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WE shall now make a study of the integral curves on the torus under the hypothesis that there are no singular points whatever. We shall use the coordinates $\varphi$ and $\omega$, where $\varphi$ is the longitude, or azimuth angle, and $\omega$ is a latitude on a meridian circle, each value of $\omega$ defining a determinate parallel on the surface. With Poincaré, we shall agree not to measure the angles $\varphi$ and $\omega$ in the ordinary way, but with the whole circumference, that is, $2\pi$ radians, as unit: in other words, to divide the ordinary measure by $2\pi$. Thus the simultaneous knowledge of $\varphi$ and $\omega$ defines a quite determinate point on it. Moreover, this point does not change if we increase either $\varphi$ or $\omega$ by any integer $n$. However, it is important to notice that, when we follow continuously any trajectory on the surface, the final values of $\varphi$ and $\omega$ at any point of that curve will be perfectly defined if such values are given at an initial point.\(^1\)

Our differential equations will be given in the form

\[
\frac{d\varphi}{dt} = \Phi, \quad \frac{d\omega}{dt} = \Omega
\]

($\Phi$ and $\Omega$ being given functions of $\varphi$ and $\omega$, which are continuous, periodic, and never vanish simultaneously).

\(^1\)It may also be convenient, sometimes, to consider $\omega'$ as equivalent to $\omega$ only after several revolutions: for example, to call $\omega$, $\omega+3$, $\omega+6$, etc., equivalent [$\omega' \equiv \omega \ (\text{mod} \ 3)$]. This amounts to considering a sort of Riemann surface on the torus, the revolution of a point being complete only when it has turned three times; and the same can be done as to $\varphi$.  

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To avoid superficial and uninteresting difficulties, we shall assume that $\Phi$ is constantly positive, so that $\phi$ is constantly increasing; and we can even admit that $\Omega$ is always positive and $\omega$ constantly increasing.¹

Now, starting with a solution curve from $\phi=0, \omega=\omega_0$, let $\phi$ increase continuously. Then $\omega$ varies continuously. When $\phi=1, 2, 3, \ldots$, let $\omega=\omega_1, \omega_2, \omega_3, \ldots$. We shall speak of $\omega_1$ as the first consequent of $\omega_0$, $\omega_2$ as the second one, and so on. Letting $\phi$ decrease, we get likewise the antecedents. As we have said, the values of the $\omega$'s are fully determinate when $\omega_0$ is given, say

\begin{equation}
\omega_1 = \psi(\omega_0),
\end{equation}

so that, to begin, we can plot them on a straight line. If (as assumed) $\Omega>0$, then we have $\omega_0<\omega_1<\omega_2\ldots$.

Our fundamental hypothesis is that two trajectories never intersect. It follows from this that:

1° $\psi(\omega)$ is a constantly increasing function of $\omega$;

2° when the initial point describes once the circumference of a meridian, its consequent will also describe once, and only once, the same circumference, so that

$$\psi(\omega_0+1) = \psi(\omega_0) + 1.$$ 

The first property can be expressed in two other forms, both of which will be of special interest for us, namely,

1° $\psi(\omega)$ gives a bi-univocal and bi-continuous correspondence between $\omega_0$ and $\omega_1$ and, likewise, there is a bi-univocal and bi-continuous correspondence between $\omega_0$ and $\omega_i$.

¹That this second hypothesis is no real restriction of generality appears immediately by considering, instead of (1), the equations

$$\frac{d\phi}{dt} = \Phi, \quad \frac{d\omega}{dt} = \Omega + l\Phi$$

(that is, changing $\omega$ into $\omega+l\phi$), where $l$ is a constant—say an integer—which can be chosen so that $\Omega + l\Phi$ is always positive.
Let us replace $\omega_0$ by another initial value $\tilde{\omega}_0$, and let us suppose that the latter lies, say, between $\omega_0$ and $\omega_1$. Then, $\tilde{\omega}_1$ will lie between $\omega_1$ and $\omega_2$; and, going on in the same way, we see that the $\tilde{\omega}$'s and the $\omega$'s will separate each other.

We have to deal with $\psi^{(2)}(\omega_0) = \psi[\psi(\omega_0)]$, $\psi^{(3)}(\omega_0) = \psi[\psi^{2}(\omega_0)]$, etc., that is, with the iteration of the function $\psi$. The principle just enunciated, shows us that, from the qualitative point of view, the result will be, to a certain extent, independent of the value of $\omega_0$.

Finally, we shall have to take account of the fact that the successive iterates can be augmented or diminished by any integer without changing the geometrical result, so that only the fractional parts $\omega'_{j}$ or, as we shall say, the "reduced values" of the successive quantities $\omega_{j}$ will finally interest us. Thus, the differential equation has disappeared from view and the problem is now of an arithmetical character.\(^1\)

But, for the present, let us still consider the exact values of the $\omega_{j}$'s (and not only their reduced values). A fundamental result is found by Poincaré, namely, that there exists a limit

\[
\mu = \lim_{j(k)} \frac{k}{j(k)},
\]

\(j(k)\) being the number of consequents of $\omega_0$ found within $k$ revolutions around the meridian circle, and, moreover, this limit $\mu$ does not depend on the value of $\omega_0$.

In an important Memoir,\(^2\) to which we shall have to return at greater length, Birkhoff has given the proof in

\(^1\)It may even happen that properly arithmetical researches, such as undertaken in recent times chiefly in connection with Riemann's theory of prime numbers, will help in solving the remaining questions in the present study.

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a simple and elegant form. Let \( \omega_i \) be the \( i^{th} \) consequent of \( \omega \) and let us consider the difference \( \omega_i - \omega \). If \( \omega \), starting from any determinate value \( \omega_0 \), increases to \( \omega_0 + 1 \), then its \( i^{th} \) consequent will increase from \( \omega_i \) to \( \omega_i + 1 \), so that the difference in question will come back to its original value, having therefore a minimum \( a_i \) and a maximum \( b_i \). On the other hand, it can never increase by 1, so that

\[
b_i - a_i < 1, \quad \text{and} \quad \frac{b_i}{i} - \frac{a_i}{i} < \frac{1}{i}.
\]

Let us now replace the integer \( i \) by another integer \( j \), so that \( a_i, b_i \) are replaced by \( a_j, b_j \). The two intervals

(4) \( \left( \frac{a_i}{i}, \frac{b_i}{i} \right) \) and \( \left( \frac{a_j}{j}, \frac{b_j}{j} \right) \)

must have a common part. This results from the consideration of the difference \( \omega_{ij} - \omega \), which must lie between \( ja_i \) and \( jb_i \), and also between \( ia_i \) and \( ib_i \).

This being the case for any two integers \( i \) and \( j \), we see that every \( a_i/i \) is less than any \( b_i/j \). As, on the other hand, the intervals (4) are infinitesimal when \( i \) or \( j \) become infinite, we see that, under these conditions, they approach a common limit \( \mu \) and this is the required limit, \( 2 \pi \mu \) being the "rotation number" (according to Birkhoff's terminology).

We also see what must take place if \( \mu \) is commensurable, say \( \mu = p/q \): then, if we take \( i = q \), we see that \( p \) is contained between \( a_q \) and \( b_q \). Therefore, in any interval of amplitude \( p \) there must lie at least one and even two \( \omega_0 \)'s such that \( \omega_q - \omega_0 = p \): therefore, two closed trajectories.

Moreover, \( p/q \) is contained within any one of the intervals (4).

Conversely, if there is a closed trajectory, there must be an \( \omega_0 \) such that \( \omega_q - \omega_0 = p \), where \( p \) and \( q \) denote two
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integers; we shall have also $\omega_n - \omega = np$ for every integer $n$, and, therefore, the limit of $(\omega_N - \omega_0)/N$ cannot be other than $p/q$. Thus, the commensurability of $\mu$ is a necessary and sufficient condition for the existence of closed trajectories.

Poincaré reaches the same results by directly considering (3). Let us assume, for instance, that there is no closed trajectory, and let us denote, as above, by $j = j(k)$ the number of consequents of $\omega_0$ contained within $k$ revolutions of the meridian circle (starting from $\omega_0$), $\omega_0$ itself being included. Under our present hypothesis, $\omega_j$ must lie beyond $\omega_0 + k$ (and not coincide with $\omega_0 + k$). Then, as we know that $j$ successive consequents of any point will be separated by the points having arguments $\omega_1, \ldots, \omega_j$, we see that any $k$ consecutive revolutions will contain either $j$ or $j - 1$ such points. Moreover, such a sequence of $k$ consecutive revolutions (whatever be their origin) will contain either $j$ or $j - 1$ consequents of $\omega_0$. Let now $l$ be another integer: the number of consequents of $\omega_0$ contained within the first $kl$ revolutions will be contained between $l[j(k) - 1]$ and $lj(k)$. But, similarly, it must be contained between $k[j(l) - 1]$ and $k(j(l))$, so that any two intervals such as

$$(5) \quad \left(\frac{j(k) - 1}{k}, \frac{j(k)}{k}\right), \quad \left(\frac{j(l) - 1}{l}, \frac{j(l)}{l}\right)$$

must have a common part. This again implies the existence of the limit $1/\mu$ which, moreover, must be contained in each interval such as (5).

Assuming now $\mu$ to be incommensurable, so that no closed trajectories exist, let us consider the circular order of the successive consequents: that is, we plot the values of the successive $\omega_i$'s on the meridian circle—and no longer on a straight line—or, what comes to the same thing, we reduce each of them to its fractional part $\omega_i'$, or “reduced
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value” by subtracting $E(\omega_i - \omega_0)$ where $E(u)$ denotes, as usual, the greatest integer contained in $u$. The successive consequent points $M_i$, under our present hypothesis that $\mu$ is incommensurable, will be infinite in number.

Now, under the same hypothesis of $\mu$ incommensurable, this integer $E(\omega_i - \omega_0)$ is in a direct relation with $\mu$; for we know:

1° that $(\omega_i - \omega_0)/j$ is included between $a_i$ and $b_i$;

2° that $a_i$ and $b_i$ never include any integer (else, by an earlier argument, there would be a closed trajectory);

3° that they include $\mu$.

Then it appears that $E(\omega_i - \omega_0)$ is nothing else than $E(\mu j)$.

We can say more, and assert that the circular order of the $\omega_i$'s (which we already know to be independent of $\omega_0$) is the same as the circular order of the numbers $\mu j - E(\mu j)$. This appears more easily from the first definition (3) of $\mu$. First, $E(1/\mu)$ is nothing else than $j - 1 = j(1) - 1$. The $j^{th}$ consequent will be, then, beyond the initial point $M_0$, but before $M_i$. Now, if, simultaneously, we consider, on another circle, the points $m_1, m_2, \ldots, m_{j(1)-1}$, the arguments of which (still measured with $2\pi$ radians as unit) are $\mu, 2\mu, \ldots, (j-1)\mu$, a $j^{th}$ point $m_i$ corresponding to the argument $j\mu$ will lie between $m_0$ and $m_1$, that is, the circular order $m_0, m_j, m_1$ will be the same as $M_0, M_i, M_1$: whence the analogous conclusion immediately ensues (on account of the above noted principle of separation $1^b$) as to the following points $M_{i+1}, \ldots$, contained within the second revolution.

Generally, let our conclusion be proved for the first $k$ revolutions, the corresponding number $j(k)$ of consequents being equal to $E(k/\mu)$. Then, starting from our initial point in the positive sense, we shall successively find points
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$M_0, M_i, M_j, M'_i, \cdots$, the numbers $i, i', i'', \cdots$, being all less than $j(k)$: the circular order of these points is therefore, under our assumptions, the same as that of $m_0, m_i, m_j, \cdots$; that is, the same as the circular order of $0, \mu_i - E(\mu_i), \mu_i' - E(\mu_i'), \cdots$. Moreover, $i, i', \cdots$, being not greater than $j(k) - 1$, our conclusion is valid for the circular order of $M_{j(k)-1}$ with respect to $M_i, M_{i'}, \cdots$, and, therefore, for $M_{j(k)}$ compared to $M_i, M_{i'}, \cdots$; and, on the other hand, it holds also for $M_{j(k)}$ compared to $M_0$, on account of (5).

Holding for $M_{j(k)}$, it will, as above, hold for $M_{j(k)+1}, \cdots$, throughout the $(k+1)$th revolution; then, our last argument extends it to the $(k+2)$th one; and so on.

All this, evidently, suggests that we consider $\omega_i$ as a (constantly increasing) function of $\mu i$. This matter is connected, as we shall see, with the structure of the set $P$ of all the consequents and antecedents of a given point.

If the trajectory from $M_0$ is not closed, these consequents and antecedents are infinite in number and, therefore, there will be a derived set $P'$. As is well known, such a derived set is closed, that is, it contains the second derived set $P''$.

The above argument shows us, under the same hypothesis that $\mu$ is incommensurable, that any arc between two points of $P$ contains other points of the same set and, therefore, points of $P'$.

But, by another method, also contained in Poincaré’s Memoir, we can prove even more, namely, that if there is no closed trajectory, any arc between two consequents

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1 Poincaré applies the argument in the text only under the hypothesis that $N$ belongs to $P'$. See p. 50 of this lecture. But it holds for any $N$, as we show in the text, and leads to the conclusions

1° that $P'$ does not depend on the choice of $M_0$;

2° that $P'$ is the same, whether deduced from the consequents of $M_0$ or from its antecedents.
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$M_i, M_{i+j}$ of $M$ must include consequents or antecedents (and even an infinite number of consequents or antecedents) of any point $N$. For, let us admit that this arc $M_i M_{i+j}$ would include only $m$ of the consequents of $N$: then the arc $M_{i+j} M_{i+j+1}$ would contain only $m' \leq m$ of them (the $j^{th}$ antecedents of which would belong to the former); the same for $M_{i+j+1} M_{i+j+2}$, and so on. If a finite number of these successive arcs (which are all directed in the positive or all in the negative sense) covered the whole circle, $N$ would have no consequent at all (which is absurd) or at most a finite number of them (so that the trajectory from $N$ would be closed). But if they do not cover the whole circle, the successive points $M_{i+j}, M_{i+j+1}, \ldots$, must tend to a determinate point $\gamma$, which, being together the limit of these points $M$ and of their $j^{th}$ consequents, must coincide with its own $j^{th}$ consequent (on account of the fact that the function $\psi(w)$ is continuous) so that such a hypothesis implies the existence of a closed trajectory.

More precisely, we see that it implies the existence of a limiting cycle, such as we met with in the case of the sphere. Treating in the same way the antecedents instead of the consequents, we should find also a limiting cycle (perhaps coinciding with the former) for negative $t$'s.

The above argument applies to the case where $N$ coincides with $M$ and gives, therefore, our former conclusion. But, in the contrary case, it immediately carries another one which is of evident importance, namely:

If there is no closed trajectory, the derived set $P'$ is independent of the choice of the initial point $M$.

Moreover, the argument shows that any arc between two consequents or antecedents of $M$ necessarily includes (if there is no closed trajectory) an infinite number of antecedents of $M$; and this settles an obvious question.
Evidently, the set $P$ consists of two parts—the antecedents of $M$ and the consequents—giving place each to a derived set. We see that these two derived sets coincide.

I do not intend to make any further study of the case of a limiting cycle or a closed trajectory; and, therefore, I shall exclude it, as a rule, in what follows.

Now, let $M$ be any point of $P$—for instance, $M_0$ itself. If $M$ belongs to $P'$, so do all its consequents (this again following from the fact that the solution is a continuous function of the initial conditions). Conversely, if $M$ does not belong to $P'$, then no point of $P$ belongs to $P'$. We have, then, only two possible hypotheses: 1° that every point of $P$ belongs to $P'$; and 2°, that no point of $P$ belongs to $P'$.

If we refer to what has already been said (see p. 23) concerning stability, we see that the present distinction is equivalent to the distinction between stability and instability in the above sense—that is, Poisson's sense—1° corresponding to stability and 2° to instability. The latter hypothesis is evidently verified in the case which we have just studied, when there is a limiting cycle.

*If (2) takes place for every trajectory, there is a limiting cycle.*

For let $M_0$ be a first point, the consequents of which give a set $P$; $N$, a point of $P'$; $Q$, the set of the consequents and antecedents of $N$. If $Q'$ coincided with $P'$, (1) would be verified for the initial point $N$, which is contrary to the hypothesis. But, if $Q'$ is distinct from $P'$, we know that we must have a limiting cycle.

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1The two cases are not distinct: if any trajectory is closed, every other trajectory must either be also closed or tend to a limiting cycle, this being seen in the same way as in the theory of centers (see p. 10; the argument is clearer if we suppose that the closure takes place after one revolution for $\varphi$ and one revolution for $\omega$, that is, $j=k=1$, but any other case can be reduced to that one, as has been remarked in footnote (1) on p. 42).
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More precisely, we see that (1) is verified (in the absence of limiting cycles) for every initial point which belongs to the derived set of the consequents of another point included in $P'$. It is easily seen that $P'$ is a perfect set.\(^1\)

We proceed to investigate the different kinds of perfect sets that may arise. There is first the possibility that $P'$ contains an arc $AB$ of the circle. We shall consider two cases here: (a) $P'$ includes the whole circle; and (b) $P'$ includes an arc $AB$, but not the whole circle.

We consider (b) first, and suppose $AB$ cannot be enlarged without enclosing points not belonging to $P'$; that is, we will not consider a smaller arc when a larger one containing it may be chosen. Consider the consequent arc $A_1B_1$. If $A_1B_1$ is enclosed in $AB$, then $A_2B_2$, the next consequent, is enclosed in $A_1B_1$, and so on. The limit $A'$ of the points $A, A_1, A_2, \ldots$, gives us a closed trajectory; so this case must be rejected.

If the arc $A_1B_1$ contains points of $AB$ and also points outside $AB$ then $AB$ is included in a larger arc, which we supposed was not the case. Thus if the points appear in the order $AA_1BB_1$, the larger arc $AB_1$ is contained in $P'$. So we put this case aside.

There remain the case that $A_1B_1$ is outside $AB$. Then all consequents of $AB$ are exterior to one another. It follows that all consequents of a point of $P$ lying in $AB$ are outside $AB$; hence no interior point of $AB$ is a point of $P'$. This is contrary to hypothesis.

In all cases, then, we have proved that (b) is impossible. Hence if $P'$ contains any arc, it contains the whole circle. Further if (a) holds for the set of consequents of a point

\(^1\)We already know that $P'$ is closed. On the other hand, if $P$ is included in $P'$, any point of $P'$ is near an infinity of points of $P$, which are points of $P'$; and, therefore, it belongs to $P''$.\)
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It holds for the consequents of any other point $N$ of the circle (if limiting cycles are excluded), as has been shown above.

If we are in this case, we can give a precise answer to one of the questions set in the above. We have seen that, starting from the initial point $M_0$ the argument of which is $\omega_0$, the various consequents of $M_0$ or, what comes to the same thing, the reduced values

$$\omega_i = \omega_i - E(\omega_i - \omega_0) = \omega_i - E(\mu i)$$

occur in the same order as the corresponding values of

$$h_i = \mu i - E(\mu i)$$

the subtracted term being the same in both formulae.

Now, let $\omega'$ be any given argument between 0 and 1. If it coincides with an $\omega_i$, we shall associate with it the corresponding value of (7). If not, let us notice that there will be an infinity of $\omega_i$ less than $\omega'$ and an infinity of $\omega_i$ greater than $\omega'$, every $\omega_i$ of the second class being therefore greater than any $\omega_i$ of the first. The same inequalities holding, as we have seen, between the corresponding values of (7), these (which, as is well known, are everywhere dense between 0 and 1) will be divided into two classes such that a Dedekind cut is obtained: by this cut a number $h = h(\omega')$ is defined.

The function $h(\omega)$ is defined over the interval $(0,1)$, and continuous along this interval (for we can choose two values $h_i$ and $h_{i'}$ of (7) including $h$ and as near as we like to $h$; and, then, the function considered will be contained between $h_i$ and $h_{i'}$ for every $\omega$ contained between $\omega_i$ and $\omega_{i'}$). It is never decreasing, in fact, under our present assumption that $P$ is everywhere dense over the circle, it is constantly increasing. Since we can find values of (7) less than 1 and arbitrarily near 1, it tends to 1 when $\omega$ tends to
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$\omega_0 + 1$, so that, if we now extend its definition to every real value of $\omega$ by subjecting it to the identity

(8) \[ h(\omega + 1) = h(\omega) + 1, \]

its continuity will be maintained throughout $(-\infty, +\infty)$. On account of (8), the difference $\theta(\omega) = h(\omega) - \omega$ will be periodic in $\omega$ (so that it admits of a continuous Fourier expansion).

As (6), (7) and (8) give us

\[ h(\omega_i) = \mu_i, \]

we have, for every $\omega_i$, and, therefore (by continuity), for every $\omega$,

(9) \[ \psi(\omega) + \theta[\psi(\omega)] = \omega + \theta(\omega) + \mu. \]

By this equation—which, under our present hypotheses, can be solved for $\psi(\omega)$—the construction of the function $\theta(\omega)$ would allow us to solve every question relating to the law of consequence.

Such are the results when the perfect set $P$ is everywhere dense. But we know (and, as we have said, Poincaré was the first to discover, though the fact was stated explicitly only by Cantor) that there are other perfect sets, which are nowhere dense. The question arises whether this might be the case for our set $P'$, the derivative of the set $P$ of consequents of $M_0$.

It is known that any perfect discontinuous set is obtained by the removal of the interior points of an infinite number of intervals (“contiguous intervals”), (everywhere dense on the circle) the extremities of which, however, belong to the set: the latter is constituted by these extremities and their limiting points. Thus, under our third hypothesis, the circle will contain three kinds of points:

1° Interior points of the contiguous arcs: these do not belong to $P'$;
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2° Extremities of contiguous arcs;
3° Limiting points of points 2°.

It is clear that the consequent of any contiguous arc is again a contiguous arc. We have seen that the set $P'$ is the same for any choice of the initial point $M_0$. But according to that choice, three kinds of trajectories are possible, two of which are such that the corresponding $P$ is included in $P'$, while under the third one—that is, when the initial point is taken within a contiguous arc—$P$ has no common point with $P'$. Such a distinction could not occur in the two preceding cases in which $P$ was entirely exterior to $P'$, whatever $M_0$ might be, or entirely included in $P'$, whatever $M_0$ might be: so that the question of its possibility is the same as the question whether the perfect set $P'$ can be discontinuous.

These two questions, equivalent to each other, were not solved by Poincaré. He did not prove the impossibility of $P'$ being perfect and discontinuous, though he pointed out some circumstances in which this impossibility would be certain.

This fundamental problem has been recently solved—and thus Poincaré’s theory brought to completion—in a quite simple and elegant manner, by a note of Denjoy in the Comptes Rendus de l’Académie des Sciences. The third hypothesis, viz., that $P'$ is perfect and discontinuous, is certainly excluded if a certain integral (which expresses the relative change in the distance between two neighboring trajectories when $\phi$ increases by $2\pi$) considered as a function of the initial position $M$, is of bounded variation; and this is always the case when the coefficients of the equation are holomorphic—or even when they admit a sufficient number of derivatives.

1Vol. CXCIV, 830, March 7, 1932.