RICE UNIVERSITY

THE DISPERSION AND GENERATION OF SEA STATES

by

Edward H. Turner

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ABSTRACT

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This paper deals with a method of generating a desired sea state in a wave tank. The time duration in which all wave groups are present at the test section and before reflected waves reach the test section is maximized for a given length of wave tank.

The duration is found to be dependent on the number of frequency bands chosen to represent the spectrum. The more frequency bands chosen, and consequently, the better the spectrum is approximated, the shorter the duration.

Each frequency band is approximated by two sine waves of equal amplitude. The frequency bands must be chosen so that they can be adequately represented by two waves.
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Edward H. Turner
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<td>C</td>
<td>Velocity of a wave group</td>
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<td>g</td>
<td>Acceleration of gravity</td>
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<td>$\lambda$</td>
<td>Wave height</td>
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<td>k</td>
<td>Wave length factor</td>
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<tr>
<td>l</td>
<td>Distance from a point</td>
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<td>L</td>
<td>Minimum distance the spectrum must travel to decay into groups of swells</td>
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<td>$L^*$</td>
<td>Distance at which all the leading edges of the wave groups coincide</td>
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<tr>
<td>m</td>
<td>Normal to the boundary</td>
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<td>N</td>
<td>Number of waves in a wave group</td>
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<td>R</td>
<td>Length of the wave tank</td>
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<td>t</td>
<td>Time of arrival</td>
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<td>T</td>
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<td>Potential energy</td>
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<td>$\eta$</td>
<td>Surface elevation</td>
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<td>$\rho$</td>
<td>Density</td>
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<td>$\omega$</td>
<td>Frequency of a wave</td>
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<td>$\omega_{AVE}$</td>
<td>Average frequency of frequency band</td>
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<td>$\omega_0$</td>
<td>Smallest frequency represented in the spectrum</td>
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<td>$\Delta t$</td>
<td>Time duration of the spectrum</td>
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<td>$\delta \omega$</td>
<td>Chosen frequency band</td>
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<td>$\Delta \omega$</td>
<td>Width of band of frequencies</td>
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I. INTRODUCTION

The purpose of this paper is to show how a condensed irregular sea spectrum can be generated for a maximum duration in a wave tank by making full use of the time available before interference by reflected waves occurs at the test section. Use is made of the fact that the wave velocity is indirectly proportional to the wave frequency. The dispersion of waves, due to the difference of velocity, is the basis of the procedure for forming the desired spectrum. The irregular sea spectrum is considered to be made up of a set of simple sine waves of various wave heights and periods. A discussion of sine waves is given in the appendix. A sample wave spectrum, which is an approximation to the Neumann spectrum for a fully developed sea, is used.

It is useful to first consider the dispersion of an irregular spectrum in order that the method for creation of a spectrum from its separate wave components may be more easily shown.
II. DISPERSION OF A SPECTRUM

Consider the complete spectrum allowed to pass a point, O, for an instant into smooth water. As the waves travel away from O, the longer waves begin to outrun the shorter waves. The group velocity of a set of waves of frequency \( \omega \) is given by

\[
C = \frac{\omega T}{4\pi} = \frac{\omega}{2\pi}.
\]

The time of arrival, \( t \), of the waves at some distance, \( l \), from the point \( O \) is

\[
t = \frac{\rho}{C}.
\]

From substitution into the previous equation this becomes

\[
t = \frac{2\omega l}{\rho},
\]

which may be rearranged to read

\[
\omega = \frac{2\pi t}{2}\rho.
\]

This shows the frequency which arrives at a particular time at the distance \( l \).

If the spectrum has a duration \( t_1 - t_o \) at the point \( O \), then the slowest wave which travels the distance \( l \) and arrives at the time \( t \) has the frequency

\[
\omega_s = \frac{\rho(t-t_o)}{2\rho}
\]  \hspace{1cm} (1)

while the frequency of the fastest wave which travels the distance \( l \) and arrives at the same time \( t \) is given by

\[
\omega_f = \frac{\rho(t-t_1)}{2\rho}.
\]  \hspace{1cm} (2)
The width of the band of the frequencies, $\Delta \omega$, that are arriving simultaneously may be found by subtracting equation (2) from equation (1):

$$\Delta \omega = \frac{\varphi \Delta t}{2\varphi}. \quad (3)$$

$\Delta t$ represents the duration of the spectrum at point $0$. As these waves travel further along, i.e. when $l$ is increased, they become less confused and the frequency band, $\Delta \omega$, passing a point at a given time becomes smaller.

At a fixed location the frequency band will continuously shift toward higher frequencies. However, distinct frequency bands can be considered passing a point at different time intervals.

It is useful to determine the minimum distance which the spectrum must travel to decay into groups of swells. As shown in the appendix, the equation of a surface composed of two waves of the same amplitude and of slightly different frequencies is

$$\eta = 2a \cos \left\{ \frac{1}{2} (k'' - k')x - \frac{1}{2} (w'' - w')t \right\} \sin \left\{ \frac{1}{2} (k'' + k')x - \frac{1}{2} (w'' + w')t \right\}.$$

The cosine part of this expression varies more slowly than the sine part. Thus the surface, representing a frequency band, $\Delta \omega = w'' - w'$, appears to be a group of sine waves of varying height. It can be assumed that a minimum of three waves passing a point is necessary to
compose a group. In this case the sine must vary at least six times as fast as the cosine. Thus
\[ 6(\frac{1}{2})(w^r - w^s) = \frac{1}{2} (w^r + w^s) \]
\[ 6\Delta w = w_{AVE} \]
\[ \left( \frac{\Delta w}{w_{AVE}} \right)_{MAX} = \frac{1}{3} \]
where \( w_{AVE} \) is the average frequency of the frequency band, \( \Delta w \). The smallest \( w_{AVE} \) must be equal to \( w_0 + \frac{1}{2} \Delta w \), where \( w_0 \) is the smallest frequency represented in the spectrum. Therefore,
\[ \frac{\Delta w_{MAX}}{w_0 + \frac{1}{2} \Delta w_{MAX}} = \frac{1}{3} \]
and
\[ \Delta w_{MAX} = \frac{2}{3} w_0 . \]
If \( L \) is defined as the minimum distance which the spectrum must travel to decay into groups of swells, it can be shown that
\[ L = \frac{\Delta t}{\Delta w_{MAX}} \]
or
\[ L = \frac{\Delta t}{\Delta w_{MAX}} = \frac{1}{2} \cdot \frac{2}{3} w_0 . \]

or
\[ L = \frac{\Delta t}{\Delta w_{MAX}} = \frac{\Delta t}{\frac{2}{3} w_0} . \]
III. GENERATION OF A SPECTRUM

In an attempt to develop a spectrum from wave components, wave groups which represent proper frequency bands must be assembled. This maximum frequency band width has already been shown.

According to equation (3) the frequency bands are independent of the frequency and all frequency bands are equal. The sum of all the frequency band widths must equal the total range of frequencies. The duration during which each frequency band is generated is directly proportional to the width of the frequency band. Since the spectrum created can only have a duration equal to the shortest duration of a wave component, a set of equal frequency bands will give the maximum spectrum duration. Therefore equal frequency bands will be used.

Due to the fact that a wave group consists of a whole number of waves, $\delta w$ must be chosen so that a whole number of waves fills the time period, $\Delta t$. The wave period, $T$, is the time it takes to produce one wave and is equal to $\frac{2\pi}{\omega}$. For a whole number of waves to fill the time period, $\Delta t$ has to be a multiple of $T$. If $\omega_1$ and $\omega_2$ are the average frequencies of the first two bands and $T_1$ is the period of the wave with average frequency, $\omega_1$ and
$N_1$ and $N_2$ are the number of waves in the wave groups representing these bands, then.

$$\frac{\Delta t}{T_i} = N_1, \quad N_1 = \frac{\Delta t \cdot w_i}{2 \pi}, \quad N_2 = \frac{\Delta t \cdot w_2}{2 \pi}$$

$$\frac{\omega_2}{\omega_i} = \frac{N_2}{N_1}, \quad \omega_2 = \frac{N_2}{N_1} \cdot \omega_i.$$

Now subtracting $\omega_i$ from both sides:

$$\omega_2 - \omega_i = \left(\frac{N_2}{N_1} - 1\right) \omega_i$$

$$\frac{\delta \omega}{\omega_i} = \frac{N_2}{N_1} - 1.$$  

If $\omega_0$ is the smallest frequency that is represented in the spectrum, then $\omega_1 = \omega_0 + \frac{1}{2} \delta \omega$. This gives

$$\frac{\delta \omega}{\omega_0 + \frac{1}{2} \delta \omega} = \frac{N_2}{N_1} - 1.$$  

$\delta \omega$ was chosen so that $N_1$ and $N_2$ would be integers, but $N_3$ must be investigated to determine whether or not it will be an integer. As shown

$$\frac{\delta \omega}{\omega_i} = \frac{N_2}{N_1} - 1,$$

and for the second frequency band

$$\frac{\delta \omega}{\omega_2 + \delta \omega} = \frac{N_2}{N_2} - 1 = \frac{\delta \omega}{\omega_2}.$$  

Substituting for $\delta \omega$ will give

$$\frac{\frac{N_2 - N_1}{N_1} \omega_i}{\omega_1 + \left(\frac{N_2 - N_1}{N_1}\right) \omega_i} = \frac{N_3 - N_2}{N_2}.$$
which may be rearranged to give

\[
\frac{N_2 - N_1}{N_1} = \frac{N_2 - N_2}{N_2} \tag{2}
\]

\[
\frac{N_1 + N_2 - N_1}{N_1} = \frac{N_2 - N_2}{N_2} \tag{3}
\]

\[
\frac{N_2 - N_1}{N_2} = \frac{N_2 - N_2}{N_2} \tag{4}
\]

\[
N_3 = 2N_2 - N_1. \tag{5}
\]

Therefore, if \( N_1 \) and \( N_2 \) are integers, \( N_3 \) is an integer and, likewise, all \( N \)'s are integers.

Wave groups are produced whose frequency range is \( \delta \omega \) in an attempt to have them recombine to form the original dispersing spectrum. Each of these wave groups is in reality a miniature spectrum which continues to disperse even as the other wave groups become superimposed on it.

The leading edge of each wave group travels with the velocity of the minimum frequency in the wave group. If \( L^* \) is defined as the distance at which all the leading edges of the wave groups coincide, then

\[
\Delta t + \frac{L^*}{C(\omega_0)} = \frac{L^*}{C(\omega_0 + \delta \omega)}. \tag{6}
\]

Substituting for the group velocities will give
\[
\frac{2L^*(\omega_0) + \Delta t}{g} = \frac{2L^*(\omega_0 + \delta \omega)}{g}
\]

or \( L^* = \frac{g \Delta t}{2 \delta \omega} \).

Since the maximum value of \( \delta \omega \) is equal to \( \Delta \omega_{max} \), the minimum value of \( L^* \) is 1.

The wave groups that recombine at \( L^* \) are the same as if they had left the original dispersing spectrum and traveled for a distance of \( 2L^* \). So instead of the original dispersing spectrum, there will be a spectrum that passes for a time \( 2\Delta t \). Therefore, at \( L^* \) each wave group consists of twice the number of waves as when compared with its initial phase.

In the appendix it is shown that the wave energy is proportional to the square of the wave height. Since there are twice the number of waves, the height of the input waves must be multiplied by the square root of two in order to have the proper wave height in the spectrum to be generated. The desired spectrum then lasts for a time \( 2\Delta t \).

The wave tank must be long enough for this process to take place without reflections reaching the point where the spectrum passes for the maximum time. Thus if \( R \) is the length of the wave tank, then

\[
R \geq L^* + C(\omega_0) \Delta t
\]
This is because the spectrum passes a point at the distance $L^*$ for a time $2\Delta t$, and the fastest wave must travel for a distance equal to twice the distance from the point at $L^*$ to the wall before reflections interfere with the spectrum at the point of maximum duration. Substituting for $L^*$ and $C(\omega_b)$ will give

$$R = \frac{2\Delta t_{\text{MAX}}}{2\delta\omega} + \frac{\delta\Delta t_{\text{MAX}}}{2\omega_0}$$

or

$$\Delta t_{\text{MAX}} = \frac{2R}{g} \left( \frac{\omega_0 \delta\omega}{\omega_0 + \delta\omega} \right).$$

The maximum time during which a spectrum representing a given frequency range can be produced is $2\Delta t_{\text{MAX}}$.

A Neumann spectrum\(^1\) of a fully developed sea with a wind speed of thirty-two knots will now be considered. The frequency range represented by this spectrum extends from $\omega = 0.353$ sec.\(^{-1}\) to $\omega = 1.26$ sec.\(^{-1}\). For convenience this range is taken to be $5\%$ to $4\%$ sec.\(^{-1}\). The maximum value of the frequency band is found by $\Delta \omega_{\text{MAX}} = \frac{\Delta \omega}{\omega_0} \to 2\%$.

In order to make this spectrum suitable for

reproduction in a wave tank, the wave height must be scaled down by a factor of proportionality, for instance 1:100. Scale up the wave frequency by the square root of this factor, which is 1:10.

The maximum time, $\Delta t_{\text{MAX}}$, for which wave groups may be produced,

$$\Delta t_{\text{MAX}} = \frac{2B}{g} \left( \frac{w_0 \delta w}{w_0 + \delta w} \right),$$

has a numerical value for the case of a 100 foot tank:

$$\Delta t_{\text{MAX}} = \frac{(2)(100 \text{ ft})}{32.2 \text{ ft/deg}^2} \frac{(\frac{\pi}{10} - \frac{\pi}{5})\text{ SEC}^2}{(\frac{\pi}{10} + \frac{\pi}{5})\text{ SEC}} = 5.56 \text{ SEC}.$$

This time interval must be filled by a whole number of waves. Thus $\omega_1$ is equal to $\frac{\pi}{10} + \frac{1}{2} \left( \frac{2\pi}{3} \right)$ which is $\frac{6\pi}{5}$. Then

$$T_1 = \frac{2\pi}{\omega_1} = \frac{10}{5} \text{ SEC}.$$

It is obvious that if the time interval, $\Delta t$, is five seconds, then three waves of period $T_1$ can be produced. For the second wave group the period is $T_2 = \frac{10}{8}$ seconds. Four waves of period $T_2$ will fill a five second $\Delta t$.

The following data is calculated from the Neumann spectrum.
\begin{tabular}{|c|c|c|}
\hline
$\omega$ & $\frac{dA^2}{d\omega}$ & $A^2$ \\
\hline
$6 \pi \over 50$ & $1.29 \times 10^4$ & $10.3 \times 10^4$ \\
$8 \pi \over 50$ & $1.88 \times 10^4$ & $15.0 \times 10^4$ \\
$10 \pi \over 50$ & $1.40 \times 10^4$ & $11.1 \times 10^4$ \\
$12 \pi \over 50$ & $0.80 \times 10^4$ & $6.37 \times 10^4$ \\
$14 \pi \over 50$ & $0.574 \times 10^4$ & $4.57 \times 10^4$ \\
$16 \pi \over 50$ & $0.274 \times 10^4$ & $2.16 \times 10^4$ \\
$18 \pi \over 50$ & $0.142 \times 10^4$ & $1.13 \times 10^4$ \\
$20 \pi \over 50$ & $0.080 \times 10^4$ & $0.637 \times 10^4$ \\
\hline
\end{tabular}
In order to get the desired scale model of a spectrum, the wave frequency is multiplied by ten and the square of the wave height is multiplied by $2 \times 10^{-4}$. The square of the wave height is changed to inches squared to obtain:

$$
\begin{align*}
\omega &\sim \text{sec}^{-1} & h^2 &\sim \text{in}^2 \\
6\pi/5 & & 3.19 \\
8\pi/5 & & 4.66 \\
10\pi/5 & & 3.44 \\
12\pi/5 & & 1.98 \\
14\pi/5 & & 1.42 \\
16\pi/5 & & 0.68 \\
18\pi/5 & & 0.35 \\
20\pi/5 & & 0.20
\end{align*}
$$

If wave groups of frequency $\omega_1, \omega_2, \ldots, \omega_8$ are produced with the square of the wave height being $h_1^2, h_2^2, \ldots, h_8^2$, an approximation of the desired scale model spectrum is obtained. The spectrum will pass for a maximum time at the distance $L^*$ where $L^* = \frac{\Delta t}{2\omega}$.

$$
L^* = \frac{32.2 \text{ft/sec}^2 \cdot \text{sec}}{2 \cdot 3.2 \pi} = \frac{32.2 \times 25}{4 \pi} = 22.2 \times 25 \approx 64 \text{ ft}
$$

The spectrum will last for a time $2\Delta t$ or ten seconds at a distance of 64 feet from the wave maker.
IV. CONCLUSION

An approximation to the desired spectrum can be generated which lasts for a time equal to twice the time duration of the input wave groups. The maximum time this spectrum can last is limited by the length of the wave tank and the maximum frequency band width which can be represented by wave groups. The largest frequency band which can be represented is given by \( \frac{2}{3} \omega_0 \) where \( \omega_0 \) is the smallest frequency represented in the spectrum.

As smaller frequency bands are taken, the approximation to the actual wave spectrum becomes more exact, but the duration of the spectrum becomes shorter.

Thus the duration available depends on the length of the wave tank, the smallest frequency in the spectrum, and the frequency band chosen.
At a given time, $\frac{g t}{2 f}$, waves arrive which represent a frequency band $\Delta \omega$. Waves of a frequency, $\omega$, arrive for a time period $\frac{2at}{2f}$. 

FIGURE 1. DISPERSION OF A SPECTRUM
FIGURE 2. APPROXIMATION OF A NEUMANN SPECTRUM
APPENDIX

The following discussion of simple sine waves is taken from Horace Lamb\(^2\). A co-ordinate system with the origin at the undisturbed water level is chosen. \(x\) is chosen to be in the direction of wave travel and \(y\) is chosen to be vertical with upward positive. If there is no variation in the direction perpendicular to the wave travel, the wave is two dimensional and the velocity potential is

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.
\]

This comes from the fact that the water was originally at rest, and is necessarily irrotational. At a fixed boundary

\[
\frac{\partial \phi}{\partial n} = 0
\]

where \(n\) is the normal to the boundary.

If the square of the velocity is neglected, then

\[
\frac{\rho}{\rho} = \frac{\partial \phi}{\partial t} - \gamma y + F(t).
\]

If \( \eta \) denotes the surface elevation, then

\[
\eta = \frac{1}{g} \left[ \frac{\partial \phi}{\partial t} \right]_{y=0}
\]

since there is uniform pressure at the surface. With only a small error it can be said that

\[
\eta = \frac{1}{g} \left[ \frac{\partial \phi}{\partial y} \right]_{y=0}.
\]

Since the direction normal to the surface is almost perpendicular to the horizontal,

\[
\frac{\partial n}{\partial t} = -\left[ \frac{\partial \phi}{\partial t} \right]_{y=0}.
\]

These equations may be combined to obtain

\[
\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial y} = 0
\]

when \( y = 0 \).

If simple-harmonic motion with a time factor \( e^{i(\omega t + \epsilon)} \) is assumed, then

\[
\omega^2 \phi = g \frac{\partial \phi}{\partial y}.
\]

It is also assumed that \( \phi = P \cos Kx \cdot e^{i(\omega t + \epsilon)} \), where \( P \) is a function of \( y \) alone. From the first equation

\[
\frac{\partial^2 P}{\partial y^2} = K^2 P = 0;
\]

therefore \( P = A e^{ky} + B e^{-ky} \).
The condition of no vertical movement at the bottom is
\[ \frac{\partial \phi}{\partial y} = 0, \]
for \( y = -h \). This gives
\[ A e^{-kh} = B e^{kh} = \frac{1}{2} C. \]
Therefore:
\[ A = \frac{1}{2} C e^{kh}, \quad B = \frac{1}{2} C e^{-kh}, \quad P = \frac{1}{2} C \cosh k(y+h). \]

\[ \phi = C \cosh k(y+h) \cos kx \cdot e^{-i(\omega t + \epsilon)}. \]

The partial derivative with respect to \( y \) is now taken to obtain
\[ \frac{\partial \phi}{\partial y} = kC \sinh k(y+h) \cos kx \cdot e^{-i(\omega t + \epsilon)}. \]

If this is introduced into the expression \( \omega^2 = \frac{g}{\rho} \frac{\partial^2 \phi}{\partial y^2} \), then
\[ \omega^2 = gk \tanh kh, \quad \omega = (gk \tanh kh)^{1/2}. \]

From the condition at the surface
\[ \eta = \frac{i \omega c}{g} \cosh kh \cos kx \cdot e^{-i(\omega t + \epsilon)} \]
or using \( a = \frac{-\omega c}{g} \cosh kh \) and retaining the real part of the expression,
\[ \eta = a \cos kx \cdot \sin (\omega t + \epsilon). \]

This represents a system of standing waves of
wave length $\lambda = \frac{2\pi}{k}$ and period $T = \frac{2\pi}{\omega}$.

For traveling waves two sets of waves,

$$\eta_1 = a \sin kx \cos \omega t$$
$$\eta_2 = a \cos kx \sin \omega t,$$

can be superimposed to give $\eta = a \sin (kx \pm \omega t)$.

The wave velocity, $U$, is given by

$$U = \frac{\omega}{k} = \left( \frac{2\pi}{\lambda} \right)^{1/2}.$$

For a wave length less than twice the depth, that is, for deep water waves, $\tanh kh \approx 1$

Therefore,

$$U = \left( \frac{2\pi}{\lambda} \right)^{1/2}.$$

Now the relationship between the velocity of a group of waves of the same wave length and the velocity of a wave in the group will be examined. Suppose two systems of waves of almost the same wave length are superimposed. The equation of the free surface will be

$$\eta = a \sin (kx - \omega t) + a \sin (k'x - \omega' t).$$

$$\eta = 2a \cos \left\{ \frac{1}{2} (k-k') x - \frac{1}{2} (\omega-\omega') t \right\} \sin \left\{ \frac{1}{2} (k+k') x - \frac{1}{2} (\omega-\omega') t \right\}.$$

The cosine part of this expression varies very slowly with $x$. The surface profile takes the form of a curve of sines in which the amplitude changes slowly from 0 to $a$. 
The distance between groups is $2\sqrt{\nu K - K'}$ and the time it takes to travel this distance is $2\sqrt{\omega - \omega'}$. The velocity of the group is then $C = \omega - \omega'_{K - K'}$, where $C$ is the group velocity. This becomes

$$C = \frac{d\omega}{dK}$$

or in another way

$$C = \frac{d}{dK} (\omega(KU)) = U - \lambda \frac{dU}{d\lambda}.$$ 

Recall the expression

$$U = \frac{\nu}{\nu K} (\tanh kh)^{1/2}.$$ 

Take the derivative of this and use the formula,

$$\sinh 2\xi = 2 \sinh \xi \cosh \xi,$$

to get

$$KU = (\nu K \tanh kh)^{1/2},$$

$$\frac{d}{dK} (\nu K \tanh kh)^{1/2} = \frac{1}{2} (\nu K \tanh kh)^{1/2} \left[ \frac{\nu \tanh kh + \nu K \text{sech}^2 kh}{\nu K \tanh (kh)^{1/2}} \right],$$

$$\frac{d^2}{dK} (\nu K \tanh kh)^{1/2} = \frac{1}{2} U \left[ 1 + \frac{K K \text{sech}^2 kh}{\tanh kh} \right],$$

$$\frac{d^2}{dK} (\nu K \tanh kh)^{1/2} = \frac{1}{2} U \left[ 1 + \frac{K K \text{sech}^2 kh}{\cosh kh \sinh kh} \right].$$
\[
\frac{d}{dk} = C = \frac{1}{2} U \left[ 1 + \frac{2k h}{\sinh 2kh} \right].
\]

For deep water this becomes

\[ C = \frac{1}{2} U. \]

Therefore, for deep water the group velocity is half the velocity of the individual waves.

\[ C = \frac{gT}{4\pi}. \]

The energy of a wave is equal to the sum of the potential and kinetic energy. The potential energy is equal to

\[ PE = \frac{1}{2} g \rho \int \frac{h^2}{\alpha} d\alpha. \]

This is equal to

\[ PE = \frac{1}{16} g \rho h^2 \lambda \sin^2 (wt + \epsilon). \]

The kinetic energy is given by

\[ KE = \frac{1}{2} \rho \int \left[ \frac{\partial \phi}{\partial x} \right] d\alpha. \]

This is equal to

\[ KE = \frac{1}{16} g \rho h^2 \cos^2 (wt + \epsilon). \]

The sum is constant and is equal to

\[ E = \frac{1}{16} g \rho h^2 \lambda. \]

This is directly proportional to the square of the height.