III

EQUATIONS OF HIGHER DEGREE
AND THE INTERVENTION OF ANALYSIS SITUS

After this discussion of centers we come back with Poincaré to the general equation of the first order

\[ f(x, y, y') = 0. \]

If \( f \) is of the first degree in \( y' \), we have the problem already treated. We have seen that the most essential feature of the results obtained is the existence, in general, of limiting cycles: every trajectory which does not end at a singular point, if not closed, is asymptotic to a closed trajectory or "limiting cycle" (and we can even consider a certain class of singular points, namely, foci, as a special kind of limiting cycles). It would appear at first sight that if \( f \) is of higher degree in \( z = y' \), we must have something entirely different, but this is not the case. In a very large class of cases, the new problem is not different from the old. It is solved in the same way and by use of the same principles; the character of the solution does not depend upon the degree of the surface either with respect to \( z \) alone, or to \( x, y, \) and \( z \). It depends upon the properties of the surface, represented by equation (1), with respect to Analysis Situs.

I hope I need not dwell long on what Analysis Situs is. It is concerned with properties which remain unchanged when arbitrary continuous transformations are made. Two surfaces (or geometric beings of any kind) which by continuous smoothing, without any tearing or fastening (think
of them as made of wax), can be transformed into each other, are said to be equivalent from the point of Analysis Situs, or, after Poincaré's terminology, homeomorphic.

Now, it is this point of view of Analysis Situs which proves to be fundamental in the study of differential equations.

This new and essential idea again bears that character of marvelous simplicity which we had to point out, in our first series of lectures. Thinking it over from our present point of view, now that Poincaré has actually taught us how to consider things, we should be tempted to wonder, not how he has been able to discover this rôle of Analysis Situs, but how geometers have had to wait on him for it, so necessary and unavoidable its introduction now appears to us to be.

Let us remember the main features of the history of Analysis Situs. You know that in the time of Euler it may, at least at first sight, have been hardly more than an occasion for rather childish problems, such as the 36 officers, the bridges of Königsberg, and so on. However, there is already one important exception, namely, Euler's classic theorem on polyhedrons, expressed by the formula

$$F + S = A + 2,$$

where $F$, $S$ and $A$ respectively mean the numbers of faces, vertices, and edges of an arbitrary polyhedron.

Now, this very example is connected with an especially curious historical fact, one of the most interesting for every one who (I must confess that such is my case) cares not for the history of Science in itself, but for what we have to learn from it, for those instances in which the past can teach us how to act or how not to act in the future. You know that Cauchy began his scientific work with a Memoir
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(1813) devoted to pure and even elementary geometry and, precisely, to properties of polyhedrons. He gives, or thinks to give, a proof of Euler's theorem (1), which proof has been accepted and taught for years. The proof intends to be absolutely general, not assuming convexity or any other special property of the polyhedron: therefore, of course, it must be and is false.

More than thirty years later, Cauchy had the honor of founding the theory of analytic functions, while Riemann was doing the same thing independently on his side. Though both discoverers attained the same general scope, in several respects our debts to them are unequal. In particular, Cauchy's theory could not have been sufficient to build up the theory of algebraic and abelian functions: one essential element, the genus, would not easily have been set in evidence, whereas, with Riemann, it appears quite intuitively from the most elementary and direct considerations of Analysis Situs. I cannot help thinking that, if Cauchy had perceived and corrected his error in 1813, he would have added this gem to his crown in 1850, instead of leaving it to his rival.

Now, the intervention of Analysis Situs in this case is really a quite general fact, the bearing of which goes far beyond the domain of algebraic or even analytic functions. Its origin is the very essence of Integral Calculus. Riemann's is the first instance in which Integral Calculus—that is, the inference of finite properties of a variety from infinitesimal ones—proves insufficient when left to its own resources exclusively. If we are given only the latter properties, something may, and will very frequently, be lacking

The authors of one of the most popular Treatises (at the time when I was a student) apparently felt some trouble with it, as they most candidly showed by literally reproducing Cauchy's own text within quotation marks, thus leaving him the entire responsibility for it.
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in our knowledge of the former, and this is supplied only by topological elements. These may even be completely independent of the properly analytic ones (which is not the case with algebraic functions). Between ordinary Euclidian space as we are accustomed to conceive it and Klein-Clifford's periodic space, there is no difference whatever in a sufficiently restricted domain: we do not know that this space has not actually the Klein-Clifford structure, and we never shall, because local properties of both spaces are rigorously identical.¹

Riemann's example ought, as we see, to have led geometers to foresee the introduction of Analysis Situs into problems of Integral Calculus, therefore into the integration of differential equations also, as an absolutely necessary thing. Nevertheless, this was not perceived until Poincaré came.

On the other hand, Analysis Situs is not to be introduced at once: another precaution is necessary, the conception of which is also due to Riemann. In fact, Riemann had to correct and improve the very definition of a function of a curve. It is by a misuse of language that we are accustomed to say that the ordinate \( y \) of a circle is a function of the abscissa \( x \): this would be true only if every given value of \( x \) gave one determinate value of \( y \), whereas it actually gives two. From this point of view (as we already noticed in our preceding Lectures), we ought to say that a non-uniform function of \( x \) is not truly a function of \( x \), as does Riemann, by considering such a function \( y \) as dependent not on the

¹A mechanical model of a Klein-Clifford space can be constructed very easily: let us describe it, for simplicity's sake, in two dimensions only. Let a vertical material axis bear a rigid piece of an arbitrary shape which can revolve freely about it and which, in its turn, shall bear the axis \( AB \) of a rigid homogeneous body of revolution which can also rotate freely around \( AB \). If each of these two independent motions is assumed to be frictionless, such a system will be governed, locally, by the same equations as a material point moving freely and frictionlessly on a horizontal plane. Nevertheless, the finite properties of the two motions will be utterly different, since, in the former, the trajectories do not go to infinity.
location of a point in a plane, but on the location of a point on a several-sheeted surface, on which, and only on which \( y \) can be said to be well defined.

An elementary and, for that very reason, a curious instance of the importance of this is given by the question of singular solutions. It is well known that, for our equation (1), the singular solution is obtained as the envelope of the general integral: and, on the other hand, it is classic that we obtain such an envelope by eliminating \( y' \) between (1) and 

\[
\frac{\partial f}{\partial y'} = 0.
\]

Consequently, it has been taught for long years that equations of the form (1) in general admit of singular integrals.

The conclusion is false. The fault lies in the fact of thinking that our equation correctly defines \( y' \) as a function of \((x,y)\), while (1), for each system of values of \( x,y \), gives several values of \( y' \) and, moreover, these cease to be regular functions of \( x,y \) precisely when (1) is satisfied, as the theorem of implicit functions fails then to be valid. Geometrically speaking, there will be several vectors at each point which represents \((x,y)\) in the plane or on the sphere. If we desire a vectorial distribution which is well defined (singular points excepted), we are not justified in considering it on a plane. We must consider it on the surface

\[
f(x, y, z) = 0.
\]

If, at \( M \), a point of the surface, we draw the plane \( dy/dx = z \), the intersection of this plane with the tangent plane gives a definite direction and we are thus provided with a field of vectors tangent to the surface, with the condition that there is a single vector at each ordinary point.
Now, and only now—that is, when the latter condition, which is fundamental, is satisfied—we can give a correct answer to the question of what happens at any point where we have the relation (1').

Geometrically, this equation defines the apparent contour of the surface (2) with respect to the $xy$-plane, that is, expresses that the tangent plane to the surface is parallel to the $z$-axis. But the plane $dy/dx = z$ is also parallel to the $z$-axis: so will be, therefore, their intersection, which is the vectorial direction associated with our point, if this direction is well determined.

If so, our point will be an ordinary point for the vectorial distribution and, consequently, for our differential equation or, more exactly, for the geometrical problem corresponding to it, when considered on our surface (2): through such a point, we shall have, in space, a perfectly determined curve satisfying the differential equation. But let us now come back to the ordinary conception of the problem, that is, project the whole diagram onto the $xy$-plane. The above curve, being projected onto a plane parallel with one of its tangents, will, as is well known, give a cusp in its projection (with a cuspidal tangent which will depend on the osculating plane to our curve in space and, in general, not lie in the tangent plane to the surface); and this is precisely the result which Darboux had proved by a special calculation, and which now becomes quite intuitive.

The only exceptional case occurs when the two planes coincide, namely,

\[(2') \quad \frac{\partial f}{\partial x} + z \frac{\partial f}{\partial y} = 0,\]

so that they no longer determine the vectorial direction. If this takes place at one point of the apparent contour, we have a singular point as previously examined and,
therefore, on the surface, a singular point of Poincaré's diagram: in general, a node, a focus or a pass: but, on the xy-plane, shapes will be different on account of the singularity of the projection.¹

Evidently, we must proceed in the same way for any general study of differential equations: such an equation must not be considered as defining a curve on the xy-plane, but a curve on the surface (2)—namely, a curve such that \( dy/dx = z \).

It may happen that this surface (2) is of genus zero, that is, topologically equivalent to a plane. Then, every result of the former theory, such as spoken of in my Lectures of 1920, will apply. For these results depend on two principles: 1° that two trajectories cannot intersect, except at a singular point; and 2° that a closed curve separates the surface into two pieces. The first principle is valid for any surface; the second is true for all surfaces which can be continuously transformed into a sphere (surfaces of genus zero). Thus, the theory in the first two parts of the Memoir is much more general than appears at first glance. It applies to a great many equations of the first order and higher than the first degree.

I shall not follow Poincaré throughout his researches concerning higher genera. The study is up to a certain point similar to that of the earlier case, but there are certain modifications. There is one case, however, in which the results are new and especially interesting: I refer to the surfaces of genus one, such as the anchor ring.

¹Walther Dyck has discussed such singularities at points satisfying (1°). As we see, the results can be foreseen at once, by projecting Poincaré's diagrams onto our horizontal plane: however complicated the new diagrams may look at first glance, the only difference from the original ones arises from the fact that every meeting point of the curve with the apparent contour gives a cusp on the projection.
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In various mathematical theories dependent upon Analysis Situs we often have to divide our surfaces into two main classes, those of genus $0, 2, 3, 4, \ldots$, and those of genus 1, the case of the genus 1 being special. At bottom this is connected with Euler's theorem. If we think of the surface as made up of (in general, curvilinear) polygons put together, we have, for a surface of genus 0,

$$F + S = A + 2,$$

where $F$, $S$, and $A$ are the number of faces, vertices, and edges respectively. If the genus is $p$, the relation

$$F + S = A + 2 - 2p$$

holds; and if $p = 1$ it becomes

$$F + S = A.$$

Poincaré points out the importance of this in the relation concerning the singular points. We again define, at each point, an index number, which is 0 at an ordinary point, which is 1 for a pass, and $-1$ for a node or focus. The index number is essentially a measure of the rotation of the vector as a small closed curve about the point is traced. On summing the index numbers for the whole surface and employing Euler's theorem, we find

$$c - f - n = A - F - S = 2p - 2,$$

where $c, f, n$ represent the number of passes, foci, and nodes, respectively. Hence, for any vectorial distribution tangent to the surface, there must be singular points, except in the case $p = 1$.

The continuation of the Memoir Sur les courbes définies par une équation différentielle, more exactly of the third part of it, deals precisely with the hypothesis that there is no singular point, the surface (2) being of genus 1 and,
therefore, reducible, without loss of generality, to a torus. We now know that no other value of the genus would be compatible with that hypothesis.

But, before reviewing this study of the torus, we must glance at other works to which Poincaré was almost unavoidably led by the above course of ideas. Knowing, as we now do, the importance of Analysis Situs, and that every passage from local to general points of view, that is, every process of integration, may be profoundly influenced by it, we must conclude that improvements in that branch of science must become, one day or another, essential to Integral Calculus: and that, just as we have been obliged to introduce orders of connection in the study of the general differential equation of the first order, we must expect to be unable to treat correctly differential systems or equations of higher order without a knowledge, as thorough as possible, of Analysis Situs in more than two dimensions. We must not wonder, therefore, that Poincaré was compelled to follow that course and that Lefschetz was able to begin his Treatise on Topology with the following lines:

"Perhaps on no branch of Mathematics did Poincaré lay his stamp more indelibly than on Topology."

On quoting this statement of Lefschetz, let us say immediately that this subject of Analysis Situs has been and continues to be most brilliantly cultivated on American soil. The names of Veblen, Alexander, Lefschetz will remain attached to the foundations of $n$-dimensional Topology, among those of the most prominent successors of Poincaré in that line. For there was much to be done, even after Poincaré's powerful impulse: there is much to be done even now. The subject is one of the most difficult in present mathematical science.

\[^{1}\text{Am. Math. Soc. Colloquium Publications, XII (1930), New York.}\]
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Let me remind you that it now contains two very distinct parts. A considerable series of profound results, which are topological in the sense of the above definition, depend on the modern theory of sets and even represent the most important geometrical applications of that theory. This branch of Topology, which has been considerably developed in recent times, begins, for instance, with the proof of Jordan's theorem. On the other hand, such theorems as Jordan's are considered as evident and admitted once and for all in Topology as Riemann understood it, that is, in what is now called Combinatory Topology.

It is this latter kind of Topology which was undertaken by Poincaré; but, even when the question is reduced to these terms, there is no comparison, as to difficulty, between the case of higher numbers of dimensions and the one which was solved by Riemann's principles. Poincaré himself, and others as well, made some mistakes. For example, Poincaré stated that there could not be a closed "one-sided" surface. Yet, the following figure, similar to a sort of fly-trap used in France, is an instance (due to Klein) of a surface of such a kind. But the mathematician does not need to construct artificially such examples, as there is one which is quite classic, no other than the projective plane.

We shall only mention some of the first and simplest discoveries of Poincaré in Topology, the more because, as we have said, the subject is a most familiar one in America.
To begin, he had to get a quite new view of what is a variety, or of what is a boundary, not excluding the possibility of counting some edges, faces, etc., several times in a boundary. It is in this sense that between several closed \((n-1)\)-dimensional varieties \(v_1, v_2, \ldots\), in an \(n\)-dimensional one \(V\), he considers "homologies," consisting in the fact that proper combinations of the \(v\)'s constitute the complete boundary of a part of \(V\).

It is well known—and this is practically all that was known before Poincaré—that the analogue of Riemann's order of connection, for \(n\)-dimensional varieties, is given by Betti numbers, of which there are \(n-1\) (that is, for \(n=3\), the "linear connection" and the "superficial connection"). Analogy with Riemann's case inclined one to believe that the knowledge of these Betti numbers, together with the number of boundaries, was sufficient to characterize the variety from the topological point of view, so that, for instance, two closed varieties with the same Betti numbers were necessarily homeomorphic. One of Poincaré's first tasks was to show that this idea was false; and, as we shall see below, further discoveries had shown more and more how complicated things are from this point of view.

Between the numbers of Betti, Poincaré discovered a beautiful relation, namely, that they are equal two by two the first one being equal to the last, the second to the last but one, and so on. Poincaré proved this by a proper application of Kronecker's index\(^1\) to intersections of \(k\)-dimensional and \((n-k)\)-dimensional varieties in \(V\); it has been noticed, since, that such a relation can be made almost intuitive by the introduction of a topological duality in polyhedral varieties. Such a duality can be understood by its simplest instance, which is given by the classic polygonal

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\(^1\)See my Lectures of 1920, p. 150, 163.
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divisions on the sphere corresponding to two conjugate regular polyhedra, each polygon of the first division being such that one point inside it (in this case its center) coincides with a vertex of the other division, and conversely.

These Betti numbers $P_i$ intervene in the $n$-dimensional generalization of Euler’s known theorem on polyhedra, the right-hand member of the equation which expresses such a generalization being $P_1 - P_2 + P_3 \cdots$. One curious consequence of this is that, for odd $n$’s, this right-hand member and, therefore, the left-hand one, vanish, on account of the equalities between the $P$’s.

But, notwithstanding the simplicity of the relations between Betti numbers, their proof by Poincaré admitted of an objection which was pointed out by Heggaard, after which Poincaré returned to the subject in five new Memoirs. The difficulty consisted in the fact that it is not allowed to divide a homology by an integer, even if the latter appears as a factor in all the coefficients. Now, this gave place to the discovery, by Poincaré, of new topological invariants, the “torsion numbers,” so that, in order that two varieties be homeomorphic, it is necessary that not only their Betti numbers, but also their torsion numbers be the same.

Even these new conditions were not sufficient, and Poincaré introduced a new element which is connected with the topological character of the variety, namely, a group. To construct this group, he introduces certain functions, $F_1$, $F_2$, $\cdots$, defined by a system of total differential equations

$$dF_1 = A_1 dx_1 + \cdots + A_n dx_n,$$

where the $A$’s are functions of the coordinates and of the $F$’s, and satisfy the conditions for total integrability.

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However, if the region is not simply connected, we get, as a result of following closed circuits, a discontinuous group of transformations on the $F$'s. Poincaré shows that to each variety there corresponds such a group; and two varieties cannot be homeomorphic if the corresponding groups are different.

Again, it might be thought that this new character, combined with the former ones was sufficient to give a complete topological description of the variety. But Alexander has shown that such is not the case, so that, on this question, the final word has not yet been said.

We thus see how abstruse the matter is. It has not ceased to give place to the works of the American school, to which we must add the names of Hermann Weyl and another young geometer, Heinz Hopf. One of the most interesting results of this recent work has been to connect the topological nature of the variety—more precisely, the right-hand member of the formula which generalizes Euler’s theorem—with the distribution of singular points of any vectorial distribution over the variety: a fact which will doubtless be of importance with respect to differential equations, as is shown by the instance of the two-dimensional case.