RICE UNIVERSITY

CONSTITUTIVE EQUATIONS FOR A SPECIAL CLASS OF GELLING HYGROSTERIC MATERIALS

by

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ABSTRACT

The object of this thesis is to investigate a set of constitutive equations which are proposed to describe the behavior of a special class of gelling hygrosteric\textsuperscript{1} materials. In recent literature,\textsuperscript{2} various relations have been suggested for the analytical description of rigid-viscous type materials which exhibit the thixotropic property of gelling. Various aspects of these relations are incorporated with a general form to which hygrosteric materials have been found to conform.

The resulting constitutive equations are then examined for physical reliability by using them to obtain the solution of simple shear flow. The results were found to meet all physically motivated anticipations and it is felt that the proposed equations offer at least a reasonable qualitative description of actual gelling materials.

\textsuperscript{1}Exhibiting both solid and fluid properties.

\textsuperscript{2}Reference (10).
**NOTATION**

- $d_{ij}$: the $ij^{th}$ component of the deformation-rate tensor, defined by Equation (I.6)
- $D_2$: the positive square root of the second invariant of the plastic deformation-rate tensor, defined by Equation (II.26)
- $\mathbf{d}a$: a differential area vector
- $d_{i}^{\alpha}$: the $i^{th}$ covariant component of $\mathbf{d}a$
- $\mathbf{d}F$: a differential force vector
- $d_{i}^{\alpha}$: the $i^{th}$ contravariant component of $\mathbf{d}F$
- $D(A_{ij}^{i})/Dt$: the total time derivative of a second-rank tensor, $A_{ij}^{i}$, defined by Equation (I.20)
- $\text{dim}(ij)$: the physical dimensions of the $ij^{th}$ component of a particular second-rank tensor
- $E$: a material parameter occurring in the defining equation for the critical shear stress
- $e_{i}$: a dimensionless unit vector
- $G$: the elastic rigidity coefficient
- $[g_{ij}]$: the metric matrix
- $[g^{ij}]$: the derived metric matrix
- $[I]$: the identity matrix
- $\sqrt{J_2}$: the positive square root of the second invariant of the deviatoric stress tensor, defined by Equation (II.23)
- $M$: a uniform shear rate
- $s_{ij}'$: the $ij^{th}$ physical stress component
- $s_{ij}$: the $ij^{th}$ tensor stress component
- $S_{ij}^{i}$: the $ij^{th}$ deviatoric stress tensor component, defined by Equation (I.19)
- $\tilde{S}_{ij}^{i}$: the $ij^{th}$ component of the Jaumann stress-rate tensor, defined by Equation (I.18)
time
\[ t \]

the \( i \)th covariant base vector component
\[ \vec{v}_i \]

the \( i \)th contravariant base vector component
\[ \overline{v}_i \]

the \( ij \)th component of the vorticity tensor, defined by Equation (I.7)
\[ \omega_{ij} \]

the \( i \)th contravariant spatial coordinate axis
\[ x^i \]

the \( i \)th contravariant material coordinate axis
\[ \dot{x}^i \]

the \( i \)th covariant derivative of \( \dot{x}^k \), defined by Equation (I.9)
\[ \dot{x}^i_{,i} \]

a material parameter occurring in the defining equation for the critical shear stress
\[ \alpha \]

a viscosity coefficient
\[ \mu \]

the critical shear stress, defined by Equation (II.24)
\[ \tau_{\text{crit}} \]

the critical shear stress for a completely "worked" material
\[ \tau_o \]

the critical shear stress for a completely "gelled" material
\[ \tau_1 \]

a square matrix
\[ [ ] \]

the transpose of a square matrix
\[ [ ]^T \]

the first derivative with respect to time
\[ ( \cdot ) \]

the second derivative with respect to time
\[ ( \cdot )'' \]

a Christoffel symbol
\[ \{ i \} \]

\[ \{ j \} \]

\[ \{ k \} \]
INTRODUCTION

Certain materials such as paints, gelatins and emulsions have been experimentally demonstrated to exhibit the thixotropic property of gelling. Various authors\(^3\) have proposed constitutive equations which analytically describe so-called rigid-viscous gelling materials which undergo no deformation in time until a certain plastic yield criterion on stress is met. Although this rigid-viscous model may closely approximate the behavior of certain materials, it is felt that a material model which would allow for elastic effects may be a more realistic choice.

The suggestion for the general form of the proposed constitutive equations was obtained from a paper by Noll.\(^4\) Strict adherence to the functional form proposed therein for hygrosteric materials was not maintained. Rather that form served as a guide by which to model the proposed set. This guide essentially suggests that the rate of deformation shall be a function of both the applied stress and rate at which stress is applied.

With this general functional form in mind, provisions must be made so that the effect of thixotropy may also be considered. Whereas many unknowns currently exist regarding the behavior of gelling materials, progress in this field is furnishing a better understanding of some of the important mechanisms in these materials. In colloidal chemistry,

\(^3\)Reference (10).
\(^4\)Reference (6).
thixotropic stiffening is visualized as slow coagulation.\(^5\) When the material is at rest the particles form a rigid "scaffolding structure" (skeleton) which accounts for the existence of a "gel" strength, i.e. the stress state required to produce non-elastic deformation. The time rate of formation and the strength of this structure, as well as the influence on the rate of formation (or "breakdown") of a superposed deformation-rate, determine the thixotropic properties of a material. Reasoning such as that given above makes it possible to arrive at physically motivated constitutive equations.

In a paper by Slibar and Paslay\(^6\) constitutive equations for the analytical description of gelling materials were proposed. In this work the formulation was based on an incompressible Bingham-type body of constant viscosity and variable critical shear stress. In accordance with experimental evidence, the current value of the critical shear stress of a material must be a combined function of the continuing recovery in rigidity and breakdown in rigidity due to past deformation. Consequently the critical shear stress during deformation will depend upon the plastic deformation-rate history and in the state of rest will vary with time.

The above understanding was reached through a generalization of the constitutive equations for similar materials. These other stress-deformation-rate relations were first proposed by Bingham\(^7\) and put into three-dimensional form by Prager and Hohenemser.\(^8\) Several generaliza-

\(^5\)Reference (4, 7).
\(^6\)Reference (10).
\(^7\)Reference (1).
\(^8\)Reference (8).
tions of these equations have been proposed and it is from their general form that the proposed set of constitutive equations is to be established.

The principal contribution to the constitutive equations already established by Slibar and Paslay is the addition of an elastic dependence of the rate of deformation upon the rate of stress. This additional dependence is physically appealing for at least the following two reasons:

1) It removes the restriction of a highly idealized rigid-viscous material model which has no deformation until the material may deform non-elasticly.

2) Elasticity effects can be considered throughout the entire deformation-rate history of the material.

As a consequence of the foregoing discussion, the proposed constitutive equations take the following tensorial form,

\[ \dot{d}_{ij} = \frac{\dot{S}_{ij}}{2G} + S_{ij} \left[ \frac{\sqrt{J_2} - \gamma_{\text{crit}}}{\sqrt{J_2}} \right] \]

where the individual terms appearing in Equation (1) are defined analytically in Appendices I and II of this thesis. Having chosen the same yield criterion for non-elastic deformation as that proposed by Slibar and Paslay, it should be noted that the portion of Equation (1) in brackets is identically zero for \( \sqrt{J_2} \) less than \( \gamma_{\text{crit}} \).

For all examples treated in this thesis, the material will be assumed incompressible, i.e. the first invariant of the deformation-rate tensor must vanish.

\[ ^9 \text{Reference (10, 11).} \]
SECTION A: STATEMENT OF AN EXAMPLE PROBLEM.

As an application of the proposed set of constitutive equations given in the preceding section, the case of simple shear flow of a gelling, incompressible, non-Newtonian material will now be given. The physical configuration for this example is shown in Figure (1) and may be described as a uniform layer of the above material existing as a simply-connected region between two parallel plates which have a normal orientation with respect to the gravity acceleration vector. The plates as well as the material layer are supposed infinite in extent with regard to dimensions normal to the material thickness. The plates are constrained so as to allow a unidirectional relative motion in the horizontal plane only.

It is further supposed that a uniform value for the critical shear stress prevails throughout the material and that this value is the maximum value which may be obtained, $\gamma_1$, i.e. the material is in a fully-gelled state.

For ease of the numerical evaluation, the body forces, for example occurring in the gravity field, are neglected.

To assist in the understanding of the role of various portions of the proposed constitutive equations, it was found that the following formulation is convenient,

$$(d_{ij})_{\text{Total}} = (d_{ij})_{\text{Elastic}} + (d_{ij})_{\text{Plastic}}$$  \hspace{1cm} (A.1)

where by definition,

$$
(d_{ij})_{\text{Elastic}} = \frac{\dot{S}_{ij}}{2G}$$  \hspace{1cm} (A.2)
In addition it should be noted that the plastic yield criterion shall be the same as that proposed by Slibar and Paslay, namely that no plastic deformation can occur unless the positive square root of the quadratic invariant of the reduced, or deviatoric, stress tensor exceeds the current value for the critical shear stress.

Hence it follows that,

\[ (d_{ij}^{\text{plastic}}) = \begin{cases} 0, & \sqrt{J_2} \leq \gamma_{\text{crit}} \\ \frac{S_{ij}}{2\mu} \left[ \sqrt{\frac{J_2 - \gamma_{\text{crit}}}{J_2}} \right], & \sqrt{J_2} \geq \gamma_{\text{crit}} \end{cases} \]  

(A.3)

Thus any deformation problem considered employing these constitutive equations, in particular the one presently being treated, can be separated into two distinct regimes. The first regime, which occurs when \( \sqrt{J_2} \leq \gamma_{\text{crit}} \), is one in which the deformation is purely of an elastic nature; whereas the second regime, which occurs when \( \sqrt{J_2} > \gamma_{\text{crit}} \), is a combined elastic and gelling-plastic phenomenon.

The reference to elastic effects may be made more meaningful by noting that the first regime is a special case of a class of materials described by Truesdell\(^{10}\) as hypo-elastic. Therefore perhaps, hypo-elastic effects might be a more appropriate description.

\(10\) Reference (12).
SECTION B. TREATMENT OF THE HYPO-ELASTIC PORTION OF THE PROBLEM UNDER CONSIDERATION BY SPECIFYING THE VELOCITY FIELD.

The treatment of the purely hypo-elastic regime of this problem will now be effected by specifying the velocity field and solving for the resulting stress-state under the assumptions of Section A.

Both the spatial and material coordinate systems are chosen to be contravariant Cartesian. Using such systems it is noted that there is no distinction between tensor and physical quantities, such as stress. This situation is a consequence of the fact that the metric matrices are the identity matrix and all Christoffel symbols vanish identically. It should be noted that the above is not usually the case and that for a general curvilinear system, identity relations between tensor and physical quantities must be determined. This problem is the subject of Appendix I. Noting the above, strict tensor notation will not be adhered to in this example for purposes of convenience.

The constitutive equations which govern this flow regime are

\[
\dot{d}_{ij} = \tilde{S}_{ij} \quad \frac{2G}{2G}.
\]

Having chosen the Jaumann definition of stress-rate, Equation (B.1) becomes, upon expansion

\[
2G \dot{d}_{ij} = D(S_{ij}) + \sum_{m} (S_{mj} \omega_{mi} + S_{im} \omega_{mj}).
\]

The flow field chosen to be analyzed was that of a uniform shear, i.e.

\[
x^1 = x^1 + Mtx^2 \quad x^2 = x^2 \quad x^3 = x^3,
\]
where \( x^i \) denotes a spatial coordinate and \( \dot{x}^i \) denotes a material coordinate. The only non-vanishing velocity gradient for this flow thus becomes

\[
\frac{\partial u_1}{\partial x^2} = M \quad (B.4)
\]

where \( u_1 \) denotes the spatial velocity of the material in the \( x_1 \)-direction.

As usual, for a Cartesian system, the following definitions are valid,

\[
d_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} \right] \quad (B.5)
\]

\[
\omega_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i} \right] \quad (B.6)
\]

Under the above definitions, the non-vanishing components of \( d_{ij} \) and \( \omega_{ij} \) are,

\[
d_{12} = d_{21} = \frac{1}{2} M \quad (B.7)
\]

\[
\omega_{12} = -\omega_{21} = \frac{1}{2} M \quad (B.8)
\]

Writing Equation (B.2) in matrix form yields,

\[
2G[d_{ij}] = [S]_{ij} + [S_{mj}]^T[\omega_{m1}] + [S_{im}]^T[\omega_{mj}] \quad (B.9)
\]

where \( S_{ij} \) denotes the total time derivative of the stress tensor as defined by Equation (I.20). For the Cartesian system now under consideration, \( S_{ij} \) reduces to the partial derivative of \( S_{ij} \) with respect to time, i.e.,

\[
\dot{S}_{ij} = \frac{\partial S_{ij}}{\partial t} \quad (B.10)
\]

Expanding Equation (B.9) yields,
From the above set of equations, three non-trivial equations result. They are,

\[ 0 = S_{11} - S_{12} M \]  \hspace{1cm} (B.12)

\[ MG = \dot{S}_{12} + (S_{11} - S_{22}) M/2 \]  \hspace{1cm} (B.13)

\[ 0 = \dot{S}_{22} + S_{12} M \]  \hspace{1cm} (B.14)

Equations (B.12) and (B.14) were then placed into the time differentiated form of Equation (B.13), yielding a linear second-order differential equation involving \( S_{12} \) only,

\[ \ddot{S}_{12} + M^2 S_{12} = 0, \]  \hspace{1cm} (B.15)

the solution of which is,

\[ S_{12} = C_1 \cos(Mt) + C_2 \sin(Mt). \]  \hspace{1cm} (B.16)

Equation (B.16) is then placed into Equations (B.12) and (B.14) in order to effect a general solution for both \( S_{11} \) and \( S_{22} \). These solutions are, respectively

\[ S_{11} = C_1 \sin(Mt) - C_2 \cos(Mt) + C_3 \]  \hspace{1cm} (B.17)

\[ S_{22} = -C_1 \sin(Mt) + C_2 \cos(Mt) + C_4. \]  \hspace{1cm} (B.18)

The arbitrary constants were determined using Equation (B.13) and the boundary conditions that \( S_{11} = S_{12} = S_{22} = 0 \) at \( t = 0 \) (reference configuration.) The value for the constants are, \( C_2 = C_3 = -C_4 = G \).
The non-trivial set of equations thus becomes,

\[ S_{12} = G \sin(Mt) \]  
\[ S_{11} = -S_{22} = G(1 - \cos(Mt)) \].  

Noting the physical configuration and the imposed flow state, it is seen that \( S_{12} \) is a positive definite quantity and the following restriction is obvious,

\[ 0 \leq S_{12} \leq G. \]  

Thus Equations (B.19) and (B.20) are only valid until \( Mt = \gamma \). For \( Mt \) greater than \( \gamma \), the physical restriction as given by Equation (B.21) is violated. This further implies that the general equations proposed are meaningful, if and only if \( \sqrt{J_2} > \gamma_{crit} \) at some time prior to \( Mt = \gamma \).

It is therefore helpful to have a full understanding of how \( \sqrt{J_2} \) varies during the imposed motion. For the case now under consideration, \( \sqrt{J_2} \) becomes,

\[ \sqrt{J_2} = \sqrt{\frac{(S_{11})^2 + (S_{22})^2 + (S_{12})^2}{2}} + \sqrt{2G(1 - \cos(Mt))^{1/2}} \].  

Equation (B.22) indicates that even though the shear component, \( S_{12} \), decreases for \( Mt > \gamma/2 \), the material can attain its plastic yield point after this value of \( Mt \) as \( \sqrt{J_2} \) continues to increase up to the limiting point \( Mt = \gamma \), at which time it attains a maximum value of \( 2G \).

Before continuing to the next analysis of this regime, the following relations should be noted for future reference,

\[ x^2(t) - x^2(0) = Mt^2 \]  
\[ Mt = \sin^{-1}(S_{12}/G), \text{ for } 0 \leq Mt \leq \gamma \].

hence,
\[ x^1(t) - x^1(0) = x^2 \sin^{-1}(S_{12}/G) \]  \hspace{1cm} (B.25)

which, together with Equation (B.20) implies that for uniform shear flow, the displacement of the material is a unique function of the stress-state and vice versa, and that both are independent of the magnitude of \( M \) per se.
SECTION C. TREATMENT OF THE HYPO-ELASTIC PORTION OF THE PROBLEM UNDER CONSIDERATION BY SPECIFYING THE STATE OF STRESS.

This section shall treat the hypo-elastic regime from the standpoint of specifying the stress state, at least in part, and deriving the flow field and unknown stress components therefrom. All assumptions made in Section A are valid for this consideration. It is interesting to show that, for this problem, if the stress state approaches an asymptotic solution by specifying the explicit form of the $S_{12}$ component, that for the identical stress state considered in Section B, the spatial displacement of the material is identical.

For $\sqrt{\frac{1}{2}} \leq \gamma_{\text{crit}}$ as before, it will be assumed that the shear component approaches some arbitrary value exponentially in time as follows,

$$S_{12} = S (1 - e^{-at}), \quad (C.1)$$

where $S$ and $a$ are constants in time. It will be seen that a solution can be formed such that

$$S_{11} = -S_{22} = N(t), \quad (C.2)$$

where the relation between $S_{11}$ and $S_{22}$ can be anticipated from the results of the previous section, further that the flow field can be given by the following,

$$u_1 = 2C(t) x^2 \quad u_2 = u_3 = 0, \quad (C.3)$$

Equations (C.3) imply that

$$d_{12} = d_{21} = C(t) \quad (C.4)$$

$$\omega_{12} = -\omega_{21} = C(t) \quad (C.5)$$
The constitutive equations take the same form as before. Hence, upon expansion it is seen that
\[
2G \begin{bmatrix} 0 & C(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \dot{S}_{11} & \dot{S}_{12} & 0 \\ \dot{S}_{22} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 \dot{S}_{12} C(t) \\ (S_{11} - S_{22}) C(t) \\ 2 \dot{S}_{12} C(t) \end{bmatrix} \quad \text{(C.6)}
\]

The three non-trivial equations of which are,
\[
\dot{S}_{11} = -S_{22} = 2 \dot{S}_{12} C(t) \quad \text{(C.7)}
\]
\[
2GC(t) = \dot{S}_{12} + (S_{11} - S_{22}) C(t). \quad \text{(C.8)}
\]

Equations (C.8) may be rewritten as,
\[
C(t) [2G + S_{22} - S_{11}] = \dot{S}_{12} = S a e^{-at} \quad \text{(C.9)}
\]
or,
\[
2G + S_{22} - S_{11} = (S a e^{-at}) [C(t)]^{-1}. \quad \text{(C.10)}
\]

The division by C(t) is presumed valid as C(t) is non-zero for all time and only tends to zero as time tends to infinity. Combining Equations (C.2) and (C.7) yields,
\[
N(t) = 2 \dot{S}_{12} C(t) = 2 SC(t) (1 - e^{-at}) \quad \text{(C.11)}
\]

Ultimately the integration of the following equation is sought,
\[
x^1 = x^1 + \int_0^\infty u_1(t) \, dt = x^1(0) + 2 x^2 \int_0^\infty C(t) \, dt \quad \text{(C.12)}
\]

where $x^1$ represents the spatial position of a material element in the direction parallel to the flow corresponding to a given stress state. This however necessitates an explicit relation for C(t).
Upon differentiating Equation (C.10) and noting Equations (C.1) and (C.7), the following expression is obtained,
\[ C(t) = \frac{4e^{at}}{a} [C(t)]^3 - \frac{4}{a} [C(t)]^3 - aC(t). \]  
(C.13)

For convenience in obtaining a solution to Equation (C.13) the following definitions are introduced,
\[ y = C(t), \quad Aa = 4. \]  
(C.14)

Hence, Equation (C.15) becomes
\[ \dot{y} = A e^{at} y^3 - Ay^3 - ay. \]  
(C.15)

Equation (C.15) is a non-linear differential equation with non-constant coefficients and is of the Bernoulli type.\(^{11}\) It may thus be linearized by the choice of a suitable substitution. Letting \( y = u^k \), it was found that the value, \( k = -1/2 \), linearized Equation (C.15) yielding
\[ \dot{u} - 2au + 2 A(e^{at} - 1) = 0. \]  
(C.16)

The complete solution of Equation (C.16) may be given by the solution to the homogeneous equation plus any particular solution. The complete solution is given as follows,
\[ u = C_1 e^{2at} + \frac{8}{a^2} e^{at} - \frac{4}{a^2}. \]  
(C.17)

where \( C_1 \) is a constant of integration. Hence an explicit relation for \( C(t) \) may now be given as
\[ C(t) = \left[ C_1 e^{2at} + \frac{8}{a^2} e^{at} - \frac{4}{a^2} \right]^{-1/2}. \]  
(C.18)

At time, \( t = 0 \),
\[ C(t) = \left[ C_1 + \frac{4}{a^2} \right]^{-1/2}. \]  
(C.19)

As \( C(t) \) is assumed to represent a finite real motion, it is obvious that

\(^{11}\)Reference (3).
\[ C_1 > -\frac{4}{a^2}. \]  

(C.20)

It is also obvious from Equation (C.18) that \( C(t) \) satisfies the physical requirement that \( C(t) \) tends to zero as \( t \) tends to infinity.

It is now possible to perform the desired integration of Equation (C.12), as

\[ x^1 = x^1(0) + 2x^2 \int_0^\infty \left[ C_1 e^{2at} + \frac{8}{a^2} e^{at} - \frac{4}{a^2} \right]^{-1/2} dt. \]  

(C.21)

To facilitate integration, the following change of variables was chosen,

\[ z = e^{at}. \]  

(C.22)

Hence Equation (C.21) becomes

\[ x^1 = x^1(0) + 2x^2 \int_1^\infty \left[ C_1 z^2 + \frac{8}{a^2} z - \frac{4}{a^2} \right]^{-1/2} \frac{dz}{z}. \]  

(C.23)

Using a standard integration formula\(^{12}\) and letting

\[ C_1 = 4 \frac{L}{a^2} \]  

(C.24)

where \( L > -1 \), Equation (C.23) in its integrated form becomes,

\[ x^1 = x^1(0) \sin^{-1} \left[ \frac{1}{\sqrt{1 + L}} \right]. \]  

(C.25)

It should be noted at this time that an even greater restriction must be placed upon the value of \( C_1 \), through its appearance in the form of \( L \) in Equation (C.25). This restriction implies that \( C_1 \geq 0 \) as \( L \geq 0 \) because of its occurrence in the argument of \( \sin^{-1} \). At this point only the lower bound for \( L \) and hence \( C_1 \) is known.

Having \( C(t) \) known explicitly it is also possible at this time to evaluate \( N(t) \) and ultimately the asymptotic values for \( S_{11} \) and \( S_{22} \).

\(^{12}\)Reference (2).
From Equation (C.11), it is seen that
\[ N(t) = \frac{2 S (1 - e^{-at})}{\left[ C_1 e^{2at} + \frac{8 e^{at}}{a^2} - \frac{4}{a^2} \right]^{1/2}} \] (C.26)

hence from Equation (C.2),
\[ S_{11} = -S_{22} = \int_0^\infty \frac{2 S (1 - e^{-at})}{\left[ C_1 e^{2at} + \frac{8 e^{at}}{a^2} - \frac{4}{a^2} \right]^{1/2}} \, dt \] (C.27)

Using the same substitutions as in the solution of Equation (C.21), and using another standard integration formula,\(^{13}\) Equation (C.27) becomes
\[ S_{11} = -S_{22} = \sqrt{1+L} - \sqrt{L}, \] (C.28)

where the above radicals denote the positive square root. Therefore the explicit relations for the asymptotic stress state are now known as
\[ S_{12} = S \]
\[ S_{11} = -S_{22} = \sqrt{1+L} - \sqrt{L}. \] (C.29)

The problem which now presents itself is that of the determination of the "arbitrary" constant \( L \), to which \( C_1 \) is uniquely related by definition. Equation (C.10) reveals some insight into this matter. Evaluation of this equation must be possible for all values of time, in particular time equals zero. This evaluation yields,
\[ 2G + 0 + 0 = S a \left[ C_1 + \frac{4}{a^2} \right]^{1/2} \]

from which the following expression is obtained,
\[ L = \left[ \frac{g}{S} \right]^2 - 1. \] (C.30)

As it has been previously pointed out that \( L \geq 0 \), it is obvious that

\(^{13}\)Reference (2).
\[ G \geq S. \quad (C.31) \]

Hence for the above equations and assumptions to be valid, the shear modulus of rigidity must be equal to or greater than the maximum value of the imposed shear stress component in the hypo-elastic regime.

Remembering the comment at the outset of this section, the correlation between the stress states and the material displacement for this section and the previous section will now be given.

First note that in both analyses, it was predicted that \( S_{12} \leq G. \)

Next note the interdependence of \( S_{11} \) and \( S_{12} \), namely from Section B,

\[ S_{11} = S_{12} \left[ \frac{G}{S_{12}} - \frac{G}{S_{12}} \sqrt{\frac{G}{S_{12}} - 1} \right] \quad (C.32) \]

or letting \( L = (G/S_{12})^2 - 1 \), Equation (C.32) reduces to the following form

\[ S_{11} = S_{12} \left( \sqrt{L+1} - \sqrt{L} \right) \quad (C.33) \]

which agrees exactly with expression (C.29) of this section.

The correlation between the displacements is now given. Equation (C.25), upon noting the identity given by Equation (C.30) becomes

\[ x^1 - x^1(0) = x^2 \sin^{-1} \left[ \frac{S}{G} \right] \quad (C.34) \]

which also agrees exactly with Equation (B.25) of Section B. As was mentioned in Section B, and is seen from Equation (C.34) the displacement is a unique function of the stress state and not a function of the rate at which this state is reached.

In retrospect, however, the conclusions were to be expected, due to the fact that the material being treated is hypo-elastic in the strict sense as defined by Truesdell.\(^{14}\) For these materials, Noll has provided

\(^{14}\)Reference (12).
a theorem\textsuperscript{15} to the effect that if a material is hypo-elastic, then, for a given initial stress, the stress at a final state depends only on the paths by which the material points reach the final state and not upon the rate at which they traverse these paths.

\textsuperscript{15}Reference (6).
SECTION D. GENERAL ANALYSIS OF THE PROBLEM UNDER CONSIDERATION.

Having treated the first regime of this flow problem, that of purely hypo-elastic motion, in each of two different manners and found results which were both physically compatible and theoretically predictable, attention will now be given to the flow state which occurs once the material has yielded "plastically." In particular it should be mentioned that this regime occurs if and only if the positive square root of the quadratic invariant of the reduced stress tensor exceeds the current value of the critical shear stress. From Section A it should also be noted that the maximum possible value for the critical shear stress, \( \tau_1 \), should be equal to or less than \( 2G \), the maximum value which \( \sqrt{J_2} \) can attain, in order to insure that plastic yielding will occur. Hence, these are some of the physical limitations which must be placed upon the applicability of the proposed equations.

With these thoughts in mind, the general constitutive equations are given by,

\[
d_{ij} = \frac{s_{ij} + S_{ij}}{2G} \left[ \frac{\sqrt{J_2} - \tau_{\text{crit}}}{\sqrt{J_2}} \right].
\] (D.1)

Upon expansion, Equation (D.1) becomes,

\[
d_{ij} = \frac{1}{2G} \left[ s_{ij} + \sum_m s_{mj} \omega_{mi} + s_{im} \omega_{mj} \right] + \frac{1}{2\mu} s_{ij} \left[ \frac{\sqrt{J_2} - \tau_{\text{crit}}}{\sqrt{J_2}} \right].
\] (D.2)

or in matrix notation,
A physical explanation for the occurrence of this regime and how it is to be treated is described below. The proposed material is at rest, existing physically as outlined in Section A. The maximum value for the critical shear stress, \( \tau_1 \), prevails uniformly throughout the material in this reference state, i.e., the material is completely gelled.

Relative unidirectional horizontal motion of the parallel plates in the form of a uniform shear is imposed. Under this motion the material thickness remains constant. From this imposed motion, functional relations for the stress components in terms of the motion may be derived. In particular until \( \sqrt{J_2} = \tau_{\text{crit}} = \tau_1 \), there is no plastic deformation and the material model is purely hypo-elastic. Section B of this thesis provides the complete solution for this portion of the flow and hence provides the boundary conditions for the general flow, once the yield criterion has been satisfied.

At the point where \( \sqrt{J_2} = \tau_1 \), the material model changes and is governed by Equation (D.1). For this time, say \( t = t^* \), all stress and stress-rate components are known from the solution of Section B. These are obviously the initial conditions for the combined elastic, gelling-plastic flow.

The set of equations, (D.3), reduces to the following non-trivial relations,

\[
0 = \frac{S_{11}}{2G} - \frac{S_{12}}{2G} + \frac{S_{11}}{2\mu} \left[ 1 - \frac{\tau_{\text{crit}}}{\sqrt{J_2}} \right]
\]

(D.4)
where it is noted that $S_{11} = -S_{22}$ as before. To effect a solution to this set there are other relations which must be considered. These relations are the defining equations for $\sqrt{J_2}$ and $\tilde{\gamma}_{\text{crit}}$.

An important aspect in the determination of $\tilde{\gamma}_{\text{crit}}$ is that the term $D_2$ must be constructed from the components of the plastic deformation-rate tensor. These components may be obtained through the use of the following relations,

$$\left( d_{ij} \right)_{\text{Plastic}} = \left( d_{ij} \right)_{\text{Total}} - \left( d_{ij} \right)_{\text{Elastic}} . \tag{D.6}$$

Hence from Equations (D.4) and (D.5) it follows that,

$$\left( d_{11} \right)_{\text{Plastic}} = -(d_{22})_{\text{Plastic}} = \frac{1}{2 \mu} (S_{12} M - \dot{S}_{11}) \tag{D.7}$$

$$\left( d_{12} \right)_{\text{Plastic}} = \frac{M}{2} - \frac{1}{2G} (S_{12} + S_{11} M) . \tag{D.8}$$

As these are the only non-vanishing components, $(D_2)_{\text{Plastic}}$ becomes,

$$\left( \left( d_{11} \right)_{\text{Plastic}}^2 + \left( d_{12} \right)_{\text{Plastic}}^2 \right)^{1/2} . \tag{D.9}$$

A question now arises concerning the existence of a steady-state flow condition. Since this has been demonstrated for the case of gelling plastic materials neglecting elastic effects, and since this analysis is to be concerned with second-order elastic effects only, it would seem physically reasonable that a steady-state should also exist for the material under consideration. With this conjecture, the steady-state flow conditions will be sought.

Due to the complexity of the given system of equations, it was felt that no purely analytical solution was possible. Hence, the system of equations were solved numerically. The basic procedure incorporated a
forward time-integration of the two constitutive equations using a modified Runge-Kutta technique. The most difficult part of this procedure was incrementing the value for the critical shear stress. A brief discussion of the numerical method is included in Appendix III of this thesis.

The data input for this problem was in the form of dimensionless ratios which were felt to be physically reasonable. Several different time increments were used to test the convergence of solutions and general agreement was obtained. The results of this investigation are shown in Figures (2) through (9). A brief discussion of each of the figures will now be given.

Figure (2): $S_{12}/\tau_o$ versus $M_t$ for the indicated dimensionless ratios. By specifying different values for the quantity $G/M\mu$, the dependence of the shear stress component upon the material viscosity only, may be observed. For all time such that $0 \leq t \leq t^*$, the curves are identical as this period represents the hypo-elastic region in which $\mu$ plays no role. According to the relative magnitudes of the material parameters, the shear component is observed to both increase and decrease, indicating the adjustment between the elastic and viscous mechanisms, as well as the almost instantaneous breakdown in rigidity and possible subsequent gelling. As was physically anticipated, the steady-state value for the shear component decreased with decreasing values for the material viscosity.

Figure (3): $S_{11}/\tau_o$ versus $M_t$ for the same ratios as were given in Figure (1). The discussion is qualitatively the same as for Figure (1).

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$^{16}$Reference (9).
Figure (4): $\tau_{\text{crit}}/\tau_o$ versus $M_t$ for the same ratios as were given in Figure (2). $\tau_{\text{crit}}$ maintains its constant magnitude of $\tau_1$, until $\sqrt{J_2}$ equals $\tau_1$, at which time the breakdown is almost instantaneous. A minimum, in this example, is reached and thereafter the material demonstrates the thixotropic phenomenon of gelling as $\tau_{\text{crit}}$ approaches a steady-state value asymptotically. This identical behavior does not always occur, e.g. Figure (7).

Figure (5): $S_{12}/\tau_o$ versus $M_t$ for the indicated dimensionless ratios. Different values for $G/M\mu$ were assigned which allowed the dependence of the shear component upon the imposed shear-rate (rate of deformation) to be noted. The discussion is qualitatively the same as for Figure (2). Again the physical expectation, that the steady-state shear component should decrease with decreasing rates of shear, is satisfied.

Figure (6): $S_{11}/\tau_o$ versus $M_t$ for the same ratios as given in Figure (5). The discussion is qualitatively the same as for Figure (5).

Figure (7): $\tau_{\text{crit}}/\tau_o$ versus $M_t$ for the same ratios as given in Figure (5). The value for $\tau_{\text{crit}}$ again remains fixed at $\tau_1$ until $\sqrt{J_2}$ equals $\tau_{\text{crit}}$. The breakdown is again almost instantaneous with the following exception being noted: the steady-state value is approached in a monotone fashion, demonstrating no "over-shoot" as was noted in Figure (4).

Figure (8): $S_{12}/\tau_o$ versus $M_t$ for the dimensionless ratios indicated. By again looking at various values for the quantity $G/M\mu$, the shear component is observed to exhibit a marked dependence upon the elastic rigidity coefficient. As $G$ dictates the rate in time at which the stress components increase as well as their possible maxima, $t^*$, the time at which $\sqrt{J_2} = \tau_{\text{crit}}$, varies from one example to the next.
$S_{12}$ varies with $G \sin(M_t)$ while $S_{11}$ varies with $G(1 - \cos(M_t))$ which implies that $\sqrt{J_2}$ will be dominated by the $S_{12}$ component for these examples. Hence for increasing values of $G$, which suggests a more "rigid" material, $t^*$ decreases and $S_{12}(t^*)$ increases. The subsequent flow profiles are similar to the cases previously discussed. As, with increasing $G$, the material becomes more "rigid", it is logical that the steady-state value for the shear component will likewise increase, which is demonstrated on this figure.

**Figure (9):** $S_{11}/C_0$ versus $M_t$ for the same ratios as were given in Figure (8). The qualitative discussion is similar to those given previously. Again physical expectations are realized upon noting that the $S_{11}$ component decreases with increasing $G$, as it is known that for a material with an infinite $G$, no normal component may be developed in shear flow.

The variations in the critical shear stress associated with Figures (8) and (9) are not included or discussed as they offer nothing new to this analysis.

A number of actual experiments were performed by the Eastman Kodak Company\(^\text{17}\) on various materials which exhibited the existence of a thixotropic mechanism. A general qualitative agreement was found between their experiments and Figures (2), (5) and (8) of this thesis.

\(^{17}\)Reference (13).
APPENDIX I. TENSOR AND PHYSICAL RELATIONS IN GENERAL AND AN ANALYSIS OF
THE CYLINDRICAL COORDINATE SYSTEM IN PARTICULAR.

The choice of a coordinate system to be used in a particular analysis will require that some thought be given to the identity relations between tensor and physical quantities. This problem is eliminated for the choice of a Cartesian system as the metric matrix is the identity matrix and all Christoffel symbols vanish identically.

In practice however, many physical configurations possess rotational symmetry which suggests that a cylindrical coordinate system should at least be considered. Typical problems falling into this category are those of Couette flow and Poiseuille flow. In Appendix II, Poiseuille flow is considered. Consequently this appendix analyzing a cylindrical system is thought to be appropriate in this thesis.

The coordinate system to be considered has the following contravariant coordinates,
\[ x^1 = r \quad x^2 = \theta \quad x^3 = z \quad \text{(I.1)} \]
Hence the metric matrix becomes,
\[ [g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ r^2 & 0 \\ 1 \end{bmatrix} \quad \text{(I.2)} \]
Note also that,
\[ [g_{ij}][g^{jk}] = [I] \quad \text{(I.3)} \]
where [I] denotes the identity matrix. Hence,
The only non-vanishing Christoffel symbols for this system are,
\[
\{ \begin{pmatrix} r \\ \theta \\ r \end{pmatrix} \} = -r \\
\{ \begin{pmatrix} \theta \\ r \\ \theta \end{pmatrix} \} = \{ \begin{pmatrix} \theta \\ \theta \\ r \end{pmatrix} \} = \frac{1}{r}
\] (I.5)

The components of the deformation-rate tensor are given by,
\[
d_{ij} = \frac{1}{2} \sum_k (g_{ik} \dddot{x}^k_{,j} + g_{jk} \dddot{x}^k_{,i})
\] (I.6)

and the components of the vorticity tensor are,
\[
\omega_{ij} = \frac{1}{2} \sum_k (g_{ik} \dddot{x}^k_{,j} - g_{jk} \dddot{x}^k_{,i})
\] (I.7)

where \( \dddot{x}^k_{,i} \) denotes the i\textsuperscript{th} covariant derivative of \( \dddot{x}^k \), expressed as a function of \( x^i \) and time. By definition,
\[
\dddot{x}^k = \frac{\partial x^k(x^i, t)}{\partial t}
\] (I.8)

therefore,
\[
\dddot{x}^k_{,i} = \frac{\partial \dddot{x}^k}{\partial x^i} + \sum_m \dddot{x}^m \left\{ \begin{pmatrix} k \\ m \end{pmatrix} \right\}
\] (I.9)

For this system, noting Equations (I.1) and (I.8), it follows that
\[
\dddot{x}^1 = \dddot{r} \\
\dddot{x}^2 = \dddot{\theta} \\
\dddot{x}^3 = \dddot{z}
\] (I.10)

Upon introducing the concept of unit vectors, Equations (I.10) become
\[
\dddot{x}^1 = u_r \\
\dddot{x}^2 = u_\theta / r \\
\dddot{x}^3 = u_z
\] (I.11)

The covariant derivatives of \( x^i \) may now be formed, giving
\[
\dddot{x}^r_{,r} = \frac{\partial u_r}{\partial r} \\
\dddot{x}^r_{,\theta} = \frac{\partial u_r}{\partial \theta} - u_\theta \\
\dddot{x}^r_{,z} = \frac{\partial u_r}{\partial z}
\] (I.12)

\[
\dddot{x}^\theta_{,r} = \frac{1}{r} \frac{\partial u_\theta}{\partial r} \\
\dddot{x}^\theta_{,\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \\
\dddot{x}^\theta_{,z} = \frac{1}{r} \frac{\partial u_\theta}{\partial z}
\]
\[
\begin{align*}
\dot{x}^r &= \frac{\partial u^r}{\partial r} \\
\dot{x}^\theta &= \frac{\partial u^r}{\partial \theta} \\
\dot{x}^z &= \frac{\partial u^z}{\partial z}
\end{align*}
\]

To form \([d_{ij}]\) and \([\omega_{ij}]\) it is necessary to obtain the following expression,

\[
[e_{ij}] [\dot{x}^j_k] = \begin{bmatrix}
\frac{\partial u_r}{\partial r} & \frac{\partial u_r - u_\theta}{\partial \theta} & \frac{\partial u_r}{\partial z} \\
\frac{r \partial u_\theta}{\partial r} & \frac{r \partial u_\phi + ru_r}{\partial \theta} & \frac{\partial u_\phi}{\partial z} \\
\frac{\partial u_z}{\partial r} & \frac{\partial u_z}{\partial \theta} & \frac{\partial u_z}{\partial z}
\end{bmatrix}
\] (I.13)

The components of \(d_{ij}\) thus become,

\[
\begin{align*}
d_{rr} &= \frac{\partial u_r}{\partial r} \\
d_{r\theta} &= \frac{1}{2} \left[ \frac{\partial u_r}{\partial \theta} + r \frac{\partial u_\theta}{\partial r} - u_\phi \right] \\
d_{r\phi} &= \frac{r \partial u_\theta}{\partial \phi} + ru_r \\
d_{rz} &= \frac{1}{2} \left[ \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right] \\
d_{zz} &= \frac{\partial u_z}{\partial z} \\
d_{\phi r} &= \frac{\partial u_\phi}{\partial r} \\
d_{\phi \theta} &= \frac{1}{2} \left[ r \frac{\partial u_\phi}{\partial \theta} + \frac{\partial u_\theta}{\partial r} \right] \\
d_{\phi z} &= \frac{1}{2} \left[ r \frac{\partial u_\phi}{\partial z} - \frac{\partial u_z}{\partial \phi} \right]
\end{align*}
\] (I.14)

The components for \(\omega_{ij}\) become,

\[
\begin{align*}
\omega_{rr} &= \omega_{r\theta} = \omega_{rz} = 0 \\
\omega_{r\phi} &= -\omega_{\phi r} = \frac{1}{2} \left[ \frac{\partial u_r}{\partial \phi} - r \frac{\partial u_\phi}{\partial r} - u_\phi \right] \\
\omega_{rz} &= -\omega_{rz} = \frac{1}{2} \left[ \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right] \\
\omega_{\phi z} &= -\omega_{\phi z} = \frac{1}{2} \left[ r \frac{\partial u_\phi}{\partial z} - \frac{\partial u_z}{\partial \phi} \right]
\end{align*}
\] (I.15)

At this time, the dimensions of the various components of \(d_{ij}\) and \(\omega_{ij}\) will be determined. It should be noted that the dimensions for the physical counterparts of these respective components should be \((\text{time})^{-1}\).

When no ambiguity exists, the following notation will be used,
\( \text{dim}(ij) = \) the physical dimensions of the \( ij \) component.  \( \text{(I.16)} \)

Hence, considering both \( d_{ij} \) and \( \omega_{ij} \) it is noted that,

\[
\begin{align*}
\text{dim}(rr) &= \text{dim}(rz) = \text{dim}(zz) = (\text{time})^{-1} \\
\text{dim}(\theta z) &= \text{dim}(r\theta) = (\text{length})(\text{time})^{-1} \\
\text{dim}(\theta \theta) &= (\text{length})^2(\text{time})^{-1}
\end{align*}
\]

\( \text{(I.17)} \)

A consideration of the stress and stress-rate tensors will now be given. In tensor form, the Jaumann stress-rate becomes,

\[
\dot{S}^{ij} = \frac{D(s^{ij})}{Dt} + \sum_{m}(s^{mj}_m + s^{im}_m) \tag{I.18}
\]

where \( s^{ij} = \sum_k s^i_k k^j \) and \( s^i_k \) is the deviatoric stress tensor component which is related to the mixed tensor component in the following manner,

\[
s^i_k = s^i_k - \frac{1}{3} \delta^i_k \sum_m s^m_m. \tag{I.19}
\]

Also note the following definition for the total time derivative of \( s^{ij} \), which is also valid for any second-rank tensor with both indices raised,

\[
\frac{D(s^{ij})}{Dt} = \frac{\partial s^{ij}}{\partial t} + \sum_k \sum_m \left[ s^{kj}_k \left\{ \begin{array}{c} 1 \\ i \end{array} \right\} + s^{ik}_k \left\{ \begin{array}{c} 1 \\ m \end{array} \right\} \right] \frac{\partial x^m(x^n,t)}{\partial t}. \tag{I.20}
\]

To obtain the dimensions of the various stress tensor components, consistent definitions must be formulated with regard to the derivation of the stress tensor using the concept of a physical contact force and a physical area of contact. Letting \( \overline{dF} \) denote a differential contact force and \( \overline{da} \) denote the differential area of contact, the following decomposition may be made,

\[
\overline{dF} = \sum_i \overline{df^i}_{\overline{v}_i} = \overline{df^1}_{\overline{v}_1} + \overline{df^2}_{\overline{v}_2} + \overline{df^3}_{\overline{v}_3} \tag{I.21}
\]

\[
\overline{da} = \sum_k \overline{da^k}_{\overline{v}_k} = \overline{da^1}_{\overline{v}_1} + \overline{da^2}_{\overline{v}_2} + \overline{da^3}_{\overline{v}_3} \tag{I.22}
\]

where \( \overline{v}_i \) and \( \overline{v}_k \) are the contravariant and covariant base-vector components respectively, for the coordinate system being considered. For the
Cartesian system, all base vectors are unity and have no dimensions. Therefore all force components, \( df^i \), and area components, \( da_k \), have the correct dimensions and represent physical quantities identically. Such is not the case for the cylindrical coordinate system now being considered. Introducing unit length dimensionless base vectors and knowing the values of the contravariant and covariant base vectors for this coordinate system allow Equations (1.21) and (1.22) to be rewritten in the following manner,

\[
\overline{\text{d}F} = df^r e_r + df^\theta e_\theta + df^z e_z \quad (\text{I.23})
\]

\[
\overline{\text{d}a} = da_r e_r + da_\theta e_\theta + da_z e_z \quad (\text{I.24})
\]

Hence,

\[
\text{dim}(\overline{\text{d}F}) = \text{dim}(df^r) = \text{dim}(df^z) = (\text{force}) \quad (\text{I.25})
\]

\[
\text{dim}(df^\theta) = (\text{force})(\text{length})^{-1}
\]

\[
\text{dim}(\overline{\text{d}a}) = \text{dim}(da_r) = \text{dim}(da_z) = (\text{length})^2
\]

\[
\text{dim}(da_\theta) = (\text{length})^3
\]

Stress tensor components are related to the above force and area components by the following equations,

\[
df^i = \sum_k s^i_k da_k \quad (\text{I.26})
\]

Upon expansion of Equations (I.26) and noting the component dimensions given by Equations (I.25), the dimensions for the tensor components of stress may be obtained. Hence for the stress tensor,

\[
\text{dim}(rr) = \text{dim}(zz) = \text{dim}(rz) = (\text{force})(\text{length})^{-2} \quad (\text{I.27})
\]

\[
\text{dim}(r\theta) = \text{dim}(\theta z) = (\text{force})(\text{length})^{-3}
\]

\[
\text{dim}(\theta \theta) = (\text{force})(\text{length})^{-4}
\]

Having these dimensions at hand, a more thorough examination of the
stress-rate tensor may now be made. In addition to the total time derivative of $S^{ij}$, the Jaumann definition is composed of products of the form, $S^{im}$ $\omega_j$.

Therefore, the following matrix product is significant to this analysis,

$$[S^{im}][\omega_j] = [S^{im}][\omega_{mk}] [g^k_j] =$$

$$\begin{bmatrix}
-(s^{12}w_{12} + s^{13}w_{13}) & \frac{1}{2}(s^{11}w_{12} - s^{13}w_{23}) & (s^{11}w_{13} + s^{12}w_{23}) \\
-(s^{22}w_{12} + s^{23}w_{13}) & \frac{1}{2}(s^{12}w_{12} - s^{23}w_{23}) & (s^{13}w_{13} + s^{23}w_{23}) \\
-(s^{23}w_{12} + s^{33}w_{13}) & \frac{1}{2}(s^{13}w_{12} - s^{33}w_{23}) & (s^{12}w_{13} + s^{23}w_{23})
\end{bmatrix}.$$  

(I.28)

Noting once again the dimensions of $S^{ij}$, $\omega_{ij}$, and $\frac{\partial x^m}{\partial t}$, it is obvious that the dimensions of the elements of Equations (I.28) and (I.20) are the same as the corresponding elements of the $S^{ij}$ tensor, differentiated once with respect to time.

It is now possible to write the appropriate tensor equations and derive therefrom the transformation identities between the corresponding tensor and physical quantities, hence

$$\frac{d_{rr}}{2G} = \frac{s^{rr}}{2G} + \frac{s_{rr}}{2\mu} \left[ 1 - \frac{\tau_{\text{crit}}}{\sqrt{J^1}} \right]$$  

(I.29)

$$\frac{d_{\theta\theta}}{2G} = \frac{s^{\theta\theta}}{2G} + \frac{s_{\theta\theta}}{2\mu} \left[ 1 - \frac{\tau_{\text{crit}}}{\sqrt{J^1}} \right]$$

$$\frac{d_{zz}}{2G} = \frac{s^{zz}}{2G} + \frac{s_{zz}}{2\mu} \left[ 1 - \frac{\tau_{\text{crit}}}{\sqrt{J^1}} \right]$$
\[
\begin{align*}
\sigma_{r\theta} &= \frac{s_{r\theta}}{2G} + \frac{s_{r\theta}}{2\mu} \left[ 1 - \frac{\nu_{\text{crit}}}{\sqrt{3}} \right] \\
\sigma_{r z} &= \frac{s_{r z}}{2G} + \frac{s_{r z}}{2\mu} \left[ 1 - \frac{\nu_{\text{crit}}}{\sqrt{3}} \right] \\
\sigma_{r z} &= \frac{s_{r z}}{2G} + \frac{s_{r z}}{2\mu} \left[ 1 - \frac{\nu_{\text{crit}}}{\sqrt{3}} \right].
\end{align*}
\]

The elements of \( S_{ij} \) are obtained from the elements of \( \tilde{S}^{ij} \) by the following operation,

\[
S_{ij} = \sum_{k} \sum_{m} g_{ik} s_{km} g_{mj}
\]

or in matrix form,

\[
[S_{ij}] = \begin{bmatrix}
S^{rr} & r^2 s_{r\theta} & s_{rz} \\
r^2 s_{r\theta} & r^4 s_{\theta\theta} & r^2 s_{r z} \\
s_{rz} & r^2 s_{r z} & s_{zz}
\end{bmatrix}
\]

Hence for the set \([S_{ij}]\),

\[
\begin{align*}
\text{dim}(rr) &= \text{dim}(rz) = \text{dim}(zz) = \text{force}(\text{length})^{-2} \\
\text{dim}(r\theta) &= \text{dim}(\theta z) = \text{force}(\text{length})^{-1} \\
\text{dim}(\theta\theta) &= \text{force}
\end{align*}
\]

Now noting relations (I.17) and (I.32) together with (I.29), it is seen that by making the appropriate division by a power of \( r \) throughout an entire equation in (I.29), the tensor equation will be reduced to the desired physical form.

In particular, the \((rr)\), \((rz)\) and \((zz)\) equations are already in the desired form. Both the \((r\theta)\) and \((\theta z)\) equations should be divided by \( r \), and the \((\theta\theta)\) equation by \( r^2 \). Letting primes denote the respective physical stress components, it follows that
\[ S_{rr}' = S_{rr} \quad S_{\theta\theta}' = S_{\theta\theta}/r \]
\[ S_{rz}' = S_{rz} \quad S_{\theta z}' = S_{\theta z}/r \]
\[ S_{zz}' = S_{zz} \quad S_{\phi\phi}' = S_{\phi\phi}/r^2 \]

(I.33)

\[ d_{rr}' = d_{rr} \quad d_{\theta\theta}' = d_{\theta\theta}/r \]
\[ d_{rz}' = d_{rz} \quad d_{\theta z}' = d_{\theta z}/r \]
\[ d_{zz}' = d_{zz} \quad d_{\phi\phi}' = d_{\phi\phi}/r^2 \]

(I.34)
APPENDIX II. FORMULATION OF THE GOVERNING STEADY-STATE RELATIONS FOR THE
PROBLEM OF POISEUILLE FLOW.

In the steady-state, if one exists, the following hypotheses may be
made regarding the velocity field,

\[ u_z = u(r) \quad u_\theta = 0 \quad u_r = 0 \quad . \quad (\text{II}.1) \]

At the outset of this analysis all stress components are believed to be
non-vanishing functions of \( r \) and \( z \), but for a material particle indepen-
dent of time. The critical shear stress should be a function of \( r \) alone.

Noting equations (II.1) and the relations for \( d_{ij} \) and \( \omega_{ij} \) in Ap-
pendix I, the only non-vanishing components of these two total tensors
are the following,

\[ d_{rz} = d_{zr} = \frac{1}{2} \frac{\partial u(r)}{\partial r} \quad (\text{II}.2) \]

\[ \omega_{zr} = -\omega_{rz} = \frac{1}{2} \frac{\partial u(r)}{\partial r} \quad . \]

The stress-rate tensor will now be examined. Again using the relations
of Appendix I, the various components of the total time derivative of
the stress tensor are seen to vanish, while the matrix product of

\[ [S^m_{ij}] [\omega^j_m] \] yields,

\[
\begin{bmatrix}
\frac{1}{2} S^{13} \frac{\partial u(r)}{\partial r} & 0 & -\frac{1}{2} S^{11} \frac{\partial u(r)}{\partial r} \\
\frac{1}{2} S^{23} \frac{\partial u(r)}{\partial r} & 0 & -\frac{1}{2} S^{12} \frac{\partial u(r)}{\partial r} \\
\frac{1}{2} S^{33} \frac{\partial u(r)}{\partial r} & 0 & -\frac{1}{2} S^{13} \frac{\partial u(r)}{\partial r}
\end{bmatrix}
\quad (\text{II}.3)
\]

Hence, \( [\hat{S}^m_{ij}] \) becomes,

\[ -32 - \]
It is now possible to form the general constitutive equations for this problem

\[
\begin{bmatrix}
\frac{s_{13}}{2} \frac{\partial u(r)}{\partial r} - \frac{1}{2} \frac{s_{23}}{s_{23}} \frac{\partial u(r)}{\partial r} + \frac{1}{2} \frac{\partial u(r)}{\partial r} (s_{33} - s_{11}) \\
\frac{1}{2} \frac{s_{23}}{s_{23}} \frac{\partial u(r)}{\partial r} - \frac{1}{2} \frac{s_{23}}{s_{23}} \frac{\partial u(r)}{\partial r} \\
\frac{1}{2} \frac{s_{23}}{s_{23}} \frac{\partial u(r)}{\partial r} (s_{33} - s_{11}) - \frac{1}{2} \frac{s_{23}}{s_{23}} \frac{\partial u(r)}{\partial r} - \frac{1}{2} \frac{s_{23}}{s_{23}} \frac{\partial u(r)}{\partial r}
\end{bmatrix}
\]

(II.4)

Note that \( r > r_{\text{transition}} \) if \( r \) and less than zero if \( r < r_{\text{transition}} \).

From (II.5) and (II.7),

\[
S_{zz} = - S_{rr}.
\]

(II.11)

From (II.6), \( S_{\theta\theta} = 0 \).

(II.12)

Noting Equation (II.11), Equation (II.10) becomes

\[
\frac{\partial u(r)}{\partial r} = \frac{1}{2G} \frac{\partial u(r)}{\partial r} (s_{zz} - s_{rr}) + \frac{s_{zz}}{2\mu} \left[ \frac{\sqrt{J_2}}{J_2} \right].
\]

(II.13)
From Equation (II.8) and (II.9) it is obvious that

\[ S_{ez} = S_{r\theta} = 0 \]  \hspace{1cm} (II.14)

The governing tensor constitutive equations, which are also the physical equations due to the absence of any \( \theta \) dependence, reduce to the following

\[ 0 = \frac{S_{rz}}{2G} \frac{\partial u(r)}{\partial r} + \frac{S_{rr}}{2\mu} \left[ \sqrt{J_2^2 - \gamma_{\text{crit}}} \right] \]  \hspace{1cm} (II.15)

\[ \frac{\partial u(r)}{\partial r} = -\frac{S_{rr}}{G} \frac{\partial u(r)}{\partial r} + \frac{S_{rz}}{\mu} \left[ \sqrt{J_2^2 - \gamma_{\text{crit}}} \right] \]  \hspace{1cm} (II.16)

where it must be remembered that \( S_{rr} = -S_{zz} \). Using the derived relations between the tensor and physical stress components and also the definition of deviatoric stress as a function of the tensor stress, the following relations may be given

\[ S_{rr} = S_{rr}' = s_{rr}' - s_{\theta\theta}' = 1/2 (s_{rr}' - s_{zz}') \]  \hspace{1cm} (II.17)

\[ S_{rz} = S_{rz}' = s_{rz}' \]

where primes denote physical quantities. The equilibrium equations are now introduced,

\[ \frac{\partial s_{rr}'}{\partial r} + \frac{1}{r} \frac{\partial s_{r\theta}'}{\partial \theta} + \frac{\partial s_{rz}'}{\partial z} + s_{rr}' - s_{\theta\theta}' = 0 \]  \hspace{1cm} (II.18)

\[ \frac{\partial s_{r\theta}'}{\partial r} + \frac{1}{r} \frac{\partial s_{r\theta}'}{\partial \theta} + \frac{\partial s_{\theta\theta}'}{\partial z} + 2s_{r\theta}' = 0 \]  \hspace{1cm} (II.19)

\[ \frac{\partial s_{rz}'}{\partial r} + \frac{1}{r} \frac{\partial s_{r\theta}'}{\partial \theta} + \frac{\partial s_{zz}'}{\partial z} + \frac{s_{rz}'}{r} = 0 \]  \hspace{1cm} (II.20)

Equation (II.19) is satisfied identically as all of the stress components vanish. Upon noting that there is complete independence of the \( \theta \) - coordinate and that \( S_{rz} \) is a function of \( r \) alone, the equilibrium equations reduce to the following form,
\begin{align}
\frac{\partial s_{rr}'}{\partial r} + \frac{1}{r} s_{rr}' - s_{\theta\theta}' &= 0 \tag{II.21} \\
\frac{\partial s_{2r}'}{\partial r} + \frac{\partial}{\partial z} s_{2z}' + s_{2r}' &= 0 \tag{II.22}
\end{align}

There are only two additional independent relations from which pertinent information may be obtained. One relation is the defining equation for the positive square root of the second invariant of the deviatoric stress tensor,

\[
\sqrt{J_2} = \sqrt{\frac{1}{2} \sum_i \sum_j s^i_j s^i_j} \tag{II.23}
\]

which for this example reduces to the following,

\[
\sqrt{J_2} = \left[ s_{rr}^2 + s_{rz}^2 \right]^{1/2} \tag{II.24}
\]

The other relation is the steady-state form of the critical shear stress.

Using the same form for the critical shear stress as proposed by Slibar and Paslay, namely

\[
\gamma_{\text{crit}} = \gamma_1 - \int_{\xi=\infty}^{\xi=\xi^+} \frac{D_2 e^{-\alpha(t-\xi)}}{E + \int_{\xi=\infty}^{\xi=\xi^+} D_2 e^{-\alpha(t-\xi)} d\xi} \left( \gamma_1 - \gamma_0 \right) \tag{II.25}
\]

the steady state form becomes

\[
\gamma(r) = \gamma_1 - \frac{(D_2)p}{E + (D_2)_p/\alpha} (\gamma_1 - \gamma_0) \tag{II.26}
\]

where

\[
(D_2)_p = \sqrt{\frac{1}{2} \sum_i \sum_j (d^i_j)_p (d^j_i)_p} \tag{II.27}
\]

which for this problem becomes \( +[ (d_{rr})_p^2 + (d_{rz})_p^2 ]^{1/2} \). The individual "plastic" components are obtained from the constitutive equations
and are

\[(d_{rr})_p = -S_{rz} \frac{\partial u(r)}{2G \frac{\partial r}{r}}, \quad \text{and} \]

\[(d_{rz})_p = \frac{1}{2} \frac{\partial u(r)}{\partial r} \left[ 1 + \frac{S_{rr}}{G} \right].\]

Hence \((D)_p\) becomes

\[\frac{1}{2} \frac{\partial u(r)}{\partial r} \left[ \left( \frac{S_{rz}}{G} \right)^2 + \left[ 1 + \frac{S_{rr}}{G} \right] \right]^{1/2} (\text{II.28})\]

and \(\Psi(r)\) may now be expressed as an explicit function of \(S_{rz}\) and \(S_{rr}\) as well as the velocity gradient \(\frac{\partial u(r)}{\partial r}\).

Therefore Equations (II.15), (II.16), (II.21), (II.22), (II.24) and the combined form of (II.26) and (II.28) comprise the system of equations which must be satisfied in the steady-state for the problem cited.
APPENDIX III. NUMERICAL TECHNIQUE EMPLOYED IN THE SOLUTION OF THE

GENERAL FLOW PROBLEM OF SECTION D.

As mentioned in Section D of this thesis, the governing set of equations describing the general flow state was of a sufficiently complex nature to necessitate the use of a numerical technique to effect a solution. The basic approach was one of forward time integration of the two constitutive equations obtained in the analysis. The particular method employed is one due to Runge-Kutta,\(^\text{18}\) which is applicable to a set of first-order differential equations. The basic procedure is outlined below.

Given two simultaneous differential equations of the following form,

\[
\begin{align*}
\frac{dx}{dt} &= f_1(x,y,t) \quad (\text{III.1}) \\
\frac{dy}{dt} &= f_2(x,y,t) \quad (\text{III.2})
\end{align*}
\]

the increments in \(x\) and \(y\) for the first finite interval in time are found from the following relations:

\[
\begin{align*}
& k_1 = f_1(x_o,y_o,t_o) \Delta t \\
& k_2 = f_1(x_o + k_1/2,y_o + m_1/2,t_o + \Delta t/2) \Delta t \\
& k_3 = f_1(x_o + k_2/2,y_o + m_2/2,t_o + \Delta t/2) \Delta t \\
& k_4 = f_1(x_o + k_3,y_o + m_3,t_o + \Delta t) \Delta t \\
& \Delta x = (k_1 + 2k_2 + 2k_3 + k_4)/6
\end{align*}
\]

\(^{18}\text{Reference (9).}\)
\[
m_1 = f_2(x_0, y_0, t_0) \Delta t \\
m_2 = f_2(x_0 + k_1/2, y_0 + m_1/2, t_0 + \Delta t/2) \Delta t \\
m_3 = f_2(x_0 + k_2/2, y_0 + m_2/2, t_0 + \Delta t/2) \Delta t \\
m_4 = f_2(x_0 + k_3, y_0 + m_3, t_0 + \Delta t) \Delta t \\
\Delta y = (m_1 + 2m_2 + 2m_3 + m_4)/6 \\
\]

The increments for succeeding intervals are computed in the same way except that \(x_0, y_0,\) and \(t_0\) are replaced by \(x_1, y_1,\) and \(t_1,\) etc. as one proceeds.

For the case considered in Section D, Equations (III.1) and (III.2) become,

\[
\begin{align*}
\dot{S}_{11} &= S_{12} - G S_{11} \left[ 1 - \frac{\tau_{\text{crit}}}{\bar{\tau}} \right] \\
\dot{S}_{12} &= G - S_{12} - G S_{12} \left[ 1 - \frac{\tau_{\text{crit}}}{\bar{\tau}} \right]
\end{align*}
\]

where the above equations have been non-dimensionalized for purposes of generalizing the solutions obtained. In Equations (III.5) and (III.6), \(\tau_{\text{crit}}/\bar{\tau}\) and \(\sqrt{J_2}/\bar{\tau}\) appear. As before, they are non-constant and assume the following form,

\[
\begin{align*}
\sqrt{J_2}/\bar{\tau} &= \sqrt{\left[ \frac{S_{11}}{\bar{\tau}} \right]^2 + \left[ \frac{S_{12}}{\bar{\tau}} \right]^2} \\
\tau_{\text{crit}} &= \tau_1 - \int_{\xi = \tau_1}^{\xi = t^*} D_2 e^{-\alpha(t - \xi)} d\xi \\
&\quad \left( \tau_1 - \tau_0 \right) \\
&\quad + \int_{\xi = \tau_1}^{\xi = t^*} D_2 e^{-\alpha(t - \xi)} d\xi \\
\end{align*}
\]
where $t^*$ denotes the time at which the plastic yield criterion is attained. As shown in Section D, $D_2$, a measure of the plastic deformation-rate, is a function of both the stress and stress-rate tensors. During any time interval used in the numerical technique, the most current value for $D_2$ is held constant so that formal integration, between $t$ and $t + \Delta t$, of Equation (III.8) might be effected.
FIGURE (1): Physical Configuration for the Problem of Shear Flow.
FIGURE (2): $S_{12}/\tau_0$ versus $Mt$ for the Indicated Values of $G/M\mu$, and the Following Dimensionless Ratios:

$\tau_1/\tau_0 = 10.0 \quad E = 0.05 \quad M/\lambda = 1.0 \quad G/\tau_1 = 1.0$
FIGURE (3): \( \frac{S_{11}}{\tau_0} \) versus Mt for the Identical Ratios as Given in Figure (2).
FIGURE (4): $\frac{\tau_{\text{crit}}}{\tau_o}$ versus $Mt$ for the Identical Ratios as Given in Figure (2).
FIGURE (5): $S_{12}/\tau_o$ versus $Mt$ for the Indicated Values of $G/M\mu$, and the Following Dimensionless Ratios:

$\tau_1/\tau_o = 10.0 \quad E = 0.01 \quad G/\alpha \mu = 10.0 \quad G/\tau_1 = 5.0$
FIGURE (6): $S_{11}/\tau_o$ versus $Mt$ for the Identical Ratios as Given in Figure (5).
FIGURE (7): $\tau_{\text{crit}}/\tau_0$ versus Mt for the Identical Ratios as Given in Figure (5).
FIGURE (8): \( \frac{S_{12}}{\tau_o} \) versus Mt for the Indicated Values of \( \frac{G}{M\mu} \), and the Following Dimensionless Ratios:

\[
\tau_1/\tau_o = 10.0 \quad E = 0.05 \quad M/\alpha = 1.0 \quad \alpha \mu/\tau_o = 5.0
\]
FIGURE (9): $S_{11}/\tau_0$ versus $M_t$ for the Identical Ratios as Given in Figure (8).
REFERENCES


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