THE THEORY OF CENTERS

We return to the equation of the first order and first degree

\[ \frac{dx}{X(x, y)} = \frac{dy}{Y(x, y)} \quad \text{or} \quad (1') \frac{dx}{dt} = X, \frac{dy}{dt} = Y. \]

We have seen\(^1\) that the problem is to be formulated in the following geometrical way on a plane or, as Poincaré often does,\(^2\) on a sphere: with each point \(M(x, y)\) on the surface (i.e., the point with cartesian coordinates \(x, y\) on the plane or a point of the sphere corresponding to it by central projection), we associate a vector issuing from \(M\) (and lying in the plane, or in the tangent plane to the sphere at \(M\)), and whose direction will be given by (1). Then, a solution of the problem is a curve of the surface whose tangent at this point is the vector corresponding to that point.

In this definition of the vector field, the direction and not the magnitude of the vector is important. Any vector with components proportional to \(X\) and \(Y\) can replace the original one. When however, \(X = Y = 0\), there is no direction defined, the vector being zero. Around such a point \(O\), the direction of the vector is likely not to be continuous, and \(O\) is a singular point of the differential equation. Following Poincaré and treating, with him, the most obvious

\(^1\)See my previous Lectures, p. 172.
\(^2\)Loc. cit., taking account of the corresponding footnote.
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and general case, in which the expansions of $X, Y$ in powers of $x, y$ contain linear terms, we have seen\(^1\) that he found four usual classes of singular points, differentiated according to the behavior of the solutions in their neighborhoods. They are sufficiently explained by the following four figures. These various cases, however, do not occur with the same generality. Case 4, together with 2, is the usual case when the equation is integrable, but it is highly exceptional in the general case. The general cases are 1, 2, and 3.

The distinction between the above cases depends on two numbers $s$ and $s'$, the roots of a certain quadratic equation which can be deduced from the linear terms of $X$ and $Y$. The node arises when the roots are real and of the same sign; the pass, when the roots are real and of opposite signs; the focus, when $s$ and $s'$ are complex (being conjugate to each other).

When $s$ and $s'$ are pure (and opposite) imaginaries, and only then, the singular point may be a center. This already shows that the case of a center is an exceptional one. But, in fact,—and this is what Poincaré now investigates—we are going to see that this condition of $s$ and $s'$ being pure imaginaries, which is necessary for having a center, is far from being sufficient.

\(^1\)Loc. cit., p. 174.
The question can be faced in the following way. Let us take the starting point \( m \) of our trajectory on a determinate direction issuing from the singular point \( O \): for instance, the direction of positive \( x' \)'s. In the present case, the curve will turn around \( O \) and intersect again the positive part of the \( x \)-axis at \( m' \), which we could call a "consequent" of \( m \). If this new point \( m' \) coincides with \( m \), the trajectory will be a closed one and if this happens for every position of \( m \) sufficiently near to \( O \), \( O \) will be a center.\(^1\) If not,\(^2\) we shall have a diagram such as that represented in fig. 1 (p. 176) of our previous Lectures, and \( O \) will be a focus,\(^3\) the trajectory either approaching \( O \) asymptotically as \( t \) increases or receding more and more from it.

This is what Poincaré is now going to discuss, under the hypothesis that \( X \) and \( Y \) are holomorphic around \( O \):—a hypothesis which, in this case, seems to be an essential one, it being doubtful whether a satisfactory solution can ever be found in the non-analytic case. The method will consist in trying to find, for the given equation, an integral

\[
F(x, y) = C,
\]

\( F \) being required to satisfy the partial differential equation

\[
\frac{\partial F}{\partial x} X + \frac{\partial F}{\partial y} Y = 0.
\]

\(^1\)In the non-analytic case, Bendixson has proposed a slightly different definition; see below, p. 24.

\(^2\)The segment \( mm' \) is a function of the length \( Om \), which becomes infinitesimal with \( Om \). It is an infinitesimal of the first order in the general case of a focus, when \( s \) and \( s' \) are complex; of higher order in the case which we are now dealing with.

\(^3\)If \( m \) is not sufficiently near to \( O \), and \( m' \) lies inside \( m \), the trajectory, instead of tending to \( O \), may admit of a limiting cycle (see my previous Lectures, p. 179–180); but, in the analytic case (Cf. p. 24), this cannot be when \( Om \) is sufficiently small.
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In the present case, it may be assumed that \(X,Y\) respectively begin with the linear terms \(y,-x\)

\[
X = y + X_2 + X_3 + \cdots, \quad Y = -x + Y_2 + Y_3 + \cdots,
\]

(where \(X_q, Y_q\) are homogeneous polynomials of degree \(q\) in \(x,y\)); then the above partial differential equation can be written

\[
y \frac{\partial F}{\partial x} - x \frac{\partial F}{\partial y} = -(X_2 + X_3 + \cdots) \frac{\partial F}{\partial x} - (Y_2 + Y_3 + \cdots) \frac{\partial F}{\partial y}.
\]

If \(F\), assumed to be holomorphic is also written in the form

\[
F = F_2 + F_3 + \cdots,
\]

\(F_q\) again denoting the homogeneous part of order \(q\) of the expansion and \(F_2\) being immediately found to be \(x^2 + y^2\), we see that \(F_q\) will appear in terms of order \(q\) in the left-hand member of (3) and only in higher terms in the right-hand member, so that the \(F\)'s can be determined in the usual way, by successive equations of the form

\[
y \frac{\partial F_q}{\partial x} - x \frac{\partial F_q}{\partial y} = H_q,
\]

where each \(H_q\) (again a homogeneous polynomial of an order denoted by its suffix) only contains those polynomials \(F\) the suffixes of which are less than \(q\), which may be assumed to have been previously calculated. Introducing polar coordinates by \(x = \rho \cos \omega, \quad y = \rho \sin \omega\), we write \(F\) in the form

\[
F = \rho^2 + F_3 \rho^3 + F_4 \rho^4 + \cdots,
\]

where \(F_q\), for the sake of simplicity, is written instead of \(F_q(\cos \omega, \sin \omega)\) and depends upon the sines and cosines
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of $q\omega$, $(q-2)\omega$, $(q-4)\omega$, \ldots. We have, on making the substitution, the successive equations

\[
\frac{dF_3}{d\omega} = H_3 = H_3(\cos \omega, \sin \omega),
\]

\[
\frac{dF_4}{d\omega} = H_4 = H_4(\cos \omega, \sin \omega), \ldots,
\]

where $H_4$ has the properties mentioned above for $F_4$. These are easy to integrate. Thus

\[
\frac{dF_3}{d\omega} = H_3 = -2(X_2 \cos \omega + Y_2 \sin \omega)
\]

\[= A \cos \omega + B \sin \omega + C \cos 3\omega + D \sin 3\omega\]

gives

\[F_3 = A \sin \omega - B \cos \omega + \frac{C}{3} \sin 3\omega - \frac{D}{3} \cos 3\omega,\]

where no constant term is to be added, as only odd multiples of $\omega$ must appear in $F_3$.

Then, we are able to construct $H_4$, viz.,

\[H_4 = -2(X_3 \cos \omega + Y_3 \sin \omega) + (3F_3 \cos \omega + \frac{dF_3}{d\omega} \sin \omega)X_2\]

\[(3F_3 \sin \omega - \frac{dF_3}{d\omega} \cos \omega)Y_2,\]

and we immediately see that this expression, when expressed as a trigonometric polynomial in $\omega$, satisfies the required condition concerning parismmetry, i.e., contains only terms corresponding to even multiples of $\omega$ (not greater than 4):

\[H_4 = C_0 + C_2 \cos 2\omega + D_2 \sin 2\omega + C_4 \cos 4\omega + D_4 \sin 4\omega.\]

But here, two cases are possible. In general, $C_0$ will not be zero. If so, the quantity $F_4$, deduced from $H_4$ by integration with respect to $\omega$, cannot be purely trigonometric: it will contain a term proportional to $\omega$ itself. We can call such a term a secular one: for such a calculation is obviously quite analogous to what is classic in Celestial
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Mechanics, where the presence of these secular terms is a well known difficulty.

If such is the case here, we see that we cannot find for \( F \) an expression of the required form. Are we, then, allowed to say that the origin is certainly not a center? At first, this seems to be doubtful. Indeed, a similar conclusion would not be legitimate in the classic case of Celestial Mechanics: it would be quite possible, there, that a term of the form \( C_\omega \), associated with other ones in \( \omega^3, \omega^5, \ldots \) (such as would appear by pursuing the calculation) could be, for instance, the expansion of

\[
\sin C_\omega = C_\omega - \frac{C_3 \omega^3}{6} + \ldots.
\]

To prove rigorously that matters stand differently in the present problem, and that the case of a center is actually excluded when \( C_0 \neq 0 \), is therefore not an insignificant difficulty. But we are able to master it thanks to the principles already introduced by Poincaré in the earlier parts of the Memoir. Indeed, under the above assumption, we can construct, around the origin, a family of contactless cycles.¹ We do this by substituting for (5) the new equation

\[
(6) \quad \frac{dG}{d\omega} = H_4 - C_0 = C_2 \cos 2\omega + D_2 \sin 2\omega + C_4 \cos 4\omega + D_4 \sin 4\omega,
\]

and noticing that this gives for \( G_4 \) a purely trigonometric function of \( \omega \). Now, the quantity

\[
(7) \quad G = \rho^2 + F_3 \rho^3 + G_4 \rho^4
\]

is a function of \( x,y \) (viz., a polynomial in \( x,y \)) which, being substituted in the left hand side of (2) will give an expression of the form

¹See my previous Lectures, p. 182.
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(7') \( C_0 \rho^4 + \text{terms of higher order} = C_0 (x^2 + y^2)^2 + \text{terms of higher order,} \)

the value of which will be constantly positive throughout a certain circle having \( O \) for its center, if, for instance, we suppose \( C_0 \) to be positive. \( G \) is what we could call an "unintegral," if the word would not look too barbarous: in fact, it has to integrals of our differential equation, the same relation as inequalities to equalities. We have already seen, in our previous Lectures how such "unintegrals" are of constant use in researches of the present kind. This is again the case here: the existence of such an "unintegral" evidently excludes the possibility of any closed trajectory in the neighborhood of the origin. Thus, the required conclusion is proved in full rigor.

It is easy to see that, under such circumstances, \( O \) must be a focus. For, as \( G \) is constantly increasing when \( t \) increases, and remains finite, it must have a limit when \( t \) tends to infinity. Moreover, \( \frac{dG}{dt} \) must tend to zero,\(^1\) which can only be, if it is the case for \( \rho \) (while on the other hand, we know that \( \omega \) is indefinitely increasing).

\(^1\)This may appear as intuitive in the present case. But, generally speaking, we have the theorem, which has been recognized by Kneser as well as myself (1896) to be of great use in these studies, that if a function \( G \) of \( t \) tends to a finite limit when \( t = \infty \) and the second derivative \( G''(t) \) remains finite, then the first derivative \( G'(t) \) tends to zero. The same theorem has been found again by Littlewood, Proc. London Math. Soc., IX (1910-11); also, in a more special case, by C. N. Moore, Trans. Am. Math. Soc., VIII (1907), in connection with arithmetical researches, after which Hardy and Littlewood and several authors, especially Landau and recently Mordell (see also my own paper in the Bull. Soc. Math. France, Proc. Verb. des séances, 1914, 68) generalized it by showing that it corresponds to inequalities between the maxima of successive derivatives of an arbitrary function of a real variable.

Of course, the condition concerning \( G'' \) is fulfilled here, as

\[
G''(t) = X \frac{\partial G'}{\partial x} + Y \frac{\partial G'}{\partial y},
\]

\( G' \) being the expression (7').
Let us now suppose that $C_0$ is zero, so that $F_4$ can be constructed, and even in an infinity of ways, as we can add an arbitrary constant $c$, giving a term $c(x^2 + y^2)$ in the value of $F$. We shall find no difficulty in the construction of $F_6$, which will be quite analogous to that of $F_4$; but an impossibility will again arise in the construction of $F_6$, if the constant term analogous to $C_0$ in the corresponding $H_6$ be different from zero: if so, we shall be able to construct an "unintegral" $G$, the derivative of which will have an expression analogous to (7') but, beginning with a term in $(x^2 + y^2)^3$, whence we can again conclude that the origin is necessarily a focus. We can go on in the same way, the calculation coming to an end if we meet with an $H$ (necessarily of an even suffix) whose constant term—in other words, whose mean value with respect to $\omega$ in the interval $(0, 2\pi)$—is different from zero, in which case we can assert that $O$ is a focus.

I shall, however, insist on one detail which Poincaré does not mention explicitly, so that it might appear to leave room for an objection to his argument, which is, in reality, perfectly valid and rigorous. We have noticed that, if $F_4$ can be found, its value contains an arbitrary constant $c$. Of course, the value of $c$, if left at first indeterminate, will occur in every further calculation; therefore, at a first glance, there seems to be no reason why the behaviour of the calculation of $F_6$ (or of $F_8, \ldots$) should not depend on that value of $c$, the constant term in $H_6$ (or $H_8$, etc.) being zero if and only if $c$ is chosen in a proper way. A similar objection would seemingly arise from the presence of arbitrary constants in $F_6$, $F_8$, etc. That such will never be the case, is a consequence of the above discussion. The possibility of constructing $F_6$, for instance, shows, as is easy to see, that the segment intercepted on a radius vector between the
initial point of a trajectory and the point which is found after turning once around $O$, i.e., the difference between the initial radius vector $\rho_0$ corresponding to $\omega = 0$ and the radius vector $\rho_1$ corresponding to $\omega = 2\pi$, is of the 7th order in $\rho_0$, while it would be of order 5 if the constant term of $H_6$ were different from zero: it is clear that this character only depends on the given question, and not on our ways of performing the calculations. But we can see the same otherwise, by noticing that, if the integral $F$ actually exists, the quantity

$$(8) \quad \bar{F} = \chi(F) = F + cF^2 + c_1F^3 + \cdots$$

will admit of the same property, whatever the function $\chi$ or, consequently, whatever the coefficients $c, c_1, \cdots$ may be; and we can say the same as concerns the fact of $F$ being an integral in the first, the second, $\cdots$ approximation, i.e., such that $\chi(F)$ begins with terms in $\rho^4, \rho^6, \cdots$. Now, the constant $c$ in expression (8) is precisely the additive constant in $F_4$ which we have considered above and denoted by the same letter. A similar observation applying to higher coefficients in the right-hand side of (8), we see that choosing arbitrarily the additive constants in the successive $F_{2n}$'s comes to the same as varying arbitrarily such coefficients, and cannot change the possibility or impossibility of the successive calculations.

We can, for example, say that we take every $F_{2n}$ without a constant term in its trigonometric expansion; or also, as Poincaré will be led to do, that we take it such that it vanishes for $\omega = 0$.

Now, what will happen, if, indefinitely, the $H_{2n}$'s are devoid of constant terms, so that every $F_{2n}$ can be constructed? A formal expression of $F$ is thus obtained. The question is whether it will be convergent or, more exactly
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(according to what we have seen above), whether we can choose the integration constants in the successive $F_{2n}$'s so as to make it convergent.

Before examining this, let us observe that the successive relations $(3')$, $(5)$ could have been found by writing our differential equation in polar coordinates, in the form

\begin{equation}
\frac{d\rho}{d\omega} = -\Psi(\rho, \omega) = -\left(\rho^2\theta_2(\omega) + \rho^3\theta_3(\omega) + \cdots\right),
\end{equation}

which gives for the partial differential equation $(2)$

\begin{equation}
\frac{\partial F}{\partial \omega} - \Psi \frac{\partial F}{\partial \rho} = 0,
\end{equation}

and that, conversely, every equation of the form $(9a)$ could be treated in the same way as $(1)$, with the only difference that the functions $F_3$, $F_4$, $\cdots$, which would appear as coefficients of the successive powers of $\rho$ in $F$ would no longer be periodic. The calculation would lead us to successive $H$'s, whence the $F_m$'s would have to be deduced let us say for $0 \leq \omega \leq 2\pi$, by integration with respect to $\omega$: for instance, we could take $0$ as the lower limit for each integration.

In particular what should be obtained by treating precisely in this way our original equation $(1)$? Such a treatment will be different from the above: for, up to now, odd terms in $(4)$ have been taken so as to have zero for their mean value, and not so as to be zero for $\omega = 0$. In the new mode of calculation, the above enunciated conditions concerning parisymmetry will not be verified, so that multiples of $\omega$ of every parity may appear in any $F$. Of course, we may again fear that the calculation thus modified might bring in secular terms and, this time, in odd coefficients as well as in even ones; but the objection can be answered
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in just the same way as before: from our integral expansion $F$ (as formerly constructed), we can deduce the new one

$$F + \gamma_1 F^{3/2} + \gamma_2 F^2 + \cdots$$

where the $\gamma$'s are again arbitrary coefficients. In a manner similar to what we have seen above, the choice of $\gamma_1$ is equivalent to the introduction of an arbitrary constant in $F_3$; this being done in a determinate way, the choice of $\gamma_2$ will be equivalent to the introduction of an arbitrary constant in $F_4$; and so on, whence we again conclude that the appearance or non-appearance of secular terms is utterly independent of the values of such constants.

We, therefore, can choose these so that every $F_q$ vanishes for $\omega = 0$; and this will allow us to apply the method of dominating functions. For, if thus conducted, the calculation of the successive $H$'s will introduce only additions, multiplications and integrations with the lower limit zero. Therefore, if we replace the expansion of $\Psi$ by a dominating one, the new expressions of the $H$'s will throughout dominate the original one.

Now, this expansion of $\Psi$ in powers of $\rho$ is assumed to be convergent, and to be so whatever $\omega$ may be (if only real) in an interval $(0, \alpha)$ of values of $\rho$. There is no loss of generality in admitting that $\alpha$ is greater than 1, as this can always be obtained by a homothetic transformation: on account of which, in (9) all the $\theta$'s will be, for every $\omega$ between 0 and $2\pi$, smaller in absolute value than a positive fixed number $K$ and the expansion (9) will be dominated\(^1\) by

$$\frac{K \rho^2}{1 - \rho}.

\(^1\)Poincaré himself assumes $X$ and $Y$ to be polynomials: which leads him to dominate (9) by $\frac{K(\rho^2 + \rho^3 + \cdots + \rho^a)}{1 - K(\rho + \rho^2 + \cdots + \rho^a)}$, from which he gets to (10). But there is no difficulty whatever in taking for $X, Y$ indefinite Maclaurin expansions.
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The question is therefore whether the differential equation

\[ \frac{d\rho}{d\omega} = -\frac{K\rho^2}{1-\rho} \]

admits an integral, or the corresponding partial differential equation

\[ \frac{\partial F}{\partial \omega} = \frac{K\rho^2}{1-\rho} \frac{\partial F}{\partial \rho}, \]

a solution which reduces to \( \rho^2 \) for \( \omega = 0 \) and which is holomorphic in \( \rho \) for all \( \omega \)'s in the interval \((0, 2\pi)\). Now, the general integral of \((11)\) is

\[ \frac{1}{\rho} + \log \rho - K\omega = \text{const.}, \]

and, therefore, a solution \( \xi \) of \((11a)\) assuming the same values as \( \rho \) when \( \omega = 0 \) will evidently satisfy the relation

\[ \frac{1}{\rho} + \log \rho - K\omega = \frac{1}{\xi} + \log \xi. \]

It is thus seen that the required integral will be

\[ \mathcal{J} = \xi^2, \]

\( \xi \) being defined by \((12)\).

Now, the question whether \( \xi \) and, consequently, \( \mathcal{J} \) are holomorphic in \( \rho \) for all sufficiently small values of the latter variable is not, at first sight, as simple as it usually is in similar problems: for the common value of both sides of \((12)\) will become infinite, and even in a rather complicated way, for \( \rho = 0 \). However, in order to resolve this question, or the more general one, which Poincaré does, for the relation

\[ \frac{1}{\rho} + \beta \log \rho + Q(\rho) - K\omega = \frac{1}{\xi} + \beta \log \xi + Q(\xi) = V, \]
with \( Q(\rho) \) holomorphic in \( \rho \) [to which we should be led by integrating similarly to (11) any analytic equation of the type
\[
\frac{d\rho}{Kd\omega} = P(\rho) = \rho^2 + \beta \rho^3 + \gamma \rho^6 + \cdots,
\]
we need only to change variables by writing
\[
\varsigma = \rho(1 + u),
\]
by which (12) becomes
\[
\frac{1}{\rho} + \log \rho - K\omega = \frac{1}{\rho} (1 - u + u^2 - \cdots) + \log \rho + u - \frac{u^2}{2} + \cdots,
\]
or
\[
(13) \quad u - u^2 + \cdots - \rho \left( u - \frac{u^2}{2} + \cdots \right) = K\rho \omega,
\]
the term \( \rho Q[(1 + u)\rho] - \rho Q(\rho) \) having to be added if we are concerned with (12'). Now, (13) will admit, for \( u \), a solution represented by a power series in \( \rho \) with a radius of convergence bounded below for every finite value of \( \omega \) and especially for \( 0 \leq \omega \leq 2\pi \), as is immediately shown by the theorem of implicit functions.

Poincaré himself does not operate exactly as we have just done: he uses two different methods, one of which consists in introducing \( u \) not in (12), but in the differential equation
\[
\frac{d\xi}{P(\xi)} = \frac{d\rho}{P(\rho)},
\]
between \( \rho \) and \( \xi \), the transformation thus obtained belonging to the class studied in the well known work of Briot and Bouquet.

Another proof rests on a direct study of the function \( \xi \) defined by (12'). We must confess that it does not seem to
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us to be entirely convincing;¹ but this is not important, as our first proof is rigorous.

The quantity \( u \) being thus calculated, we go on as usual in the "Calcul des limites," by noticing that

\[
\mathcal{J} = \xi^2 = \rho^2(1 + u)^2
\]

is necessarily the result which we should obtain if we treated the partial differential equation (11a) in the way described above (every integration with respect to \( \omega \) being carried out from the lower limit zero). The corresponding power series in \( \rho \) is convergent, and so is, a fortiori the analogous one obtained by dealing in the same way with (9a).

On the other hand, we know that the latter expansion, under our assumptions,—i.e., if (9a) is obtained from our original equation (1)—will be entirely trigonometric in \( \omega \). Such an expansion does not yet give us a satisfactory answer to our question because it does not satisfy the conditions that the coefficient of \( \rho^q \) only introduces multiples of \( \omega \) with the same parity as \( q \); or, in other terms, it does not remain invariant by changing \( \rho \) into \( -\rho \) and \( \omega \) into \( \omega + \pi \). But this difficulty is easily solved: for such an invariance belongs to the quantity \( \Psi \) which intervenes in the partial differential equation (9a), so that, \( F(\rho, \omega) \) being a solution of that equation, another solution will be

\[
F(-\rho, \omega + \pi),
\]

and a third one

\[
\frac{1}{2}[F(\rho, \omega) + F(-\rho, \omega + \pi)].
\]

¹Poincaré writes (Jour. de Math., 4th ser., I, 187): "Il faut démontrer ensuite que \( \xi \) revient à la même valeur quand \( \rho \) décrit dans son plan, et \( \omega \) dans le sien, un contour suffisamment petit enveloppant le point zéro. Or, dans ces conditions, le premier et, par conséquent, le second membre de l'équation augmenteront d'un multiple de \( 2\pi \), ce qui fera décrire au point \( \xi \) un contour fermé enveloppant le point zéro." This would be valid if \( \xi \) were a uniform function of \( V \), the common value of the two sides of the equation: which he does not prove.
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This one possesses the invariance in question and satisfies, therefore, all the conditions of the problem, which was to be proved.

We see therefore that in the above treated case of \( X, Y \) holomorphic in \((x, y)\) about the origin, the vanishing of all the \( C_0 \)'s is a necessary and sufficient condition in order that this point be a center.\(^1\)

The difficulty will be, in general, to verify the infinite number of conditions thus obtained. Poincaré points out a simple case where their existence can be asserted \textit{a priori}, viz., when the differential equation is symmetrical with respect to the axis of \( x \), which precisely occurs in some problems suggested by Celestial Mechanics.

Let me say that this is one of the first occasions on which Poincaré introduces stability questions; stability being said to take place in the case of a center, where every trajectory is closed, at least inside a certain region \( R \). Poincaré shows that if \( R \) does not include the whole plane, its boundary necessarily goes through a singular point of the equation. On the contrary, the case of a focus, where trajectories indefinitely approach the origin or more and more recede

\(^1\)Bendixson, in his Memoir of the \textit{Acta Math.}, XXIV (see below) solved the question in another and quite direct way. He does not start from the partial differential equation, but from the original one, written in polar coordinates

\[
\frac{d\rho}{d\omega} = h_2 \rho^2 + h_3 \rho^3 + \cdots .
\]

If we again suppose that the data are holomorphic, \( \rho \) will be (see below) a holomorphic function of \( \omega \) and \( \rho_0 \), the latter denoting the initial value of \( \rho \) for \( \omega = 0 \). As, moreover, \( \rho \) must vanish with \( \rho_0 \), we shall have

\[
\rho = \rho_0 + h_2(\omega)\rho^2.
\]

Now, every successive \( k \) will be determined by an integration with respect to \( \omega \), in terms of the preceding one and the \( k \)'s. Things will behave as in Poincaré's discussion, the question being, each time, whether \( k \) will be of a trigonometric form or, on the contrary, will contain a secular term. In this method, there is no difficulty as to convergence, \( \rho \) being known \textit{a priori} to be holomorphic with respect to \( \rho_0 \).
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from it until they leave the region where expansions in powers of \( x \) and \( y \) are (in general) available, corresponds to instability.

But Poincaré also deals with stability or instability as concerns trajectories not contained in the immediate neighborhood of one point by choosing proper instances of equations (of the first order, but not of the first degree) belonging to the type

\[
\frac{d\rho}{d\omega} = f(\rho).
\]

This study interests us as having led him, for the first time, to the consideration of the Poisson stability, which will play an important part in his further researches: as he finds, it may happen that a trajectory, issuing from any given point \( P \), will return an infinity of times to an infinitely close neighborhood of \( P \) (though, perhaps, after having gone, meanwhile, farther and farther away from it, to more and more remote regions of the plane), which is the definition of Poisson stability.

This study of singular points of a differential equation has been pursued by several authors\(^1\) in the direction of higher singularities. These treat the problem rather from the analytic point of view, in the complex field. Such is not the case with Bendixson,\(^2\) especially in his paper in the *Acta Mathematica*, XXIV (1900). Bendixson arrives at


\(^2\) We also mention a paper of O. Perron (*Math. Zeitschrift*) where Poincaré’s theory is resumed with the minimum number of hypotheses concerning the functions \( X, Y \).
Theory of Centers

general theorems valid for any sufficiently regular form of the functions \(X, Y\), even including, at least in the first part of his Memoir, non-analytic ones. The conclusions show a remarkable generalization of what happens in the first case where \(X\) and \(Y\) begin with linear terms. If the singular point \(O\) is not a focus, there exist a certain number of trajectories abutting at \(O\), which divide the vicinity of \(O\) into several regions, in some of which the differential equation behaves as it would in the neighborhood of a pass, while in others,—which are called nodal regions—things take place as around a node, i.e., every trajectory from a point within such a region abuts on \(O\). It may happen that trajectories of the latter kind end at \(O\) only for \(t = + \infty\) or only for \(t = - \infty\), such nodal regions being then called open ones; but we may also find closed nodal regions, which are such that every trajectory through them ends at \(O\) as well for \(t = + \infty\) as for \(t = - \infty\), the trajectories enclosing each other as shown in the diagram below.

Regions of this kind, if they do not include the whole surface, must have for their boundary a trajectory which contains another singular point, as is the case for the region around a center where trajectories are closed.

The only circumstance showing a notable difference from the previous discussion is the possibility of a pass-like region.
Later Work of Poincaré

including in its inside one or several nodal regions $R_1$, $R_2$, \ldots, the trajectories within such a pass-like region behaving as shown in the diagram on the preceding page.

The reader of this paper will notice a difference between Poincaré’s and Bendixson’s definitions of centers: for a singular point $O$ is called by Bendixson a center if there is an infinity of closed trajectories surrounding $O$ and indefinitely near it, while, in Poincaré’s work, a point is a center only if every trajectory passing within a sufficiently close neighborhood of $O$ is closed. But this is an instance in which analytic equations and non-analytic ones may be profoundly different. If $X, Y$ are holomorphic around $O$, either all trajectories passing sufficiently near $O$ are closed or none of them is.

This results from the fundamental theorems, completed by Poincaré (as we have previously seen)\(^1\) by the study of the influence of initial conditions. Let us take our given equation in polar coordinates, with $\omega$ as the independent variable, with respect to which it will behave regularly. Then, $\rho_0$ denoting the initial value of $\rho$ for $\omega = 0$, the value $\rho_1$ assumed by $\rho$ for another given value $\omega_1$ of $\omega$ will be a holomorphic function\(^2\) of $\rho_0$ around $\rho_0 = 0$ (vanishing with $\rho_0$). Let us take especially $\omega_1 = 2\pi$: the difference $\rho_1 - \rho_0$, being represented by a convergent power series in $\rho_0$, will either be identically zero or vanish for no $\rho_0$ between zero and a certain positive value.

A second part of the same paper is more especially devoted to the analytic case. In order to see in what direction

\(^1\)See my previous Lectures, p. 176.

\(^2\)This is, in the first place, proved inside a certain interval of values of $\omega$ around $\omega = 0$ or, more generally, around $\omega_0$, if we should take an initial value $\omega_0$ other than zero. But classical arguments show that the amplitude on such an interval has a lower limit independent of $\omega_0$, so that only a finite number of them is required in order to cover the interval $(0, 2\pi)$: therefore, successive applications of the principle of analytic continuation in a finite number will give the conclusion in the text.
Poincaré’s original discussion has to be generalized for higher singularities, let us mention the precise result obtained by Bendixson as to “nodal” regions. He succeeds in obtaining an upper limit for their number if $X$ and $Y$ begin with terms of order $m$; there cannot be more than $2m$ nodal regions around $O$.

We must not leave the subject of equations of the first order and the first degree without reverting to the first part of the theory as contained in the first two parts of Poincaré’s Memoir and spoken of in my previous Lectures: indeed, we must mention that this part of this theory has been given, in recent times, a quite new and remarkable development. On account of the very boldness and profoundness of Poincaré’s ideas, it could be imagined that he worked for the far future. In fact, some time elapsed before the various parts of his work could be continued, though this was done in several, but always theoretical, directions. Now, it is the more remarkable that, for some years, the methods in the Memoir *Sur les courbes définies par une équation différentielle* have been applied in a quite effective and practical way, for engineering purposes. The question arises on account of the fact that, when treating real and practical dynamical problems, it is absolutely impossible to abstract from dissipative forces, and these completely upset the classic theory of oscillations around a position of equilibrium. Periodic motions occur under quite other circumstances than those classically known, namely when there are auxiliary forces which depend on speed, so that the equations can no longer be assumed to be linear.

The very curious phenomena thus arising and which theory as well as experiment show to be very different from our previous ideas, were, for the first time, discovered by
Van der Pol;¹ these have been studied rather thoroughly as well for natural as for forced oscillations. Now, such a study, especially in the latter case, depends on Poincaré’s methods, the question being to investigate the limiting cycles; and Poincaré’s principles have been used for actual, numerical calculation.²

¹See Phil. Mag., 7 ser., II (1926), 978; Jahrb. der drahtlosen Telegr. und Telephonie, XXVIII (1926), 178; etc.
²See papers of Cartan, Lienard, Le Corbeiller, and, more recently (in the Archiv für Elektrotechnik), Andronon.