AN EXPERIMENTAL AND ANALYTICAL INVESTIGATION
OF THE EFFECT OF TRAY LIQUID
ON THE VIBRATION BEHAVIOR OF BUBBLE TOWERS

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ABSTRACT

Study of field observations indicates that tray liquid affects the vibration behavior of a bubble tower as would a tuned dynamic vibration absorber. In this thesis an analytical investigation and experimental model studies were carried out to the conclusion that tray liquid does indeed act as a dynamic absorber.
Acknowledgments are to be made to Dr. James Woodburn, chairman of the Mechanical Engineering Department, who allotted funds for the construction of the apparatus, and to Mr. R. Guidry and Mr. R. Martin who performed the necessary machine work. Credit is also due Professor W. B. Diboll for his helpful suggestions regarding the design of the apparatus. Most important, it is to Dr. Peter A. Szego that a debt of gratitude is owed for without his patient guidance and assistance, this thesis could never have been written.
INTRODUCTION

Throughout the chemical and petroleum industries bubble towers are used to bring liquids and vapors into intimate contact. A bubble tower is a slender cylindrical tank mounted with its axis vertical; the interior of the tower is filled with closely spaced horizontal trays. A typical tower might be 5 feet in diameter and 100 feet high with 30 equally spaced trays. When a bubble tower is in operation, each tray carries a 3 to 6 inch layer of liquid and arrangements are made to bubble vapor up through the liquid on each tray.

Externally a bubble tower looks somewhat like a smokestack, and the resemblance actually goes beyond overall appearances; bubble towers, like smokestacks, are subject to wind-induced vibrations. The particular case which gave rise to this investigation involved a tower 8 feet in diameter and 145 feet high. Under the influence of a steady wind at 30 miles per hour, the tower was found to sway in a direction perpendicular to the wind with a frequency of 40 cycles per minute. A check of the natural frequencies of the tower indicated that the tower was vibrating as a cantilever beam in the first mode, and application of an approximate formula (Den Hartog, p. 373)* indicated that excitation resulted from the von Karman vortex effect.

The curious thing about the described situation is that the resonant condition developed only when the trays were completely empty. One might be tempted to believe that addition of liquid reduced vibration tendencies.

* Complete references are to be found in the bibliography.
by changing the natural frequency of the tower through addition of mass. However a study of the actual situation showed that if the liquid is treated as added mass, the natural frequency is lowered by less than five per cent. This change would only cause resonance to occur at some lower wind velocity, but it was observed that the presence of liquid on the trays of the tower prevented excessive vibrations for all wind velocities. There is a definite likelihood that some of the effect of the added liquid is evidenced in energy dissipation; however, the interesting facet of the problem lies in the dynamic interrelation between liquid and the tower. Under the disturbing influence of tower motion, the liquid will exhibit motion with respect to the tower. The liquid may slosh like coffee in a cup. For a tray similar to those in the tower, the lower modes of liquid motion were found by calculation to have frequencies in a range including 40 cycles per minute. The addition of tray liquid to a tower may in a rough sense be tantamount to attaching a tuned spring-mass system to the tower, the effect being one of dynamic absorption (Den Hartog, pp. 112-132).

The initial objective of the work described in this thesis was verification, both analytical and experimental, of the contention that tray liquid influences the motion of a bubble tower as would a dynamic absorber. The complexity of mathematical studies dictated that the initial investigation deal with a column and a single tray. The source of excitation and physical dimensions for the experimental model were governed by experimental factors to be discussed in Section 4.
The program of investigation then involved the following phases: choice of a simple system for study of the phenomenon, development of analytical and experimental methods for studying that system, and determination of the dynamic behavior of the system by the two methods.

It is to be pointed out that complete understanding of a simple system will yield a certain insight into the nature of the problem in an actual tower. Furthermore an analytical study of a simple case may aid in the developments of approximate methods for describing more complex situations, and at the same time, the experimental techniques used for the elementary investigation can be extended to substantiate the validity of those approximate methods as applied to more complex systems.
SECTION 1

FORMULATION OF THE PROBLEM

The system under study is a uniform circular column which supports a circular tray of liquid. The top end of the column is free, but the bottom end is built in to a base which is subject to a simple harmonic displacement, as shown in figure 1. The bottom of the tray is normal to the column; while the sides of the tray are concentric with and parallel to the column. It is the aim of this analysis to describe the effect of liquid in the tray upon the motion of the tower when the tray is located at some arbitrarily specified point between the top and the bottom.

Figure 1

\[ a_0 \sin \omega t \]
AN ANALYTICAL INVESTIGATION OF THE MOTION OF THE
TRAY AND CONTAINED LIQUID AS A SEPARATE PROBLEM

Analysis of the system as a whole requires treatment of liquid motion, and it is desirable to develop methods for describing fluid motion without burdening the procedures with the complexities of the entire system. The system to be considered in this section is an annular tray with a horizontal bottom and vertical sides. The tray is given a horizontal displacement which can be described by $b_0 \sin \omega t$. And for purposes of this analysis, the fluid in the tray is assumed to be incompressible, inviscid, and subject only to irrotational motion. Further assumptions made are that the amplitude of surface waves is small and that $b_0$ is small. Cylindrical coordinates are chosen as indicated in figure 2.

Figure 2
On the basis of assumptions made, fluid motion must satisfy the following differential equations throughout the domain:* 

\[ \frac{\partial \Phi}{\partial t} - F - \frac{|\nabla|}{2} - \frac{P}{\rho} = 0, \]  

(2-1)

and 

\[ \nabla^2 \Phi = 0, \]  

(2-2)

where 

\( \Phi \) = velocity potential function 

\( F \) = field force function 

\( P \) = pressure 

\( \rho \) = density of liquid 

\( \nabla \) = velocity of liquid, given by \( \nabla = -\nabla \Phi \).

**Boundary conditions at outer and inner walls**

The velocity of the tray is \( b_0 \omega \cos \omega t \) since displacement is described by \( b_0 \omega \sin \omega t \). At the outer and inner walls, the velocity of the liquid in the radial direction must be equal to the radial component of the velocity of the wall. This condition may be expressed as 

\[ -\frac{\partial \Phi}{\partial r} = b_0 \omega \cos \omega t \cos \theta \]  

(2-3)

at \( r = r_1 \) (inner radius) and \( r = r_0 \) (outer radius) for all values of \( z \) from bottom to surface.

* Streeter, pp. 24-26, where \( \nabla = i q, F = \Omega \), and \( F(t) \) is incorporated into \( \Phi \).

~ In the equation designations, the first number or letter denotes the section or appendix and the second number specifies the equation.
Boundary conditions at bottom

At the bottom of the tray the liquid can have no velocity in the z direction;

at \( z = 0 \), \[ \frac{\partial \Phi}{\partial z} = 0 \] \hspace{1cm} (2-4)

Boundary conditions at surface

The assumption that gravity is the only field force leads to \( F = gz \).

If velocities are assumed small, and if the pressure at the surface is set equal to zero, equation (2-1) becomes

\[ \frac{\partial \Phi}{\partial t} - gz_0 = 0 \] \hspace{1cm} (2-5)

at the surface where \( z_0 \) describes the position of the surface.

Equation (2-5) may be rewritten as

\[ z_0 = \frac{1}{g} \frac{\partial \Phi}{\partial t} \] \hspace{1cm} (2-6)

The velocity of the surface in the z direction may be described by

velocity = \( \frac{\partial z_0}{\partial t} \) \hspace{1cm} (2-7)

For small waves the velocity of the surface may also be described by

velocity = \( -\frac{\partial \Phi}{\partial z} \) \hspace{1cm} (2-8)

where \( \frac{\partial \Phi}{\partial z} \) is evaluated at the equilibrium surface.

From equations (2-6), (2-7), and (2-8),

\[ \frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} = 0 \] \hspace{1cm} (2-9)

at \( z = h \).
Solution for the motion

A separation of variables is assumed:

\[ \Phi = \Theta(\theta) T(t) \varphi(r,z) . \]

Equation (2-3) then becomes

\[ - \Theta(\theta) T(t) \frac{\partial \varphi(r,z)}{\partial r} = b_0 \omega \cos \omega t \cos \theta \]  \hspace{1cm} (2-10)

at \( r = r_1, \ r = r_0, \) and for \( 0 < z < h. \)

From equation (2-10)

\[ \Theta(\theta) = \cos \theta , \]  \hspace{1cm} (2-11)

\[ T(t) = b_0 \omega \cos \omega t , \]  \hspace{1cm} (2-12)

and

\[ - \frac{\partial \varphi(r,z)}{\partial r} = 1 \]  \hspace{1cm} (2-13)

at \( r = r_1, \ r = r_0, \) and for \( 0 < z < h. \)

Equation (2-4) becomes

\[ \frac{\partial \varphi}{\partial z} = 0 \]  \hspace{1cm} (2-14)

at \( z = 0. \)

Equation (2-9) becomes

\[ \omega^2 \varphi = g \frac{\partial \varphi}{\partial z} \]  \hspace{1cm} (2-15)

at \( z = h. \)

And equation (2-2) becomes

\[ \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} - \frac{1}{r^2} \varphi + \frac{\partial^2 \varphi}{\partial z^2} = 0 \]  \hspace{1cm} (2-16)

since

\[ \frac{\partial^2 \Phi}{\partial \theta^2} = - \Theta T \varphi . \]
If a further separation, \( \phi(r,z) = R(r) Z(z) \), is assumed, equation (2-16) may be rewritten

\[
\frac{1}{R} \left( R'' + \left( \frac{1}{r} \right) R' - \frac{R}{r^2} \right) - \frac{1}{Z} Z'' = 0 .
\] (2-17)

In order for equation (2-17) to hold throughout the domain,

\[
\frac{1}{Z} Z'' = -k^2 , \quad \text{a constant} .
\] (2-18)

It should be noted that for \( k = 0 \), the solution of equation (2-18) is

\[
Z = A' + B' z .
\]

From equation (2-14), \( Z' = 0 \) at \( z = 0 \); therefore \( B' = 0 \).

From equation (2-15), \( \omega^2 Z = g Z' = 0 \); therefore \( A' = 0 \).

The solution from \( k = 0 \), \( Z \equiv 0 \), is not of immediate interest.

From equations (2-17) and (2-18),

\[
Z'' + k^2 Z = 0 ,
\] (2-19)

and

\[
R'' + \left( \frac{1}{r} \right) R' - \left( k^2 + \frac{1}{r^2} \right) R = 0 .
\] (2-20)

Equation (2-19) has the solution

\[
Z = A \cos(kz) + B \sin(kz) .
\] (2-21)

Equation (2-20) has the solution

\[
R = 6 I_1(\kappa r) + D K_1(\kappa r)
\] (2-22)

where \( I_1 \) indicates a Bessel function of the first kind, of order 1, with imaginary arguments and \( K_1 \) indicates a Bessel function of the second kind, of order 1, with imaginary arguments.

From equation (2-14), \( Z' = 0 \) at \( z = 0 \); therefore \( B = 0 \).

And

\[
Z = A \cos(kz) .
\] (2-23)
From equations (2-15) and (2-23),

\[ \omega^2 Z R = g R Z^* \]

at \( z = h \), and

\[ (\omega^2 h/g) \cos(kh) + kh \sin(kh) = 0 \]

(2-24)

Define:

\[ \xi = kh \]

\[ \lambda = \omega^2 h/g \]

Equation (2-24) becomes

\[ u(\xi) = \xi \sin \xi + \lambda \cos \xi = 0 \]

(2-25)

The roots of equation (2-25), designated by \( \xi \), \( \xi_2 \), \( \xi_3 \), \( \cdots \) \( \xi_n \), may be complex and are discussed in appendix A. Each \( \xi \) determines a \( k_n \).

Since equations (2-14), (2-15), and (2-16) are linear, the solution for \( \phi \) is

\[ \phi = \sum_n [C_n I_1(k_n z) + D_n K_1(k_n z)] \cos(k_n z) \]

(2-26)

From equation (2-26), equation (2-13) becomes

\[ - \sum_n [C_n I_1(k_n z) + D_n K_1(k_n z)] \cos(k_n z) k_n = \xi \quad n, \]

(2-27)

and

\[ - \sum_n [C_n I_1(k_n z) + D_n K_1(k_n z)] \cos(k_n z) k_n = 1 \]

(2-28)

for \( 0 < z < h \) and where \( I_1(\ ) \) and \( K_1(\ ) \) are taken to mean the derivatives with respect to the entire argument.

It is assumed that there exists a set of \( \alpha_n \) such that

\[ \sum_n \alpha_n \cos(k_n z) = 1 \]

(2-29)

for \( 0 < z < h \).

Then

\[ - k_n [C_n I_1(k_n z) + D_n K_1(k_n z)] = \alpha_n \]

and

\[ - k_n [C_n I_1(k_n z) + D_n K_1(k_n z)] = \alpha_n \]

(2-30)
From equations (2-30),

\[
C_n = \frac{\left| \begin{array}{cc}
- \alpha_n / k_n & K_n'(k_n r_o) \\
- \alpha_n / k_n & K_n'(k_n r_1)
\end{array} \right|}{\left| \begin{array}{cc}
I_n'(k_n r_o) & K_n'(k_n r_o) \\
I_n'(k_n r_1) & K_n'(k_n r_1)
\end{array} \right|},
\]

\[
D_n = \frac{\left| \begin{array}{cc}
I_n'(k_n r_o) & - \alpha_n / k_n \\
I_n'(k_n r_1) & - \alpha_n / k_n
\end{array} \right|}{\left| \begin{array}{cc}
I_n'(k_n r_o) & K_n'(k_n r_o) \\
I_n'(k_n r_1) & K_n'(k_n r_1)
\end{array} \right|}.
\]

From equations (2-11), (2-12), and (2-26)

\[
\Phi = b \rho \omega \cos \omega t \cos \theta \sum_n [C_n I_n(k_n r) + D_n K_n(k_n r)] \cos(k_n z) \quad (2-31)
\]

**Solution for net horizontal wall force**

If wall force is assumed positive in the direction \( \theta = 0 \),

\[
\text{net wall force} = \int_0^h \int_0^{2\pi} P(r_o, z, \theta, t) \cos \theta \, dz \, r_o d\theta
\]

\[
- \int_0^h \int_0^{2\pi} P(r_1, z, \theta, t) \cos \theta \, dz \, r_1 d\theta.
\]

For the sake of simplicity the integral is evaluated for a general \( r \).
From equation (2-1) and \( F = g z \),

\[
P = -\rho g z - \rho \frac{|v|^2}{2} + \rho \frac{d^2 \phi}{dt^2}.
\]

The integral of each term is considered separately.

\[
- \int_0^h 2\pi \int_0^{2\pi} \rho g z \cos \theta \, dz \, d\theta = 0 \quad \text{since} \quad \int_0^{2\pi} \cos \theta \, d\theta = 0.
\]

\[
|v|^2 = \theta^2 \left[ T \frac{d^2 \phi}{dr^2} \right]^2 + \left( \frac{d\theta}{d\phi} \right)^2 \left[ T \frac{d^2 \phi}{dz^2} \right]^2.
\]

Since \( \theta^2 = \cos^2 \theta \) and \( \left( \frac{d\theta}{d\phi} \right)^2 = \sin^2 \theta \),

and since \( \frac{2\pi}{0} \int \cos^2 \theta \, d\theta = 0 \)

and \( \frac{2\pi}{0} \int \sin^2 \theta \, d\theta = 0 \),

\[
- \int_0^h 2\pi \int_0^{2\pi} \rho \frac{|v|^2}{2} \cos \theta \, dz \, d\theta = 0.
\]

From equation (2-31),

\[
\frac{d^2 \phi}{dt^2} = -b_0 \omega^2 \sin \omega t \cos \theta \sum \left[ C_n l_n (k \cdot r) + D_n k_n (k \cdot r) \right] \cos (k \cdot z).
\]

\[
\frac{h}{2\pi} \int \int F(r, z, \theta, t) \cos \theta \, dz \, d\theta = -b_0 \rho \omega^2 \sin \omega t \sum \left[ C_n l_n (k \cdot r) + D_n k_n (k \cdot r) \right] \frac{\sin \frac{k \cdot z}{c}}{c}.
\]
And equation (2-32) becomes

net wall force

\[ = - b_0 \rho \omega^2 \sin \omega t \sum_n \frac{r_n}{n} \left[ C_n I_1(k_n r_o) + D_n K_1(k_n r_o) \right] \sin \frac{S_n}{S_o} \]

\[ + b_0 \rho \omega^2 \sin \omega t \sum_i \frac{r_i}{n} \left[ C_i I_1(k_i r_i) + D_i K_1(k_i r_i) \right] \sin \frac{S_i}{S_o} , \]

or net wall force \( = - b_0 \omega^2 (\rho \omega r_o) \sin \omega t \)

\[ \sum_n \frac{C_n I_1(k_n r_o) - \frac{r_i}{r_o} I_1(k_n r_i)}{n} \left[ C_n I_1(k_n r_o) + D_n K_1(k_n r_o) - \frac{r_i}{r_o} K_1(k_n r_i) \right] \sin \frac{S_n}{S_o} \]

**Solution for net horizontal wall force for a simply connected tray**

In the case of certain models it is desirable to facilitate numerical calculations by neglecting the effect of the inner wall, which brings about a substantial, though not unreasonable, simplification. That this simplification is not unreasonable is born out by two considerations. First, the actual wall force exerted on the inner wall is small compared to the total since the area of the inner wall will usually be much smaller than the outer wall area. Also of importance to this reasoning is the fact that the nonsymmetrical modes of fluid motion, which result in a net wall force, have nodal points at the center. And second, for the modes of fluid motion which are of interest in the case to be studied herein, the calculated natural frequency is changed by less than 5% when the inner wall is neglected.*

* The general method of determining frequency is given in Lamb, Articles 191 and 257. Roots of the transcendental frequency equation may be found in articles by Dwight and by Truell
If the tray is considered simply connected, equation (2-26) becomes

$$\phi = \sum_n [C_n I_1(k_n r) \cos(k_n z)] . \quad (2-34)$$

Equation (2-27), which results from the boundary condition at the outer wall, now becomes

$$- \sum_n [C_n I_1'(k_n r_o) \cos(k_n z)] = 1 \quad (2-35)$$

where

$$C_n = \frac{-\zeta_n}{k_n I_1'(k_n r_o)} , \quad (2-36)$$

the solution for $\zeta_n$ remaining as presented in appendix B. As a result of equation (2-34), (2-31) is changed to

$$\Phi = b_0 \omega \cos \omega t \cos \theta \sum_n [C_n I_1(k_n r) \cos(k_n z)] . \quad (2-37)$$

Net wall force is then given by

$$\text{net wall force} = - b_0 \omega^2 \sin \omega t \left( \rho h r_o \right) \sum_n [C_n I_1(k_n r_o) \frac{\sin \zeta_n}{\zeta_n}] .$$

Furthermore from equations (2-36) and (B-5),

$$\text{net wall force} = - b_0 \omega^2 \sin \omega t \left( \rho h r_o \right) \frac{\sin \zeta_n}{\zeta_n} \frac{I_1(\zeta_n)}{I_1'(\zeta_n)} , \quad (3-38)$$

where $\sigma = r_o/h$ and as before $\lambda = \omega^2 h/\rho$, $\zeta_n = k_n h$, and $u(\zeta) = \zeta \sin \zeta + \lambda \cos \zeta$, the summation being taken over all roots of $u(\zeta)$. 
AN ANALYTICAL INVESTIGATION OF THE COMPLETE SYSTEM

\[ M = \text{mass of the tray} \]
\[ m = \text{mass per unit length of the column} \]

In order to analyze the complete system described in section 1, coordinates are chosen as indicated in figure 3. The displacement of the tray and its rotation about a horizontal axis must be considered small to comply with conditions set up on the motion of the tray in section 2.
It is noted that
\[ v_1 = a_0 \sin \omega t - v_1^*, \quad \frac{d^2 v_1}{dx_1^2} = -\frac{d^2 v_1^*}{dx_1^2}, \]
and
\[ v_2 = a_0 \sin \omega t + v_2^*, \quad \frac{d^2 v_2}{dx_2^2} = +\frac{d^2 v_2^*}{dx_2^2}. \]

Application of the elementary beam formula to the bottom part of the column yields
\[
EI \frac{d^4 v_1^*}{dx_1^4} = q_1 = m \frac{d^2 v_1}{dt^2},
\]
and
\[
\frac{d^4 v_1}{dx_1^4} + \frac{m}{EI} \frac{d^2 v_1}{dt^2} = 0. \tag{3-1}
\]

If \( v_1 = X_1(x_1)a_0 \sin \omega t, \) equation (3-1) becomes
\[
\frac{d^4 X_1}{dx_1^4} - \frac{m}{EI} \frac{d^2 X_1}{dt^2} = 0 \tag{3-2}
\]

If \( j \) is defined by, \( j^4 = \frac{EI}{M_0} \), equation (3-2) has the solution
\[
X_1 = E_1 \sin(jx_1) + F_1 \cos(jx_1) + G_1 \sinh(jx_1) + H_1 \cosh(jx_1) \tag{3-3}
\]

If \( v_2 = X_2(x_2)a_0 \sin \omega t, \) similar considerations result in
\[
X_2 = E_2 \sin(jx_2) + F_2 \cos(jx_2) + G_2 \sinh(jx_2) + H_2 \cosh(jx_2) \tag{3-4}
\]
Boundary conditions on lower part

At \( x_1 = 0 \), \( w_1 = a_0 \sin \omega t \) so that \( X_1 = 1 = F_1 + H_1 \);

and therefore \( H_1 = 1 - F_1 \) \hspace{1cm} (3-5)

Also at \( x_1 = 0 \), \( \frac{\partial w_1}{\partial x_1} = 0 \) so that \( 0 = E_1 + G_1 \);

and therefore \( G_1 = -E_1 \) \hspace{1cm} (3-6)

From equations (3-5), and (3-6),

\[
X_1 = E_1 [\sin(jx_1) - \sinh(jx_1)] + F_1 [\cos(jx_1) - \cosh(jx_1)] \cdot \cosh(jx_1).
\]

\hspace{1cm} (3-7)

Boundary conditions on top part

At \( x_2 = 0 \), \( \frac{\partial^2 w_2}{\partial x_2^2} = 0 \) so that \( 0 = -F_2 + H_2 \);

and therefore \( H_2 = F_2 \) \hspace{1cm} (3-8)

Also at \( x_2 = 0 \), \( \frac{\partial^3 w_2}{\partial x_2^3} = 0 \) so that \( 0 = -E_2 + G_2 \);

and therefore \( G_2 = E_2 \) \hspace{1cm} (3-9)

From equations (3-8), and (3-9),

\[
X_2 = E_2 [\sin(jx_2) + \sinh(jx_2)] + F_2 [\cos(jx_2) + \cosh(jx_2)] \cdot \cosh(jx_2).
\]

\hspace{1cm} (3-10)
Continuity conditions at tray

At the tray $x_1 = L_1$, and $x_2 = L_2$.

Shown in figure 4 are the positive directions for moment and shear in accord with the choice of coordinates in figure 3. The direction of wall force is as assumed in section 2.

\[
\begin{align*}
\frac{\partial^2 v_2}{\partial t^2} - M_1 \quad & \quad \text{wall force} \\
- \frac{M_1}{t^2} & \quad \text{Figure 4}
\end{align*}
\]

It is convenient to let $p_1, a, b, c, d, e, f$, be defined by the following equations:

\[
p_1 = JL_1 \\
a = \sin p_1 - \sinh p_1 \\
b = \cos p_1 - \cosh p_1 \\
c = -\sin p_1 - \sinh p_1 \\
d = -\cos p_1 - \cosh p_1 \\
e = \cosh p_1 \\
f = \sinh p_1
\]
In terms of the above symbols $X_1$ and its derivatives become

$$X_1 = [E_1a + F_1b + e]$$

$$\frac{dX_1}{dx_1} = j [E_1b + F_1c + f]$$

$$\frac{d^2X_1}{dx_1^2} = j^2 [E_1c + F_1d + e]$$

$$\frac{d^3X_1}{dx_1^3} = j^3 [E_1d + F_1a + f]$$

at $x_1 = L_1$.

It is also convenient to let $p_2, a', b', c', d'$ be defined by the following equations:

$$p_2 = jL_2$$

$$a' = \sin p_2 + \sinh p_2$$

$$b' = \cos p_2 + \cosh p_2$$

$$c' = -\sin p_2 + \sinh p_2$$

$$d' = -\cos p_2 + \cosh p_2$$

(3-13)
In terms of the above symbols $X_2$ and its derivatives become

$$X_2 = [E_2a^i + F_2b^i]$$

$$\frac{d X_2}{d x_2} = j [E_2b^i + F_2c^i]$$

$$\frac{d^2 X_2}{d x_2^2} = j^2 [E_2c^i + F_2d^i]$$

$$\frac{d^3 X_2}{d x_2^3} = j^3 [E_2d^i + F_2e^i]$$

at $x_2 = L_2$.

In figure 4, D'Alembert's principle has been used to present the dynamic problem as a static one so that the sum of all horizontal forces acting on the tray must be zero. Then from figure 4,

$$V_1 - V_2 = \text{wall force} - M \frac{d^2 w_2}{d t^2}.$$  \hspace{1cm} (3-18)

But

$$V_1 = - EI \frac{d^3 x_1}{d x_1^3} a_o \sin \omega t,$$

$$V_1 = - EIj^3 [E_1d + F_1a + f] a_o \sin \omega t.$$  \hspace{1cm} (3-19)
and
\[ V_2 = EI \frac{d^3x_2}{dx_2^3} a_o \sin \omega t, \]

\[ V_2 = EIj^3 \left[ E_2 a' + F_2 a' \right] a_o \sin \omega t. \quad (3-20) \]

\[ M \frac{\partial^2 v_2}{\partial t^2} = M x_2 \frac{d^2T}{dt^2}, \]

\[ M x_2 \frac{d^2T}{dt^2} = -\omega^2 M \left[ E_2 a' + F_2 b' \right]. \quad (3-21) \]

If it is noted that \( b_o \) in equation (2-33) is the same as \( a_o x_2(L_2) \) in the above derivation, equation (2-33) becomes

\[ \text{wall force} = -\omega^2 \left[ E_2 a' + F_2 b' \right] \left( \rho \mu r_o \right) \frac{\mathcal{F}}{h} \left[ \ldots \right] a_o \sin \omega t. \quad (3-22) \]

Then from equations (3-19), (3-20), (3-21) and (3-22), equation (3-18) can be rewritten

\[ E_1 d + F_1 a + E_2 a' + F_2 a' \]

\[ + \left[ E_2 a' + F_2 b' \right] \frac{\omega^2}{EIj^3} \left[ M - \left( \rho \mu r_o \right) \frac{\mathcal{F}}{h} \left[ \ldots \right] \right] = 0. \quad (3-23) \]

For the situation where \( r_i \) is considered zero, the wall force is given by equation (2-38). Expressed in terms of the above symbols,

\[ \text{wall force} = -\omega^2 \left[ E_2 a' + F_2 b' \right] \left( \rho h^2 \pi r_o \lambda \right) \frac{\mathcal{F}}{h} \left[ \ldots \right] a_o \sin \omega t, \]

and equation (3-23) should be altered to read
$$E_1 d + F_1 a + E_2 a' + F_2 a'$$
$$+ [E_2 a' + F_2 b'] \frac{\omega^2}{EI_{j}} [M - (\rho h^2 \tau_0 \lambda) \Sigma_n \cdots ] = 0. \text{ (3-24)}$$

For further use of equation (3-23) it is desirable to introduce these abbreviations:

$$q = \frac{\omega^2}{EI_{j}} [M - (\rho h^2 \tau_0 \lambda) \Sigma_n \cdots ] , \text{ (3-25)}$$

and $$q^* = \frac{\omega^2}{EI_{j}} [M - (\rho h^2 \tau_0 \lambda) \Sigma_n \cdots ] . \text{ (3-26)}$$

**Solution for the response of the system**

The amplitude of the top of the tower relative to the base is taken as the criterion for study of the response of the system. The relative amplitude of the top of the tower is given by $w^2_2$ at $x_2 = 0$.

Inasmuchas $$w^2_2 = v_2 = a_o \sin \omega t ,$$

and $$v_2 = \chi_2(x_2) a_o \sin \omega t ,$$

it follows that $$w^2_2 = |\chi_2(x_2) - 1| a_o \sin \omega t .$$

At $x_2 = 0$, $\chi_2(x_2) = 2F_2$. Then in dimensionless form the response of the system is expressed as the amplitude ratio

$$\frac{w^2_2(0)}{a_o \sin \omega t} = [2F_2 - 1] . \text{ (3-27)}$$
Equations (3-15), (3-16), (3-17) and (3-23) may be solved by use of determinants to find $F_2$. In the solution for $F_2$ it is convenient to introduce the following abbreviations:

\[
p = p_1 + p_2
\]

\[
[1] = 4(\cos p + \cosh p)
\]

\[
[2] = -2(\cos p_1 - \cosh p_1)
\]

\[
[3] = 8(1 + \cos p \cosh p)
\]

\[
a'[4] + b'[5] = 4 \cosh p_2 \left( \sin p_2 - \sin p \cosh p_1 \right)
\]

\[
+ 4 \cos p_2 \left( -\sinh p_2 + \cos p_1 \sinh p \right)
\]

\[
+ 4 \left( \cos p_1 \sinh p_1 - \sin p_1 \cosh p_1 \right)
\]

$F_2$, in terms of the above abbreviations, is given by

\[
F_2 = \frac{[1] + q(a'[2])}{[3] + q(a'[4] + b'[5])}
\]

where for a simply connected tray $q^*$ replaces $q$.

As a matter of interest, the moment in the column at $x_1 = 0$ is

\[
M = \frac{1}{EI} \left[ 2F_1 - 1 \right]
\]

And

\[
[2F_1 - 1] = \frac{[1] + q(a'[2] + b'[3])}{[4] + q(a'[5] + b'[6])}
\]
where

\[ [1] = -8(\sin p \sinh p) \]
\[ [4] = 8(1 + \cos p \cosh p) \]
\[
 a'[2] + b'[3] = -4 \sin p_1 (\cosh p_1 + \cos p_2 \cosh p) \\
 4 \sinh p_1 (\cos p_1 + \cos p \cosh p_2) \\
 a'[5] + b'[6] = 4 \cosh p_2 (\sin p_2 - \sin p \cosh p_1) \\
 + 4 \cos p_2 (- \sinh p_2 + \cos p_1 \sinh p) \\
 + 4(\cos p_1 \sinh p_1 - \sin p_1 \cosh p_1) .
\]

\( F_2 \) is to be used in section 6 to calculate the response of a test model.
Design and construction of apparatus

At the time that development of the experimental procedures was begun, the nature of the model to be tested was well established as a column with a single tray. The first problem to be attacked was that of providing a means for exciting vibrations in the column. Soon to be discarded was the idea of simulating the field situation by inducing vibrations with an air stream; however it was decided that a distributed excitation system was needed rather than a localized excitation. An acceleration field was chosen as the source of excitation, and it was decided that this field should be created by subjecting the base of the column to an oscillatory horizontal motion. Thus the problem materialized as presented in section 1.

Vibration testing tables are available commercially, but none seemed to offer the control and regulation characteristics deemed necessary for the tests in question. This situation with respect to commercial machines led to the necessity for designing and building a machine to meet the needs of this investigation. As a secondary objective of the design program, an attempt was made to construct a table which could be adapted to a wide variety of investigations. Thus it was desired that the machine be made to accommodate models much larger than those of immediate interest and that the speed range extend beyond that to be used for the model tower.

At an early stage in the design of the apparatus, it was decided that the machine should be mounted on an isolated 1800 pound slab
available in the Mechanical Engineering Department at Rice. The over-all size of the framework of the vibration table was governed in part by the pattern of mounting holes in the slab. And complete design of the framework was influenced by the following stiffness considerations: it was considered necessary that the deflections of the table itself be of much smaller magnitude than table amplitude and that for all operating frequencies of the machine, the framework be stiff enough to prevent resonant vibrations within the table. Pendulum and leaf spring supporting systems having been judged to constrain motion inadequately, a ball or rooler bearing arrangement which would constrain the table to a linear path was decided upon. It was possible to purchase ball type bearings which accomodate axial motion of a shaft. The actual support system was made by attaching bearings at the four corners of the table, and allowing the bearings to ride on shafts fastened to the slab as shown in figures 5 and 6 at the end of the section. In the fabrication of the apparatus, extensive precautions were taken to insure that the axes of the support bearings were parallel; then arrangements were made to adjust the axial alignment of the support shafts for free motion of the table.

To induce the motion of the table, a reaction drive system was chosen in preference to a direct mechanical or electromagnetic drive. The effective force on the table results from two eccentric weights rotating on parallel shafts and producing a net force in the direction of the table motion. In as much as the magnitude of the net force and hence the magnitude of the table is proportional to the amount of unbalance on the shafts, amplitude is varied by changing the size of the unbalance weights. As expected from design calculations a range of unbalance of
0.4 to 1.2 pound-inches results in an amplitude range of 0.004 to 0.012 inches. Care was taken to locate the line of action of the reaction force so that it passed through the center of mass of the table in order to minimize tendencies of the table to deviate from the linear path imposed by the constraint of the support system. To supply power to the shafts at variable speed, a 1/3 horsepower direct current motor was chosen because of the erratic speed regulation of variable-speed mechanisms. The motor was mounted in the table with due consideration to its effect on the effect on the center of mass of the table, and a silent chain type drive was arranged to transmit torque to the shafts. Actually the machine, as it exists, has two speed ranges which result in a total speed range from 50 to 7000 revolutions per minute. For low speed operation, 50 to 450 rpm., the drive chain applies torque to only one of the shafts and the two shafts are geared together as shown in figure 5. Figure 6 shows the arrangement for high speed operation with the chain driving both shafts and the gears removed. The latter arrangement was chosen because of the almost prohibitive cost of gears for high speeds.

The shafts and bearing blocks were designed and bearings chosen using large factors of safety in order to permit long life and, most important, to minimize the danger of fatigue failures at high speeds. All in all no attempt was made to conserve materials or to minimize weight. Completely assembled, the moving portion of the machine weighs approximately 200 pounds. This might appear to be a gross overdesign when the size of the model under study is considered, but it should be noted that the effect of model behavior on table characteristics is
reduced when the mass of the table is relatively large. During the
tests to be described, it was possible to maintain surprisingly steady
motion of the table even at the resonance peaks of the model.

Design of test model

The prime consideration in the design of a test model to simulate,
at least in essence, the field situation was the matching of the frequency
of the first nonsymmetrical mode of tray liquid vibration with the lowest
column frequency. After a systematic study of possible combinations of
dimensions, the following choices were made:

<table>
<thead>
<tr>
<th>Inside diameter of tray</th>
<th>1.875 inches</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depth of liquid</td>
<td>approximately 1 inch</td>
</tr>
<tr>
<td>Liquid</td>
<td>water</td>
</tr>
<tr>
<td>Diameter of column</td>
<td>0.250 inches</td>
</tr>
<tr>
<td>Height of the column</td>
<td>approximately 30 inches</td>
</tr>
<tr>
<td>Column material</td>
<td>steel</td>
</tr>
</tbody>
</table>

The above choices were made with fabrication problems in mind, and these
problems were consequently minimized. Drill rod was used for the column
and the tray was made by soldering a short length of thin-walled steel
tubing to a premachined bottom, the completed tray weighing 0.3531 pounds.

For attachment of the tray to the column and for mounting of the column,
tapered thread collets were made. Details of the model are to be seen
in figures 5 and 6. It was possible to determine the natural frequency
of the fluid analytically and the system was tuned experimentally by
changing the length of the column so as to make the natural frequency of
the empty tower approximately equal to that of the liquid to be added.
In the tests conducted the tray was positioned at the top of the column to simplify numerical work in the corresponding analytical solution. Tuning of the column with the tray at the top resulted in a column height of 26.5 inches. To make the tests with liquid, 45.0 milliliters of water were added to the tray.

**Measurement techniques**

The relative displacement of the top of the column with respect to the table was determined most conveniently by measuring the amplitude of the table and the absolute amplitude of the top of the column. With a dial indicator it was possible to measure the amplitude of the table, but measurement of the column amplitude involved the use of a cathetometer or optical micrometer. The cathetometer was mounted on the slab so that it could be focused on a target at the top of the column, and oriented so that the travel of the instrument was parallel to the motion of the column. The target was made by cementing two pieces of razor blade together to project above the top of the tower. When seen through the telescope of the cathetometer, the target appeared as a vertical band with well defined parallel edges. The target was made so that these edges were the cutting edges of a razor blade. At high speed it was not possible to measure amplitude using the hairline of the cathetometer and a more elaborate scheme had to be employed. A shield was attached to the moving part of the instrument to block off a vertical band in the center of the field of vision. In order to measure amplitude, the instrument's micrometer screw was turned until one of the extreme positions of the target was hidden by the shield. The micrometer screw was then turned slowly in the opposite direction until a flicker.
of the moving target was observed at the outside edge of the shield. The micrometer reading at this point combined with a similar reading for the other extreme position of the target determined the peak-to-peak absolute amplitude of the model tower. Zero reading corrections were applied to the difference in micrometer readings to allow for the width of the target and shield. It was possible using the above described techniques to measure table peak-to-peak amplitude to the nearest 0.01 millimeter and the peak-to-peak absolute amplitude of the top of the tower to the nearest 0.05 millimeter. To measure frequency, a mechanical tachometer was used. Worth mention is that the maximum range of frequency readings for any test point was of the order of 0.3 radians per second.
SECTION 5

EXPERIMENTAL DETERMINATION OF RESPONSE CURVES FOR MODEL

Performance of tests

The experimental investigation involved gathering sufficient data to plot the amplitude response curves for the model; one series of determinations was made for the model with liquid in the tray and another series without liquid in the tray. As an initial step in the investigation, measurements were taken at arbitrary frequencies throughout the range of interest. A plot of the results of these first measurements was then utilized to select suitable frequencies at which additional measurements were taken. This procedure made it possible to plot the curves rather well with a minimum number of measurements. For each frequency setting, several minutes were allowed to permit the apparatus to assume steady behavior. Column amplitude measurements were the most time consuming; therefore table amplitude and frequency were checked repeatedly to insure that conditions did not change while amplitudes were being measured at a frequency setting. It was found, in preliminary observations of the behavior of the model tower, that the model tended to exhibit noticeable vibration perpendicular to the motion of the table, and it was necessary to minimize this motion to obtain steady vibration of the column. To reduce perpendicular motion a straightedge was clamped to the table about 8 inches above the base of the column so that the length of the straightedge was in the direction of table motion. The straightedge was placed next to the tower with one of its edges barely touching the column. With this arrangement the
undesirable sidewise vibration was essentially eliminated, and measurements of column amplitude taken with and without the restraint of the straightedge showed that column amplitude was not decreased appreciably by friction between the column and the straightedge.

Processing of data

To determine relative amplitude from the measurements taken, it was necessary to subtract table amplitude from the absolute tower amplitude with due consideration of phase angles. Actually the phase angles depend upon the damping in the system, but in the absence of information about the amount of damping, the results were determined on the assumption that there was no damping present. That this assumption was not totally unreasonable was verified in the following manner. For the case with the tray empty, the experimental amplitude response curve was plotted (figure 7) and compared with the response curves of damped one-degree-of-freedom systems (Den Hartog, p. 66) in order to estimate the damping present. Comparison of two response curves may be made by comparing the width of the peaks at half the height of the peaks. From the above considerations and from phase angle curves for the one-degree-of-freedom system, it was concluded that damping in the model without liquid had a negligible effect of phase angles. Considered next was the response curve for the case with liquid in the tray (figure 7) for which the two peaks were studied separately and comparisons made as above. It was determined for this case that, although damping was not negligible, phase angles were not altered greatly by damping except near the resonance frequencies. Even near resonance the nature of the response
curve was not altered greatly by consideration of damping, in spite of significant phase angle changes, because here the magnitude of tower amplitude is large as compared to table amplitude. It must be admitted that at the minimum between the two peaks, the above considerations break down and results here do not deserve confidence.
Experimental results and curves

Included here are the observed measurements and the results of the experimental investigation. Also included are response curves for the model both with and without liquid in the tray.

Experimental Data from Test of Model Without Water in Tray

<table>
<thead>
<tr>
<th>frequency (rad/sec)</th>
<th>peak-to-peak absolute amplitude (mm)</th>
<th>peak-to-peak absolute amplitude of table (mm)</th>
<th>peak-to-peak relative amplitude of tower (mm)</th>
<th>ratio of relative amplitude of tower to table amplitude</th>
</tr>
</thead>
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<td>0.66</td>
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<td>0.99</td>
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*relative amplitude = absolute amplitude - (cosθ)table amplitude, where θ = phase angle between tower and table displacement.
Experimental Data from Test of Model With Water in Tray

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<th>frequency</th>
<th>peak-to-peak absolute amplitude of tower</th>
<th>peak-to-peak absolute amplitude of table</th>
<th>peak-to-peak relative amplitude of tower*</th>
<th>ratio of relative amplitude of tower to table amplitude</th>
</tr>
</thead>
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<tr>
<td>rad/sec.</td>
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<td>mm.</td>
<td>mm.</td>
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EXPERIMENTAL RESPONSE CURVES FOR MODEL WITH AND WITHOUT LIQUID IN TRAY

Figure 7
SECTION 6

NUMERICAL CALCULATION OF RESPONSE CURVES FOR MODEL

Description of method

From equation (3-27) the ratio of relative amplitude of tower to table amplitude is
\[
\frac{\nu^*}{a_0 \sin \omega t} = 2F_2 - 1,
\]
and from equation (3-28)
\[
F_2 = \frac{[1] + q^*a'[2]}{[3] + q^*(a'[4] + b'[5])}
\]
where \(q^*\) is used because the model tray is simply connected and the symbols are defined:
\[
a' = \sin p_2 + \sinh p_2
\]
\[
[1] = 4(\cos p + \cosh p)
\]
\[
[2] = -2(\cos p_1 - \cosh p_1)
\]
\[
[3] = 8(1 + \cos p \cosh p)
\]
\[
a'[4] + b'[5] = 4 \cosh p_2 (\sin p_2 - \sin p \cosh p_1)
\]
\[
+ 4 \cos p_2 (-\sinh p_2 + \cos p_1 \sinh p)
\]
\[
+ 4(\cos p_1 \sinh p_1 - \sin p_1 \cosh p_1)
\]
Model dimensions essential to the numerical calculations are:

- column diameter: 0.250 inches
- column height: 26.5 inches
- weight of tray: 0.3531 pounds
- inside diameter of tray: 1.875 inches
- water added to tray: 45.0 milliliters

For the model then,

\[ L_1 = 26.5 \text{ inches} \]
\[ L_2 = 0 \]
\[ M = 0.3531 \text{ pounds} \]
\[ m = 0.03139 \text{ pounds per inch} \]
\[ j = 0.899 \times 10^{-2} \times \sqrt{\omega} \text{ 1/inches} \]
\[ p_1 = p = 0.236 \sqrt{\omega} \]
\[ p_2 = 0 \]
\[ \sigma = 0.9447 \]

Since \( p_1 = p \) and \( p_2 = 0 \), the terms in the solution for \( F_2 \) can be expressed as follows:

\[ a' = 0 \]
\[ [1] = 4(\cos p - \cosh p) \]
\[ [3] = 8(1 + \cos p \cosh p) \]
\[ a'[4] + b'[5] = 8(\cos p \sinh p - \sin p \cosh p) \]

It is noted that [2] does not enter into the calculation since \( a' = 0 \).

Equation (3-26) states that

\[ q^* = \frac{\omega^2}{EI_j^3} \left[ M - (\rho h^2 \tau \lambda) \sum \right] \]
where from appendix D

\[
\sum_n = \frac{a}{\lambda} - 2\alpha^2 \sum_{m=1}^\infty \frac{\tanh(j_m'/\alpha)}{[j_m'(j_m'^2 - 1)]} \cdot \frac{1}{(j_m'/\alpha)\tanh(j_m'/\alpha) - \lambda},
\]

the \( j_m' \) being roots of \( J_1(j_m') \).

For the model with liquid in the tray, \( q^* \) reduces to

\[
q^* = 0.157\sqrt{\omega}
\]

\[
+ 3.050 \times 10^{-4} \times \omega^{5/2} \sum_{m=1}^\infty \frac{\tanh(j_m'/\alpha)}{[j_m'(j_m'^2 - 1)]} \cdot \frac{1}{(j_m'/\alpha)\tanh(j_m'/\alpha) - \lambda},
\]

and for the model without liquid in the tray,

\[
q^* = 0.221\sqrt{\omega}.
\]

At a given frequency, calculation of amplitude ratio involves the following steps: computation for \( q^* \) from equations (6-1) and (6-2), computation of expressions \([1], [3]\) and \( a'[4] + b'[5] \), and computation of \( F_2 \) and \( 2F_2 - 1 \).

Roots of \( J(J) \) as tabulated by Smith, Rodgers, and Traub are

\[
\begin{align*}
J_1' & = 1.8412 & J_5' & = 14.8636 \\
J_2' & = 5.3314 & J_6' & = 18.0155 \\
J_3' & = 8.5363 & J_7' & = 21.1644 \\
J_4' & = 11.7050 & J_8' & = 24.3113
\end{align*}
\]

However for the present calculation sufficient accuracy is obtained using only two terms of the summation for \( q^* \). Estimates of the remainder in the frequency range considered indicated that two terms lead to results good to three significant figures.
Numerical results and curves

Included here are the results of numerical calculation of the response of the model tower both with and without liquid in the tray. As in the experimental study, amplitude is measured at the top of the tower. Also included are curves plotted from these results.

Results of Numerical Calculation of
Response of Model Without Liquid in Tray

<table>
<thead>
<tr>
<th>frequency in rad/sec</th>
<th>ratio of relative amplitude of tower to table amplitude</th>
<th>frequency in rad/sec</th>
<th>ratio of relative amplitude of tower to table amplitude</th>
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<tbody>
<tr>
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<td>.35</td>
<td>28</td>
<td>23.57</td>
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<td>16</td>
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<tr>
<td>26</td>
<td>5.11</td>
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<tr>
<td>27</td>
<td>12.70</td>
<td>42</td>
<td>2.11</td>
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</table>
Results of Numerical Calculation of Response of Model With Liquid in Tray

<table>
<thead>
<tr>
<th>frequency in rad./sec.</th>
<th>ratio of relative amplitude of tower to table amplitude</th>
<th>frequency in rad./sec.</th>
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THREE CURVES FOR RESPONSE WITH AND WITHOUT LIQUID IN TRAY

FIGURE 8

AXIS RATIO

FREQUENCY (rad/sec)
SECTION 7

COMPARISON OF ANALYTICAL AND EXPERIMENTAL RESULTS

Immediately noticeable in figure 9, included in this section, is the similarity between the response curves for the model without liquid in the tray as obtained by analytical and experimental methods. The principal difference between the two curves is that the peak of the experimental curve is shifted about 5 radians per second to the left by damping. This change in the resonance frequency of the system because of damping parallels the effect of damping on an analogous one-degree-of-freedom system. Were it necessary to determine the amount of damping in the model, comparison of the resonance change with that produced by damping in a one-degree-of-freedom system might be a means of finding the damping in the model.

Figure 10, also included in this section, shows theoretical and experimental amplitude response curves for the model with liquid in the tray. It is to be observed that here damping shifts the first peak to the left by 2 radians per second and shifts the second peak to the left by 6 radians per second. The location of the peaks of the experimental curve is explained by consideration of two effects. First, as in the case of the model without tray liquid, damping within the column tends to move the peaks of the experimental curve to the left. And second damping in the liquid causes the two peaks to be moved together, which is in accord with theory for a two-degree-of-freedom system wherein damping between the absorbing mass and the main system corresponds to liquid damping in the model. The general shape of the curves agrees well with those for a two-degree-of-freedom system (Den Hartog, p.123).
EXPERIMENTAL AND THEORETICAL RESPONSE CURVES FOR MODEL WITHOUT LIQUID IN TRAY

FIGURE 9
EXPERIMENTAL AND THEORETICAL RESPONSE CURVES FOR MODEL WITH LIQUID IN TRAY

FIGURE 10

AMPLITUDE RATIO

FREQUENCY (rad./sec.)
CONCLUSIONS

The objective of this thesis was to substantiate the hypothesis that liquid on the trays of a bubble tower acts as a dynamic vibration absorber. To achieve this objective, analytical and experimental methods were developed for studying the behavior of a simple model which simulated a bubble tower. As a part of the experimental phase of the program, a 200-pound vibration testing table was designed and constructed and successfully operated at speeds ranging from 50 to 7000 revolutions per minute with amplitudes up to 0.012 inches. Experimental tests were carried out and numerical calculations made in order to plot the amplitude response curves of a model consisting of a column and a single tray. Using both experimental and analytical methods, response curves were plotted for the model without liquid in the tray and with liquid in the tray.

The prediction of the response curves of the model by analytical means has been substantiated experimentally; comparison of these response curves with due consideration of the effect of damping indicated that the results as obtained by the two methods are in good agreement. Both methods of study show that addition of liquid to the tray of the model results in two resonance peaks as would be expected from two-degree-of-freedom theory. The experimental curves offer even stronger evidence in support of the original contention concerning the effect of tray liquid on the vibration behavior of a bubble tower. Addition of liquid to the model tray reduced the peak amplitude by a factor of 6.2.
As presented in this thesis the investigation of the problem is by no means complete. Many possible extensions to this analysis present themselves. Of interest would be further study of the model system, using the methods described herein, to determine the effect of changes in the governing parameters. Also, the discussion in Section 7 regarding the effect of damping indicates that a more realistic analysis must include consideration of damping. A beginning step toward study of an actual tower would be development of an approximate theory, which takes damping into account, to describe the behavior of a system with many trays. Just as analytical study of the model system required a description of the liquid motion, so application of an approximate theory to an actual tower would necessitate study of liquid motion in an actual tray. This study of liquid motion might well lead to an involved analytical and experimental investigation because of the complicated geometry of a real tray and because damping should be considered in a rational theory. The ultimate goal of the above program would be the development of a design criterion to permit the phenomenon of tray liquid dynamic absorption to be considered in the design of bubble towers.

One final observation to be made is that consideration of the beneficial effect of tray liquid on the vibrations of a bubble tower suggests the possibility of utilizing this phenomenon to improve the vibration characteristics of other structures.


SYMBOLS

\( a_0 \)  \hspace{1em} \text{amplitude of motion of base of column}
\( \omega \)  \hspace{1em} \text{frequency of motion of base of column in radians per second}
\( t \)  \hspace{1em} \text{time}
\( b_0 \)  \hspace{1em} \text{amplitude of motion of tray}
\( r \)  \hspace{1em} \text{coordinates describing position in tray}
\( \theta \)  \hspace{1em} \text{coordinate describing position along bottom part of column}
\( z \)  \hspace{1em} \text{coordinate describing position along top part of column}
\( \phi \)  \hspace{1em} \text{velocity potential function of liquid}
\( F \)  \hspace{1em} \text{field force function of liquid}
\( P \)  \hspace{1em} \text{pressure in liquid}
\( \rho \)  \hspace{1em} \text{density of liquid}
\( v \)  \hspace{1em} \text{velocity of liquid}
\( r_i \)  \hspace{1em} \text{inner radius of tray}
\( r_o \)  \hspace{1em} \text{outer radius of tray}
\( g \)  \hspace{1em} \text{gravitational acceleration}
\( z_o \)  \hspace{1em} \text{position of liquid surface}
\( h \)  \hspace{1em} \text{equilibrium depth of liquid}
\( k \)  \hspace{1em} \text{a separation constant}
\( \zeta \)  \hspace{1em} \text{a dimensionless parameter} = kh
\( \lambda \)  \hspace{1em} \text{a dimensionless parameter} = \omega^2 h/g
\( \zeta_n \)  \hspace{1em} \text{defined by equation (2-29)}
\( \sigma \)  \hspace{1em} \text{a dimensionless parameter} = r_o/h
\( x_1 \)  \hspace{1em} \text{coordinate describing position along bottom part of column}
\( x_2 \)  \hspace{1em} \text{coordinate describing position along top part of column}
deflection of bottom part of column

$w_1^*$
deflection of top part of column

$q_1$
effective load per unit length on bottom part of column

$q_2$
effective load per unit length on top part of column

$v_1$
absolute displacement of bottom part of column

$v_2$
absolute displacement of top part of column

$L_1$
length of bottom part of column

$L_2$
length of top part of column

$m$
mass per unit length of column

$M$
mass of tray

$E$
modulus of elasticity of column material

$I$
column cross section area moment of inertia taken about a diameter

$M_1$
moment in bottom part of column

$M_2$
moment in top part of column

$V_1$
shear in bottom part of column

$V_2$
shear in top part of column

$j$
a separation parameter $= \left[ \frac{m_0^2}{EI} \right]^{1/4}$

$p_1$
a parameter $= jL_1$

$p_2$
a parameter $= jL_2$
APPENDIX A

It is necessary to investigate the roots of the equation

\[ \zeta \sin \zeta + \lambda \cos \zeta = 0 \quad (A-1) \]

where \( \zeta = \xi + i\eta \).

It can be shown that if \( \zeta \) is a root of equation (A-1), then \(-\zeta, \overline{\zeta}, \) and \(-\overline{\zeta}\) are also roots. For purposes of discussion it therefore suffices to study only the first quadrant of the \( \zeta \) plane and the axes, \( \zeta = + \xi \) and \( \zeta = + i\eta \).

Consideration of roots such that \( \zeta = + \xi \)

For \( \zeta = + \xi \), equation (A-1) becomes

\[ \xi \sin \xi + \lambda \cos \xi = 0. \quad (A-2) \]

It is to be noted that \( \cos \xi \neq 0 \) since any root such that \( \cos \xi = 0 \) requires either \( \xi = 0 \) or \( \sin \xi = 0 \), which is in contradiction.

Similarly \( \xi 
eq 0 \) since \( \xi = 0 \) requires that \( \lambda \cos \xi = 0 \), which again is in contradiction; therefore all roots are greater than zero. With these conditions imposed on the roots of equation (A-2), it may be rewritten

\[ \tan \xi = -\frac{\lambda}{\xi}. \quad (A-3) \]

From figure 11 it is concluded that

\[ \frac{\pi}{2} \leq \xi < \pi, \]

\[ 3\frac{\pi}{2} \leq \xi < 2\pi \text{ etc.}, \]

and as \( n \to \infty \), \( \xi_n \to n\pi \).
Consideration of roots such that \( \xi = + \imath \eta \)

For \( \xi = + \imath \eta \), equation (A-1) becomes

\[
-\eta \sinh \eta + \lambda \cosh \eta = 0. \tag{A-4}
\]

Since \( \cosh \eta > 1 \), equation (A-4) may be rewritten as

\[
\eta \tanh \eta = \lambda. \tag{A-5}
\]

For \( \omega > 0, \lambda > 0 \), so that \( \eta = 0 \) is not a possible solution of equation (A-5), and \( \eta \tanh \eta > 0 \). It is noted that

\[
\frac{d}{d\eta} (\eta \tanh \eta) = \tanh \eta + \eta \text{sech}^2 \eta.
\]

Since \( \eta > 0 \),

\[
\frac{d}{d\eta} (\eta \tanh \eta) > 0,
\]

and \( \eta \tanh \eta \) is therefore an increasing function.
From figure 12, it is concluded that there is only one pure imaginary root for equation (A-1).

Consideration of roots such that $\xi = \xi + i\eta$, $\xi > 0$, $\eta > 0$.

For $\xi = \xi + i\eta$, $\xi > 0$, $\eta > 0$, equation (A-1) becomes

\[
(\xi + i\eta)[\sin\xi \cosh \eta + i \cos \xi \sinh \eta] \\
+ \lambda [\cos \xi \cosh \eta - i \sin \xi \sinh \eta] = 0. \tag{A-6}
\]

A separation of equation (A-6) into real and imaginary parts yields

\[
\xi \sin \xi \cosh \eta - \eta \cos \xi \sinh \eta + \lambda \cos \xi \cosh \eta = 0, \tag{A-7}
\]

and

\[
\xi \cos \xi \sinh \eta + \eta \sin \xi \cosh \eta - \lambda \sin \sinh \eta = 0. \tag{A-8}
\]

For $\eta > 0$, $\sinh \eta$ and $\cosh \eta$ are both greater than zero. In equation (A-7), any value of $\xi$, such that $\cos \xi = 0$, requires $\xi = 0$ which is in contradiction. In equation (A-8), any value of $\xi$, such that $\sin \xi = 0$, requires that $\xi = 0$ which is excluded as a possible solution in the discussion of the case, $\xi = + \xi$. Equations (A-7) and (A-8) may be rewritten.
\[ \xi \tan \xi - \eta \tanh \eta + \lambda = 0, \quad (A-9) \]

and
\[ \xi \cot \xi + \eta \coth \eta - \lambda = 0. \quad (A-10) \]

If equations (A-9) and (A-10) are added,
\[ \xi (\tan \xi + \cot \xi) + \eta (\coth \eta - \tanh \eta) = 0. \quad (A-11) \]

Since
\[ \tan \xi + \cot \xi = \frac{\sin^2 \xi + \cos^2 \xi}{\sin \xi \cos \xi} = \frac{2}{\sin(2 \xi)}, \]

and
\[ \coth \eta - \tanh \eta = \frac{\cosh^2 \eta - \sinh^2 \eta}{\sinh \eta \cosh \eta} = \frac{2}{\sinh(2 \eta)}, \]

equation (A-11) becomes
\[ \frac{2 \xi}{\sin(2 \xi)} + \frac{2 \eta}{\sinh(2 \eta)} = 0. \quad (A-12) \]

Limit \( n \to 0 \), \( \frac{2 \eta}{\sinh(2 \eta)} = 1. \)

Limit \( n \to \infty \), \( \frac{2 \eta}{\sinh(2 \eta)} = 0. \)

For \( \eta > 0 \), \( \frac{d}{d \eta} \left[ \frac{2 \eta}{\sinh(2 \eta)} \right] < 0. \)

Then \( 0 < \frac{2 \eta}{\sinh(2 \eta)} < 1, \)

and from equation (A-12), \( \left| \frac{2 \xi}{\sin(2 \xi)} \right| < 1. \)
It is then required that \( |2 \xi| < \sin(2 \xi) | < 1 \).

\( |2 \xi| < 1 \) requires that \( 2 \xi < 1 \), which means that

\[
\frac{2 \xi}{\sin(2 \xi)} > 0,
\]

and that equation (A-12) cannot be satisfied. It is concluded that there can be no roots such that \( \xi > 0 \) and \( \eta > 0 \).

**Verification of the simplicity of roots \( \xi = \xi^\star \).**

If \( \xi^\star \) is a multiple root,

\[
u(\xi^\star) = \xi^\star \sin \xi^\star + \lambda \cos \xi^\star = 0, \quad \text{(A-13)}
\]

and

\[
u'(\xi^\star) = \sin \xi^\star + \xi^\star \cos \xi^\star - \lambda \sin \xi^\star = 0. \quad \text{(A-14)}
\]

Equations (A-13) and (A-14) may be rewritten as

\[
\tan \xi^\star = -\lambda/\xi^\star,
\]

and

\[
\tan \xi^\star = -\xi^\star/(1 - \lambda).
\]

Then

\[
\lambda = \frac{\xi^\star}{1 - \lambda} \quad \text{or} \quad \xi^\star^2 = (1 - \lambda).
\]

This is possible only if \( \lambda < 1 \) since \( \lambda = 1 \) would require \( \xi^\star = 0 \).

For \( \lambda < 1 \), \( \xi^\star^2 < 1 \), but it has been shown that \( \xi, > \pi/2 \). There is therefore no solution, \( \xi^\star \).
Verification of simplicity of roots $\xi = i\eta_o$.

If $i\eta_o$ is a multiple root,

$$u(i\eta_o) = -\eta_o \sinh \eta_o + \lambda \cosh \eta_o = 0, \quad (A-15)$$

and

$$u'(i\eta_o) = \sinh \eta_o + \eta_o \cosh \eta_o - \lambda \sinh = 0. \quad (A-16)$$

Equations (A-15) and (A-16) may be rewritten as

$$\tanh \eta_o = \lambda / \eta_o,$$

and

$$\tanh \eta_o = -\eta_o / (1 - \lambda).$$

Since $\tanh \eta_o < 1$,

$$\frac{\lambda}{\eta_o} < 1 \quad \text{or} \quad \eta_o > \lambda,$$

and

$$\frac{\eta_o}{\lambda - 1} < 1 \quad \text{or} \quad \eta_o \ll \lambda - 1.$$

There is a contradiction in this last statement and it is concluded that $\eta_o$ must be single valued.
DETERRMINATION OF $\alpha_n$

$\alpha_n$ is defined by equation (2-29):

$$\sum_{n} \alpha_n \cos(k_n z) = 1$$  \hspace{1cm} (B-1)

for $0 < z < h$.

If $s$ is defined by $s = z/h$, equation (B-1) becomes

$$\sum_{n} \alpha_n \cos(s \zeta_n) = 1$$  \hspace{1cm} (B-2)

for $0 < s < 1$.

Figure 13

Considering the $\zeta$ plane as shown in figure 13, define the integral $I$: 

\[ I = \frac{1}{2\pi i} \oint_{c} \frac{F(\xi)}{u(\xi)} \, d\xi \]  

(B-3)

where \( c \) is taken to mean the closed contour indicated by a dotted line in figure 13, \( u(\xi) = \xi \sin \xi + \lambda \cos \xi \), and \( F(\xi) \) is an analytic function of \( \xi \) except at the origin:

\[ F(\xi) = \frac{-\lambda}{\xi}. \]  

(B-4)

It should be noted that the contour passes through no roots of \( u(\xi) \).

As shown in appendix A all roots of \( u(\xi) \) are simple roots. Consequently all singularities of \( 1/u(\xi) \) lead to simple poles with residues given by

\[ \sum_{n} \frac{F(\xi_{n}) \cos(s\xi_{n})}{u'(\xi_{n})}. \]

The singularity of \( F(\xi) \) also leads to a simple pole with residue given by

\[ \frac{-\lambda \cos(0)}{u(0)} = -1. \]

Limit \( I = \) sum of all residues,
\[ n \to \infty \]
\[ a \to \infty \]

\[ = -1 + \sum_{n} \frac{-\lambda \cos(s\xi_{n})}{u'(\xi_{n})}. \]

It is shown in appendix C that

Limit \( I = 0. \)
\[ n \to \infty \]
\[ a \to \infty \]

Therefore

\[ \sum_{n} \left[ \frac{-\lambda \cos(s\xi_{n})}{u'(\xi_{n})} \right] = 1. \]

By comparison with equation (B-2),

\[ \alpha_{n} = \frac{-\lambda}{\xi_{n}u'(\xi_{n})} \]  

(B-5)
THE INTEGRAL  \[ I = \frac{1}{2\pi i} \oint_C \frac{\Lambda \cos(s\xi)}{s\xi u(s)} \, ds \]

It is desired to show that the integral,

\[ I = \int_C \frac{\Lambda \cos(s\xi)}{s\xi u(s)} \, ds, \]

becomes zero as the contour \( C \) becomes infinite suitably. From appendix B, \( C \) is the contour indicated by the dotted line in figure 13; \( \Lambda \) is a constant for a given forcing frequency; and \( u(s) = \xi \sin \xi + \Lambda \cos \xi \).

Investigation shows that the integrand is odd with respect to \( \xi \). It is therefore sufficient to consider only that part of the contour which is in the first quadrant.

Consideration of the right side of the contour

For the line where \( \xi = n\pi + \pi/2 \) and \( 0 < \eta < a \), the following integral is investigated:

\[ I_1 = \int_0^a \frac{\cos(s\xi)}{s\xi u(s)} \, ds. \]

From equation (A-6), if \( \xi = (n\pi + \pi/2) + i\eta \),

\[ u(s) = (-1)^n \left[ (n\pi + \pi/2)\cosh \eta + i(\eta \cosh \eta - \Lambda \sinh \eta) \right], \]

and

\[ |u(s)| \gg n\cosh \eta. \quad (C-1) \]

It is also noted that \[ |\xi| = |(n\pi + \pi/2) + i\eta| > n, \quad (C-2) \]

and

\[ |\cos(s\xi)| \leq 1/2 \left\{ |e^{is(\xi + i\eta)}| + |e^{is(\xi + i\eta)}| \right\} . \]
Then since $0 \leq \gamma \leq 1$,
\[ |\cos(s^\gamma)| \leq \cosh(\gamma) \geq \cosh(1) \cdot \tag{C-3} \]

From relationships (C-1), (C-2), and (C-3),
\[ |I_1| \leq \int \frac{a \cosh^2 \gamma}{n(n \cosh \gamma)} d\gamma, \]
and
\[ |I_1| \leq \frac{a}{n^2} \cdot \tag{C-4} \]

Consideration of the top of the contour

For the line where $\gamma = a$ and $0 \leq \xi \leq n\pi + \pi/2$, the following integral is investigated:
\[ I_2 = \int \cos(s^\xi) \frac{d\xi}{\xi u(\xi)}. \]

From equation (A-6), if $\xi = \xi + ia$ and $a$ is considered very large,
\[ u(\xi) = \cosh a \left[ 1 + o(e^{-2a}) \right] \]
\[ \left[ \xi \sin \xi + (\lambda - a)\cos \xi \right] + i[\xi \cos \xi - (\lambda - a)\sin \xi] \cdot \]

Therefore,
\[ |u(\xi)| = \cosh a \left[ 1 + o(e^{-2a}) \right] \left[ \xi^2 + (\lambda - a)^2 \right]^{1/2}, \]
and
\[ |u(\xi)| \geq \cosh a(a)[1 + o(1/a)] \cdot \tag{6-5} \]

It is also noted that $|\xi| > a$, \tag{G-6}
and
\[ |\cosh(s^\xi)| \leq \cosh a \cdot \tag{G-7} \]
From relations (C-5), (C-6), and (C-7),

\[ |I_2| \leq \int_{0}^{\pi} \frac{\cosh a}{a(\cosh a)(1 + o(1/a))} \, d\phi, \]

and

\[ |I_2| \leq \frac{2\pi n}{a^2} [1 + o(1/a)] \]  \hspace{1cm} (C-8)

if \( a = n \),

\[ \lim_{n \to \infty} |I_1| = 0, \]

and

\[ \lim_{n \to \infty} |I_2| = 0. \]

Therefore the integral over the entire contour is zero.
APPENDIX D

EVALUATION OF THE SUM:

\[ S = \sum_{n} \frac{\sin \xi_n}{(\xi_n)^3 u'(\xi_n)} \frac{I_1(\sigma \xi_n)}{I_1'(\sigma \xi_n)} , \]

where symbols are as defined in section 2.

Consideration is given to the integral

\[ I = \int_{c} \frac{\sin \xi}{(\xi)^3 u(\xi)} \frac{I_1(\sigma \xi)}{I_1'(\sigma \xi)} d\xi , \quad (D-1) \]

where the contour is, as in appendix B, indicated by the dotted line in figure 13 (appendix B). The roots of \( u(\xi) \) are single valued; consequently, the resulting singularities in the integrand of \( I \) lead to residues given by:

\[ \sum_{n} \frac{\sin \xi_n}{(\xi_n)^3 u'(\xi_n)} \frac{I_1(\sigma \xi_n)}{I_1'(\sigma \xi_n)} = S . \quad (D-2) \]

Another group of residues will result when \( I'(\sigma \xi) = 0 \). \( I_1(\sigma \xi) \) and \( I_1'(\sigma \xi) \) may be rewritten as \( iJ_1(-i\sigma \xi) \) and \( J_1'(-i\sigma \xi) \) respectively; then the roots of \( I_1'(\sigma \xi) \) may be expressed as \( j_m^1 \) where \( j_m^1 \) satisfies

\[ J'(j_m^1) = 0 \quad \text{and} \quad j_m^1 = (-i\sigma \xi_m) , \]

and \( m \) is an integer such that \( 1 \leq m \leq \infty \). This group of residues may be represented by
\[
\sum_{m=1}^{2} \frac{\sin(ij^{'}/\sigma)}{(ij^{'}/\sigma)^3[\lambda \cos(ij^{'}/\sigma) + (ij^{'}/\sigma)\sin(ij^{'}/\sigma)]} \frac{I_1(ij^{'})}{I''_1(ij^{'})}.
\]

It should be noted that the above sum covers only positive \( m \) since the terms of the sum are even in \((j^{'})^3\). \( I''_1(\sigma\xi) = -iJ''_1(-\sigma\xi) \), and the sum may therefore be rewritten,

\[
\sum_{m=1}^{2} \frac{\sinh(j^{'}/\sigma)}{(j^{'}/\sigma)^3[\lambda \cosh(j^{'}/\sigma) - (j^{'}/\sigma)\sinh(j^{'}/\sigma)]} \frac{J_1(j^{'}/\sigma)}{J''_1(j^{'}/\sigma)};
\]

but

\[
J''_1(j^{'}/\sigma) + (1/j^{'})J_1(j^{'}) + [1 - (1/j^{'})^2]J_1(j^{'}) = 0,
\]

so that

\[
\frac{J_1(j^{'}/\sigma)}{J''_1(j^{'}/\sigma)} = -\frac{j^{'}/2}{j^{'2} - 1}.
\]

The sum now becomes

\[
\sum_{m=1}^{2} \frac{\tanh(j^{'}/\sigma)}{[j^{'}/(j^{'2} - 1)]} \cdot \frac{1}{(j^{'}/\sigma)\tanh(j^{'}/\sigma) - \lambda}.
\]

In addition to the two above discussed groups of residues, there is a single residue arising when \( \xi = 0 \).

Since

\[
\sin \xi = \xi + o(\xi^3),
\]

\[
I_1(\sigma\xi) = (\sigma/2)\xi + o(\xi^3),
\]

and

\[
I'_1(\sigma\xi) = 1/2 + o(\xi^2),
\]

the integrand of equation (D-1) leads to a simple pole. It is noted that for \( \xi = 0 \),

\[
\frac{\sin \xi I_1(\sigma\xi)}{\xi^2} = \frac{\sigma}{2}.
\]
\[ L_1'(\sigma \xi) = \frac{1}{2}, \]

and

\[ u(\xi) = \lambda; \]

then the residue is

\[ \frac{\sigma}{\lambda}. \]  

\[ (D-4) \]

It is shown in appendix F that \( I \) becomes zero, as the contour becomes infinite in a suitable fashion; therefore the sum of all residues is zero, and from equations (D-2), (D-3), and (D-4)

\[ S = \frac{-\sigma}{\lambda} - 2\sigma^2 \sum_{m=1}^{\infty} \frac{\tanh(j_m' / \sigma)}{j_m-j_m'^2-1 \sqrt{(j_m'^2-1)}} \cdot \frac{1}{(j_m'/\sigma) \tanh(j_m'/\sigma) - \lambda} \]  

\[ (D-5) \]
It is desired to show that the integral,
\[ I = \int_c \frac{\sin \xi}{\xi^3 u(\xi)} \frac{I_1(\alpha \xi)}{I_1(\alpha \xi)} \, d\xi, \]
becomes zero as the contour \( c \) becomes infinite in a suitable fashion.

From appendix B, \( c \) is the contour indicated by the dotted line in figure 13. Other symbols are as indicated in section 2. Since the integrand is odd with respect to \( \xi \), it is sufficient to consider only that part of the contour which is in the first quadrant.

**Consideration of the right side of the contour**

For the line where \( \xi = n\pi + \pi/2 \) and \( 0 \leq \gamma \leq a \), the following integral is investigated:
\[ I_1 = \int_0^a \frac{\sin \xi}{\xi^3 u(\xi)} \frac{I_1(\alpha \xi)}{I_1(\alpha \xi)} \, d\xi. \]

It has been shown in appendix C that \( |u(\xi)| > n \cosh \gamma \), and \( |\xi| > n \).

And from the considerations studied in appendix C,
\[ |\sin \xi| \leq \frac{1}{2} \left[ |e^{i\xi}| + |e^{-i\xi}| \right], \]
\[ |\sin \xi| \geq \cosh \gamma. \]

Hence,
\[ \left| \frac{i \sin \xi}{\xi^3 u(\xi)} \right| < \frac{1}{n^4}. \]  (E-1)

According to Watson (p. 203, equation 2).
\[ I_1(w) = \frac{e^w - ie^{-w}}{[2\pi i]^\frac{1}{2}} [1 + o(1/w)] , \]

and
\[ I_1'(w) = \frac{e^w + ie^{-w}}{[2\pi i]^\frac{1}{2}} [1 + o(1/w)] , \]

for \( 0 \leq \arg w \leq \pi/2 \).

Therefore,
\[ \frac{I_1(w)}{I_1'(w)} = \frac{e^w - ie^{-w}}{e^w + ie^{-w}} [1 + o(1/w)] . \]

For large \( n \),
\[ \frac{I_1(n)}{I_1'(n)} = 1 + o(1/n) \quad (E-2) \]

From equations (E-1) and (E-2),

the integral, \[ I_1 = \int_0^a \left( 1/n^4 \right) [1 + o(1/w)] \, d\xi , \]

the integral \[ I_1 = (a/n^4) [1 + o(1/w)] . \quad (E-3) \]

**Consideration of the top of the contour**

For the line \( \eta = a \) and \( 0 \leq \xi \leq n\pi + \pi/2 \), the following integral is investigated:
\[ I_2 = \int_0^{n\pi + \pi/2} \frac{\sin \xi}{\xi^2 u(\xi) \xi^2} \frac{I_1(\xi \xi)}{I_1'(\xi \xi)} \, d\xi . \quad (E-4) \]

From appendix C and above,
\[ |u(\xi)| \geq a(cosh a)[1 + o(1/a)], \]
\[ |\sin \xi| < \cosh a, \]
\[ |\xi| > a. \]

Therefore
\[ \left| \frac{\sin \xi}{\xi^3 u(\xi)} \right| < (1/a^4)[1 + o(1/a)]. \quad (E-5) \]

For large \( a \),
\[ \frac{I_1(w)}{I_0(w)} = \frac{e^w - ie^{-w}}{e^w + ie^{-w}} [1 + o(1/a)]. \]

In order that the contour not pass through singularities, the singularities arising from \( I_1'(w) = 0 \) must be investigated and \( a \) specified to avoid these points. Where \( w = u + iv \), singularities exist only if \( u = 0 \) since from appendix D all roots of \( I_1'(w) \) are pure imaginary; therefore singularities exist where
\[ e^w + ie^{-w} = (\cos u + \sin v) + i(\sin u + \cos v) = 0. \]

Finally singularities exist for
\[ \tan v = -1, \]
\[ v = 3\pi/4, \quad 7\pi/4, \ldots \quad (m - 1/4)\pi. \]

If \( a \) is specified as \( (m + 1/4) \), the contour will pass between singularities.

\[ \left| \frac{e^w - ie^{-w}}{e^w + ie^{-w}} \right| = \frac{e^u(\cos v + i\sin v) - e^{-u}(\sin v + i\cos v)}{e^u(\cos v + i\sin v) + e^{-u}(\sin v + i\cos v)}. \]

but for \( v = (m + 1/4) \), \( \cos v = \sin v. \)
From relations (E-5) and (E-6),

\[
\left| \frac{e^w - ie^{-w}}{e^w + ie^{-w}} \right| = \left| \frac{e^u - e^{-u}}{e^u + e^{-u}} \right| < 1. \tag{E-6}
\]

the integral, \(|I_2| < \int_0^{\pi + \frac{\pi}{2}} \left( \frac{1}{\sqrt{a^2}} \right) \left[ 1 + o(1/a) \right] d\xi\),

where \(a = (m + 1/4)\pi\).

If \(m = n\),

\[
\operatorname{Limit}_{n \to \infty} |I_1| = 0,
\]

and

\[
\operatorname{Limit}_{n \to \infty} |I_2| = 0.
\]

Therefore the integral over the entire contour is zero.