THE paradoxical circumstances thus met with as to our differential equations must, of course, correspond to analogous facts concerning point transformations, for we know that the two topics are closely connected.

This is, nowadays, especially clear when we consider point transformations in their whole domains of definition. Indeed, the subject was proposed by the Academy of Sciences of Paris as a prize question for the year 1918, which gave rise to the works of three young geometers (two of whom we have since lost prematurely): Fatou, Lattès, and Julia. It was found that in the case which may be considered the simplest of all—transformations of one complex variable—transformations as simple as

\[ z_1 = 2z^k + 1 \quad (k \text{ an integer}), \quad z_1 = \frac{z + z^2}{2}, \quad z_1 = \frac{3z - z^3}{2}, \]

the most subtle and abstruse considerations of the modern theory of sets are necessary in order to define the regions in which the initial point must be taken in order that its successive consequents approach some one invariant point rather than another. The boundaries of such regions can by no means be considered as "lines," from the older and usual point of view: their study no longer depends on "combinatory Analysis Situs," but on "Analysis Situs of point sets."

\[^1\text{See p. 37.}\]
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We can foresee from these facts that something of this complexity must penetrate even into the local domain surrounding an invariant point; and this appears clearly in some later Chapters of Poincaré himself as well as in important works of other geometers.

We have seen that the most difficult and obscure case occurs when the moduli of $r_1$ and $r_2$ are no longer different from unity. In fact there is one subcase where these difficulties are most easily foreseen: namely, when our transformation is of the form

$$x_1 = x + \cdots, \quad y_1 = y + \cdots,$$

(dots standing, as usual, for terms of higher order), in other words, when its linear part reduces to the identical transformation, so that the distance between any point near the origin and its transform is an infinitesimal of the second order. It is evident that the cumulative and final effect of iterating such an operation will be more difficult to ascertain than in any other case. A fundamental question is the following: will this transformation be stable or unstable? That is, a point $P$ being taken in the neighborhood of $O$, will its successive consequents remain also in the neighborhood of $O$ or will they not?

It is clear that no answer to this question follows from what we have said thus far. But an answer has been given in two short Notes of Levi-Civita in the *Comptes Rendus de l'Académie des Sciences* of Paris, in which the proof is afforded that, apart from exceptional cases, the transformation is unstable. The result may depend, as often happens, on the special choice of $P$ in the vicinity of $O$; but we can indicate a certain angle $A$ at $O$ such that, if $P$ lies within $A$,

1*Les Méthodes nouvelles de la Mécanique Céleste*, Vol. III, Ch. XXIII.
2July 9 and 16, 1900.
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so will all its consequents, as long as they have not emerged from a certain corresponding circular sector S, from which they must necessarily emerge, and that only by crossing its circumference.

From this statement on point transformations, we can, of course, deduce a corresponding one concerning certain classes of periodic solutions.

But the most important improvements upon Poincaré's work, in that line as in several others, are due to Birkhoff, whose Memoir of the Acta Mathematica Vol. XLIII, cited above, is mainly devoted to this question of point transformations about a singular point, under the hypothesis that there exists an integral invariant, a restriction which, as we have seen, does not represent any diminution in the interesting character of the dynamical results.

The starting point of Birkhoff's method is the connection, pointed out by Poincaré, as we have seen above, between a point transformation and a continuous motion. We have noticed that the former can be deduced from the latter; but, conversely, when we are given a point transformation $T$, we can try to regard it as belonging to a continuous family, each point thus moving on a certain line between its original and its final position, say when the time $t$ varies from 0 to 1, further integral values of $t$ corresponding to the iteration of our given transformation if positive, and of its inverse if negative. Birkhoff does not investigate at all how this can actually be done in the most general way and constructs only one continuous set of such transformations, the equations of which can be explicitly written in terms of those of $T$. One of the most striking features of his argument—also connected, as we shall see below, with Poincaré's principles—is that the construction is purely
formal, that is, it gives series which are, as a rule, divergent and that, nevertheless, it enables him to obtain quite effective and rigorous results.

This is obtained by a kind of "on and back" method, which proves to be rather general. Indeed, it can be said to be the origin of practically every result in this kind of question. But it is Birkhoff himself who has put into full evidence the very nature and gist of it, especially when he applied it to such problems as equations in finite differences.¹ Let us suppose that $T$ is unstable, so that, conversely, the iteration of $T^{-1}$ will carry an arbitrary point $P$ (at least if chosen within a proper region $R$) into the immediate neighborhood of the invariant point $O$. Let us suppose, moreover, that we know an approximation $T'$ of $T$, that is, a transformation $T'$ which differs from $T$ only by terms of sufficiently high order in the neighborhood of $O$, and a line $L'$ invariant under $T'$ and located in $R$. Then, the transform $P_k$ of $P$ by the operation $T^k$ will (if $T'$ is a sufficiently close approximation to $T$) approach a limiting position $p$ for $k = \infty$, and the locus of $p$, when $P$ describes $L'$, is a line invariant under $T$.

In the first place, he thus obtains a new solution for the hyperbolic case ($s_1 > 1 > s_2$) previously treated (see p. 66) by Poincaré and Lattès. Choosing the variables so that $T'$ reduces to $^2 x_1 = s_1 x, y_1 = s_2 y = y/s_1$ so that $L$ is the $x$-axis, he finds that the series which give $P_k$ admit a fixed dominating one which is convergent, and that they, therefore, admit of a limiting integral series.

So far, we have only a new and elegant solution of a problem already solved in other ways. But the results are

²In Birkhoff's cases, on account of the existence of the integral invariant, we always have $s_1 s_2 = 1$. 
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of a quite new and remarkable nature when we attack Levi-Civită’s case

\[ x_1 = x + \cdots, \quad y_1 = y + \cdots. \]

Again submitting the diagram to a preliminary transformation so that the approximate invariant line becomes the \( x \)-axis,\(^1\) we find, if the order \( m \) of approximation (that is, the order of contact between \( T' \) and \( T \)) is sufficiently great,\(^2\) that the on-and-back method again gives the desired limiting invariant curves which are the only ones possible.\(^3\) But, this time, nothing allows us to assert that this curve will be analytic at the invariant point: \( y \) may, and usually will, be a function of \( x \) belonging to the class which Birkhoff calls hypercontinuous, its expansion in an integral series being only an asymptotic one, with the radius of convergence zero. That such can be the case, is easily shown by the very simple example:

\[ (T) \quad u_1 = \frac{u}{1 + u}, \quad v_1 = (1 + u)^2(v + u^2), \]

the direct iteration of which gives, for every integral \( k \),

\[ u_k = \frac{u}{1 + ku}, \quad v_k = (1 + ku)^2\left\{v + u_2\left(1 + \frac{1}{(1+u)^4} + \cdots + \frac{1}{[1+(k-1)u]^4}\right)\right\}. \]

This can be written, with the help of the eulerian function \( \Gamma \) and its logarithmic derivative

\(^1\)It proves more convenient, in this case, to do this so as to remove the invariant point to infinity in the \( x \) direction.

\(^2\)The order \( m \) must generally be taken greater than 5, while, in the problem of Poincaré and Lattès, it was sufficient to take \( m = 2 \).

\(^3\)The number of such real curves may be either one or three, their tangents being determined by a cubic equation.
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\[ v_x = (1 + ku)^2 \left\{ v + \frac{1}{6u^2} \left[ \psi''' \left( \frac{1 + ku}{u} \right) - \psi''' \left( \frac{1}{u} \right) \right] \right\}, \]

\[ \psi(z) = \frac{d}{dz} \log \Gamma(z), \]

since \( \psi(z + 1) = 1/z + \psi(z) \). It is then found that the existence of an analytic and regular invariant function or an analytic and regular invariant curve would be contradictory to the fact that the classic asymptotic series for the expansion of \( \log \Gamma(1/u) \) (or any derivative of it) in powers of \( u \) is never convergent.

Poincaré foresaw in the *Méthodes nouvelles de la Mécanique Céleste* not only that such asymptotic expansions must intervene, but that their investigation might lead to effective and important results. We see that this is fully confirmed by Birkhoff’s Memoir.

Further important and delicate researches are pursued by Birkhoff in the same Memoir, and are likely to shed a new light upon the integration of systems of the second order about a periodic solution.

The above considerations—not only the last part of Poincaré’s Memoir but also the aforementioned continuations contributed by his successors—allow us to grasp the general importance of periodic solutions: an importance which has proved greater and greater on account of the results contained in such works as the prize Memoir of the *Acta Mathematica* and the *Méthodes nouvelles de la Mécanique Céleste*. Until a few years ago, these results even inclined us to believe that periodic solutions might generate all other ones—at least, those remaining in a finite domain—in the same way that rational numbers generate irrational ones by a limiting process. This, however, is not exact, as Birkhoff’s works indicate: this rôle does not, in the general case, belong to periodic motions, but to more general
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ones, the "recurrent motions" of Birkhoff. A striking fact, recently proved by Franklin, is the way in which this new and fundamental class of motions is connected with Bohr's almost periodic functions mentioned in the beginning (see p. 5).

But we must not forget that these new conceptions could hardly have arisen without having been prepared by the previous principles of Poincaré, whose sentence\(^1\) remains true: Les solutions périodiques se sont montrées "la seule brèche par laquelle nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable."

J. Hadamard.

\(^1\)Les Méthodes nouvelles de la Mécanique Céleste, I, 82.