



RICE UNIVERSITY

THE OUTER BOUNDARY OF THE NON-EQUILIBRIUM
REGION OF THE PRANDTL-MEYER EXPANSION FAN
IN AN IDEAL DISSOCIATING GAS

by

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Thesis Director's signature:

A handwritten signature in black ink, appearing to read "J. W. ...", is written over a horizontal line. There are small marks below the line, possibly indicating a correction or a specific part of the signature.

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ABSTRACT

An approximate method for the explicit numerical calculation of the outer boundary of the non-equilibrium region in the Prandtl-Meyer expansion fan of an ideal dissociating gas has been presented. Solution for the lowest order approximation was carried out for a particular initial state: degree of dissociation = 0.52427, dimensionless temperature = 0.06, dimensionless velocity = 0.74276, dimensionless density = 10^{-7} . Curves showing the shape of the outer boundary and the variation of the state and motion variables along this outer boundary are presented.

The results obtained using this approximation are generally in agreement with those to be expected along the outer boundary between the frozen and equilibrium wave heads. Also, as expected, the lowest order approximation fails to maintain accuracy at angular locations of greater than about four degrees into the expansion fan.

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LIST OF SYMBOLS

a_m	function which depends linearly on the values of the integrand on the strip boundaries
B_1	function which determines the radial decay distance
C	reaction rate parameter containing the (backward) specific rate constant
G	typical integrand in the Method of Integral Relations
h	enthalpy per unit mass of mixture
L_i	known function of its arguments in the Method of Integral Relations
m	mass of atoms per unit mole (gm/mol)
M	Mach number
n	number of equations in a system
N	number of strips into which the integration region is divided
p	pressure of the mixture
P_i	known function of its arguments in the Method of Integral Relations
q	velocity
Q_i	known function of its arguments in the Method of Integral Relations
r	radial distance from the corner
t	time
T	temperature of the mixture
u	radial component of velocity
v	tangential component of velocity
w_j	dependent variables in the Method of Integral Relations
x	degree of dissociation

α_m^i	function of Q_i evaluated on the strip boundaries
β_m^i	function of L_i evaluated on the strip boundaries
ϵ	internal energy per unit mass of mixture
θ	angular coordinate
λ	inner boundary of the integration region; inner boundary of the non-equilibrium region
μ	outer boundary of the integration region; outer boundary of the non-equilibrium region
τ	reaction rate parameter containing the (backward) specific rate constant

Subscripts

d	denotes a dimensional quantity
f	denotes "frozen" value of a quantity
i	denotes the particular equation in a system of equations
k	denotes evaluation on the k^{th} line
m	summation index in an interpolation polynomial
o	denotes evaluation in the initial state

Superscripts

i	denotes the particular equation in a system of equations
*	denotes a characteristic (or non-dimensionalizing) quantity

I. INTRODUCTION

Prandtl-Meyer, or centered-expansion, flow is the flow of a gas from a uniform, steady, supersonic state parallel to a boundary which changes direction abruptly at a sharp corner (see Figure 1). The solution of this problem is used widely for the step-wise solution of more complicated problems.

When a gas mixture is experiencing no chemical reaction (i.e., frozen composition) or is undergoing reaction in thermodynamic equilibrium, there is no characteristic length involved explicitly or implicitly in the problem. This implies a complete independence of scale, or that quantities which specify the motion and state of the gas mixture are constant along radial lines emanating from the sharp corner. This dependence only upon angular position is just as in the case of flow of non-reacting gases, and under this condition these types of Prandtl-Meyer flow are readily solved, although numerical means generally must be used.

These two special cases of flow with chemical reactions are called "frozen" flow and "equilibrium" flow. For the case of frozen flow, the mass fraction of each species comprising the mixture is constant over the entire flow field. Instantaneous chemical equilibrium is accomplished in equilibrium flow, and there is no net production rate for any given species. The mass fractions of the mixture species are related to the other local thermodynamic state variables by means of equilibrium constants. For these two special cases, an

exact numerical solution exists (1, 2).*

When chemical reactions do take place at finite reaction rates in the gas as it is expanding, characteristic lengths, the distances required for the reactions to go to completion, are inherent in the problem. Under this condition, the simplifying circumstance of independence of scale is no longer applicable.

The "ideal dissociating gas" was introduced by M. J. Lighthill (3); this type of gas is one in which the translational and rotational modes of each molecule and atom are completely excited but the vibrational degrees of freedom are excited to only half the classical value. In extending Lighthill's work, N. C. Freeman (4) derived a reaction rate equation for an ideal dissociating gas under non-equilibrium conditions.

The Method of Characteristics can be used to obtain a numerical solution to the non-equilibrium flow, but an explicit solution cannot be established in terms of the boundary conditions and independent coordinates because the characteristic equations are irreducible. The numerical procedures used to solve the characteristic equations are time consuming and complex. The method employed by J. P. Appleton (5) uses a first order finite difference approximation to the characteristic equations and involves the use of a characteristic grid in the Cartesian plane. Advancement of the solution requires a trial-and-error iteration scheme, and the rate of convergence of the solution depends upon the accuracy of the first guess at each point.

An approximate solution which is somewhat better suited to

*Numbers in parentheses refer to references listed on page 32.

automatic digital computation is presented in this thesis. The Method of Integral Relations as formulated by Dorodnitsyn (1, 6) is employed to replace the governing partial differential equations by a system of ordinary differential equations. The technique is also well-suited to problems, such as the Prandtl-Meyer flow of a reacting gas, in which one of the flow boundaries must be determined as a part of the solution. Thus, the Method of Integral Relations allows the outer boundary of the non-equilibrium reaction region in such a flow to be explicitly introduced into the problem. A first approximation of this boundary shape has been calculated in this thesis by solving the system of differential equations by numerical means on an IBM 1620 Data Processing System. The numerical technique uses the Runge-Kutta method, as modified by S. Gill (7), as a starter for the predictor-corrector integration scheme devised by R. W. Hamming (7).

II. EQUATIONS GOVERNING THE FLOW

1. Description of the Thermodynamic State

The thermodynamic representation of a dissociating gas is very complex, even when the gas model is composed of simple diatomic molecules. With the aid of several assumptions, M. J. Lighthill (3) introduced the concept of the ideal dissociating gas. These assumptions allow a considerable simplification of the thermodynamic description of the gas mixture. The thermodynamic relations for an ideal dissociating gas, in terms of non-dimensional variables, are

law of mass action (equilibrium composition)

$$\frac{x^2}{(1-x)} = \frac{1}{f} \exp\left(-\frac{1}{T}\right), \quad (2.1)$$

equation of state

$$p = (1+x)fT, \quad (2.2)$$

internal energy equation

$$\epsilon = 3T + x, \quad (2.3)$$

where x is the mass fraction of atoms, f is the density of the gas mixture, T is the temperature, p is the pressure, and ϵ is the internal energy. If \mathcal{P} is any non-dimensional variable, then $\mathcal{P} = \mathcal{P}_d / \mathcal{P}^*$, where the subscript d indicates a dimensional quantity and the superscript $*$ indicates a characteristic variable for dissociation. The non-dimensionalizing quantities are listed in Table I.

The specific enthalpy expression is

$$h = \epsilon + p/\rho = (4 + x) T + x . \quad (2.4)$$

For non-equilibrium processes, the law of mass action must be replaced by a reaction rate equation. Freeman (4) has proposed a reaction rate equation for an ideal dissociating gas. The form of this equation is such that under equilibrium conditions it reduces to the law of mass action. Freeman's reaction rate equation is

$$\frac{dx}{dt} = CT^{-s} \rho \left[(1 - x) \exp\left(-\frac{1}{T}\right) - \rho x^2 \right] , \quad (2.5)$$

where s is a constant, and the variation of C , the reaction rate parameter, is probably small compared to the variation of $\exp(-\frac{1}{T})$ for the temperature ranges of interest. The non-equilibrium parameter, C , determines the time scale of the dissociation process since it contains the (backward) specific rate constant.

2. Gas Dynamic Equations

The fundamental equations governing the flow of a dissociating gas are the equations expressing mass, energy, and momentum conservation, the equations expressing the thermodynamic state of the mixture, and the reaction rate equation, which expresses species conservation. The flow is here considered to be steady, adiabatic, and everywhere supersonic; dissipative effects, such as viscosity, heat conduction, and mass diffusion are neglected. Under these assumptions these several equations can be written in the two-dimensional polar coordinates of Figure I in the following manner:

continuity

$$\frac{\partial(\rho ru)}{\partial r} + \frac{\partial(\rho v)}{\partial \theta} = 0, \quad (2.6)$$

momentum conservation, radial direction

$$u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} - \frac{v^2}{r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0, \quad (2.7)$$

momentum conservation, tangential direction

$$u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + \frac{uv}{r} + \frac{1}{\rho r} \frac{\partial p}{\partial \theta} = 0, \quad (2.8)$$

energy equation

$$h + \frac{1}{2} (u^2 + v^2) = H = \text{constant}, \quad (2.9)$$

equations of state

$$h = h(p, T, x), \quad (2.10)$$

$$\rho = \rho(p, T, x), \quad (2.11)$$

reaction equation

$$u \frac{\partial x}{\partial r} + \frac{v}{r} \frac{\partial x}{\partial \theta} = \frac{f(p, \rho, x)}{\tau}, \quad (2.12)$$

where u is the radial component of velocity, v is the tangential component of velocity, and τ is a reaction rate parameter. It may be noted that in dimensional form τ_d has units of time.

By virtue of the continuity equation (2.6), equations (2.7), (2.8), and (2.12) can be rearranged to

$$\frac{\partial [r(p + \rho u^2)]}{\partial r} + \frac{\partial(\rho uv)}{\partial \theta} = p + \rho v^2, \quad (2.13)$$

$$\frac{\partial (r^2 \rho_{uv})}{\partial r} + \frac{\partial [r (\rho + \rho v^2)]}{\partial \theta} = 0, \quad (2.14)$$

$$\frac{\partial (r \rho_{ux})}{\partial r} + \frac{\partial (\rho_{vx})}{\partial \theta} = \frac{\rho_{rf} (p, \rho, x)}{\tau}. \quad (2.15)$$

Equations (2.6), (2.13), (2.14), and (2.15) are in the Standard Divergence Form:

$$\begin{aligned} & \frac{\partial P_i (r, \theta; w_1, w_2, \dots, w_j, \dots)}{\partial r} + \frac{\partial Q_i (r, \theta; w_1, \dots, w_j, \dots)}{\partial \theta} \\ & = L_i (r, \theta; w_1, \dots, w_j, \dots) \quad , \end{aligned} \quad (2.16)$$

where P_i , Q_i , L_i are functions of the dependent variables w_1, w_2, \dots, w_j , and r, θ are the independent variables.

III. METHOD OF INTEGRAL RELATIONS

The Method of Characteristics allows an exact formulation for computing the solution to the type of flow under consideration; however, it possesses the undesirable trait of being difficult to program for digital computation. Dorodnitsyn (1, 6) has proposed a general method of numerical solution for non-linear hydrodynamics problems which is particularly well-suited for computer use.

The scheme of Dorodnitsyn is, basically, an approximate solution to a system of partial differential equations in two independent variables when expressed in the Standard Divergence Form:

$$\frac{\partial P_i (r, \theta; w_1, \dots, w_j, \dots, w_n)}{\partial r} + \frac{\partial Q_i (r, \theta; w_1, \dots, w_n)}{\partial \theta} = L_i (r, \theta; w_1, \dots, w_n) \quad , \quad (3.1)$$

where $i = 1, 2, \dots, n$; the w_j are unknown functions of r and θ ; and P_i, Q_i, L_i are known functions of their arguments.

The integration region in polar coordinates (see Figure 2) is $\theta_0 \leq \theta \leq \theta_1$, $\lambda(\theta) \leq r \leq \mu(\theta)$. The procedure consists of dividing the region into N strips by means of drawing equidistant lines $r = r_k$, then integrating the system of n equations along an arbitrary radial line from $r = \lambda$ to the boundary of each strip, r_k , where $k = 1, 2, \dots, N$. The equidistant lines are defined by the equation

$$r_k = \lambda(\theta) + \left[\frac{N - (k-1)}{N} \right] \left[\mu(\theta) - \lambda(\theta) \right] \quad , \quad k = 1, 2, \dots, N+1 \quad , \quad (3.2)$$

where $r = r_1 = \mu(\theta)$ is the outer boundary and $r = r_{N+1} = \lambda(\theta)$ is

the inner boundary.

Upon integration along a radial line from $r = \lambda(\theta)$ to $r = r_k(\theta)$, equation (3.1) becomes a system of N independent integral relations of the form

$$\begin{aligned} (P_i)_k - (P_i)_{N+1} + \frac{d}{d\theta} \int_{\lambda(\theta)}^{r_k(\theta)} Q_i dr + (Q_i)_{N+1} \frac{d\lambda}{d\theta} \\ - (Q_i)_k \frac{dr_k}{d\theta} = \int_{\lambda(\theta)}^{r_k(\theta)} L_i dr, \end{aligned} \quad (3.3)$$

where $k = 1, 2, \dots, N$ and the indices k and $N+1$ indicate evaluation on the k^{th} line and inner boundary, respectively.

The integrands in (3.3) are now approximated by interpolation polynomials of the form

$$G(r, \theta) = a_{N+1}(\theta) + \sum_{m=1}^N a_m(\theta) \left[\frac{r - \lambda(\theta)}{\mu(\theta) - \lambda(\theta)} \right]^m, \quad (3.4)$$

where $G(r, \theta)$ is a typical integrand of (3.3) and the $a_m(\theta)$ depend linearly on the values of the integrand on the strip boundaries; that is to say the $a_m(\theta)$ are evaluated by

$$(G)_k - a_{N+1}(\theta) = \sum_{m=1}^N a_m(\theta) \left[\frac{N - (k-1)}{N} \right]^m, \quad k = 1, \dots, N, \quad (3.5)$$

with $(G)_{N+1} = a_{N+1}(\theta)$.

Upon substitution of (3.4) into (3.3), a system of nN ordinary

differential equations is obtained with the flow variables (considering equations (2.6), (2.13), (2.14), and (2.15)) evaluated at the interpolation points as the dependent variables and θ as the independent variable. Integration of the equations in the region $\theta_0 \leq \theta \leq \theta_1$ and application of the appropriate boundary conditions will then yield the desired solution.

If the following interpolation formulae, which are of the form (3.4),

$$Q_i = \alpha_{N+1}^i(\theta) + \sum_{m=1}^N \alpha_m^i(\theta) \left[\frac{r - \lambda(\theta)}{\mu(\theta) - \lambda(\theta)} \right]^m, \quad (3.6.a)$$

$$L_i = \beta_{N+1}^i(\theta) + \sum_{m=1}^N \beta_m^i(\theta) \left[\frac{r - \lambda(\theta)}{\mu(\theta) - \lambda(\theta)} \right]^m, \quad (3.6.b)$$

are substituted into (3.3), these relations become, upon performing the indicated integration,

$$\begin{aligned} & (P_i)_k - (P_i)_{N+1} - \left[(Q_i)_k - (Q_i)_{N+1} \right] \frac{d\lambda}{d\theta} - \left\{ \sum_{m=1}^N \frac{m}{m+1} \alpha_m^i \left[\frac{N-(k-1)}{N} \right]^{m+1} \right\} \frac{d(\mu - \lambda)}{d\theta} \\ & + \left\{ \sum_{m=1}^N \frac{1}{m+1} \left[\frac{N-(k-1)}{N} \right]^{m+1} \left(\frac{d\alpha_m^i}{d\theta} - \beta_m^i \right) \right. \\ & \left. + \left[\frac{N-(k-1)}{N} \right] \left(\frac{d\alpha_{N+1}^i}{d\theta} - \beta_{N+1}^i \right) \right\} (\mu - \lambda) = 0, \quad \begin{matrix} k=1,2,\dots,N \\ i=1,2,\dots,n \end{matrix} \quad (3.7) \end{aligned}$$

where

$$(Q_i)_k - \alpha_{N+1}^i(\theta) = \sum_{m=1}^N \alpha_m^i(\theta) \left[\frac{N-(k-1)}{N} \right]^m, \quad (3.8.a)$$

$$(Q_i)_{N+1} = \alpha_{N+1}^i(\theta), \quad (3.8.b)$$

$$(L_i)_k - \beta_{N+1}^i(\theta) = \sum_{m=1}^N \beta_m^i(\theta) \left[\frac{N-(k-1)}{N} \right]^m, \quad (3.8.c)$$

$$(L_i)_{N+1} = \beta_{N+1}^i(\theta). \quad (3.8.d)$$

For a chosen value of N , equations (3.7) then become the system of ordinary differential equations to be used in place of the original partial differential equations (3.1).

IV. THE OUTER BOUNDARY OF THE NON-EQUILIBRIUM FLOW

1. Introduction

Prandtl-Meyer flow is that flow which results when a gas flows in uniform, supersonic, steady motion along a boundary which changes direction suddenly at a sharp corner. With the use of the Prandtl-Meyer flow solution, a number of more complex flows may be solved (e.g., design of supersonic nozzles) in a step-wise manner; accordingly, it is one of the important, elementary problems treated in the study of non-reacting gas flows.

When there are no chemical reactions in the flowing gas, or if it is undergoing either equilibrium or frozen flow, no characteristic length is inherent explicitly or implicitly in the problem, thus implying a complete independence of scale. From this it may be inferred that the state and motion variables governing the flow are constant along any radial line emanating from the sharp corner, and, in fact, solutions of this nature do exist (2).

If chemical reactions of finite rate are taking place while the gas is expanding, the independence of scale simplification is no longer effective. Characteristic lengths, the distances required for completion of the reactions, are inherent in the problem. Under this condition the occurrence of a solution, such as that for either frozen or equilibrium flow, is not expected.

The non-equilibrium Prandtl-Meyer flow can be solved numerically by the Method of Characteristics if the necessary boundary conditions

are known. Because the characteristic equations are irreducible, explicit solution of the problem in terms of the boundary conditions and independent coordinates cannot be effected.

In this chapter, the Method of Integral Relations, as described in the previous chapter, is used to replace the system of governing partial differential equations by a system of ordinary differential equations. The solution of these equations is obtained numerically by the use of the predictor-corrector integration scheme devised by R. W. Hamming (7). The Runge-Kutta method, as modified by S. Gill (7), is used as a starter for the predictor-corrector scheme.

2. Lowest Order Approximation to the Flow

The most elementary approximation using the Method of Integral Relations is that in which the region is divided into one strip (i.e., $N = 1$). For this case, if the subscript $N+1$ is replaced by f , equations (3.7) and (3.8) become

$$\begin{aligned} & (P_i)_1 - (P_i)_f - \alpha_1^i \frac{d\lambda}{d\theta} - \frac{1}{2} \alpha_1^i \frac{d(\mu - \lambda)}{d\theta} \\ & + \left[\frac{1}{2} \left(\frac{d\alpha_1^i}{d\theta} - \mathcal{B}_1^i \right) + \left(\frac{d\alpha_f^i}{d\theta} - \mathcal{B}_f^i \right) \right] (\mu - \lambda) = 0, \end{aligned} \quad (4.1)$$

$$\alpha_f^i = (Q_i)_f, \quad (4.2.a)$$

$$\alpha_1^i = (Q_i)_1 - (Q_i)_f, \quad (4.2.b)$$

$$\mathcal{B}_f^i = (L_i)_f, \quad (4.2.c)$$

$$\mathcal{B}_1^i = (L_i)_1 - (L_i)_f. \quad (4.2.d)$$

The inner boundary of the flow is assumed to be a frozen flow, and it is assumed that this inner boundary, $r = \lambda(\theta)$ is at the corner; then $\lambda \equiv 0$. With this assumption and using equations (4.1) and (4.2), equations (2.6), (2.13), (2.14), and (2.15) become

$$\frac{d(\mathcal{F}_1 v_1)}{d\theta} - (\mathcal{F}_1 v_1 - \mathcal{F}_f v_f) \frac{d \ln \mu}{d\theta} = -2 \mathcal{F}_1 u_1 - \frac{d(\mathcal{F}_f v_f)}{d\theta} \quad (4.3)$$

$$\begin{aligned} \frac{d(\mathcal{F}_1 u_1 v_1)}{d\theta} - (\mathcal{F}_1 u_1 v_1 - \mathcal{F}_f u_f v_f) \frac{d \ln \mu}{d\theta} = \\ -p_1 - 2 \mathcal{F}_1 u_1^2 + \mathcal{F}_1 v_1^2 + p_f + \mathcal{F}_f v_f^2 - \frac{d(\mathcal{F}_f u_f v_f)}{d\theta} , \end{aligned} \quad (4.4)$$

$$\frac{d(p_1 + \mathcal{F}_1 v_1^2)}{d\theta} = -2 \mathcal{F}_1 u_1 v_1 , \quad (4.5)$$

$$\begin{aligned} \frac{d(\mathcal{F}_1 v_1 x_1)}{d\theta} - (\mathcal{F}_1 v_1 x_1 - \mathcal{F}_f v_f x_f) \frac{d \ln \mu}{d\theta} = \\ -2 \mathcal{F}_1 u_1 x_1 + \frac{\mathcal{F}_1 \mu^{f_1}}{\tau} - \frac{d(\mathcal{F}_f v_f x_f)}{d\theta} . \end{aligned} \quad (4.6)$$

In addition to the energy equation (2.9), one more relation is needed to complete the system of equations. This relation must furnish information concerning the outer boundary. The most obvious statement which can be made about the outer boundary is that, since it is the outer boundary of the non-equilibrium region, the flow along this outer boundary must be in equilibrium, or that $f_1(p_1, \mathcal{F}_1, x_1) = 0$. With this assumption the flow is entirely specified, and the system of equations is completed.

The system of equations to be solved is, then, equations (4.3) - (4.5) and

$$\frac{d(\mathcal{f}_1 v_1 x_1)}{d\theta} - (\mathcal{f}_1 v_1 x_1 - \mathcal{f}_f v_f x_f) \frac{d \ln \mu}{d\theta} = -2 \mathcal{f}_1 u_1 x_1 - \frac{d(\mathcal{f}_f v_f x_f)}{d\theta}, \quad (4.7)$$

which is the revised form of (4.6), and

$$\left(\frac{\partial \mathcal{f}_1}{\partial \mathcal{f}_1} \right)_{p,x} \frac{d\mathcal{f}_1}{d\theta} + \left(\frac{\partial \mathcal{f}_1}{\partial p_1} \right)_{\mathcal{f},x} \frac{dp_1}{d\theta} + \left(\frac{\partial \mathcal{f}_1}{\partial x_1} \right)_{\mathcal{f},p} \frac{dx_1}{d\theta} = 0, \quad (4.8)$$

which is a differentiated form of the equation expressing equilibrium along the outer boundary, and

$$\left(\frac{\partial h_1}{\partial \mathcal{f}_1} \right)_{p,x} \frac{d\mathcal{f}_1}{d\theta} + \left(\frac{\partial h_1}{\partial p_1} \right)_{\mathcal{f},x} \frac{dp_1}{d\theta} + \left(\frac{\partial h_1}{\partial x_1} \right)_{p,\mathcal{f}} \frac{dx_1}{d\theta} + u_1 \frac{du_1}{d\theta} + v_1 \frac{dv_1}{d\theta} = 0, \quad (4.9)$$

which is a differentiated form of the energy equation (2.9). These equations form a first approximation to the flow and can be expected to produce results only for the equilibrium flow along the outer boundary. The value of this approximation is that it reveals information about the nature of the outer boundary of the non-equilibrium region of the Prandtl-Meyer flow of an ideal dissociating gas.

3. Decay of the Wave Head

Along the outer boundary, the reaction rate equation (2.5) reduces to the law of mass action (2.1). Comparison of equations (2.5) and (2.12) shows

$$f(p, \mathcal{f}, x) = (1-x) \exp \left[\frac{-\mathcal{f}(1+x)}{P} \right] - \mathcal{f} x^2. \quad (4.10)$$

The reaction parameter was chosen to correspond to that used by J. P. Appleton (5), which is

$$\tau_d = \frac{m^2 T_d^2 f}{(8.4 \times 10^{14}) (3500^\circ\text{K}) (1 + x) f_d^2}, \quad (4.11)$$

where the subscript d indicates that this quantity is not non-dimensional.

Across the first radial characteristic (i.e., the frozen wave head) of the relaxing flow, the derivatives of the flow variables must experience a finite jump. Glass and Kawada (8) and Appleton (5) show that this gradient attenuates with increasing distance from the corner. A radial decay distance can be defined as the distance along the wave head at which the gradients of the flow variables have decreased to some designated non-zero values. F. A. Wierum, in unpublished notes, suggests that the radial length

$$r_{od} = \frac{10}{B_1}, \quad (4.12)$$

where, for an ideal dissociating gas,

$$B_1 = \frac{x_o^2 (1 + x_o - 3T_o)^2}{6(4 + x_o)(1 + x_o) T_o^2} \cdot \frac{f}{a_{od} \tau_{od} \sqrt{1 - \frac{1}{M_o^2}}}, \quad (4.13)^*$$

and the subscript o indicates evaluation in the initial state, be used as the distance from the corner (along the wave head) at which the disturbance be considered completely attenuated. His notes indicate that the corner disturbance is 99.95 percent attenuated at this distance.

* This expression is also given by Glass and Kawada (8), Glass and Takano (10), and Appleton (5).

Beyond this point the fluid will not experience the influence of the corner. This radial length then is the initial value of the outer boundary as measured by $\mu(\theta)$ of Chapter III.

4. Solution of a Particular Problem

The equations (4.3) - (4.5), (4.7) - (4.9) have been programmed for solution on an IBM 1620 Data Processing System. The Runge-Kutta method, as modified by S. Gill (7), is used as a starter for a predictor-corrector numerical integration scheme devised by R. W. Hamming (7). The input to the program includes the initial state of the gas and the elements needed for the frozen flow solution (9) at the corner.

So that a comparison might be made with the Method of Characteristics solution, the initial state was chosen to correspond to that used by J. P. Appleton (5), namely,

$$\text{degree of dissociation} = x_o = 0.52427 ,$$

$$\text{temperature} = T_o = 0.06 ,$$

$$\text{density} = \rho_o = 10^{-7}$$

$$\text{pressure} = p_o = 0.91456 \times 10^{-8}$$

$$\text{velocity} = q_o = 0.74276 .$$

If the characteristic length is chosen equal to r_{od} of equation (4.12), then μ_o will be unity and the characteristic time must be chosen to be the ratio of this characteristic length to the characteristic velocity listed in Table I (that is $\tau^* = r_{od}/q^*$).

Glass and Takano (10) indicate that the radial attenuation distance should increase from its value at the (frozen) wave head to infinity when the flow has turned through an angle such that the radial

line coincides with the equilibrium wave head. According to this the shape of the outer boundary ($\mu(\theta)$) should increase, as θ increases, from r_{od} ($\mu_0 = 1$) to infinity after a nominal expansion. The formulation employed here predicts the shape of this boundary and indicates that μ does increase rapidly but does not go to infinity.

The calculated shape of the outer boundary in the immediate vicinity of the frozen wave head is shown in Figure 3. This shape is in essential agreement with the expected shape discussed previously. Figures 4, 5, 6, 7, and 8 show the variation of pressure, density, temperature, dissociation fraction, and velocity, respectively, in the region of the wave head. All of these quantities should be constant along the outer boundary in this region (that is, between the frozen and the equilibrium wave heads). As shown in Figures 4 and 5, the pressure and the density, as calculated by this one-strip approximation, show a decrease in value. This behavior appears to be a result of the rather extreme nature of the one-strip assumption which requires properties to vary linearly between the corner and the outer boundary along a radial line.

Figure 9 shows the variation of pressure with radial distance from the corner. The "actual" variation depicted in this figure is a qualitative representation of the variation which would be obtained from a Method of Characteristics solution, such as that of Glass and Takano (10). These authors obtain such a property variation along an η characteristic line for the several different cases they investigated. It is seen that the linear variation as assumed in the one-

strip approximation is expected to lie below that of the actual variation.

Figure 10 shows, in polar coordinates, the calculated shape of the outer boundary, μ . The figures 3 - 10 were plotted from results calculated using a step size (for the independent variable, θ) of one-half of a degree.

The stability of the numerical technique employed was found to be very sensitive to the step size used. Since the predictor-corrector scheme is stable for essentially any type of function, the instability is thought to have originated in the Runge-Kutta portion of the program. Increments greater than one-half of a degree led to a fluctuation of calculated values when the results of the starter program were fed into the predictor-corrector program. However, step sizes of one-half of a degree and smaller led to a stable solution.

V. CONCLUSIONS

The results of this investigation seem to indicate that the one-strip analysis will yield only limited information about the shape of and property variation along the outer boundary of the non-equilibrium region of a Prandtl-Meyer expansion fan in an ideal dissociating gas. This simple model for the non-equilibrium flow region is seen to be useful for obtaining the shape of the outer boundary of this region between the frozen and the equilibrium wave heads. The analysis does indicate that the use of the Method of Integral Relations to obtain a description of the flow can readily be accomplished and that a two-strip or three-strip analysis may be expected to yield significant information about the outer boundary and the property variations within the flow field.

Preliminary investigation of a two-strip analysis has been undertaken. The results to date indicate, as in the one-strip analysis, that the size of the increment (of the independent variable, θ) used is important to the stability of the program. It is suggested that the increment size in the Runge-Kutta portion of the program be made small and that the step size used in the predictor-corrector scheme be made variable. The solution also appears to be quite sensitive to variations in the initial state of the gas. Investigation of the stability of the solution, as related to variations in the initial radius of the outer boundary, is also recommended. This problem does not arise in the one-strip analysis.

TABLE I

Characteristic Dissociation Quantities	Oxygen	Nitrogen
T^* K	59,000	113,000
ρ^* gm/cc	150	130
p^* atmos.	2.3×10^7	4.1×10^7
h^* Kcal/gm	8.67	8.02
q^* Km/sec	3.9	5.8

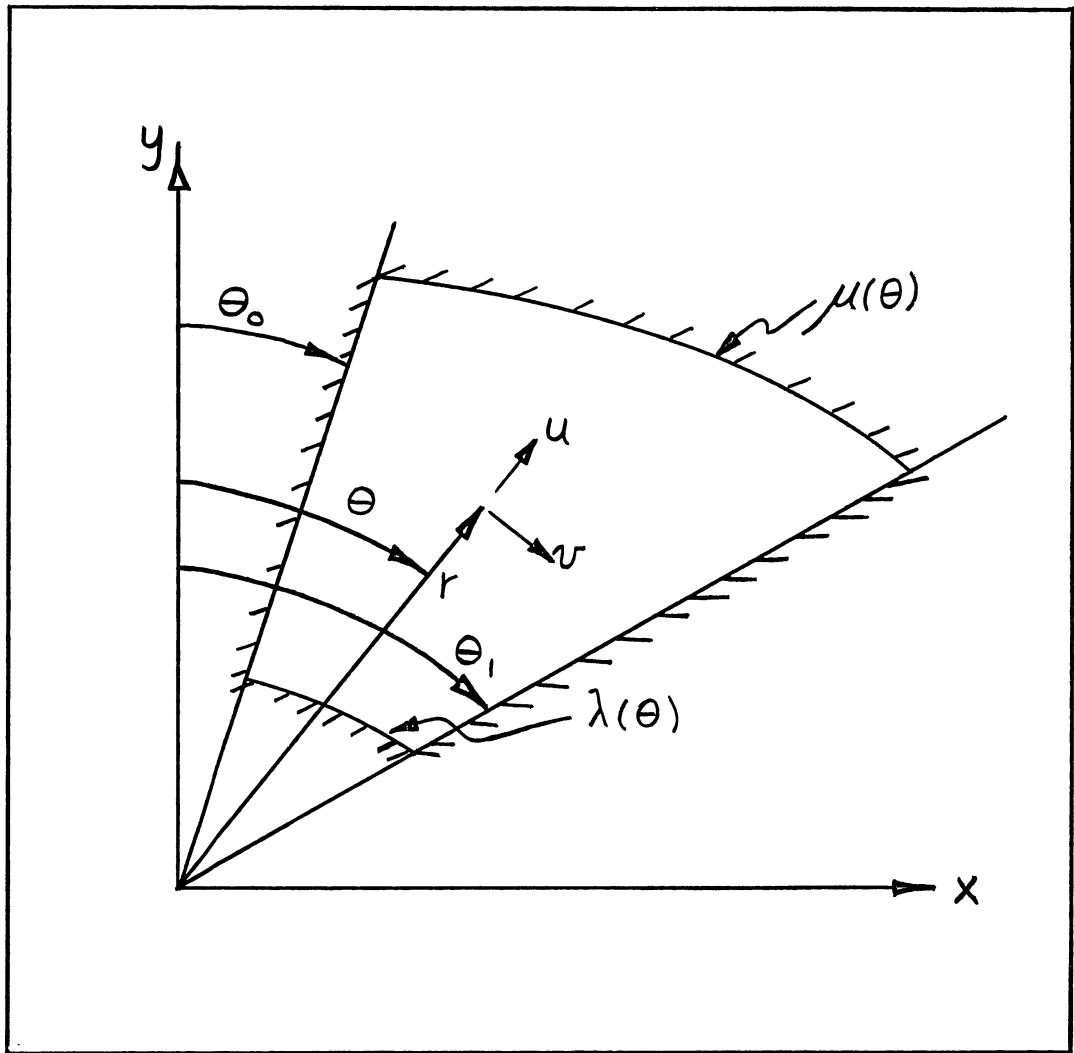


FIGURE 2

Region of Integration for the Method of
Integral Relations

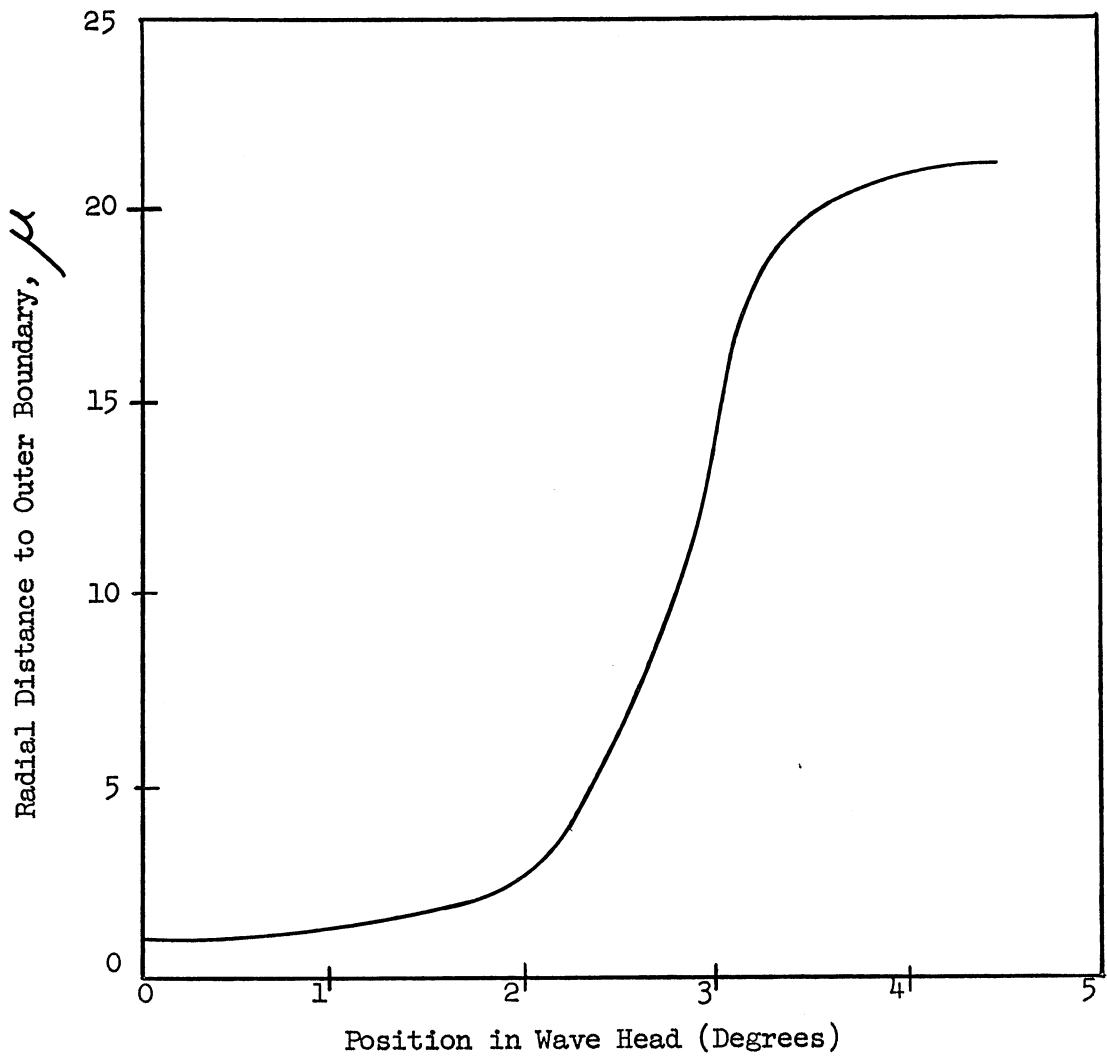


FIGURE 3
Variation of the Outer Boundary in the Region
of the Wave Head

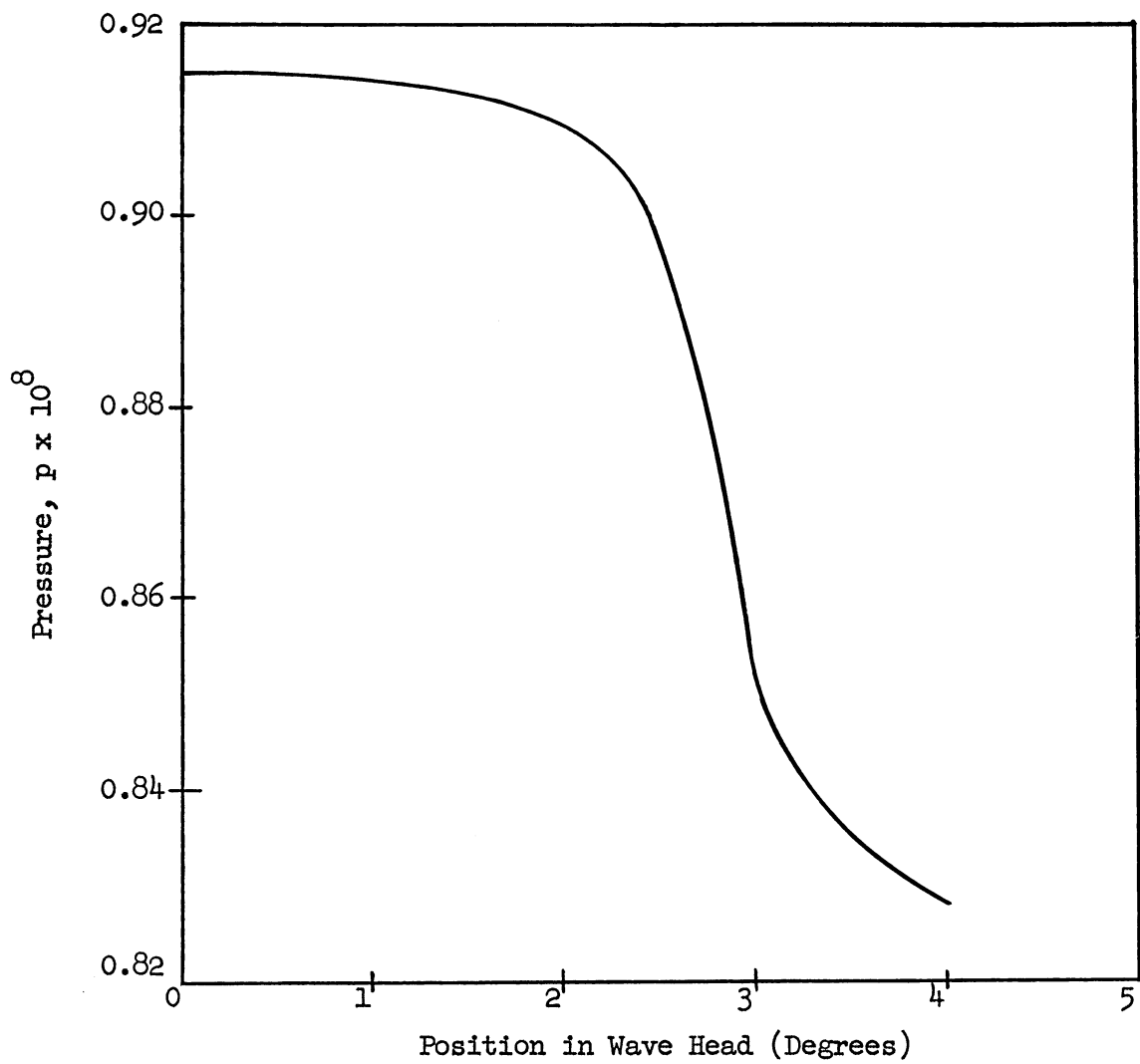


FIGURE 4
Variation of Pressure in the Region of
the Wave Head

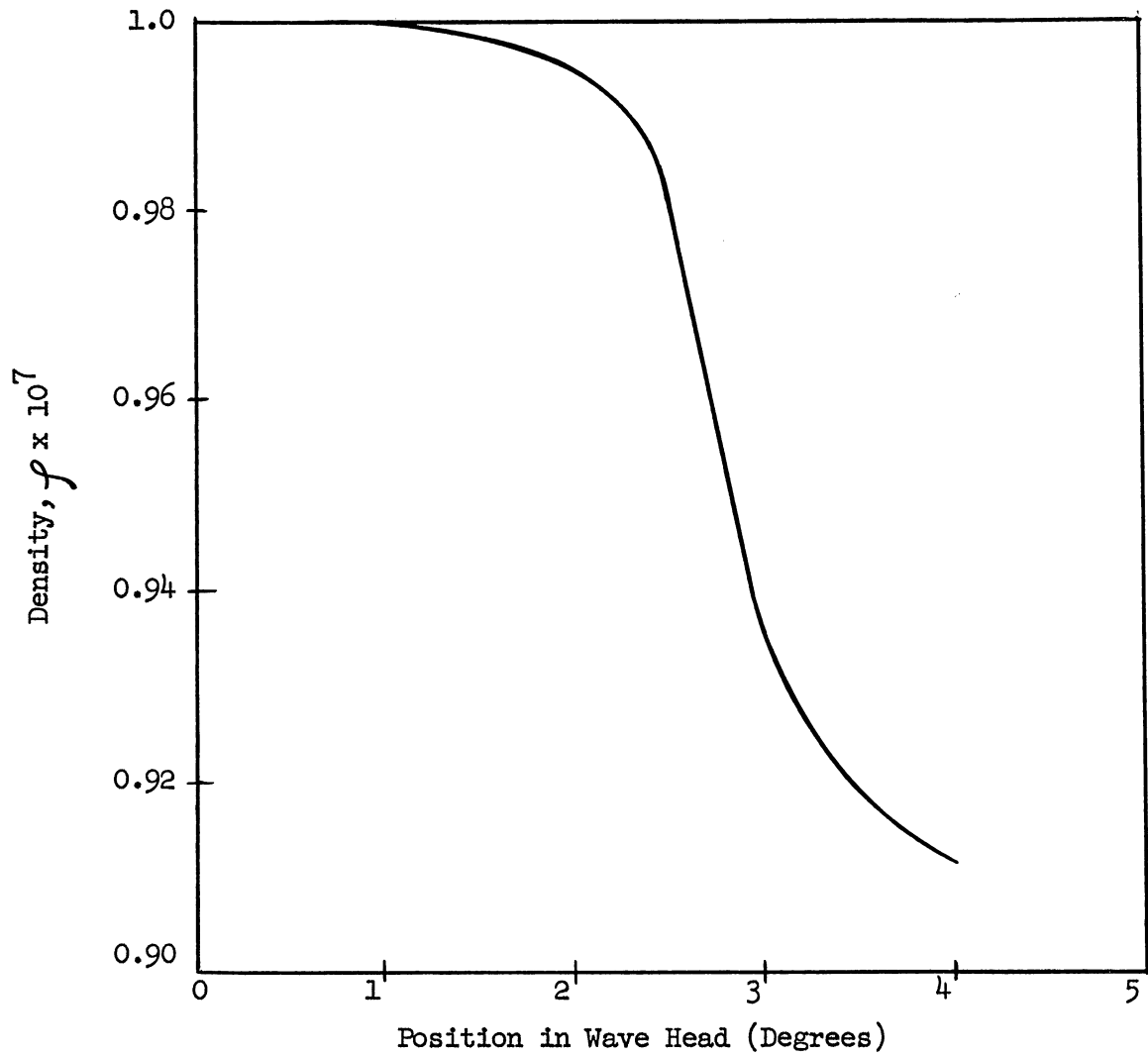


FIGURE 5
Variation of Density in the Region of
the Wave Head

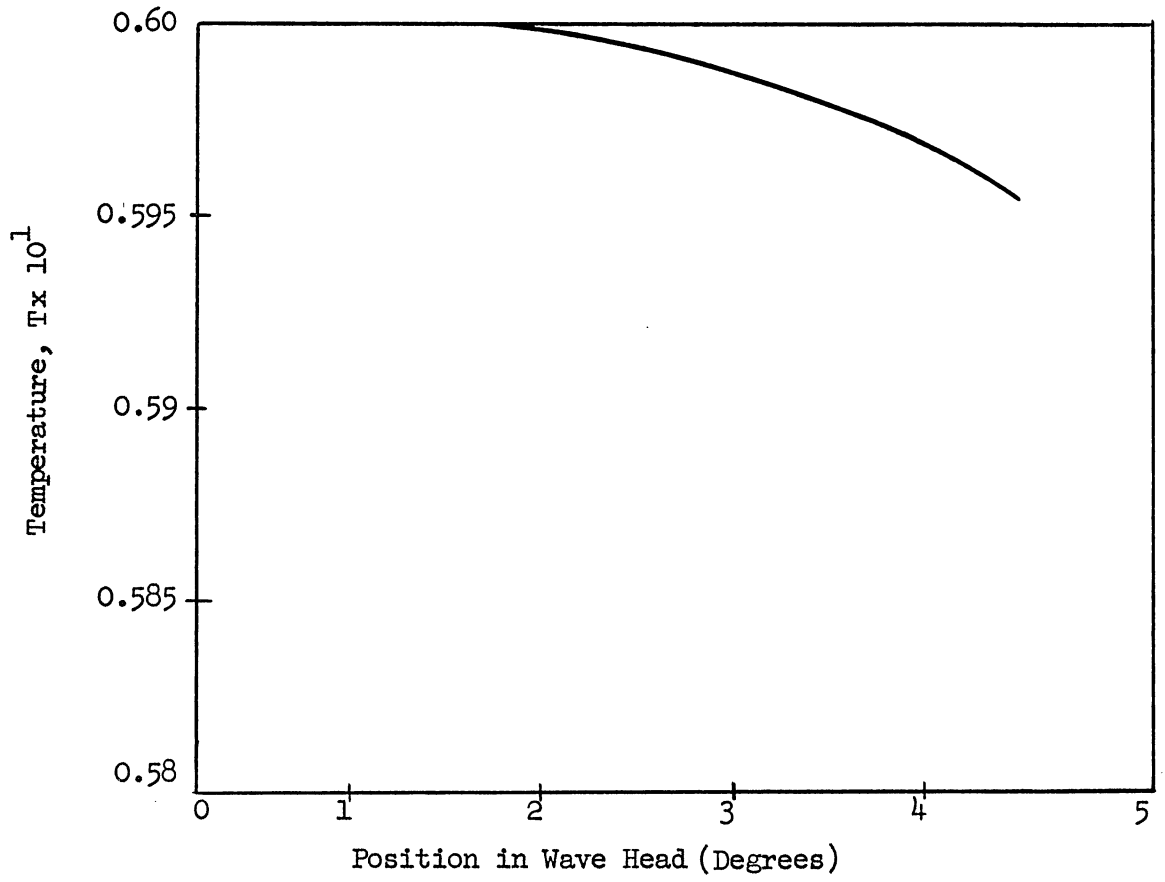


FIGURE 6
Variation of Temperature in the Region of
the Wave Head

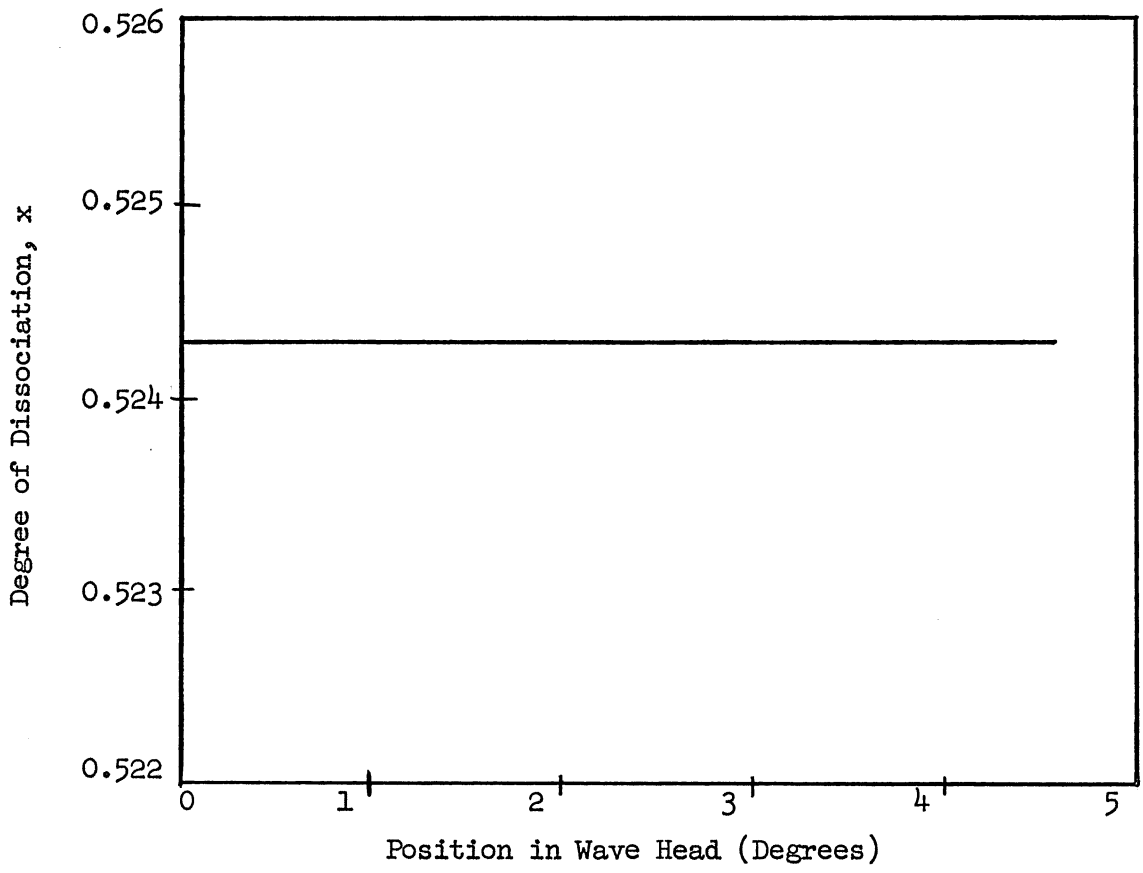


FIGURE 7
Variation of Degree of Dissociation in the
Region of the Wave Head

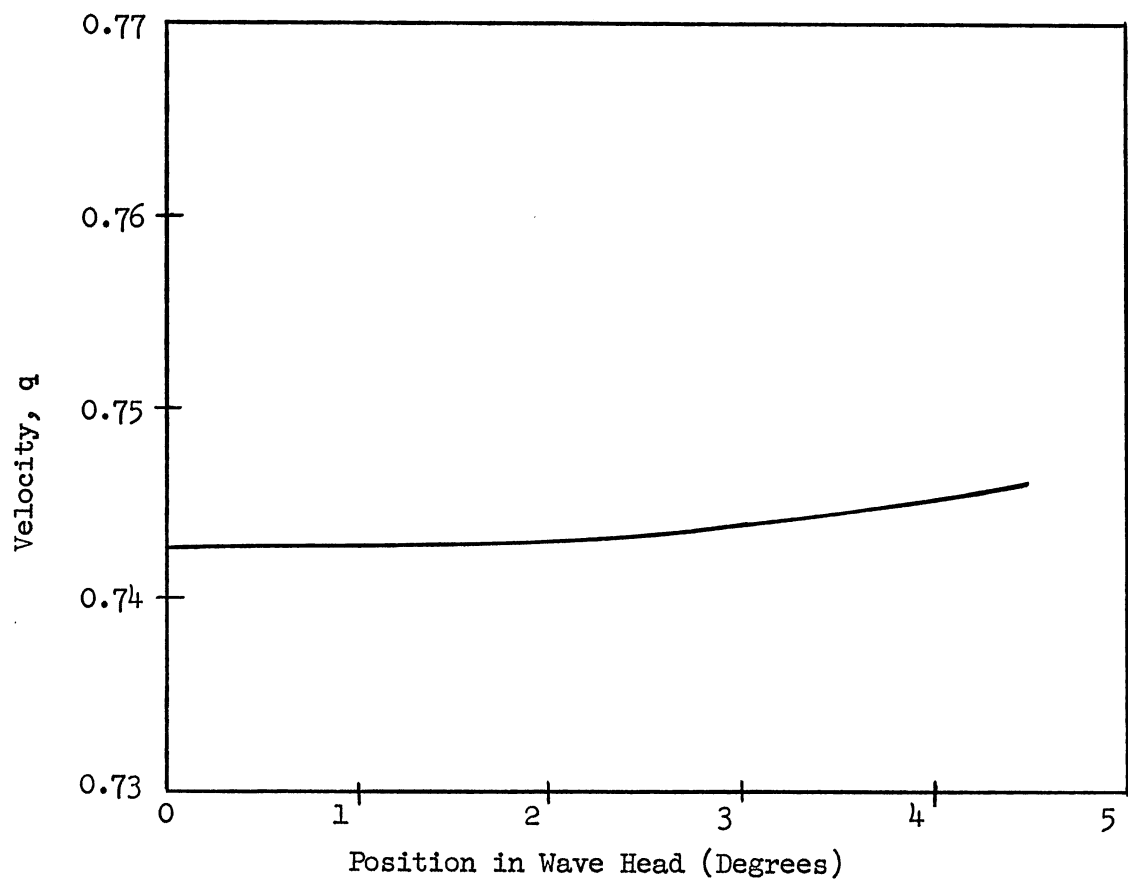


FIGURE 8
Variation of Velocity in the Region of the
Wave Head

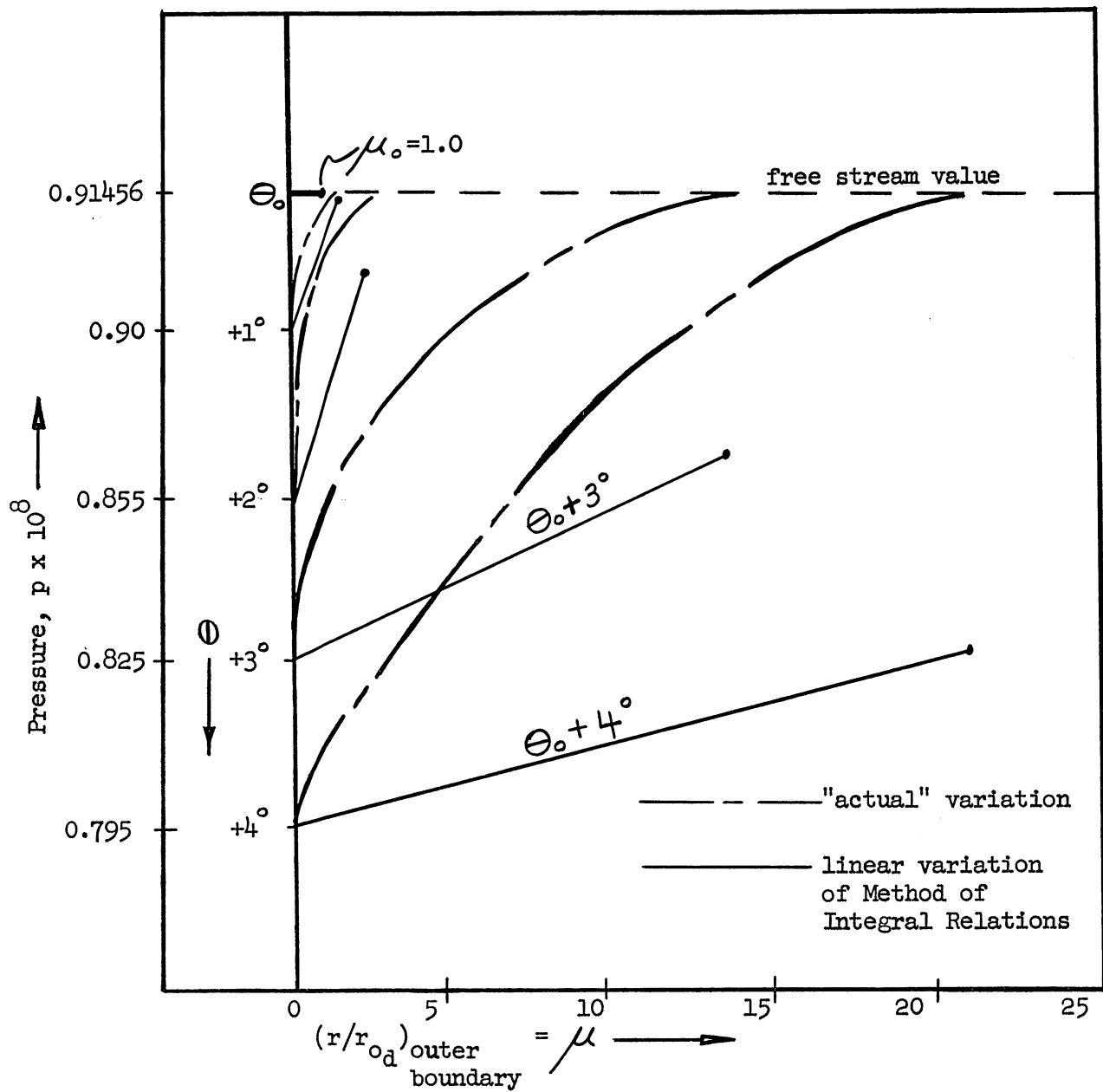


FIGURE 9

Variation of Pressure with Radial Distance from the Corner (along the outer boundary).

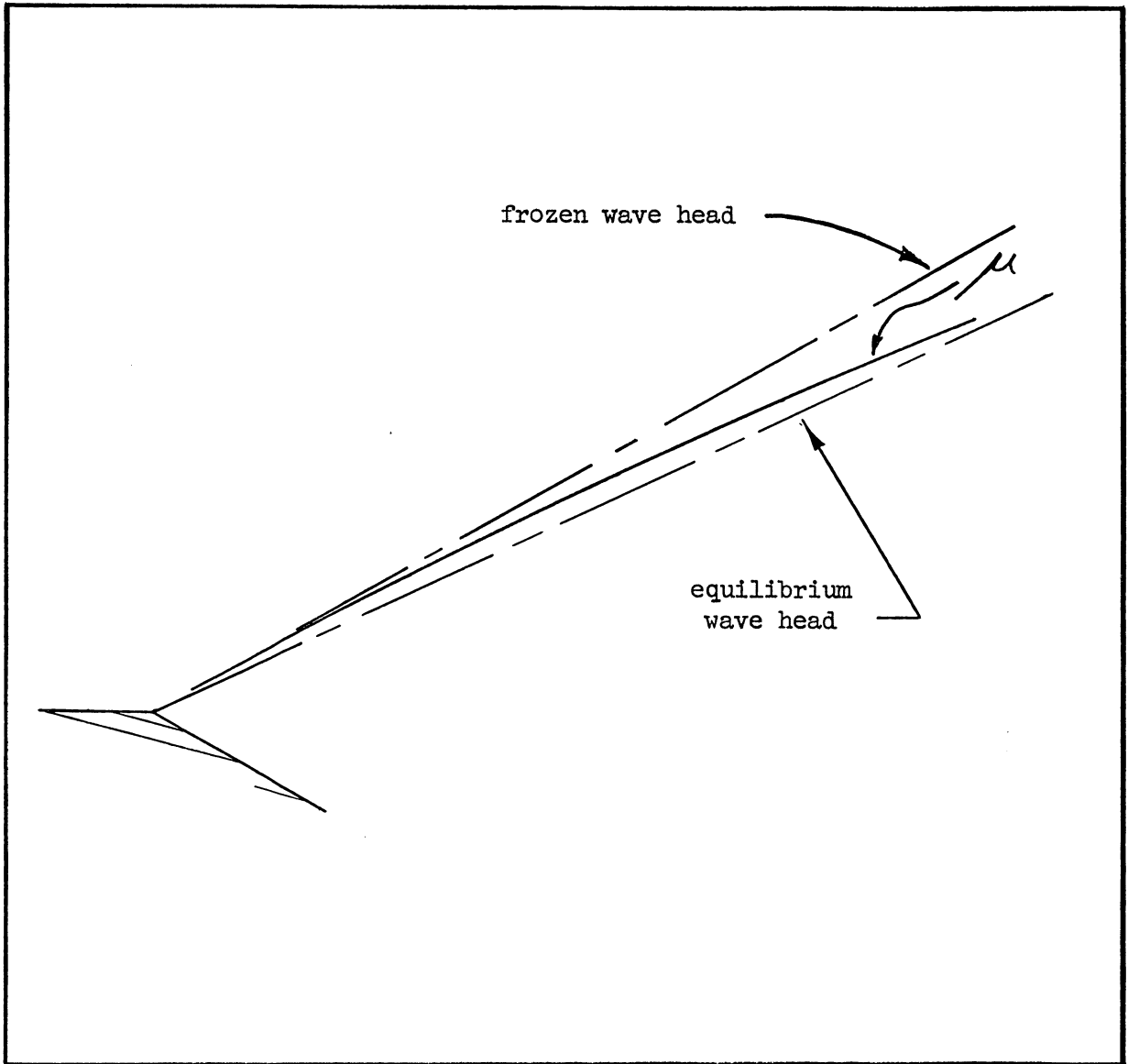


FIGURE 10
The Outer Boundary of the
Non-equilibrium Region

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