THE DYNAMIC STRESSES GENERATED IN A FINITE ELASTIC
BODY OF REVOLUTION BY LONGITUDINAL IMPACT
WITH A RIGID WALL

by

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ABSTRACT

General statements concerning phase and group velocities of stress waves propagating in elastic media where at least one dimension is infinite are valuable for separating the distinct characteristics of the various operating phenomena. The problem of impact of a finite body, however, is so greatly complicated by the inherent reflections, reinforcements, and cancellations of the stress pulses that little use can be made of these general statements in describing the overall transient response of such a body. Presented here is a finite difference numerical method for integrating over space and time the appropriate equations of motion and boundary conditions for determining the transient stress state in a finite elastic body of revolution resulting from longitudinal impact against a rigid wall. Also presented here are the results obtained from programming this numerical procedure for the Rice University digital computer and using this program to evaluate the transient stress states in elastic bodies of nine distinct boundary configurations.
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### NOTATION

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<td>Coefficients in the equations for satisfying the zero stress boundary conditions along the transverse boundary.</td>
</tr>
<tr>
<td>$A$, $B$, $C$, $D$</td>
<td>Matrices containing the coefficients $A_i$, $B_i$, $C_i$, $D_i$, $i = 1, 2, \ldots, 12$, for each of $N-1$ pseudo nodes along $r = b(z)$.</td>
</tr>
<tr>
<td>$b(z)$</td>
<td>Transverse boundary curve, cubic equation</td>
</tr>
<tr>
<td>$b'(z)$</td>
<td>Tangent to the transverse boundary curve</td>
</tr>
<tr>
<td>$B(r) = h/2r$</td>
<td>Radial lattice parameters</td>
</tr>
<tr>
<td>$B'(r) = \frac{B(r)}{B(r)}$</td>
<td></td>
</tr>
<tr>
<td>$B''(z) = h/2b(z)$</td>
<td></td>
</tr>
<tr>
<td>$c_b = (E/\rho)^{1/2}$</td>
<td>Velocity at which longitudinal waves propagate in thin bars.</td>
</tr>
<tr>
<td>$c_e = (\mu/\rho)^{1/2}$</td>
<td>Velocity at which equivoluminal (distortion) waves propagate in infinite elastic media</td>
</tr>
<tr>
<td>$c_i = (\gamma/\rho)^{1/2}$</td>
<td>Velocity at which irrotational (dilatation) waves propagate in infinite elastic media</td>
</tr>
<tr>
<td>$D$</td>
<td>Permissible deviation of the residuals. The residuals must all be made smaller than $D$ in order to acquire approximate equilibrium throughout the body at some instant $t_j$.</td>
</tr>
<tr>
<td>$E$</td>
<td>Young's elastic modulus</td>
</tr>
<tr>
<td>$h$</td>
<td>Unit spacing between nodes of the finite difference network</td>
</tr>
<tr>
<td>$k$</td>
<td>Number of time steps between stress evaluations</td>
</tr>
<tr>
<td>$K$</td>
<td>Overrelaxation parameter, the optimum value of which was found in practice to be unity</td>
</tr>
<tr>
<td>$L$</td>
<td>Length of the impacting body</td>
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M  Maximum radial dimension of the finite difference network in unit spacings h

\( m \geq T/\Delta t \)  Number of time steps required to span the total impact cycle duration

\( m_i, n_i \)  Nondimensional coordinates of pseudo nodes along \( r = b(z) \)

\( N = L/h \)  Number of unit spacings into which the length has been divided

\( n' \)  Number of regular nodes lying interior to the boundary

\( n'' \)  Number of pseudo nodes lying on or exterior to the boundary

\( n = n' + n'' \)  Total number of nodes in the finite difference network

0  Subscript denoting the particular node being investigated

\( R_i, Z_i \)  Residuals corresponding to the \( i^{th} \) node, indicating the amount by which \( u_i \) and \( w_i \) deviate from their equilibrium values with respect to displacements at neighboring nodes

\( R, Z \)  Matrices containing the \( n' \) residuals as defined by the displacements in \( U, W, U', W', U'', W'' \).

\( \bar{R}, \bar{Z} \)  The average magnitudes of the residuals in \( R, Z \)

\( |\bar{R}|, |\bar{Z}| \)  The average magnitudes of the residuals in \( R, Z \)

\( R_{RR}, Z_{ZZ}, \theta\theta, RZ \)  Matrices containing the stresses \( \sigma_{rr}, \sigma_{zz}, \sigma_{\theta\theta}, \sigma_{rz} \), respectively, for points corresponding to the spaces between the finite difference nodes

\( r, z, \theta \)  Cylindrical coordinates

\( T_{nn}, T_{nt} \)  Normal and tangential tractions, respectively, applied to the boundary whose normal is \( \bar{n} \)

\( T \)  Total duration of the impact cycle, measured from the initial instant \( t_0 \) to the time at which a stress reversal (compression to tension) occurs along the impact face
**t**  
Time coordinate

\[ \Delta t \]  
Unit time interval between successive time steps

\[ t_j = j \Delta t, \]  
Discrete positions in time at which the displacements are calculated

\[ j = 1, 2, \ldots, m \]

\[ u, w \]  
Components of the infinitesimal displacements in the r, z-directions, respectively

\[ (u_i, w_i), (u_i', w_i'), (u_i'', w_i'') \]  
Displacement components corresponding to the \( i^{\text{th}} \) node at time \( t_j, t_{j-1}, t_{j-2} \), respectively

\[ U, W, U', W', U'', W'' \]  
Matrices containing the n displacements \( u_i, w_i, u_i', w_i', u_i'', w_i'' \), respectively

\[ V \]  
Striking velocity

\[ W \]  
Weight of impacting body

\[ x \]  
General independent variable, representing either r, z, or t

\[ y \]  
General dependent variable, representing either u or w

\[ y^{(k)} \]

\[ y_0 \]  
Total \( k^{\text{th}} \) derivative of the function \( y(x) \) evaluated at the point \( x_0 \).

\[ y'_{ijk} \]  
Partial derivative of the function \( y(i, j, k, \ldots) \) with respect to the independent variables represented by \( i, j, k \)

\[ e_{rr}, e_{zz}, e_{\theta\theta}, e_{rz} \]  
Relevant components of the strain tensor

\[ \sigma_{rr}, \sigma_{zz}, \sigma_{\theta\theta}, \sigma_{rz} \]  
Relevant components of the stress tensor for axially symmetric problems in cylindrical coordinates

\[ \delta_{ij} \]  
The Kronecker delta, defined to have the value 1 if \( i = j \), and value 0 if \( i \neq j \).

\[ \nu \]  
Poisson's ratio
\[ \lambda = \nu \frac{E}{(1+\nu)(1-2\nu)} \]

Lame's elastic constants

\[ \mu = \frac{E}{2(1+\nu)} \]

Mass density of the elastic material

\[ \gamma = \lambda + 2\mu = \frac{(1-\nu)}{1+\nu} \left( \frac{E}{1-2\nu} \right) \]

Convenient combinations among \( \lambda, \mu, \gamma \), used to simplify the governing residual equations

\( \rho' = \rho \left( \frac{h}{\Delta t} \right)^2 \)

Fraction of the unit spacing \( h \) between the nearest pseudo node and the transverse boundary in the radial direction

\( \gamma' = \frac{(\lambda + \mu)}{4} \)

Convenient combinations among \( \lambda, \mu, \nu \), \( B'' \), and \( b' \) used to simplify the equations which express the transverse boundary conditions

\[ \gamma'' = 2(\lambda + 3\mu) + \rho' \]

\[ \xi = \frac{2B''\lambda(1-b'^2)}{(\gamma - \lambda b'^2)} \]

\[ \beta = \frac{(\lambda - \gamma b'^2)}{(\gamma - \lambda b'^2)} \]

\[ \alpha' = b'(\alpha - 2B'') \frac{\lambda}{\mu} \]

\[ \beta' = b'(\beta \lambda - \gamma) \frac{1}{\mu} \]
INTRODUCTION

When the motion of a bounded, continuous body, composed of isotropic elastic material and initially in static equilibrium under zero external loads, is abruptly disturbed over certain parts of its surface, there results the generation of a continuously time-varying state of stress throughout the body. The more prominent features of such disturbances, called stress waves, have been found to propagate at certain distinct velocities depending on the mechanical properties of the constituent material.\(^1\) Most theoretical studies of stress waves in the past have been restricted to simple one-dimensional cases involving thin rods or infinite media,\(^2\) because of the complexity of obtaining solutions to the governing equations for finite bodies in three dimensions.

The present investigation is directed to the problem of determining the transient stress conditions in a finite elastic body of revolution which results from having the body impact longitudinally against a rigid wall with an initial striking velocity \(V\). The solution to this problem is thought to closely approximate the stress conditions produced in any

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1 See pages 13, 19, and 42 of Kolsky, reference (40) in the Bibliography, and page 454 of Timoshenko and Goodier (7).

2 An extensive list of references on stress wave propagation is included in the Bibliography.
geometrically similar body of revolution as a result of its impacting with a second, comparable body, where \( V \) is measured relative to the center of mass of the two bodies.

The problem originated in connection with an analysis of the performance of a percussion drilling tool, the fundamental operating principles of which are stated as follows. Compressed air was used to accelerate an elastic hammer to a downward velocity \( V \); the major portion of the resulting kinetic energy of the hammer was then transferred to an elastic anvil by means of longitudinal impact such as described above. To the lower end of the anvil was attached a rock bit, which in turn absorbed the energy of the anvil by fracturing some of the rock upon which it rested. By frequently repeating this process high penetration rock drilling was accomplished.\(^1\)

During the analysis of the percussion drilling tool, it was conjectured that the amount of penetration per impact blow was dependent upon the masses and configurations of the hammer and anvil, a supposition supported by experimental evidence.\(^2\) Breakage of hammers and anvils was attributed to the large vibrational stresses generated during the brief period of contact immediately following the impact collision. The magnitudes of these stresses were likewise considered to depend upon the relative sizes and configurations of the hammer and anvil.

\(^1\) See references (1) and (2).

\(^2\) See references (3) and (4).
The present study is intended to be a preliminary investigation into this latter problem, and its specific purpose is therefore twofold. (1) The necessary theory will be developed to integrate the governing equations of motion by a numerical procedure, with the intention of using a high-speed digital computer to facilitate the actual computations required for each solution. (2) The computer program so developed will be used to provide results serving to relate the configuration of the impacting body with the transient stress state in the body. To aid in the design of impact hammers, the overall maximum stresses developed for each of nine boundary configurations (see Figure 15) will be computed and compared, and the normal stresses along the impact face together with the duration of the impact cycle will be used to express the effectiveness of the impact collision.

Throughout this analysis the constituent material is assumed to be homogeneous, isotropic, and linearly elastic; that is, the stresses always remain below the yield point, and the effects of temperature and of stress and strain rates are assumed negligible. Likewise it is assumed that contact between the impacting body and the rigid wall is uniform and occurs instantaneously over the entire contact area, a condition which is difficult to realize in practice, and to which is usually attributed the discrepancies which appear between experiment and theory.¹

¹ See page 451 of Timoshenko and Goodier (7), and pages 98 and 267 of Goldsmith (38).
It is hoped that the analysis and results presented here are general enough to be applicable to problems of impact other than that discussed above.
Figure 1

Impact situation showing cutaway of typical body configuration.
Figure 2

Finite difference network imposed on typical half section, showing boundary conditions.
THE BOUNDARY

An example of a typical configuration of the class of elastic bodies being studied is depicted in Figure 1. This diagram shows a body of revolution symmetric about a central axis $z$, having plane ends $z = 0$ and $z = L$, and a transverse boundary $b(z)$, which in any cross-sectional plane is represented by the cubic polynomial in $z$

$$b(z) = B_0 + B_1 z + B_2 z^2 + B_3 z^3, \quad 0 \leq z \leq L, \ldots \ldots \ldots \ldots \ldots \ldots (1)$$

where the coefficients $B_i, i = 0, 1, 2, 3$, are evaluated in terms of configurational requirements as specified below. The tangent to this boundary curve is

$$b'(z) = B_1 + 2B_2 z + 3B_3 z^2, \quad 0 \leq z \leq L. \ldots \ldots \ldots \ldots \ldots \ldots (2)$$

The cubic polynomial was chosen as the functional form for the transverse boundary because it is the lowest order (smoothest) polynomial which provided the degree of flexibility required for the configuration tests — a major objective of this investigation — and because of the relative simplicity of manipulating low order polynomials.

The coefficients $B_i$ are determined by specifying the boundary radius $b(z)$ and the boundary slope $b'(z)$ at the two points $z_0 = 0$ and $z_1 = L$. By expanding equations (1) and (2) in terms of $z_0$ and $z_1$ and solving for $B_i, i = 0, 1, 2, 3$, the following expressions are readily obtained.
\[ B_0 = b(0), \]
\[ B_1 = b'(0), \]
\[ B_2 = \frac{[2b'(0) + b'(L)]}{L} - \frac{3[b(0) - b(L)]}{L^2}, \ldots \ldots \ldots (3) \]
\[ B_3 = \frac{[b'(0) + b'(L)]}{L^2} + \frac{2[b(0) - b(L)]}{L^3}. \]
THE INITIAL AND BOUNDARY CONDITIONS

The last section considered the class of boundary configurations to which the ensuing analysis is devoted. This section is concerned with stating explicitly the conditions of stress and displacement which must be satisfied over this boundary, and the initial conditions which will insure the occurrence of the impact phenomenon described in the introduction. Let the points of the body be referred to a cylindrical coordinate system, such that the z-axis coincides with the axis of symmetry, the impact face lies in the plane \( z = 0 \), and the upper face lies in the plane \( z = L \), as shown in Figure 1. Since the problem is one of complete symmetry it is necessary to consider only a planar half-section bounded by the lines \( r = z = 0 \), \( z = L \), and \( r = b(z) \), as shown in Figure 2.

Consider a deformation of the body from its unstressed reference state, characterized by infinitesimal displacements of the points in the plane of each half-section, so that all motion can be referred to the functions \( u(r,z,t) \) and \( w(r,z,t) \) which represent the components of these displacements in the \( r \)- and \( z \)-directions respectively. By virtue of the displacements being infinitesimally small, each point remains essentially at its original position and the rigid wall can be thought of as likewise coinciding with \( z = 0 \), so that the proximity of the body to the rigid wall is determined by the \( w \)-displacements along the impact face \( (z=0) \). Let the displacements thus defined generate a state of
stress (σ_{rr}, σ_{zz}, σ_{θθ}, σ_{rz}) throughout the region occupied by the body according to the linear theory of infinitesimal elasticity, and let the tractions (T_{nn}, T_{nt}) act over its surface.¹

Along the boundary z = 0. Since the present analysis is concerned only with the period during which the elastic body remains in contact with the rigid wall, the w-displacements along z = 0 may be defined as zero over all time. The u-displacements along z = 0 may be determined either by defining them as zero over all time, corresponding to the case of high frictional forces between the body and the rigid wall, or by specifying the T_{rZ}-tractions to vanish for all time, corresponding to the case of zero friction between the body and the rigid wall. The latter condition is the closer approximation to the actual physical situation, and will be used throughout this analysis.

Along the central axis of symmetry r = 0. In order that the material not separate along the central axis, it is necessary that the u-displacements along r = 0 vanish for all time. Likewise, by symmetry the shearing stresses σ_{rZ} must vanish along this central axis for all time.

Along the boundary z = L. It is required that no external loads act upon the body other than those encountered along the impacting face z = 0. Therefore, the tractions T_{zz} and T_{rZ} must vanish along z = L for all time.

¹ See references (5), (6), and (7).
Along the transverse boundary \( r = b(z) \). Again it is required that no external forces act upon the body. To meet this requirement, the normal and tangential components of the applied tractions, \( T_{nn} \) and \( T_{nt} \), respectively, must vanish along the boundary expressed by the cubic equation \( r = b(z) \) for all time.

**Initial conditions.** Consider the necessary conditions on the displacements so that the body — initially in elastic equilibrium — possesses an initial velocity \( V \) with respect to the "rigid wall" (defined to be \( w = 0 \) along \( z = 0 \)), as required to simulate the phenomenon of impact. Such conditions require that the displacements \( u \) and \( w \), and therefore also the stresses, be zero at the initial instant \( t_0 = 0 \), and that the time-rate change of particle displacements be zero in the radial direction, \( u_{,t} = 0 \), and be equal to the impact velocity in the longitudinal direction, \( w_{,t} = -V \), throughout the body at time \( t_0 \).
APPROXIMATE SOLUTION BY NUMERICAL METHODS

From the situation depicted in Figure 1, it is obvious that the present problem is one of axial symmetry and that it will be completely determined by solving for the elastic stresses and infinitesimal displacements occurring in a single half-section, as shown in Figure 2. Standard solutions to certain problems in the theory of elasticity having axially symmetric stress distributions can be found in the literature. These problems are limited to certain classes for which solutions in terms of stress functions have been found. The limitations imposed on these classes of problems usually include the requirements that the boundary of the elastic body coincide with the constant coordinate surfaces and that the conditions imposed on these boundaries be specified solely in terms of either applied tractions or displacements. Moreover, dynamic elastic problems in more than one-dimension are inherently more difficult to solve than the corresponding static problems because of the increased complexity of the governing equations.

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1 See for instance Timoshenko and Goodier (7), articles 26, 27, 29, 30, of chapter 2, and chapter 13.

2 This restriction is a definite limitation since there exists only a finite number of orthogonal coordinate systems to which the classical analytical techniques may be applied. See Morse and Feshbach (16), chapter 5.
Nevertheless, there have been concerted efforts to obtain formal solutions to problems which are similar to, or direct simplifications of, the problem considered in this study. These problems generally fall into one of two categories: (1) those which consider physically realizable boundary conditions, but which simplify the governing equations of motion, as in the Hertz theory,¹ the one-dimensional wave theory,² and the improved one-dimensional theories;³ and (2) those which take into account the "exact" three-dimensional equations of motion, but which require boundary conditions which are at best approximations to physical situations. Such boundaries are usually restricted to infinite⁴ or semi-infinite⁵ cylindrical rods. Until now experimental investigations of the propagation of elastic waves have been limited chiefly to recording the transient strains and displacements along the surfaces of cylindrical rods which are usually long in comparison to their radial dimensions.⁶ Likewise, it has been

1 For a general discussion, see chapter 4 of Goldsmith (38). See also articles 125-127, chapter 13, of Timoshenko and Goodier (7), and article 139 of Love (5).

2 See chapter 15, articles 141-143, of Timoshenko and Goodier (7). See also references (42) and (43).

3 See references (35), (43), (47), (48), (49), and (54).

4 See article 201, page 289, of Love (5). See also references (34), (35), (37), (44), (45), (47), (50), and (51).

5 For an elegant solution, see Skalak, reference (53). See also Curtis, reference (41), for an extension and generalization of Skalak's solution.

6 See references (35), (56), (57), and (61).
found difficult to apply the methods of photoelasticity to such problems.\footnote{1} The interested reader is referred to Goldsmith (38) for an extensive bibliography related to impact and stress wave phenomena.

Since the concern of this investigation is the stress solution of a finite elastic body involving a general transverse boundary, mixed boundary conditions, and dynamical equilibrium, the only recourse left is to consider an approximate solution using numerical techniques. An approximate solution is defined here to be any sufficiently dense set of values for the unknown stresses and displacements, corresponding to discrete points throughout the elastic body, such that some smooth curve passing through these values satisfies the system of governing equations to within a prescribed degree of accuracy. It will be noted that such an approximate solution to the given problem represents a third category which can be compared with the two categories of problem simplifications described on the preceding page.

In general, the best method for devising a numerical solution of elasticity problems involving mixed boundary conditions and/or dynamical equilibrium is by writing the governing equations exclusively in terms of finite differences in the displacements,\footnote{2} and this

\footnote{1}{See pages 255 of Goldsmith (38); also see references (11), (58), (59), (65), and (66).}

\footnote{2}{See Fox (11), and article 94 of Allen (8).}
method will be used throughout the present investigation. Consider the two-dimensional region in the r, z-plane occupied by the half-section shown in Figure 2, and impose upon this region a network of n nodes, based on a square lattice having a constant unit spacing $h = L/N$, such that the network coincides with the three straight sides of the boundary, and such that it extends to all lattice points which include the eight nearest neighbors of those points interior to $r = b(z)$. Assign unknown infinitesimal displacements $u_1(t_j)$ and $w_1(t_j)$ to each of these nodes, where $i = 1, 2, \ldots, n'$ for the $n'$ regular nodes interior to the region considered, where $i = n' + 1, n' + 2, \ldots, n$ for the remaining $n''$ pseudo nodes coinciding with and immediately exterior to the boundary, and where $j = 1, 2, \ldots, m \geq T/\Delta t$ for each of the m discrete instants of time $t_j = j\Delta t$. Here T is the total duration of the impact cycle, as defined in the Notation, and $\Delta t$ is the unit time interval separating each discrete position in time.

Consider the two partial differential equations in the displacements $u$ and $w$, which govern the dynamic equilibrium of forces throughout an elastic body having an axis of symmetry. These equations can be written in finite difference form, which involves replacing the space and time derivatives with approximating expressions in terms of displacements at neighboring nodes and at corresponding nodes evaluated at previous positions in time; two such equations can be assigned to each regular interior node. Likewise, equations for the boundary
conditions can be written in terms of displacements, again using finite
differences to approximate derivatives, and can be solved for the dis-
placements at the pseudo nodes in terms of displacements at the regu-
lar nodes.\textsuperscript{1} The above arguments serve to transform the problem of
satisfying a set of linear differential equations throughout a continuous
region and along its boundary at each instant of time into the approxi-
mating problem of satisfying a system of $2n$ linear algebraic equations
in $2n$ unknown displacements for each of $m$ successive instants of time.
The inclusion of the dynamical aspect does not appreciably complicate
the approximate problem, as it merely adds a constant term to each
of the $2n$ equations depending upon the equilibrium configurations at
preceding instants of time.

Once a sufficiently accurate set of values for the displacements
$u, w$ at each of the $n$ nodes has been determined for a particular instant
of time, a corresponding set of values for the relevant stresses $\sigma_{rr}$,
$\sigma_{zz}$, $\sigma_{\theta\theta}$, $\sigma_{rz}$ may be obtained through Hookean expressions for
stresses in terms of finite differences in the displacements. The to-
tality of sets of values for these stresses evaluated over the specified
time interval $t_m \geq T$ constitutes the complete approximate solution to
the problem defined for this investigation. Approximate solutions to

\textsuperscript{1} For methods of approximating specific boundary value problems
other than the type considered here, the reader is referred to
Allen (8) and Southwell (18), and to the bibliography on related
works given in (8).
systems of simultaneous linear equations are readily obtained by any of various numerical procedures,¹ and if programmed for an electronic digital computer such solutions can be obtained with great speed and precision. An algorithm incorporating the above numerical scheme was devised and programmed for the Rice University megacycle computer,² and the results obtained from this program, a series of approximate solutions as described above, comprise the major contribution of the present study.

Numerical solutions have various shortcomings, mostly related to their inflexibility, and are legitimately criticized on such grounds. An argument raised against numerical solutions to problems of this type is that each particular solution (such as solved by Whitworth³) requires the expenditure of the same time and effort as the initial solution.⁴ This argument applies chiefly to solutions by hand computation, since the major effort devoted to machine solutions is in the initial programming. The generality of a computer program is limited only by the speed and

¹ See for instance chapters 2 and 3 of Householder (13). The first chapter of (13) provides an interesting discussion of sources of errors to be expected in numerical solutions.

² See reference (17).

³ See reference (18).

⁴ See page 1, reference (9).
memory capabilities of the particular machine\(^1\), and by the knowledge, ingenuity, and incentive of the programmer.

A second argument concerns the indefinite accuracy of most numerical solutions, and another closely related argument questions the legitimacy of computing the unknown quantities only at discrete points throughout a region. The logical reply to these arguments is that all mathematical models are more or less idealizations and approximations to some more complicated situation. The numbers used in mathematical formulae are all approximations to some extent, particularly those based upon empirical data — data measured from and applied to a physical problem can never be exact nor complete. If bounds can be specified on the permissible exactitude or completeness that certain information related to a given problem requires in order to be of value, then a numerical procedure can often be formulated to supply the solution to such a problem. "From the practical standpoint a function is sufficiently determined when its values are known within tolerable limits of uncertainty and at a number of points which is large enough to define the trend of its values elsewhere."\(^2\)

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\(^1\) This is a restriction of decreasing significance with modern machines, many of which have clock times of less than \(10^{-6}\) second and immediate access memory capacities exceeding \(10^4\) words.

\(^2\) Southwell (18), page 9.
FINITE DIFFERENCE APPROXIMATIONS

The ensuing analysis is based upon the validity of replacing the actual differential relationships, which govern the displacements throughout the region considered to be an elastic body, by polynomial approximations called finite differences, evaluated at discrete points on a square network of nodes. Various approximate expressions for the derivatives at a point in terms of neighboring points, as well as various polynomial interpolation and extrapolation formulae can be readily developed by means of truncated Taylor's series. The advantages which Taylor's series affords over other methods include the directness in deriving the required expressions and the fact that the remainder terms in the series give an indication of the magnitude of the truncation errors to be expected.¹

A general method is presented below for obtaining finite difference approximations for the kth derivative \( y_0^{(k)} = \frac{d^k}{dx^k} y(x_0) \) of an analytic function \( y = y(x) \) in terms of N values \( y_n = y(x_n) \) located at the points \( x_n = x_0 + nh, \ n = a_1, a_2, \ldots, a_N, \) on an equally spaced network about \( x_0 \) as shown in Figure 4, where \( a_i \) are distinct nonzero integers and where \( k < N. \)

¹ For a further discussion of polynomial interpolation and numerical differentiation see chapters 2 and 3 of Kopal (14).
The Taylor’s series expression for $y_n$ in terms of $y_0 = y_0^{(0)}$ and the derivatives $y_0^{(m)}$, $m = 1, 2, 3, \ldots$, at the point $x_0$ is given by

\[
y_n = y(x_0 + nh) = \sum_{m=0}^{\infty} (nh)^m y^{(m)}/m! = y_0 + y_0'(nh) + y_0''(nh)^2/2 + y_0'''(nh)^3/6 + \ldots . \tag{4}
\]

The general linear form in $y_n = y_{a_i}$, $i = 1, 2, \ldots, N$, can be written

\[
\sum_{i=1}^{N} A_i y_{a_i} = \sum_{i=1}^{N} A_i \sum_{m=0}^{\infty} (a_i h)^m y_0^{(m)}/m! = \sum_{m=0}^{\infty} y_0^{(m)}(h^m/m!) \sum_{i=1}^{N} A_i (a_i)^m
\]

\[
= y_0 \sum_{i=1}^{N} A_i + y_0' h \sum_{i=1}^{N} A_i a_i + y_0''(h^2/2) \sum_{i=1}^{N} A_i (a_i)^2 + \ldots . \tag{5}
\]

Consider the case where $y_0$ is not among the values $y_n$. In order that $y_0$ and $y_0^{(m)}$, $m = 1, 2, \ldots, k-1, k+1, \ldots, N$, do not enter into the expression for $y_0^{(k)}$ it is required that

\[
\sum_{i=1}^{N} A_i = \sum_{i=1}^{N} A_i a_i = \sum_{i=1}^{N} A_i (a_i)^2 = \ldots = \sum_{i=1}^{N} A_i (a_i)^{k-1}
\]

\[
= \sum_{i=1}^{N} A_i (a_i)^k = \ldots = \sum_{i=1}^{N} A_i (a_i)^{N-1} = 0.
\]

This set of $N-1$ linear algebraic equations can be solved for the relative magnitudes of the $N''A_i''$ using Cramer’s rule.\(^1\) Thus

---

\(^1\) For most commonly used finite difference approximations $N$ is a small integer, and Cramer's rule can be used satisfactorily to solve for the $A_i$, as indicated above. In the case that $N$ exceeds four or five, however, any of several other standard elimination techniques, such as the Gauss-Jordan reduction, can be used to obtain the solution to the above set of homogeneous algebraic equations.
where \( i = 1, 2, \ldots, N \).

Having values for \( A_i \), an expression can be derived for \( y_0^{(k)} \) from equation (5) as

\[
y_0^{(k)} = \left[ \frac{k!}{h^k \sum_{i=1}^{N} A_i(a_i)^k} \right] \left[ \sum_{i=1}^{N} A_i y_{a_i} - \sum_{m=0}^{\infty} y_{0}^{(m)} \left( \frac{h^m}{m!} \right) \sum_{i=1}^{N} A_i(a_i)^m \right]
\]

\[= \left[ \frac{k!}{h^k \sum_{i=1}^{N} A_i(a_i)^k} \right] \sum_{i=1}^{N} A_i y_{a_i^i}, \quad \ldots \quad \ldots \quad \ldots \quad . \quad (7)
\]

Interpolation and extrapolation formulae for \( y_0 \) are obtained by setting \( k = 0 \) in equations (6, 7).

For the case where \( y_0 \) is included among the values \( y_n \), the expression for \( A_i \) is obtained similarly to equation (6) and is given by
\[ A_i = \alpha_i \]

where \( \alpha_i = (-1)^{i+1} (a_1 a_2 a_3 \ldots a_{i-1} a_{i+1} \ldots a_N) \) and \( i = 1, 2, \ldots, N \), and the derivative \( y_0^{(k)} \) is given by the formula

\[
y_0^{(k)} = \left[ \frac{k!}{h^k \sum_{i=1}^{N} A_i (a_i)^k} \right] \left[ \sum_{i=1}^{N} A_i y_{a_i} - y_0 \sum_{i=1}^{N} A_i - \sum_{m=N+1}^{\infty} y_0^{(m)} \left( \frac{h^m}{m!} \right) \sum_{i=1}^{N} A_i (a_i)^m \right]
\]

As an example of the use of equations (6-9) consider deriving the four point forward difference approximation for \( y_0' \) in terms of \( y_0, y_1, y_2, \) and \( y_3 \), so that \( k = 2, N = 3, a_1 = 1, a_2 = 2, \) and \( a_3 = 3 \). Since \( y_0 \) is included among \( y_n \), equations (8, 9) must be used. To obtain the coefficients \( A_i, i = 1, 2, 3 \), the above values for \( a_i \) are substituted into equations (8).
\[ A_1 = (-1)^2(a_2a_3) \begin{vmatrix} 1 & 1 \\ a_2 & a_2 \end{vmatrix} = (2)(3)(9 - 4) = 30. \]

\[ A_2 = (-1)^3(a_1a_3) \begin{vmatrix} 1 & 1 \\ a_1 & a_2 \end{vmatrix} = -(1)(3)(9 - 1) = -24. \]

\[ A_3 = (-1)^4(a_1a_2) \begin{vmatrix} 1 & 1 \\ a_1 & a_1 \end{vmatrix} = (1)(2)(4 - 1) = 6. \]

To obtain the expression for \( y''_0 \) these values of \( A_i \) are then introduced into equations (9).

\[
y''_0 = \left[ \frac{2!}{h^2 \sum_{i=1}^{3} A_i (a_i)^2} \right] \left[ -y_0 \sum_{i=1}^{3} A_i + \sum_{i=1}^{3} A_i y a_i - \sum_{m=4}^{\infty} y_0^{(m)} \frac{h^m}{m!} \sum_{i=1}^{3} A_i (a_i)^m \right] 
\]

\[
= h^{-2} \left[ 2y_0 - 5y_1 + 4y_2 - y_3 + (11/12)h^4y^{'''}_0 + \ldots \right] 
\]

\[
\cong h^{-2} \left[ 2y_0 - 5y_1 + 4y_2 - y_3 \right]. 
\]

By proceeding in a similar manner any such finite difference expressions can be obtained. Below are listed some of the more commonly used of such expressions, ones which will find application throughout the remainder of this analysis.

1. **First derivative** \( y'_0 \):

\[
y'_0 = (y_1 - y_{-1})/2h - h^2y^{'''}_0/6 - \ldots \cong (y_1 - y_{-1})/2h. \quad \ldots \ldots \quad (10) 
\]

\[
y'_0 = (3y_0 - 4y_{-1} + y_{-2})/2h + h^2y^{'''}_0/3 + \ldots \quad \ldots \ldots \quad (11) 
\]

\[
\cong (3y_0 - 4y_{-1} + y_{-2})/2h. 
\]
2. Second derivative $y''_0$:

\[ y''_0 = \frac{(y_1 - 2y_0 + y_{-1})}{h^2} - \frac{h^2 y''_0}{12} + \ldots \]
\[ \approx \frac{(y_1 - 2y_0 + y_{-1})}{h^2}. \]  

\[ y''_0 = \frac{(y_0 - 2y_{-1} + y_{-2})}{h^2} + \frac{h y''_0}{12} + \ldots \]
\[ \approx \frac{(y_0 - 2y_{-1} + y_{-2})}{h^2}. \]  

\[ y''_0 = \frac{(2y_0 - 5y_{-1} + 4y_{-2} - y_{-3})}{h^2} + \frac{(11/12)h^2 y''_0}{12} + \ldots \]
\[ \approx \frac{(2y_0 - 5y_{-1} + 4y_{-2} - y_{-3})}{h^2}. \]

3. Extrapolation formulae for $y_0$:

\[ y_0 = 3y_{-1} - 3y_{-2} + y_{-3} + \frac{h^3 y'''_0}{3} + \ldots \]
\[ \approx 3y_{-1} - 3y_{-2} + y_{-3}. \]

\[ y_0 = 6y_{-2} - 8y_{-3} + 3y_{-4} + \frac{4h^3 y'''_0}{3} + \ldots \]
\[ \approx 6y_{-2} - 8y_{-3} + 3y_{-4}. \]

\[ y_0 = 4y_{-1} - 6y_{-2} + 4y_{-3} - y_{-4} + \frac{h^4 y''''_0}{5} + \ldots \]
\[ \approx 4y_{-1} - 6y_{-2} + 4y_{-3} - y_{-4}. \]
In connection with the boundary conditions along the transverse boundary \( b(z) \), it shall be required to evaluate the first derivative \( y' \) of displacement at a point lying a nonintegral distance \( \xi h, \ 0 \leq \xi \leq 1 \), from a node \( x_0 \) on a network having unit spacing \( h \), in terms of the displacements \( y_0, y_{-1}, y_{-2} \) at nodes on either side of \( x_\xi = x_0 - \xi h \). This situation is depicted by Figure 5.

By using a method similar to that presented above for points coinciding with the net points, the following expressions are obtained,

\[
y'_\xi = \left[ (3-2\xi)y_0 - 4(1-\xi)y_{-1} + (1-2\xi)y_{-2} \right]/2h + (2-6\xi + 3\xi^2)h^2y''/6 + \ldots
\]

\[
\approx \left[ (3-2\xi)y_0 - 4(1-\xi)y_{-1} + (1-2\xi)y_{-2} \right]/2h.
\]

\[
y_\xi = (2y_{-1} - y_{-2}) - \xi(y_{-1} - y_{-2}) + (1-\xi)(2-\xi)h^2y'/2 + \ldots
\]

\[
\approx (2-\xi)y_{-1} - (1-\xi)y_2.
\]

Expressions for finite difference approximations to partial derivatives in two or more independent variables are found, in general, by combining the corresponding approximations to
derivatives in single independent variables. As an example consider deriving the expression for the cross derivative $y_{rz}$ to be approximated at the node 0 in terms of neighboring nodes, as designated in Figure 6. Such an expression is found by applying the two-point formula (10) successively to the $r$- and $z$-derivatives as follows:

$$y_{rz} \approx \left[ y_y, r \right]_{z} \approx \left[ (y_y, r)_2 - (y_y, r)_4 \right]/2h$$

$$\approx \left[ (y_5 - y_6)/2h - (y_8 - y_7)/2h \right]/2h \quad \ldots \ldots \ldots \quad (20a)$$

$$\approx (y_5 - y_6 + y_7 - y_8)/4h^2.$$  

$$y_r \approx (y_1 - y_3)/2h, \quad \ldots \ldots \ldots \ldots \quad (20b)$$

$$y_z \approx (y_2 - y_4)/2h, \quad \ldots \ldots \ldots \ldots \quad (20c)$$

$$y_{rr} \approx (y_1 - 2y_0 + y_3)/h^2, \quad \ldots \ldots \ldots \ldots \quad (20d)$$

$$y_{zz} \approx (y_2 - 2y_0 + y_4)/h^2, \quad \ldots \ldots \ldots \ldots \quad (20e)$$

Likewise, the forward difference approximation for the second time derivative is given, in analogy with equation (13), by

$$y_{tt} \approx (y_0 - 2y_0' + y_0'')/(\Delta t)^2, \quad \ldots \ldots \ldots \ldots \quad (20f)$$

where $y_0$ is the value of $y(t)$ evaluated at node 0 and at the current instant of time $t_j$, and where $y_0'$ and $y_0''$ are the values of $y$ corresponding to node 0 at the two previous instants $t_{j-1}$ and $t_{j-2}$, respectively.
THE GOVERNING EQUATIONS

The arguments presented in this section comprise a derivation of the governing equations of motion in terms of finite differences in the displacements for axisymmetric stress distributions in bodies of revolution. The equations of equilibrium in terms of stresses and the equations expressing the strains in terms of displacements in cylindrical coordinates for axial symmetry are to be found in the literature. Using the notation introduced previously, these expressions may be written

\[ \sigma_{rr},r + \sigma_{rz},z + (\sigma_{rr} - \sigma_{\theta\theta})/r = F_r, \]
\[ \sigma_{rr},r + \sigma_{zz},z + \sigma_{rz}/r = F_z, \]

and

\[ e_{rr} = u_r, \quad e_{zz} = w_z, \]
\[ e_{\theta\theta} = u/r, \quad e_{rz} = (u_z + w_r)/2. \]

Using the generalized Hooke's relationships between stresses and strains for an isotropic elastic material,\(^{2}\)

\[ \sigma_{ij} = 2\mu e_{ij} + \lambda \delta_{ij} e_{kk}, \]

where \( \lambda \) and \( \mu \) are Lame's elastic constants, \( \delta_{ij} \) is the Kronecker

---

1 See, for instance, expressions (177, 178), page 343, of Timoshenko and Goodier (7).

2 See equation (22.3), page 66, Sokolnikoff (6).
delta, and $e_{kk}$ is the dilatation, together with equations (22) above, the following expressions are obtained for the stresses in terms of displacements.

\[
\sigma_{rr} = (\lambda + 2\mu) e_{rr} + \lambda (e_{zz} + e_{\theta\theta}) = \gamma u_{,r} + \lambda (w_{,z} + u_{/r}),
\]

\[
\sigma_{zz} = (\lambda + 2\mu) e_{zz} + \lambda (e_{rr} + e_{\theta\theta}) = \gamma w_{,z} + \lambda (u_{,z} + u_{/r}),
\]

\[
\sigma_{\theta\theta} = (\lambda + 2\mu) e_{\theta\theta} + \lambda (e_{rr} + e_{zz}) = \gamma u_{/r} + \lambda (u_{,r} + w_{,z}),
\]

\[
\sigma_{rz} = 2\mu e_{rz} = \mu (u_{,z} + w_{,r}).
\]

Here $\gamma = \lambda + 2\mu$ for brevity in writing the equations.

Substituting equations (24) into equations (21), and replacing the body forces $F_r$ and $F_z$ by "inertia forces", the equations of motion are obtained in terms of displacements,

\[
\gamma [(ru)_{,r}/r]_{,r} + 4\gamma' w_{,rz} + \mu u_{,zz} = \rho u_{,tt},
\]

\[
\gamma w_{,zz} + 4\gamma' (ru)_{,z},_{r}/r + \mu (rw_{,r})_{,r}/r = \rho w_{,tt},
\]

where $\gamma' = (\lambda + \mu)/4$ and $\rho$ is the mass density of the material.

Equations (25) are essentially the same as those given by equations (52), page 143, of Love,\(^1\) as follows.

\[
(\lambda + 2\mu) \partial \Delta / \partial r + 2\mu \partial \omega / \partial z + \rho F_r = \rho f_r,
\]

\[
(\lambda + 2\mu) \partial \Delta / \partial z - (2\mu/r) \partial (r\omega) / \partial r + \rho F_z = \rho f_z,
\]

\(^1\) Reference (5).
where \( \Delta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} \) is the dilatation and \( 2 \varpi = \frac{\partial u}{\partial z} \) - \( \frac{\partial w}{\partial r} \) is the rotation, where \( f_r = \frac{\partial^2 u}{\partial t^2} \) and \( f_z = \frac{\partial^2 w}{\partial t^2} \) are the accelerations, and where the body forces \( F_r \) and \( F_z \) can be neglected.

The equations of motion (25) may be transformed into expressions containing finite difference approximations for a node 0 on the square network numbered as shown in Figure 6. Substituting for the appropriate expressions from among the finite difference approximations (20), equations (25) become

\[
R_0 = \gamma (1+B)u_1 + \gamma (1-B)u_3 + \mu (u_2 + u_4) - (\gamma'' + 4\gamma B^2)u_0 \\
\quad + \rho'(2u_0' - u_0'') + \gamma'(w_5 - w_6 + w_7 - w_8),
\]

\[
Z_0 = \mu (1+B)w_1 + \mu (1-B)w_3 + \gamma (w_2 + w_4) - \gamma''w_0 \\
\quad + \rho'(2w_0' - w_0'') + \gamma'(u_5 - u_6 + u_7 - u_8) + 4\gamma B \gamma'(u_2 - u_4),
\]

where \( \gamma'' = 2(\lambda + 3\mu) + \rho' \), \( \rho' = \rho (h/\Delta t)^2 \), and \( B = h/2r \). The quantities \( R_0 \) and \( Z_0 \) are defined to be residuals, the absolute values of which, in order to obtain approximate equilibrium among the displacements at nodes surrounding a node 0, must be reduced to less than some prescribed permissible deviation \( D \).

In discussing the Gauss-Seidel iteration process, which is a systematic procedure for liquidating \( R_0 \) and \( Z_0 \), it will be necessary to consider the relationships for varying \( R_0 \), and therefore also \( R_1 \), \( R_2 \), \( R_3 \), \( R_4 \), by perturbing the value of \( u_0 \); and likewise the relationships
for varying $Z_0$, and therefore also $Z_1$, $Z_2$, $Z_3$, $Z_4$, by perturbing the value of $w_0$ (see Figure 6, for the relative locations of the nodes).

The expressions for the variation dependence of these quantities are found from equations (26) to be

\[
\begin{align*}
\Delta R_0 &= -KR_0, \\
\Delta u_0 &= KR_0/(\gamma'' + 4\gamma B^2), \\
\Delta R_1 &= \gamma (1-B)\Delta u_0, \\
\Delta R_2 &= \Delta R_4 = \mu \Delta u_0, \\
\Delta R_3 &= \gamma (1+B)\Delta u_0, \\
\Delta Z_0 &= -KZ_0, \\
\Delta w_0 &= KZ_0/\gamma'', \\
\Delta Z_1 &= \mu (1-B)\Delta w_0, \\
\Delta Z_2 &= \Delta Z_4 = \gamma \Delta w_0, \\
\Delta Z_3 &= (1-B)\Delta w_0.
\end{align*}
\]

where $K$ is an overrelaxation parameter ($K \approx 1$).

It should be emphasized that the foregoing governing residual equations (26) apply only for regular interior nodes (nodes having at least eight nearest neighbors). For pseudo nodes (see Figure 2), values for the displacements must be supplied from boundary condition equations and from parabolic extrapolation formulae expressed in terms of displacements at interior nodes, as derived in the next section.

Expressions for evaluating the stresses. The finite difference expressions necessary to evaluate the stresses at the fictitious points 0, corresponding to the spaces between the finite difference nodes as shown in Figure 7, from

Figure 7
a set of equilibrium displacements, are derived as follows. Special approximations involving only four nodes are used in the interest of localizing as much as possible the propagation of disturbances in both the space and time dimensions.

\[ 2h y, r \approx y_1 - y_2 - y_3 + y_4, \]
\[ 2h y, z \approx y_1 + y_2 - y_3 - y_4, \]
\[ 4h y_0 \approx y_1 + y_2 + y_3 + y_4. \]

Substituting these expressions into the equations (24) for stresses in terms of displacements, the required expressions for the stresses are obtained.

\[ 2h \sigma_{rr} \approx (\gamma + \lambda B') (u_1 + u_4) - (\gamma - \lambda B') (u_2 + u_3) + \lambda (w_1 + w_2 - w_3 - w_4), \]
\[ 2h \sigma_{zz} \approx \lambda (1 + B') (u_1 + u_4) - \lambda (1 - B') (u_2 + u_3) + \gamma (w_1 + w_2 - w_3 - w_4), \]
\[ 2h \sigma_{\theta \theta} \approx (\lambda + \gamma B') (u_1 + u_4) - (\lambda - \gamma B') (u_2 + u_3) + \lambda (w_1 + w_2 - w_3 - w_4), \]
\[ 2h \sigma_{rz} \approx \mu (u_1 + u_2 - u_3 - u_4 + w_1 - w_2 - w_3 + w_4), \text{ where } B' = B_2 / (1 + B_2). \]
THE BOUNDARY CONDITION EQUATIONS

In this section finite difference expressions will be derived for satisfying approximately the boundary conditions — stated explicitly in a previous section — at the points on the boundary nearest the pseudo nodes, by expressing the displacements at these pseudo nodes exclusively in terms of displacements at regular interior nodes. For the straight boundaries, \( r = 0, \ z = 0, \) and \( z = L, \) the pseudo nodes lie on the boundary and therefore coincide with the points at which the boundary conditions are to be satisfied. The order of presenting the derivations in what follows is such as to prevent ambiguous or meaningless expressions for displacements at the pseudo nodes.

Along \( z = 0. \) Designate the nodes adjacent to a pseudo node 0 lying on the boundary \( z = 0 \) as shown in Figure 8. The conditions which must be satisfied along \( z = 0 \) are that \( w = 0 \) and \( T_{rz} = 0. \) The first condition, \( w = 0, \) can be satisfied directly at the node 0 since the dependent variables of the problem are the displacements. The latter condition, \( T_{rz} = \sigma_{rz0} = 0, \) can be satisfied by substituting finite difference approximations, involving the numbered nodes of Figure 8, for the derivatives appearing
in equation (24d), and setting the resulting expression to zero, as

\[
\frac{2h(u)}{j} \sigma_{rz0} = 0 = 2h(u) + 2h(w, r) = 3u_0 - 4u_3 + u_4.
\]

Therefore:

\[
w_0 = 0, \quad u_0 = \frac{(4u_3 - u_4)}{3}, \quad \ldots \ldots \ldots \ldots (30)
\]

for each pseudo node 0 lying along \( z = 0, \quad 0 \leq r \leq b(0) \).

**Along \( r = 0 \).** Designate the nodes adjacent to a pseudo node 0 lying on the boundary \( r = 0 \) as shown in Figure 9. The boundary conditions along \( r = 0 \) are \( u = 0 \) and \( \sigma_{rz} = 0 \), so that the required finite difference expressions, obtained similarly as those above are

\[
u_0 = 0, \quad w_0 = \frac{(4w_3 - w_4)}{3}, \quad \ldots \ldots \ldots \ldots (31)
\]

for each pseudo node 0 lying on \( r = 0, \quad 0 < z < L \).

**Along \( z = L \).** Designate the nodes adjacent to a pseudo node 0 lying on the boundary \( z = L \) as shown in Figure 10. The boundary conditions along \( z = L \) are \( T_{zz} = T_{rz} = 0 \), and can be satisfied, as above, by
introducing suitable finite difference approximations involving the
nodes of Figure 10 into the expressions (24b) and (24d), and equat-
ing the resulting expressions to zero.

For the specific node A along the central axis, \( u_A = 0 \), so
that \( w_A \) can be obtained from (24b) as

\[
\frac{2h}{Y} \sigma_{zz}^A = 0 = 2h(w, z)_A + 2h(\lambda/Y)(u_r + u/r)_A
\]
\[\approx 3w_A - 4w_4 - w_8.\]

Therefore: \( u_A = 0, \quad w_A \approx (4w_4 - w_8)/3. \quad \cdots \cdots \quad (32) \)

For the specific node B, the following finite difference ap-
proximations can be derived using equations (10) and (11).

\[
2h(u, r)_B \approx 2(2hu, r)_4 - (2hu, r)_8 \approx 2u_3 - u_7, \]
\[2h(w, r)_B \approx 2(2hw, r)_4 - (2hw, r)_8 \approx 2(w_3 - w_5) - (w_7 - w_9), \]
\[2h(y, z)_B \approx 3y_B - 4y_4 + y_8. \]

Substituting these expressions into equations (24b) and (24d),

\[
\frac{2h}{Y} \sigma_{zz}^B = 0 = (2hw, z)_B + 2(\lambda/Y)(hu, r + u)_B
\]
\[\approx 3w_B - 4w_4 + w_8 + (\lambda/Y)(2u_3 - u_7 + 4u_4 - 2u_8), \]
\[
\frac{2h}{\mu} \sigma_{rz}^B = 0 = (2hu, z)_B + (2hw, r)_B
\]
\[\approx 3u_B - 4u_4 + u_8 + 2(w_3 - w_5) - (w_7 - w_9), \]

and solving for \( w_B \) and \( u_B \),
$w_B \simeq \left[ 4w_4 - w_8 + \left( \lambda / \gamma \right)(u_7 + 2u_8 - 2u_3 - 4u_4) \right] / 3,$

$u_B \simeq \left[ 4u_4 - u_8 + w_7 - 2w_3 + 2w_5 - w_9 \right] / 3.$

For the remaining pseudo nodes $0$ along $z = L$, the following finite difference approximations are assumed.

$$2h(u, r)_0 \simeq 2(3u_4 - 4u_5 + u_6) - (3u_8 - 4u_9 + u_{10}),$$

$$2h(w, r)_0 \simeq 3w_0 - 4w_1 + w_2,$$

$$2h(y, z)_0 \simeq 3y_0 - 4y_4 + y_8.$$  

Substituting these approximations into equations (24b) and (24d),

$$\left( \frac{2h}{\gamma} \right) \sigma_{zz_0} = 0 = (2hw, z)_0 + \left( \frac{\lambda}{\gamma} \right)(2hu, r + 2hu/r)_0$$

$$= 3w_0 - 4w_3 + w_8 - \left( \frac{\lambda}{\gamma} \right)(3 + 4B)(2u_4 - u_8)$$

$$- \left( \frac{\lambda}{\gamma} \right)(4u_9 - u_{10} - 8u_5 + 2u_0),$$

$$\left( \frac{2h}{\mu} \right) \sigma_{rz_0} = 0 = (2hu, z)_0 + (2hw, r)_0$$

$$= 3u_0 - 4u_4 + u_8 + 3w_0 - 4w_1 + w_2,$$

and solving for $w_0$ and $u_0$,

$$w_0 \simeq \left[ 4w_4 - w_8 + \left( \lambda / \gamma \right)(3 + 4B)(u_8 - 2u_4) \right.$$  

$$+ \left( \lambda / \gamma \right)(8u_5 - 2u_6 - 4u_9 + u_{10}) \right] / 3,$$

$$u_0 \simeq \left[ 4u_4 - u_8 - 3w_0 + 4w_1 - w_2 \right] / 3,$$

for each pseudo node $0$ on $z = L$, $3h \leq r \leq b(L)$.
Along \( r = b(z) \). Consider a triangular element of volume of unit thickness, as shown in Figure 11, whose sides \( S_1 \) and \( S_2 \) lie in the \( r,z \)-directions and whose hypotenuse \( S \) coincides with a portion of the transverse boundary \( b(z) \). Since the boundary conditions on \( b(z) \) require that the normal and tangential tractions, \( T_{nn} \) and \( T_{nt} \), must vanish, all forces acting on \( S \) can be ignored.

From force equilibrium, equations are obtained which relate the stresses and the slope of the transverse boundary. Summing forces in the \( r \)- and \( z \)-directions, respectively,

\[
\sum F_r = 0 = S_1 \sigma_{rz} - S_2 \sigma_{rr} = (S \sin \psi) \sigma_{rz} - (S \cos \psi) \sigma_{zz},
\]

\[
\sigma_{rr} = \tan \psi \sigma_{rz} = b' \sigma_{rz}, \quad \ldots \ldots \ldots \ldots (35)
\]

and

\[
\sum F_z = 0 = S_1 \sigma_{zz} - S_2 \sigma_{rz} = (S \sin \psi) \sigma_{zz} - (S \cos \psi) \sigma_{rz},
\]

\[
\sigma_{rz} = \tan \psi \sigma_{zz} = b' \sigma_{zz}, \quad \ldots \ldots \ldots \ldots (36)
\]

Here it has been observed that \( b' = \tan \psi \). Combining (35) and (36), a third expression is obtained,

\[
\sigma_{rr} = (b')^2 \sigma_{zz}. \quad \ldots \ldots \ldots \ldots (37)
\]
By writing equations (36) and (37) in terms of displacements and solving the resulting expressions for $u, r$ and $w, r$, it will always be possible to derive finite difference expressions for $u_0$ and $w_0$ at the nearest pseudo node, since the transverse boundary is a single-valued function of $z$ and its pseudo nodes are therefore uniquely accessible along radial strings. Substituting equations (24a) and (24b) into equation (37),

$$\gamma u, r + \lambda w, z + \lambda u/b = (b')^2(\gamma w, z + \lambda u, r + \lambda u/b),$$

or

$$-u, r = \beta w, z + \alpha u/h,$$

or

$$(38)$$

where $\alpha = 2B''(1-b'^2)/(\gamma - \lambda b'^2), \beta = (\lambda - \gamma b'^2)/(\gamma - \lambda b'^2)$, and $B'' = h/2b$. Substituting equations (24b) and (24d) into equation (36),

$$\mu (w, r + u, z) = b'(\gamma w, z + \lambda u, r + \lambda u/b),$$

and eliminating $u, r$ by substituting from equation (38),

$$-w, r = u, z + \beta' w, z + \alpha' u/h,$$

or

$$(39)$$

where $\alpha' = b'/(\alpha - 2B'')$, $\beta' = b'/(\beta \lambda - \gamma)/\mu$. 

In general, the pseudo nodes $[r_0, z_0]$ do not coincide with the boundary $b(z)$. It was implied, however, in deriving (38) and (39) that the derivatives be evaluated at the boundary points $[r = b(z_0) = r_0 - \xi h, z = z_0]$ defined by the intersection of $b(z)$.
with radial strings of the finite difference network, as shown in Figure 12. Finite difference approximations to these derivatives must likewise be evaluated at the points \([b(z_0), z_0]\), and, to be unique, must involve \(u_0\) and \(w_0\) and only those displacements among \(u_i\) and \(w_i\), \(i = 1, 2, \ldots, 12\), which lie interior to the transverse boundary.

For the finite difference expressions to contain only displacements at \([r_0, z_0]\) and at nodes interior to the boundary, it is necessary to recognize several distinct boundary classifications depending upon which of the twelve nodes shown in Figure 12 lie inside the boundary curves \(z = 0\), \(z = L\), and \(r = b(z)\). Five such categories, denoted by \(a, b, c, d, e\), are defined according to the following logic diagram, where \(m_i, n_i\),
i = 0, 1, 2, 3, 4, are integers defined to be the radial and axial coordinates, respectively, in unit spacings h, of the pseudo nodes, as shown in Figure 13.

To determine into which particular category a certain pseudo node belongs, it is necessary merely to evaluate the logical sequence shown in Figure 14. An arrow extending horizontally from a magnitude comparison indicates the path to be taken if the comparison is fulfilled.

The required finite difference expressions are obtained by substituting finite difference approximations relevant to each particular category into equations (38) and (39), and solving for \( u_0 \) and \( w_0 \). The finite difference approximations,
which are variants of equations (18) and (19), apply for each category.}

Upon inserting the above expressions into equations (38) and (39), the following equations are obtained.

\[
\begin{align*}
\mathbf{u}_0 & = a_7 \mathbf{u}_7 + a_{11} \mathbf{u}_{11} + b_7' (-2hw, z)_{7} + b_{11}' (2hw, z)_{11}, \\
\mathbf{w}_0 & = c_7 \mathbf{u}_7 + c_{11} \mathbf{u}_{11} + c_7' (-2hu, z)_{7} + c_{11}' (2hu, z)_{11}, \\
& \quad + d_7 \mathbf{w}_7 + d_{11} \mathbf{u}_{11} + d_7' (-2hw, z)_{7} + d_{11}' (2hw, z)_{11},
\end{align*}
\]

where the coefficients are given by

\[
\begin{align*}
a_7 &= \frac{[4(1-\xi) - 2\alpha(2-\xi)]}{(3-2\xi)}, \\
a_{11} &= \frac{[-1 + 2\xi + 2\alpha(1-\xi)]}{(3-2\xi)}, \\
c_7' &= \frac{(2-\xi)}{(3-2\xi)}, \quad c_{11}' = \frac{(1-\xi)}{(3-2\xi)}, \\
b_7' &= \mathcal{B} c_7', \quad b_{11}' = \mathcal{B} c_{11}', \\
d_7' &= \mathcal{B}' c_7', \quad d_{11}' = \mathcal{B}' c_{11}', \\
c_7 &= -2\alpha' c_7', \quad c_{11} = 2\alpha' c_{11}', \\
d_7 &= 4c_7', \quad d_{11} = -(1-2\xi)/(3-2\xi).
\end{align*}
\]

Following are listed the finite difference approximations for \((y, z)_{7}\) and \((y, z)_{11}\), and the final expressions for \(u_0\) and \(w_0\) obtained by substituting these approximations into equations (40), for each of the five categories discussed above.
(a) Neutral slope: \((-2hy, z)_{7} \approx y_{6} - y_{8}; (2hy, z)_{11} \approx y_{12} - y_{10}\).

\[ u_{0} \approx a_{7}u_{7} + a_{11}u_{11} + b_{7}^{1}(w_{6} - w_{8}) + b_{11}^{1}(w_{12} - w_{10}), \]

\[ w_{0} \approx c_{7}u_{7} + c_{11}u_{11} + c_{1}^{1}(u_{6} - u_{8}) + c_{11}^{1}(u_{12} - u_{10}) \ldots \ldots (41) \]

\[ + d_{7}w_{7} + d_{11}w_{11} + d_{7}^{1}(w_{6} - w_{8}) + d_{11}^{1}(w_{12} - w_{10}). \]

(b) Weak negative slope: \((hy, z)_{7} \approx y_{7} - y_{6}; (hy, z)_{11} \approx y_{11} - y_{10}\).

\[ u_{0} \approx a_{7}u_{7} + a_{11}u_{11} + 2b_{7}^{1}(w_{6} - w_{7}) + 2b_{11}^{1}(w_{11} - w_{10}), \]

\[ w_{0} \approx (c_{7} - 2c_{7}^{1})u_{7} + (c_{11} + 2c_{11}^{1})u_{11} + 2c_{1}^{1}u_{6} - 2c_{11}^{1}u_{10} \ldots \ldots (42) \]

\[ + (d_{7} - 2d_{7}^{1})w_{7} + (d_{11} + 2d_{11}^{1})w_{11} + 2d_{1}^{1}w_{6} - 2d_{11}^{1}w_{10}. \]

(c) Weak positive slope: \((hy, z)_{7} \approx y_{8} - y_{7}; (hy, z)_{11} \approx y_{12} - y_{11}\).

\[ u_{0} \approx a_{7}u_{7} + a_{11}u_{11} + 2b_{7}^{1}(w_{7} - w_{8}) + 2b_{11}^{1}(w_{12} - w_{11}), \]

\[ w_{0} \approx (c_{7} - 2c_{7}^{1})u_{7} + (c_{11} - 2c_{11}^{1})u_{11} - 2c_{1}^{1}u_{8} + 2c_{11}^{1}u_{12} \ldots \ldots (43) \]

\[ + (d_{7} - 2d_{7}^{1})w_{7} + (d_{11} - 2d_{11}^{1})w_{11} - 2d_{1}^{1}w_{8} + 2d_{11}^{1}w_{12}. \]

(d) Strong negative slope:

\[ (-2hy, z)_{7} \approx -3y_{7} + 4y_{6} - y_{5}; (2hy, z)_{11} \approx y_{12} - y_{10}. \]

\[ u_{0} \approx a_{7}u_{7} + a_{11}u_{11} + b_{7}^{1}(-3w_{7} + 4w_{6} - w_{5}) + b_{11}^{1}(w_{12} - w_{10}), \]

\[ w_{0} \approx (c_{7} - 3c_{7}^{1})u_{7} + c_{11}u_{11} + c_{1}^{1}(4u_{6} - u_{5}) + c_{11}^{1}(u_{12} - u_{10}) \]

\[ + (d_{7} - 3d_{7}^{1})w_{7} + d_{11}w_{11} + d_{7}^{1}(4w_{6} - w_{5}) + d_{11}^{1}(w_{12} - w_{10}). \]
(e) Strong positive slope:

\[-2hy, z \] \( \simeq 3y_7 - 4y_8 + y_9; \ (2hy, z) \}_{11} \simeq y_{12} - y_{10}. \]

\[ u_0 \simeq a_7u_7 + a_{11}u_{11} + b'_7(3w_7 - 4w_8 + w_9) + b'_{11}(w_{12} - w_{10}), \]

\[ w_0 \simeq (c_7 + 3c'_7)u_7 + c_{11}u_{11} + c'_7(u_9 - 4u_8) + c'_{11}(u_{12} - u_{10}) \]

\[ + (d_7 + 3d'_7)w_7 + d_{11}w_{11} + d'_7(w_9 - 4w_8) + d'_{11}(w_{12} - w_{10}). \] (45)

It will be noted that (41) through (45) are linear algebraic expressions for \( u_0 \) and \( w_0 \) at the pseudo nodes \( [r_j, z_j] = jh, j = 1, 2, \ldots , N-1 \), in the displacements \( u_i \) and \( w_i \) at the regular nodes \( [r_i, z_i], i = 1, 2, \ldots , 12 \), whose locations relative to \( [r_0, z_0] \) are indicated in Figure 12. Therefore,

\[ u_0 \simeq \sum_{i=1}^{12} (A_iu_i + B_iw_i), \quad w_0 \simeq \sum_{i=1}^{12} (C_iu_i + D_iw_i), \ldots \ldots \] (46)

for each of the \( N-1 \) pseudo nodes \( [r_0, z_0] \) along \( r = b(z) \), where the coefficients \( A_i, B_i, C_i, D_i \) of \( u_i \) and \( w_i \) are determined from (41) through (45). Equations (46) represent the relationships, second order correct wherever possible, necessary for satisfying the zero stress conditions along the transverse boundary.

**Extrapolation formulae.** The residual equations (26) involve, in general, not only the pseudo nodes \( [r_0, z_0] \) immediately exterior to \( r = b(z) \), but often involve pseudo nodes \( [r_a, z_a] \) and \( [r_b, z_b] \) located radially beyond \( [r_0, z_0] \) (see Figure 12), especially for points where the boundary slope \( b'(z) \) is large. The additional relationships
for providing values of displacements at the pseudo nodes \([r_a, z_a]\) and \([r_b, z_b]\) are obtained by considering parabolic extrapolation formulae similar to equations (15) and (16). Thus,

\[
y_a \approx 3(y_0 - y_7) + y_{11}, \quad y_b \approx 6y_0 - 8y_7 + y_{11}, \quad \ldots \ldots (47)
\]

for each of the 2(N+1) pseudo nodes \([r_a, z_a]\) and \([r_b, z_b]\) along \(r = b(z)\). Here, as in previous derivations, \(y\) represents a general dependent variable, and can be taken to mean either \(u\) or \(w\).

The problem of satisfying force equilibrium along a general curved boundary in two-dimensions in terms of displacements at the nodes of a square finite difference network was discussed by Leslie Fox.\(^1\) The equations derived above for satisfying the zero-stress boundary conditions along \(r = b(z)\) are direct extensions of the methods described in Fox's paper.

\(^1\) Reference (11).
THE NUMERICAL PROCEDURE

The last two sections presented the mathematical formulation necessary to obtain, for a given instant of time \( t_j \), approximate dynamical equilibrium at each of the \( n' \) regular nodes (by satisfying the residual equations (26)) and at each of the \( n'' \) pseudo nodes (by satisfying the boundary condition equations (30-34, 46, 47)). This approximate formulation was found to consist of two linear algebraic equations for the displacements \( u \) and \( w \) at each of the \( n = n' + n'' \) nodes of the finite difference network (see Figure 2) in terms of the displacements at neighboring nodes. The present section describes the procedure by which the closed system of \( 2mn \) linear equations, which results from considering the total \( n \) nodes for each of the total \( m \) time steps, can be satisfied simultaneously.

To facilitate handling the dependent variables and the coefficients of the boundary condition equations, the following matrices will be used. The elements of the matrices containing dependent variables are considered to correspond directly to the nodes of the finite difference network, which was defined to have a total of \( N + 1 \) nodes in the longitudinal direction and a maximum of \( M + 1 \) nodes in the radial direction.

1. Consider six \((M+1) \times (N+1)\) matrices \( U, W, U', W', U'', W'' \), each of whose \( n \) nonzero elements represent the
displacements \( u_i, w_i, u'_i, w'_i, w''_i \), respectively at each of the finite difference nodes.

2. Consider two \((M-1) \times (N-1)\) matrices \( R \) and \( Z \), each of whose \( n' \) nonzero elements represent the residuals \( R_i \) and \( Z_i \), respectively, at each of the regular nodes, as defined by equations (26).

3. Consider four \( M \times N \) matrices \( RR, ZZ, \theta \theta, RZ \), each of whose approximately \( n \) nonzero elements represent the stresses \( \sigma_{rr}, \sigma_{zz}, \sigma_{\theta \theta}, \sigma_{rz} \), respectively, at each of the spaces between the finite difference nodes, as defined by equations (29).

4. Consider four \( 12 \times (N-1) \) matrices \( A, B, C, D \), each of whose \( N-1 \) nonzero rows contain the coefficients \( A_i, B_i, C_i, D_i \), respectively, of the transverse boundary condition equations (46) corresponding to each of the pseudo nodes along \( r = b(z) \).

The boundary condition equations are independent of time and the residual equations depend only upon the sets of equilibrium displacements at the two preceding instants. Therefore, the complete solution of the title problem can be obtained by progressing successively from one position in time to the next, solving the system of \( 2n \) algebraic equations for the elements of \( U \) and \( W \) in terms of the elements of \( U', W', U'', \) and \( W'' \) at each of the time positions, until
the total interval \( t_m = m\Delta t \) has been spanned. The elements of \( U \) and \( W \) at each position in time will be considered sufficiently correct if they simultaneously satisfy the approximate boundary condition equations and provide elements of \( R \) and \( Z \) whose magnitudes are all smaller than some permissible deviation \( D \). The stresses need be calculated at only those instants of time \( t_j^k = j\Delta t, j = 1,2,\ldots,m/k \), where \( k \) is a positive integer, which are sufficiently separated to provide a descriptive pattern of successive stress states.

The procedure for determining the approximate solution as described above is defined systematically by the following sequence of events.\(^1\) Assume that values have been prescribed for all the constants, including the elements of \( A, B, C, D \), required for operating on the dependent-variable matrices, such that these given constants determine a system of linear equations in \( u \) and \( w \) that is stable and will converge to the proper solution under the procedure outlined below.\(^2\)

1. Specify the initial conditions. A beginning for the numerical procedure is provided by specifying the sets of displacements in

\(^1\) See section 91, page 127 of Allen (8) for a description of an equivalent relaxation procedure.

\(^2\) The problems of determining stability criterion and estimates for convergence rates and error bounds will not be discussed here. The interested reader is referred to the appropriate sections of Hildebrand (12), Householder (13), Kopal (14), and Lowan (15), and especially to the comprehensive survey by Forsythe and Wasow (10).
U', W', and U'', W'', for the instants \( t_0 = 0 \) and \( t_{-1} = -\Delta t \), respectively, according to the initial conditions described on page 11, and proceeding to solve for the displacements in U and W corresponding to the time \( t_1 = \Delta t \). The initial conditions require that \( u = w = u_t = 0 \) and \( w_t = -V \) at time \( t_0 \), and these conditions can be satisfied approximately by letting

\[
\begin{align*}
  u' &= w' = u'' = 0, \\
  w'' &= V\Delta t,
\end{align*}
\]  

(48)

for each of the elements of \( U', W', U'', \) and \( W'' \), respectively. As an initial estimate for the displacements contained in the matrices U and W, all their elements are likewise set to zero.

(2) Satisfy the boundary conditions. (a) Solve for \( u_0 \) and \( w_0 \) at each of the pseudo nodes along \( z = 0 \) by using equations (30). (b) Solve for \( u_0 \) and \( w_0 \) at each of the pseudo nodes along \( r = 0 \) by using equations (31). (c) Solve for \( u_A \) and \( w_A \), \( u_B \) and \( w_B \), and \( u_0 \) and \( w_0 \) at each of the pseudo nodes along \( z = L \) by using equations (32), (33), and (34), respectively. (d) Solve for \( u_0 \) and \( w_0 \) at each of the pseudo nodes immediately exterior to \( r = b(z) \) by using equations (46), whose coefficients are contained in the matrices A, B, C, D. (e) Solve for \( u_a \), \( w_a \), and \( u_b \), \( w_b \) at each of the pseudo nodes one and two network spacings exterior to those encountered in (d) by using the parabolic extrapolation formulae (47). If step (4) immediately preceded step (2), transfer to step (5).
(3) Assign and check the R-residuals. By using equation (26a), calculate the values of the residuals \( R_i \) for each of the regular nodal elements of \( R \), and compute the average magnitude \( |\overline{R}| = \frac{1}{n_1} \sum_{i=1}^{n_1} |R_i| \) of these residuals. If the magnitudes \( |R_i| \) of the residuals so calculated are all less than the permissible deviation \( D \), transfer to step (7).

(4) Liquidate the R-residuals. Equations (27) were derived in order to obtain approximate equilibrium throughout the body by liquidating the residuals \( R_i \) and \( Z_i \), which are originally large by virtue of the approximate initial estimates for \( u_i \) and \( w_i \). By proceeding systematically from one element to another throughout the matrices \( U \) and \( R \), varying \( u_0 \) at each element by an amount depending upon \( R_0 \), and obtaining the corresponding variations of \( R_0 \), \( R_1 \), \( R_2 \), \( R_3 \), and \( R_4 \) (see Figure 6) according to equations (27a), the magnitude of every element in \( R \) will tend toward zero. Repeat this process until the magnitudes of all elements of \( R \) are less than ten percent of the average magnitude \( |\overline{R}| \) calculated in step (3), and return to step (2).

(5) Assign and check the Z-residuals. By using equation (26b), calculate the value of the residual \( Z_i \) for each of the regular nodal elements of \( Z \), and compute the average magnitude \( |\overline{Z}| = \frac{1}{n_1} \sum_{i=1}^{n_1} |Z_i| \) of these residuals. If the magnitudes \( |Z_i| \) of the residuals so calculated are all less than the permissible deviation \( D \), transfer to step (7).
Liquidate the Z-residuals. By proceeding systematically from one element to another throughout the matrices W and Z, varying \( w_0 \) at each element by an amount depending upon \( Z_0 \) and obtaining the corresponding variations of \( Z_0, Z_1, Z_2, Z_3, \) and \( Z_4 \), according to equations (27b), the magnitude of every element of Z tends toward zero. Repeat this process until the magnitudes of all elements of Z are less than ten per cent of the average magnitude \( |\bar{Z}| \) calculated in step (5), and return to step (2).

Transfer the displacements back one time step and determine whether the stresses are to be calculated. Replace the elements of \( U'' \) and \( W'' \) by the corresponding elements of \( U' \) and \( W' \), and replace the elements of \( U' \) and \( W' \) by the corresponding elements of \( U \) and \( W \). Retain the current values in \( U \) and \( W \) to be used as initial estimates for calculating the displacements at the next instant of time. Add \( \Delta t \) to \( t_j \), and if \( j = t_j / \Delta t \) is not an exact multiple of \( k \) return to step (2).

Calculate the stresses. By applying equations (29) to each group of four nearest neighboring elements in \( U \) and \( W \), values are obtained for corresponding elements of \( RR, ZZ, \theta \theta, \) and \( RZ \). If \( j = t_j / \Delta t \) is not equal to \( m \), return to step (2); if \( j = m \), an adequate number of stress states has been evaluated, so that the normal stress along the impact face has changed from compressive to tensile, and the approximate solution to the problem is therefore complete.
Among the various techniques known for solving the required system of linear algebraic equations, a modification of the Gauss-Seidel point iteration method (as expressed by steps (4-7) above) was selected as best utilizing the speed and memory capabilities of the Rice University computer. The method of relaxation developed for hand computation by Southwell and his contemporaries is also a variation of the Gauss-Seidel method. Recent advances in numerical techniques, such as the successive overrelaxation method and the alternating directions method, which have been developed specifically to solve certain classes of multidimensional partial differential equations by means of high-speed computing machines, could

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1 See chapter 2 of Householder (13). A brief comparative evaluation of the various numerical manipulations applicable to systems of linear equations is presented on pp. 81-83.

2 For a brief description of the Gauss-Seidel method, as applied to problems in the theory of elasticity, see sections 123-125, pp. 442-465, of Sokolnikoff (6).

3 See references (8), (12), (18), for descriptions of the relaxation method. The interested reader is referred to chapter 9 of reference (8) for the application of relaxation to simultaneous differential equations, and to its bibliography for works concerned with relaxation.

4 Such techniques were originally expounded in the references (24), (26), (30), and (31).

5 For articles devoted to the successive overrelaxation method, see references (30) and (32).

6 For articles related exclusively to the alternating directions method, see references (20), (21), (22), and (24).
perhaps have been programmed to solve this problem more effec-
tively (with higher speed and better accuracy). The Gauss-Seidel
procedure was selected over these methods, however, on the basis
of its simplicity and conciseness, and on the basis of its maintaining
a close analogy to the physical situation.
RESULTS

The numerical procedure outlined in the preceding section was programmed for the Rice University digital computer,¹ and this program was used to investigate the transient stress states in elastic bodies having the nine configurations and using the grid spacings shown in Figure 15. The calculations were performed using the following values for each of the parameters involved: $E = 30 \times 10^6$ psi, $\nu = 0.300$, $\rho = 732 \times 10^{-6}$ lb-sec$^2$/in$^4$, $L = 12$ in, and $V = 30$ fps. The resulting transient stresses are depicted by a series of graphs in Appendices B and C. Appendix B presents plots of the instantaneous maximum magnitude of each of the relevant stress components versus time, including the normal stress along the impact face ($\sigma_{zz}(z=0)$), and shows the instants at which each of these stresses becomes an overall maximum. Appendix C presents profiles, along the centerline, of the longitudinal stress ($\sigma_{zz}(r=0)$) for various instants of time throughout each impact cycle, and shows the progression of the initial dilatation wave and of successive reflected waves through the body for each of the test cases.

To correlate the configuration of an impacting body with its effectiveness as a hammer, the results of this investigation were consolidated, in nondimensional form, into the following graphs.

¹ See reference (17).
Figure 16 shows the overall maximum of each of the stress components \( \sigma_{ij}^{\text{max}} \) for each of the nine test cases. Figures 17a, 17b, and 17c show, respectively, the duration \( T \) of the impact cycle, the area \( A \) of the impact face, and the weight \( W \) of the body, for each of the test cases. In general, the test cases providing the smallest overall maximum tensile stresses \( \sigma_{ii}^{(+)} \text{max} \), where \( ii = rr \) or \( zz \) would be those least likely subject to breakage from impact. Likewise, the test cases exhibiting the greatest compressive forces transmitted across their impact faces \( A \) times \( \sigma_{zz}^{(z=0) \text{max}} \) and having the longest durations \( T \) would be those most favorable to impact applications. From these observations a figure of merit was devised for the effectiveness of an impact collision, being directly proportional to the overall maximum compressive stress along the impact face, the impact face area, and the impact duration, being inversely proportional to \( \sigma_{ii}^{(+)} \text{max} \), and normalized with respect to the weight of the body. Figure 17d shows this figure of merit for each of the test cases.

Appendix A presents an infinitesimal theory for the propagation of the initial dilatation pulse in impacting elastic bodies of revolution. For the parameter values specified above, \( \lambda = 17.3 \times 10^6 \) psi, \( \mu = 11.54 \times 10^6 \) psi, \( Y = 40.4 \times 10^6 \) psi, and \( c_i = 235,000 \) ips, so that the initial dilatation stresses, expressed by equations (49), become \( \sigma_{zz}^{1} = 61,900 \) psi, \( \sigma_{rr}^{1} = \sigma_{\theta\theta}^{1} = 26,500 \) psi, and \( \sigma_{rz}^{1} = 0 \).
Comparison of these theoretical dilatation stresses and of their propagating velocity $c_i$ with the propagation of the initial dilatation waves depicted in Appendix C, indicates excellent agreement. The primary discrepancies between the infinitesimal theory and the present discrete computational solution arise from the tendency of the numerical solution to smooth the abrupt stress pulse predicted by the infinitesimal theory.
Figure 15. Cross Sections of Specific Body Shapes Investigated Showing the Finite Difference Grid Spacings.
FIGURE 16. MAXIMUM STRESS COMPONENTS FOR EACH OF THE VARIOUS TEST CASES
FIGURE 17a. DURATION OF IMPACT CYCLE

FIGURE 17b. AREA OF IMPACT FACE

FIGURE 17c. WEIGHT OF IMPACTING BODY

FIGURE 17d. IMPACT EFFECTIVENESS FIGURE OF MERIT FOR EACH OF THE VARIOUS TEST CASES
CONCLUSIONS

The computer program based on the preceding numerical analysis was successful in obtaining approximate solutions to the title problem, and it appears that this finite difference method could be extended to other problems of wave propagation in continuous, deformable, finite bodies. Among the problems which could be investigated are those involving various geometrical and loading configurations, and those involving nonhomogeneous, anisotropic, and anelastic material properties. It is felt that the computed transient stresses obtained from the present study (using a maximum of only 375 net points), although exhibiting comparatively distorted wave shapes, were entirely adequate for the limited purposes of design. Numerical solutions to the present or related problems possessing significantly greater accuracy and detail than that obtained here will probably have to wait for the appearance of faster computing machines and for more refined numerical techniques.

The information shown in Figure 16, indicating the variation of the overall maximum stresses with the configurations shown in Figure 15, may be used to aid in the design of machine members whose service is to impact longitudinally against other members, as exemplified by the percussion drilling tool described in the introduction. The mode of failure of a hammer, such as the development of a longitudinal or circumferential crack, will suggest which of the
stress components has caused the failure. Figures 15 and 16 can then be used to determine the necessary changes of configuration to reduce that particular stress component below its minimum yield value, so that the hammer could be redesigned to eliminate the failure.

The test cases which provide the largest values of the impact-effectiveness figure of merit shown in Figure 17d, are considered to represent those configurations which are most favorable to impact applications. It is seen that test cases 5, 6, 1, and 2, in ascending order, provide the largest values of this figure of merit. Reference to Figure 15 indicates, therefore, in general, that long thin bodies are to be preferred to short stocky ones, that a moderately top-heavy body is to be preferred to one that is perfectly cylindrical, and that a perfectly cylindrical body is to be preferred to one that is bottom-heavy. Extreme deviations from cylindrical are unfavorable for both the top-heavy and bottom-heavy configurations, as demonstrated by test cases 3 and 7, respectively.\(^1\)

\(^1\) For case 3, the numerical solution became unstable, apparently due to the accumulation of roundoff errors, just prior to completion, as observed from Figure 20.
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APPENDIX A

Infinitesimal Theory of Wave Propagation in a General Body of Revolution for Small Time

Consider an elastic body of revolution impacting against a rigid wall with a striking velocity $V$, as shown in Figure 1. Since the relative particle velocity, upon impact, is initially in the longitudinal direction only, the effective elastic modulus is $\gamma = \lambda + 2\mu$, the lateral modulus is $\lambda$, and the velocity of propagation is $c_i = (\gamma/\rho)^{1/2}$.

By reasoning similar to that which Kolsky used in determining the stress intensity of displacement waves propagating in long thin rods, the initial stresses at a point $P'(r, 0, 0)$ along the impact face are found to be

\[
\sigma_{zz}^i = \gamma w, z = -\rho V c_i = -V(\rho \gamma)^{1/2}, \\
\sigma_{rr}^i = \sigma_{\theta\theta}^i = \lambda w, z = -\lambda V(\rho / \gamma)^{1/2}, \\
\sigma_{rz}^i = 0.
\]

These stresses are manifested instantaneously at the moment of impact, and remain constant until, presumably, an unloading wave traveling at dilatation velocity arrives at the point in question on the impact face from the free transverse boundary.

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1 See page 43 of reference (40).
Consider a point \( P(r, z, t) \) interior to the body, and denote its projection on the impact face at time \( t = 0 \) by \( P'(r, 0, 0) \). Let \( P''(b_0, 0, 0) \) be the point which forms the corner between the impact face and the transverse boundary at time \( t = 0 \). Initially every point \( P'(r, 0, 0), 0 \leq r < b_0 \), assumes the constant stresses given by equations (49). For positive time, a plane square wavefront carries these stresses in the longitudinal direction from point \( P' \) toward point \( P \) with the dilation velocity \( c_i \), and, simultaneously, an unloading wave travels inward from point \( P'' \) toward point \( P \), also with the dilation velocity.

The small-time history of any point \( P(r, z, t), 0 \leq r < b_0, 0 \leq z < L \), is therefore given by infinitesimal theory as follows. For \( t < z/c_i \), the material is undisturbed and bears zero stress. At \( t = z/c_i \), the initial dilation wave arrives at \( P \) from the point \( P' \), and the stresses instantaneously jump to the values expressed by equations (49), and remain at these constant values until \( t = [(b_0 - r)^2 + z^2]^{1/2}/c_i \), at which time the unloading wave arrives at \( P \) from point \( P'' \). For \( t > [(b_0 - r)^2 + z^2]^{1/2}/c_i \), the stresses at point \( P \) are hopelessly muddled by the effects of reflected dilation waves and by the arrival of waves traveling at the slower shear and surface wave velocities.

It will be noted that the region carrying the constant dilation stresses is roughly the shape of an inverted cone and that for \( t > b_0/c_i \), the thickness of this region decreases with increasing \( t \). For a long
bar, therefore, the thickness of this region should eventually decrease to atomic dimensions, causing these dilatation stresses thereby to disappear.
APPENDIX B

Plots of Maximum Stresses Against Time

for Each of the Test Cases
Figure 18. Maximum stresses versus time for Case 1.
Figure 19. Maximum stresses versus time for Case 2.
Figure 20. Maximum stresses versus time for Case 3.
Figure 21. Maximum stresses versus time for Case 4.
Figure 22. Maximum stresses versus time for Case 5.
Figure 23. Maximum stresses versus time for Case 6.
Figure 24. Maximum stresses versus time for Case 7.
**Figure 25. Maximum stresses versus time for Case 8.**
Figure 26. Maximum stresses versus time for Case 9.
APPENDIX C

Longitudinal Stress Profiles Along the Centerline
for Various Times and for Each of the Test Cases
Figure 27. Longitudinal stress profiles along centerline for Case 1.
Figure 28. Longitudinal stress profiles along centerline for Case 1.
Figure 29. Longitudinal stress profiles along centerline for Case 2.
Figure 30. Longitudinal stress profiles along centerline for Case 2.
Figure 3.1: Longitudinal stress profiles along centerline for Case 2.
Figure 32. Longitudinal stress profiles along centerline for Case 3.
Figure 33. Longitudinal stress profiles along centerline for Case 3.
Figure 34. Longitudinal stress profiles along centerline for Case 3.
Figure 35. Longitudinal stress profiles along centerline for Case 4.
Figure 36. Longitudinal stress profiles along centerline for Case 4.
Figure 37. Longitudinal stress profiles along centerline for Case 5.
Figure 38. Longitudinal stress profiles along centerline for Case 5.
Figure 39. Longitudinal stress profiles along centerline for Case 6.
Figure 40. Longitudinal stress profiles along centerline for Case 6.
Figure 4.1. Longitudinal stress profiles along centerline for Case 6.
A number accompanying a curve indicates the constant time (msec) at which the curve was evaluated. A distance travelled by a pulse in 4 msec at the velocity C.

Figure 42. Longitudinal stress profiles along centerline for Case 7.
Figure 43. Longitudinal stress profiles along centerline for Case 7.
Figure 44. Longitudinal stress profiles along centerline for Case 8.
Figure 4.5. Longitudinal stress profiles along centerline for Case 8.
Figure 46. Longitudinal stress profiles along centerline for Case 9.
Figure 47. Longitudinal stress profiles along centerline for Case 9.