

where A = dx/dy denotes the function

$$A(C,\sigma,\tau,\lambda,y) = \sqrt[3]{[2y/(C + \sigma y/\tau^3 - \lambda y^2)]} \tag{41}$$

Therefore, the extremal arc is determined providing C, λ, and x_o are known. To find C, λ, and x_o, the continuity requirement between the regular shape and the zero-slope shape is invoked to yield the abscissa of the corner point

$$x_o = \int_{\beta}^{1} A\,dy \tag{42}$$

Then, Eqs. (33) and (41) are combined to obtain the relationship

$$C = 3\int_{\beta}^{1} (y/A^2)dy + 2\beta^2/\tau^2 \tag{43}$$

Finally, for solutions of Class I, the following relation holds:

$$x_o = 1 \tag{44}$$

For solutions of Class II, Eqs. (24) and (37-2) are combined, with y = 1 at the corner, to give

$$C = \lambda - \sigma/\tau^3 \tag{45}$$

Relationships (40) through (45) have been solved with the aid of an IBM 7040 computer to obtain the functions

$$\lambda = \lambda(\sigma, \tau, \beta), \quad C = C(\sigma, \tau, \beta), \quad x_o = x_o(\sigma, \tau, \beta) \tag{46}$$

which are shown in Figs. 2 through 4 for $\sigma = 4 \times 10^{-3}$ and several values of τ and β. After λ, C, and x_o are known, Eqs. (40) can be used to obtain the geometry of the optimum shapes, which are shown in Figs. 5 through 7 for $\sigma = 4 \times 10^{-3}$ and several values of τ and β. Incidentally, the case $\beta = 0$ corresponds to the sharp-nosed solutions of Ref. 4.

In order to determine the limit of validity of the solutions, we define the critical thickness ratio τ_c as the value of τ for which Eqs. (44) and (45) are simultaneously valid. This yields the relationship

$$\tau_c = \tau_c(\sigma, \beta) \tag{47}$$

which is plotted in Fig. 8 for $\sigma = 4 \times 10^{-3}$. The line (47) divides the $\beta\tau$-plane into two regions: the upper region $\tau > \tau_c$ corresponds to the solutions of Class I, and the lower region $\tau < \tau_c$ corresponds to the solutions of Class II.

Finally, in order to check that the solutions obtained are truly optimum, a comparison has been made with truncated cones having the same endpoints as the optimum shapes. In Fig. 9, the quality coefficient I for the truncated cones is plotted versus τ for $\sigma = 4 \times 10^{-3}$ and several values of β. Comparison of Figs. 2 and 9 shows that, in every case, the quality coefficient of the extremal solution is lower than that of the truncated cone. As an example, for $\sigma = 4 \times 10^{-3}$ and $\tau = 1/4$, the relative difference is 26.9% if $\beta = 0$, 18.5% if $\beta = 0.2$, and 28.5% if $\beta = 0.4$.

6. Discussion and Conclusions

The problem of minimizing the ballistic factor of a slender, axisymmetric missile having given length, base radius, and specific weight is formulated under the assumptions that the pressure coefficient is modified Newtonian and the surface-averaged skin-friction coefficient is constant. It is shown that, depending on the values of the parameters σ, β, τ, the optimum shape can be either a regular shape or a regular shape followed by a cylindrical afterbody. For given σ and β, increasing τ decreases the ballistic factor; while, for given σ and τ, increasing β increases the ballistic factor. Comparison of the extremal solution with a truncated cone shows that the ballistic factor of the former is considerably lower than that of the latter.

With slight alterations, the theory developed in this thesis can be made valid for blunt-nosed bodies which have a finite radius of curvature at the nose. Specifically, one must assume that the skin-friction drag of the nose is negligible with respect to the pressure drag of the remainder of the body and that the volume of the nose is negligible with respect to the volume of the remainder of the body.

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LIST OF CAPTIONS

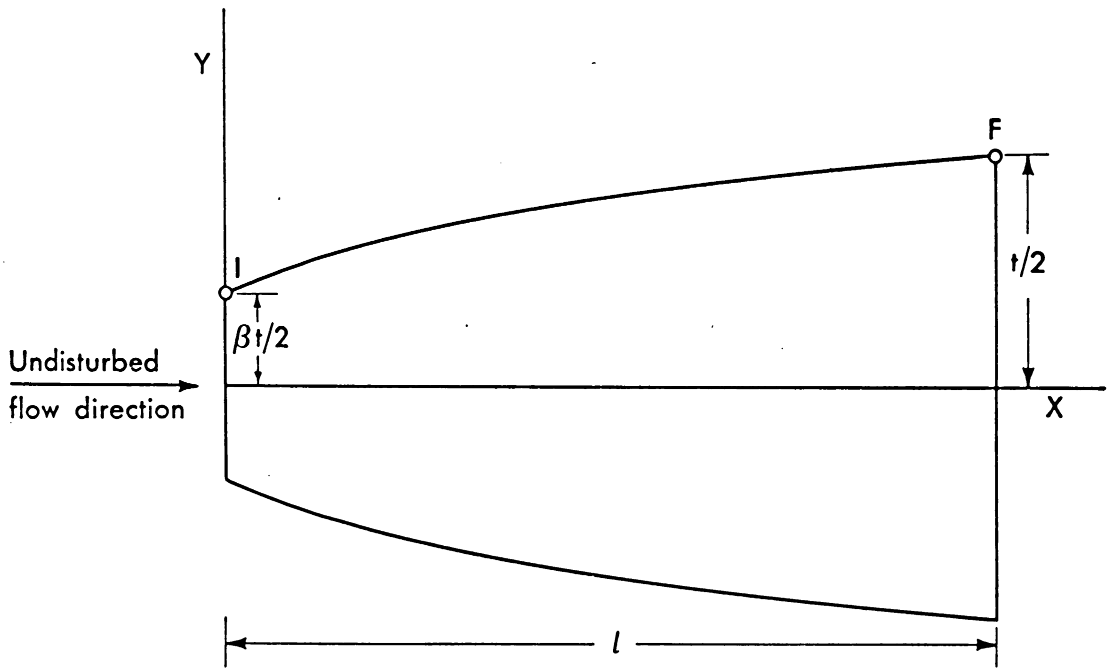

Fig. 1 Coordinate system.

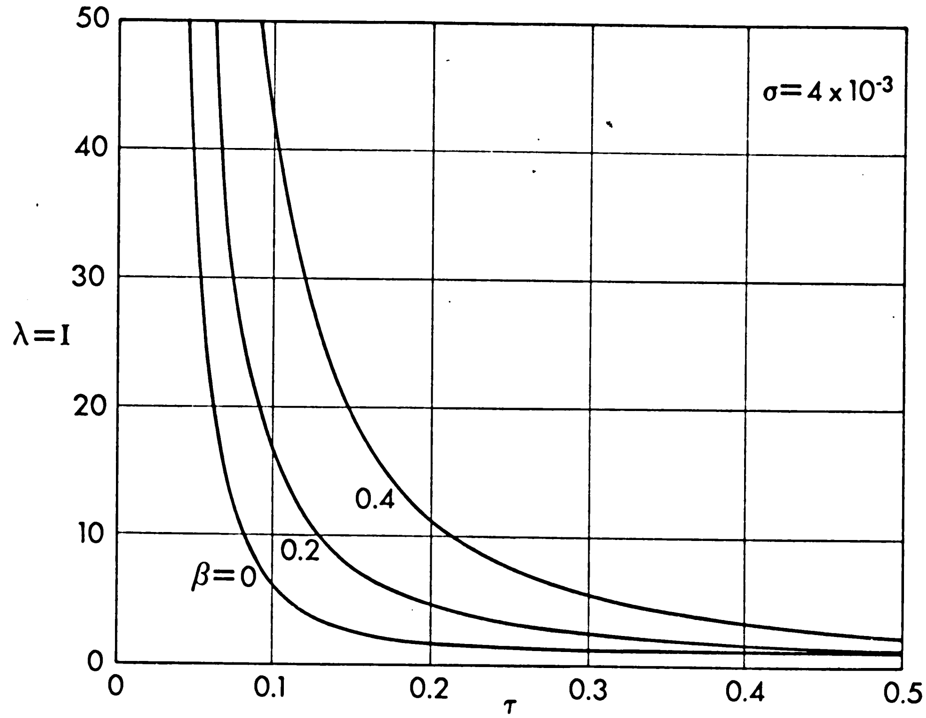

Fig. 2 Minimum quality coefficient.

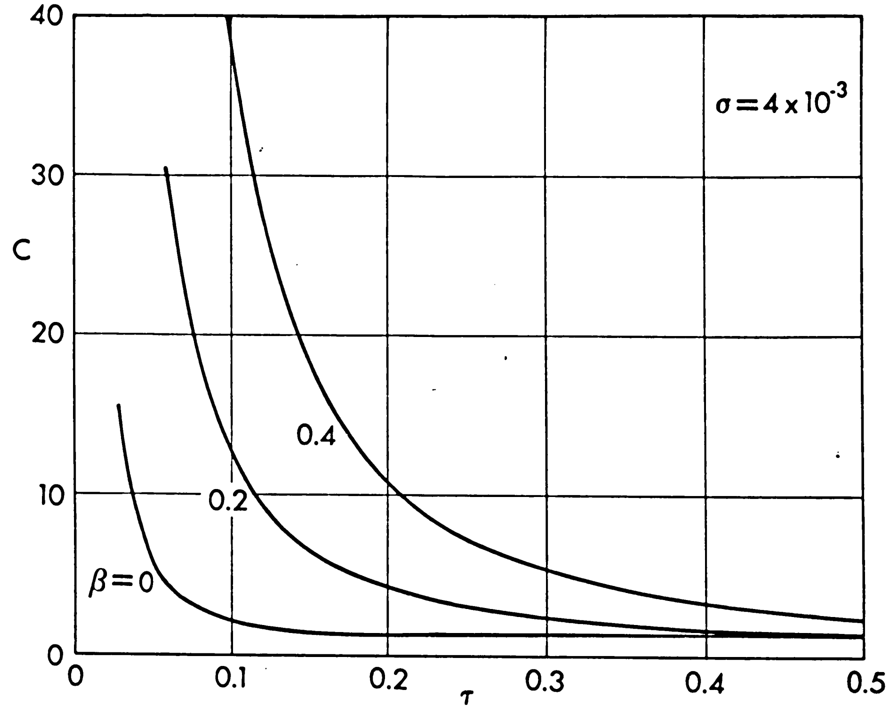

Fig. 3 The constant C.

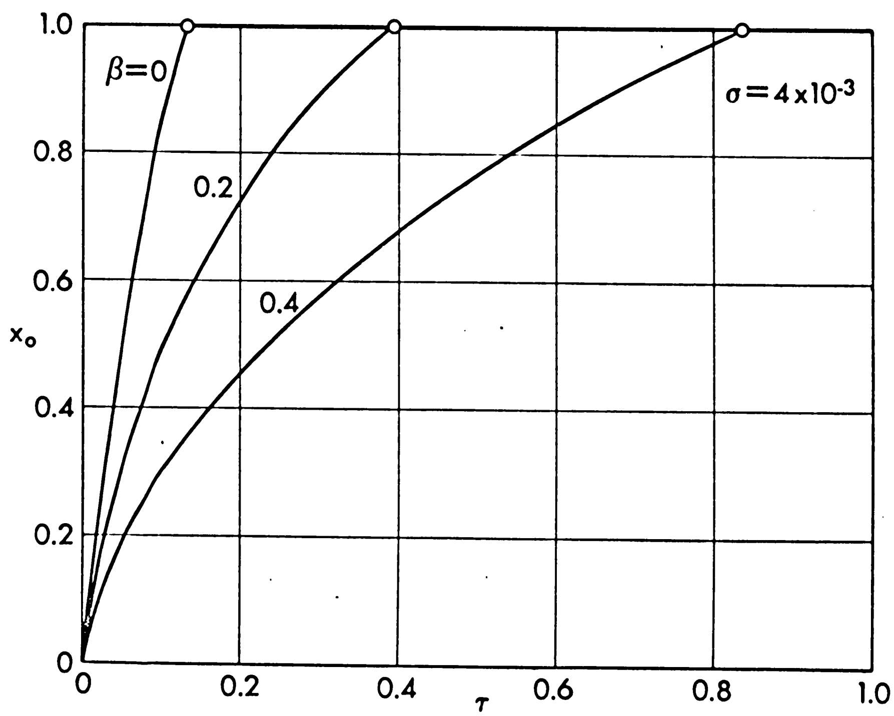

Fig. 4 Abscissa of the corner point.

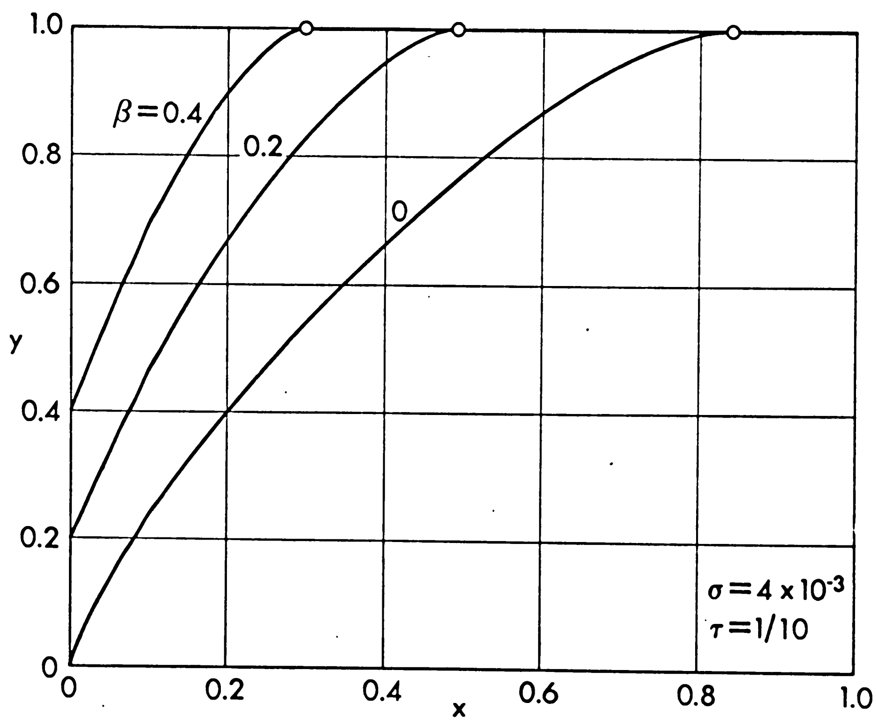

Fig. 5 Optimum bodies for $\tau = 1/10$.

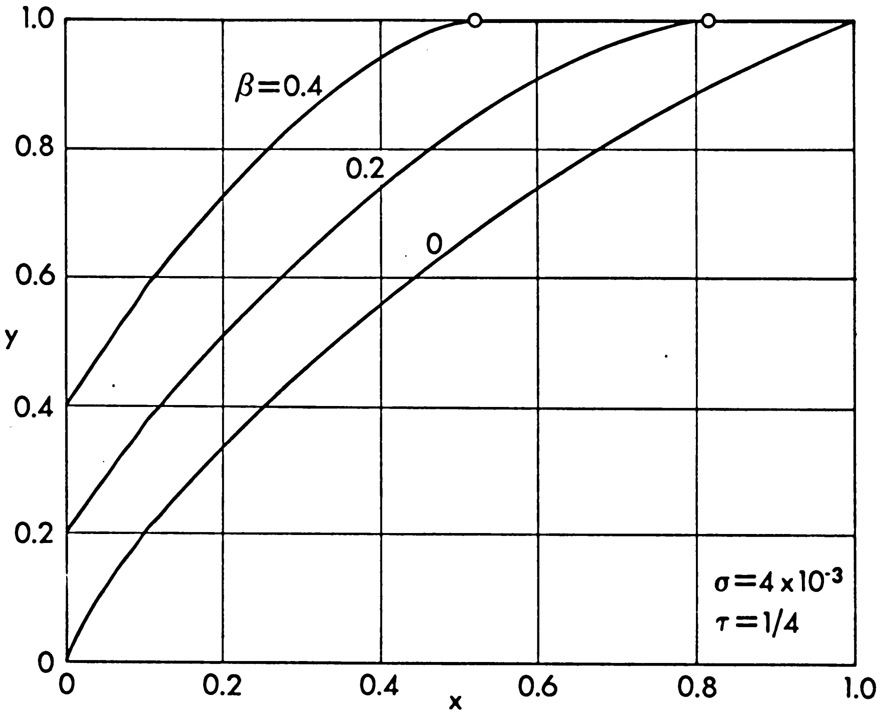

Fig. 6 Optimum bodies for $\tau = 1/4$.

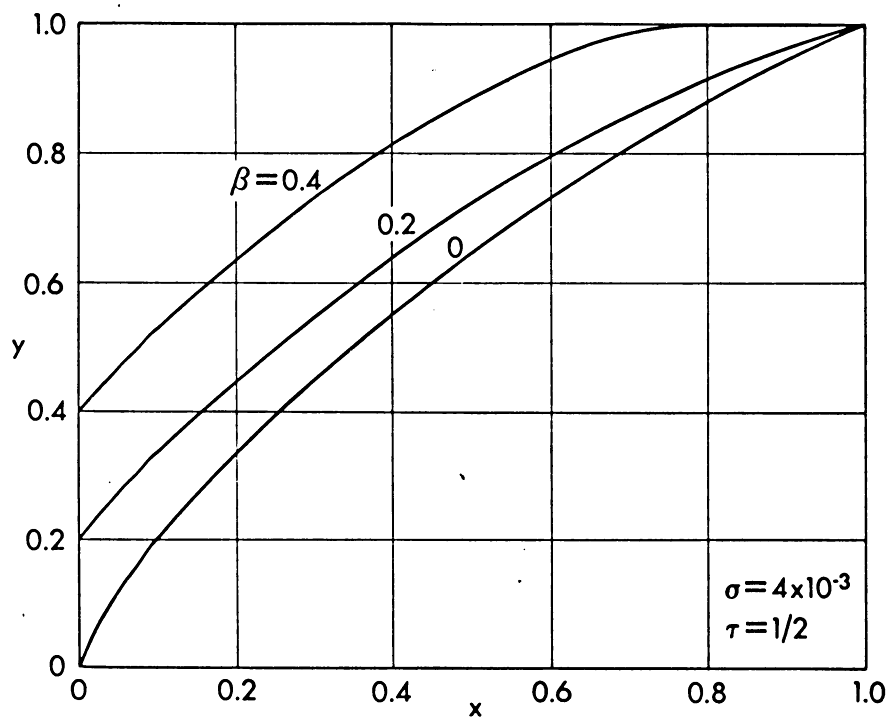

Fig. 7 Optimum bodies for τ = 1/2.

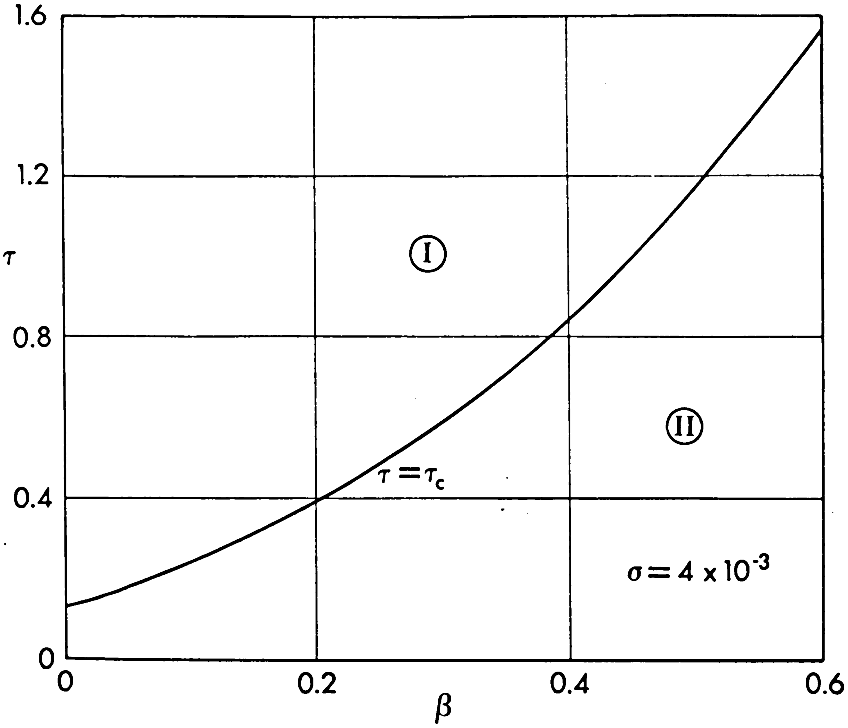

Fig. 8 Critical thickness ratio.

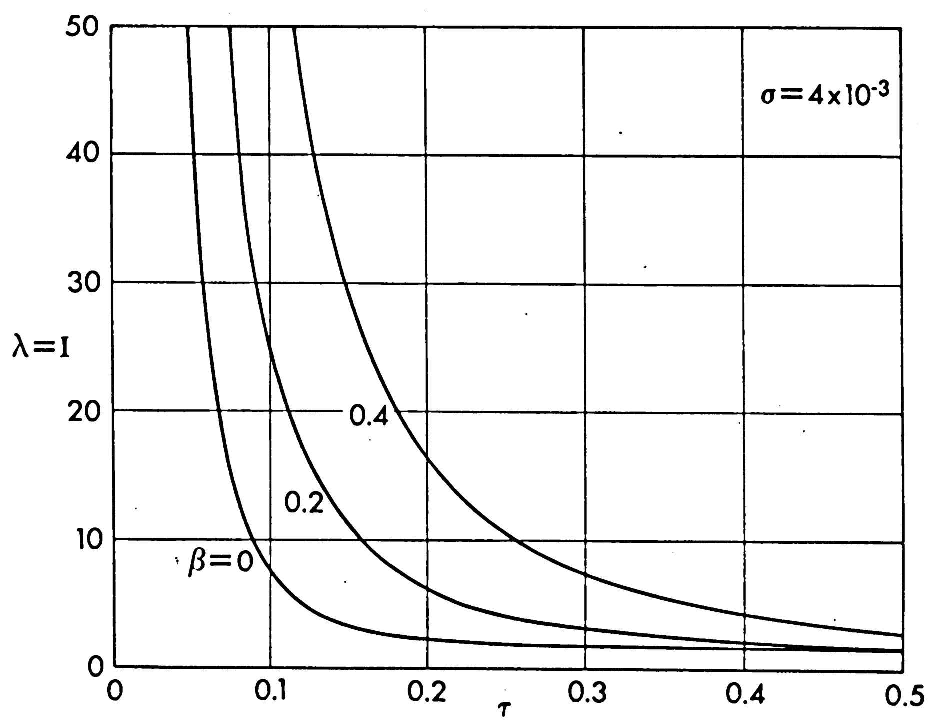

Fig. 9 Quality coefficient of truncated cones.