Toric fibrations
and models of universal torsors

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We study smooth projective threefolds fibered by toric surfaces over the projective line. We show that for certain families of degree 6 del Pezzo and quadric surface bundles the universal torsor corresponding to the generic fiber extends to a smooth model over the base. It respects the action of model for the Néron-Severi torus and induces the Abel-Jacobi map from the space of sections. This corresponds to the map from a set of rational points on the generic fiber to the Galois cohomology group of torsors under the Néron-Severi torus. For the model of the latter we also compute corresponding groups of torsors over the base.
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Introduction

The Jacobian of a smooth complex curve is a classical example of bringing transcendental methods from complex analysis into the algebraic setting. Originally defined as a space of periods on a complex curve, the Jacobian admits a purely algebraic interpretation as an abelian variety – the group of degree zero line bundles and the Albanese variety of a curve. Construction of the latter exists over an arbitrary base field (see, e.g. [Mil86]) and hence, this object, along with its universal property, exists for a smooth algebraic curve over an arbitrary field.

To a smooth complex projective threefold $X$ which does not admit a nonzero holomorphic 3-form one can naturally associate its Intermediate Jacobian $IJ(X)$, the abelian variety obtained as the quotient

$$IJ(X) = H^2(X, \Omega^1) / H^3(X, \mathbb{Z}).$$

The group $Z_1$ of algebraic 1-cycles on $X$ is the group generated by curves. Any 1-cycle homologous to zero is the boundary of a 3-chain $\gamma$. Integration along $\gamma$ is a functional on the vector space $H^1(X, \Omega^2)$, hence, by duality, it defines an element in $H^2(X, \Omega^1)$. This induces a map from the subgroup of 1-cycles homologous to zero $Z_{1,\text{hom}}$ to the Intermediate Jacobian – the Abel-Jacobi map $AJ$ ([Tur72]):

$$Z_{1,\text{hom}} \xrightarrow{AJ} IJ(X).$$

Let $\pi : \mathcal{X} \to B$ be a smooth toric surface fibration over a curve. What can we say about the sections of $\pi$? In this setting the Abel-Jacobi map appears naturally as sections form a subset of 1-cycles on $\mathcal{X}$. The difference of two sections representing the same homology class $\beta \in H_2(\mathcal{X}, \mathbb{Z})$ is a homologically trivial cycle. For given $\beta$
fix a basepoint $\sigma_0 \in \text{Sect}(\pi, \beta)$ in the space of sections corresponding to class $\beta$. Then the Abel-Jacobi map on cycles induces a map from the space of sections $\text{Sect}(\pi, \beta)$ to the Intermediate Jacobian:

$$\text{Sect}(\pi, \beta) \to \text{IJ}(\mathcal{X}).$$

$$\sigma \mapsto \text{AJ}((\sigma - \sigma_0))$$

This gives a way to analyze the sections by considering fibers of the above morphism. For quadric surface bundles Hassett and Tschinkel introduced the notion of height ([HT12]) and obtained structural results about the fibers of the above map for sections of a fixed height.

What if $\mathcal{X}$ is defined over an arbitrary field? Does there exist a natural analogue to the Intermediate Jacobian and the Abel-Jacobi map? Suppose $\mathcal{X}$ is defined over $\mathbb{Q}$. In his talk [Maz14] Barry Mazur discussed the phantom variety $J_p(\mathcal{X})_{\mathbb{Q}}$, the abelian variety over $\mathbb{Q}$ obtained from the twisted Galois representation on the group $H^3_{\text{et}}(X \times \overline{\mathbb{Q}}, \mathbb{Q}_l(1))$. If it exists, then over the complex numbers the Intermediate Jacobian becomes an obvious candidate. Moreover, at the places $p$ of good reduction for $\mathcal{X}$ the phantom would define abelian varieties over fields $\mathbb{F}_p$ which naturally become an analogue to the Intermediate Jacobian over finite fields. When does this abelian variety exist in general and how does one construct it?

In recent work, Achter and Casalaina-Martin show that this phantom abelian variety exists for threefolds over $\mathbb{Q}$ whose Chow group of points is supported on a curve, as well as for conic bundles over $\mathbb{P}^2_{\mathbb{Q}}$ ([ACM14]). The work of Harris, Roth and Starr [HRS02] suggests that the origin of these abelian varieties might be Albanese varieties for certain spaces parametrizing curves on a threefold $\mathcal{X}$. In particular, they show that the Intermediate Jacobian of a cubic threefold is the MRC-quotient for varieties of rational curves on it of degrees up to 5. The classical construction of the Intermediate Jacobian of a cubic threefold as an Albanese variety of the space of lines on it is due to Clemens and Griffiths ([CG72], see also the paper of Turin [Tur72]). Certain families of threefolds fibered in degree $\leq 5$ del Pezzo surfaces over $\mathbb{P}^1$ have the Prym-type construction of their Intermediate Jacobian, which is due to Kanev ([Kan89]).
We now consider how this is related to rational points on the generic fiber. The properness of the map $\pi$ implies that there is 1-1 correspondence

$$\text{Sect}(\pi) = X(K)$$

between the sections of $\pi$ and the $K$-rational points on the generic fiber $X$, where $K = k(B)$ is the function field of the $k$-curve $B$. This observation provides another approach: producing sections becomes equivalent to the arithmetic problem of producing rational points on the generic fiber. If $B = \mathbb{P}^1$ over an algebraically closed field $k = \overline{k}$, the existence of a rational point follows from the Tsen’s theorem ([Sha94]); for a finite field $K$ the existence of a rational point follows from the Chevalley-Warning theorem ([Ser96]). In general, two necessary conditions for the existence of a rational point on $X$ are

1. $X_p(K_p) \neq \emptyset$ for every $p \in B$;

We notice that degree 6 del Pezzo and quadric surfaces considered in this work satisfy Hasse principle over the global field $K$ (see, e.g., Theorem 2.1 in [VA13], or [CT72] for an elementary proof when $K$ is the number field); in particular, for them condition 1 becomes sufficient.

The geometric Picard group of the generic fiber is a Galois module. It corresponds to a torus over the field $K$, the Néron-Severi torus $T_{NS}$. Colliot-Thélène and Sansuc introduce a special class of principal homogeneous spaces under tori $T_{NS}$ – universal torsors ([CTS87b]). These are varieties $U$ over $X$ with an action of $T_{NS}$. In particular, every rational point on $X$ can be lifted to a point on some unique universal torsor inducing the partition of the set $X(K)$ by the classes of universal torsors. Over the base $B$, the Néron-Severi torus $T_{NS}$ admits a model $\mathcal{T}$. Does the universal torsor $U$ extend to a smooth model $\mathcal{U}$ over $B$ such that $\mathcal{T}$ still acts? If such model exists it induces a non-canonical pull-back map from the space of sections to the group of $\mathcal{T}$-torsors over $B$:

$$\text{Sect}(\pi) \to H^1_{\text{ét}}(B, \mathcal{T}).$$
Over an algebraic closure $\overline{k}$ of the base field $k$ this can be considered as an algebraic analogue to the Abel-Jacobi map. It describes the geometry of all sections, but also descends to a map over $k$. A similar approach in characteristic 0 was discussed by Yi Zhu ([Zhu11]) in the setting of homogeneous fibrations over curves with the relative Picard number equal to 1. Zhu constructs the Abel-Jacobi map from the space of sections $\text{Sect}(\pi)$ to the Jacobian $J(D)$ of the curve $D$, the étale cover of the base obtained as the unique connected component of the relative Picard functor $\text{Pic}_{X|B}$.

In this work we consider toric fibrations over the projective line $\mathbb{P}^1$ and the corresponding Néron-Severi tori $\mathcal{T}$ of their generic fibers. Our main interest is in the construction of examples for families of fibrations and computing the groups of torsors $H^1(\mathbb{P}^1, \mathcal{T})$. We work with two classes of toric fibrations: degree 6 del Pezzo and quadric surface bundles. Using the toric construction given by Colliot-Thélène and Sansuc, we construct the universal torsor and its model. In general, the explicit construction of the universal torsors for the non-toric case is more subtle (see, e.g. [Der07] and [Has04]).

The work is organized as follows. In Chapter 1 we outline the background material on principal homogeneous spaces under algebraic tori. We give the definition of Colliot-Thélène and Sansuc of the universal torsor and discuss how the existence of its model over $\mathbb{P}^1$ leads to a map into a group of torsors.

In Chapter 2 we construct an explicit family of quadric bundles, compute its cohomology and construct the model of its universal torsor.

Chapter 3 is devoted to degree 6 del Pezzo fibrations. After giving a short review of del Pezzo surfaces, we discuss singular fibers and the local monodromy. Then we construct an example of a fibration and compute groups of torsors for different cases of Galois groups acting on the generic fiber. For the quadratic splitting we also construct the model of the universal torsor. For the case of maximal monodromy we show how Kanev’s construction of the Intermediate Jacobian leads to the same abelian variety as the identity component of the group of torsors.
Chapter 1

Toric surface fibrations

In this chapter we discuss a model for the Néron-Severi torus associated to a threefold fibered by toric surfaces over the projective line. We first outline the general theory of algebraic tori and their principal homogeneous spaces. Then we describe the notion of universal torsor introduced by Colliot-Thélène and Sansuc. Finally, we discuss how the model of the universal torsor over the base of fibration induces the map from the space of sections.

1.1 Algebraic tori and their torsors

An algebraic torus over a field $K$ is an algebraic group $T$ which becomes isomorphic to a direct product of multiplicative groups $\mathbb{G}_m$ after passing to an algebraic closure $\overline{K}$ of the ground field. The groups of characters of $\overline{T} = T \times_K \overline{K}$ and $\mathbb{G}_m^d$, therefore, are isomorphic: they are equal to $\mathbb{Z}^d$, where $d$ is the dimension of torus. However, in general, characters might have different structure as Galois modules. In particular, every torus $T$ over the field $K$ is defined by its $G$-module of characters $\widehat{T}$ ([Bor91]), where $G$ is the Galois group of the field $K$. This fact reflects the duality between the categories of algebraic tori over $K$ and finitely generated free $G$-modules. The equivalence is given by the contravariant functor $T \rightarrow \widehat{T}$.

A principal homogeneous space under a torus $T$, or simply a $T$-torsor, is a $K$-variety
which admits an action of $T$ such that $\overline{Y}$ with the induced action of $\overline{T}$ becomes isomorphic to the torus $\mathbb{G}_m^K$ acting on itself. The set of isomorphism classes of torsors under a torus $T$ is given by the étale cohomology group $H^1(K, \mathcal{T})$ ([Vos98], 3.5).

The module of characters $\hat{T}$ is called a permutation module if, as a group, it has a basis that is permuted by the action of $G$. The corresponding torus in this case is called quasi-split. It is known that for quasi-split tori $H^1(K, \mathcal{T}) = 0$, i.e. every principal homogeneous space is isomorphic to torus $T$ itself. In [CTS76] Colliot-Thélène and Sansuc define a special class of tori whose character group $\hat{T}$ satisfies the property $H^1(\mathcal{H}, \text{Hom}_{\mathbb{Z}}(\hat{T}, \mathbb{Z})) = 0$ for all subgroups $\mathcal{H}$ of $G$.

These tori are called flasque and the main interest in defining them comes from the fact that for every torus $T$ there exists a flasque resolution

$$1 \to F \to E \to T \to 1 \quad (1.1)$$

where $F$ is a flasque torus and $E$ is quasi-split torus. This fact was established independently by Voskresenskii ([Vos69]) and Colliot-Thélène and Sansuc ([CTS87a]; see also the survey [Kun07]).

A normal algebraic variety $X$ over field $K$ is called toric if it contains a dense open subset $Y \subset X$ isomorphic to a principal homogeneous space under a certain $K$-torus $T$, and $X$ admits an action $T \times X \to X$ extending the action of $T$ to its principal homogeneous space.

Let $X$ be a smooth toric surface – a toric variety of dimension 2. In this case the Picard group $\text{Pic}(\overline{X})$, the group of divisors on $\overline{X}$ modulo linear equivalence, is a finitely generated free abelian group. It admits a $G$-action and by the duality of categories corresponds to a certain algebraic torus $T_{\text{NS}}$ over $K$, the Néron-Severi torus of $\overline{X}$. In other words, $T_{\text{NS}}$ is a torus whose module of characters is exactly $\text{Pic}(\overline{X})$. The torus $T_{\text{NS}}$ is flasque ([Vos77], 4.48) and fits into the resolution (1.1) which can be constructed from the exact sequence of the dual modules

$$0 \to \hat{T} \to \mathbb{Z}^{\text{NS}} \to \text{Pic}(\overline{X}) \to 0 \quad (1.2)$$
This sequence has the following geometric interpretation: the module $\mathbb{Z}^\Sigma$ as a group is generated by the curves $D$ lying in the complement of $\overline{Y}$ in $\overline{X}$. They are permuted by the Galois action and are mapped to corresponding classes of divisors $[D] \in \text{Pic}(\overline{X})$. The character $\lambda \in \hat{T}$ defines a birational map $\lambda : \overline{X} \to K^\times$. In the above sequence it is mapped to the corresponding divisor $\text{div}(\lambda) \in \mathbb{Z}^\Sigma$ which is supported on the set of curves $D$.

The dual sequence of tori becomes

$$1 \to T_{\text{NS}} \to T_\Sigma \xrightarrow{\varphi} T \to 1$$  \hspace{1cm} (1.3)

In particular, this endows $T_\Sigma$ with a left $T_{\text{NS}}$-action $T_{\text{NS}} \times_T T_\Sigma \to T_\Sigma$ satisfying the property that the map

$$T_{\text{NS}} \times_T S \to S \times_T S$$
$$ (g, s) \mapsto (gs, s)$$

is an isomorphism. Consequently $S$ is called a $T_{\text{NS}}$-torsor over $T$. This generalization of torsors over fields to torsors over an arbitrary base was developed by Colliot-Thélène and Sansuc ([CTS87b]). They show that the group of isomorphism classes of these torsors is in one-to-one correspondence with elements of the Čech cohomology group $\check{H}^1(T, T_{\text{NS}})$. This can be extended further up to arbitrary, possibly noncommutative, group schemes over a general base, but then the classifying cohomology “group” becomes only a pointed set (see, e.g. [Poo15]).

Within this theory, $S$ corresponds to a class in $\check{H}^1(T, T_{\text{NS}})$. The embedding $T \to X$ gives the natural map

$$\check{H}^1(X, T_{\text{NS}}) \to \check{H}^1(T, T_{\text{NS}})$$  \hspace{1cm} (1.4)

which is surjective given the flasqueness of $T_{\text{NS}}$. The surjectivity follows from the result of Colliot-Thélène and Sansuc ([CTS87a]) which can be restated by saying that any torsor over an embedded torus under a flasque group can be extended to a torsor over the whole variety $X$. This leads to the torsor $\tilde{S}$ containing $S$ and corresponding to some class $[\tilde{S}] \in \check{H}^1(X, T_{\text{NS}})$ and extending the map $\varphi$ from (1.3) to $\varphi : \tilde{S} \to X$. In the last section of this chapter, we use the result from ([CTS87b]) to construct $\tilde{S}$ and its model. Given a rational point $a \in X(K)$ the pullback $\tilde{S}_a = \tilde{S} \times_X K$ is a $T_{\text{NS}}$-torsor over $K$:
Therefore, any torsor \( S \in \tilde{\text{H}}^1(X, T_{NS}) \) induces a natural map from the set of rational points of a toric variety to the group of \( K \)-torsors under the Néron-Severi torus:

\[
X(K) \to H^1(K, T_{NS})
\]  

(1.5)

Considering pre-images under this map induces the equivalence relation on \( X(K) \) and hence a partition of the set of rational points.

Fix a torsor \( S \in \tilde{\text{H}}^1(X, T_{NS}) \). Then any element \( \lambda \in \text{Pic}\overline{X} \) defines a map \( T_{NS} \xrightarrow{\lambda} \mathbb{G}_{m, \overline{K}} \) since \( \lambda \) is a character. This induces the composite map

\[
S \in H^1(X, T_{NS}) \to H^1(X, T_{NS}) \xrightarrow{\lambda} H^1(X, \mathbb{G}_{m, \overline{K}}) = \text{Pic}\overline{X}
\]

Varying \( \lambda \in \text{Pic}\overline{X} \) we obtain the map \( \chi : \tilde{\text{H}}^1(X, T_{NS}) \to \text{End}_G(\text{Pic}\overline{X}) \) which fits into the exact sequence

\[
0 \to H^1(K, T_{NS}) \to H^1(X, T_{NS}) \xrightarrow{\chi} \text{End}_G(\text{Pic}\overline{X})
\]  

(1.6)

The torsor \( S \) is called \textit{universal} if \( \chi(S) = \text{id} \). In particular, (1.6) shows that, if nonempty, the set of universal torsors is itself a principal homogeneous space under the group \( H^1(K, T_{NS}) \). The main interest in universal torsors comes from the following fact:

**Theorem.** (Colliot-Thélène, Sansuc, 1976) \textit{The equivalence relation on the set} \( X(K) \) \textit{defined by a universal torsor is stronger than any other defined by arbitrary} \( M \)-torsor \textit{for arbitrary} \( K \)-torus \( M \).

In the following we will consider universal torsors for toric surfaces in more detail.
1.2 Toric surface fibrations and models of tori

Let $k$ be an algebraically closed field and consider a proper morphism

$$\pi : \mathcal{X} \to \mathbb{P}^1$$

to the projective line from a smooth threefold $\mathcal{X}$. Assume that a general fiber of $\pi$ is isomorphic to a smooth projective toric surface. The fiber $X$ over the generic point $\text{Spec} k(t)$ in this setting is also a smooth toric surface, but over the field $K = k(t)$. As described in the previous section, it contains as a dense open subset a principal homogeneous space under some algebraic torus $T$. The threefold $\mathcal{X}$ is a smooth model of $X$ over the Dedekind scheme $\mathbb{P}^1_k$. We are interested in constructing two models: $\mathcal{T}$ for the associated torus $T_{NS}$ and, if a universal torsor $S$ for $T_{NS}$ exists, its model $\mathcal{U}$. The latter should satisfy the property that the action of $\mathcal{T}$ extends to it:

$$T_{NS} \cap S \quad \mathcal{U} \cap \mathcal{T} \quad ?$$

Motivation comes from the fact that properness of $\pi$ implies that sections of the fibration are in one to one correspondence with the set of rational points $X(K)$ of the generic fiber. If both models exist then pulling back the universal torsor by a section $\sigma \in \text{Sect}(\pi)$ gives an element $\sigma^* \mathcal{U} \in H^1_{\text{et}}(\mathbb{P}^1, \mathcal{T})$. This would be compatible with the map (1.5) making the following diagram commute

$$\text{Sect}(\pi) \to H^1_{\text{et}}(\mathbb{P}^1, \mathcal{T})$$

$$= X(K) \to H^1(K, T_{NS})$$

where the right vertical map is the pullback to the generic point.

The torus $T_{NS}$ is flasque. For $K$ finitely generated over the prime field Colliot-Thélène
and Sansuc show that the group $H^1(K, T_{\text{NS}})$ is finite (([CTS76])). Then the partition induced by torsor $S$ is also finite.

Often $H^1(\mathbb{P}^1, \mathcal{T})$ has an abelian variety as the identity component. When $k$ is the field of complex numbers the map

$$\text{Sect}(\pi) \to H^1(\mathbb{P}^1, \mathcal{T})$$

is expected to coincide with the Abel-Jacobi map to the Intermediate Jacobian of the threefold $\mathcal{X}$ ([Tur72]). In the following we construct families of toric bundles and compute the group of torsors for corresponding tori.
Chapter 2

Quadric surface bundles

A quadric surface over the field $K$ is a projective variety in $\mathbb{P}^3_K$ given by a homogeneous polynomial $p(x, y, z, w) = 0$ of degree 2. Passing to a fixed algebraic closure $\overline{K}$ allows one to diagonalize the form $p$ by transforming it into a sum of squares using an appropriate change of coordinates. The diagonal form with four squares

$$x^2 + y^2 + z^2 + w^2 = 0$$

defines a smooth quadric surface. The diagonal form with three squares

$$x^2 + y^2 + z^2 = 0$$

defines a quadric surface with one singular point $[x : y : z : w] = [0 : 0 : 0 : 1]$ which is an ordinary double point. In this case, the surface is referred to as the singular quadric cone. It is known that a smooth quadric surface over an algebraically closed field is isomorphic to the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ (see, e.g. [Har95]). In particular, the surface contains two families of lines (rulings) coming from each copy of $\mathbb{P}^1$ and its (geometric) Picard group is isomorphic to $\mathbb{Z}^2$ generated by these rulings.

Consider a quadric surface $X$ over the field $K = k(t)$ given by the equation

$$x^2 - f(t)y^2 = zw,$$

where

$$f(t) = (t - a_1)(t - a_2) \cdots (t - a_{r-1})$$

and

$$[x : y : z : w] \in \mathbb{P}^3_K$$

(2.1)
is a polynomial of odd degree $r - 1$ with distinct roots. It admits a model over $\mathbb{P}^1$: in one chart with local coordinate $t$ the model is defined by the same equation. In the other chart with point at infinity $t = \infty$ where the local coordinate is $s = 1/t$ equation (2.1) becomes

$$x^2 - s(1 - sa_1)\ldots(1 - sa_{r-1})\left(\frac{y}{s^{r/2}}\right)^2 = zw$$

The change of variable $\overline{y} = y/s^{r/2}$ gives a glueing into the complete smooth model $\mathcal{X}$. The projection $\pi : \mathcal{X} \to \mathbb{P}^1$ is given by the coordinate $t$. The model has $r$ singular fibers with ordinary double points over the points $t = a_1, \ldots a_{r-1}, a_r = \infty$. The fact that $f(t)$ has distinct roots is referred to by saying that the model $\mathcal{X}$ has a square free discriminant.

The generic fiber of this model is the surface $X$ over the non-algebraically closed field $K$. It splits after passing to the degree 2 field extension

$$L = K(\sqrt{(t - a_1)\ldots(t - a_{r-1})})$$

and becomes isomorphic to $\mathbb{P}^1_L \times \mathbb{P}^1_L$. The Galois group $\mathcal{G} = \mathbb{Z}/2\mathbb{Z}$ exchanges the rulings making $\text{Pic}X$ a permutation module. This implies that $H^1(K, T) = 0$ and hence the principal homogeneous space inside the quadric surface is isomorphic to a torus $T$ acting on it. In terms of the defining equation (2.1), this torus corresponds to the open subset $z \neq 0, w \neq 0$.

First we compute the Néron-Severi torus $T_{\text{NS}}$ corresponding to $T$. By definition it is the torus dual to the $\mathcal{G}$-module $\text{Pic}X$. It can be computed (see e.g. [Man86], Section 8.8) from the group ring $K[\text{Pic}X]$ as

$$T_{\text{NS}} = \text{Spec}(K[\text{Pic}X]^\mathcal{G}).$$

The group ring $K[\text{Pic}X]$ is generated by the elements $a, b, \frac{1}{a}, \frac{1}{b}$ where the variables $a$ and $b$ come from classes of two rulings. The Galois action permutes $a$ and $b$. Then the subring of invariants is generated by

$$x = \sqrt{f(t)}(a - b), \quad y = a + b, \quad z = -4abf(t), \quad z^{-1} = \frac{-1}{4abf(t)}$$

which satisfy two relations

$$x^2 - f(t)y^2 = z, \quad zz^{-1} = 1 \quad (2.2)$$
These are the defining equations for $T_{NS}$ over $K$. Since $f(t) \in K[t]$ is a polynomial, this can be considered a model over the open subset of $\mathbb{P}^1$. Similar to the construction of $\mathcal{X}$, this glues up at the point at infinity $t = \infty$ producing the group scheme $\mathcal{T}$, the $\mathbb{P}^1$-model of $T_{NS}$. We notice that the fiber of $\mathcal{T}$ over the general point is isomorphic to $\mathbb{G}_m \times \mathbb{G}_m$ while at the $r$ points corresponding to singular quadric surfaces the fiber is isomorphic to the algebraic group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{G}_a$.

Let $\mathcal{T}_0$ be $\mathbb{P}^1$-torus given locally by the equation

$$x^2 - f(t)y^2 = 1.$$  

Then we have the following exact sequence

$$1 \to \mathcal{T}_0 \to \mathcal{T} \to \mathbb{G}_m \to 1 \quad (2.3)$$

where the map $\mathcal{T}_0 \to \mathcal{T}$ is $(x, y) \mapsto (x, y, 1)$ and the map $\mathcal{T} \to \mathbb{G}_m$ is $(x, y, z) \mapsto z$. This gives the long exact sequence

$$1 \to \pm 1 \to k^\times \to k^\times \to H^1(\mathbb{P}^1, \mathcal{T}_0) \to H^1(\mathbb{P}^1, \mathcal{T}) \to \text{Pic } \mathbb{P}^1 \to 1$$

where the map $k^\times \to k^\times$ is squaring. This shows that cohomology groups of both tori fit into the exact sequence

$$0 \to H^1(\mathbb{P}^1, \mathcal{T}_0) \to H^1(\mathbb{P}^1, \mathcal{T}) \to \mathbb{Z} \to 0. \quad (2.4)$$

Both cohomology groups appearing in (2.4) are computed in the next section.
2.1 Cohomology computations

Over the open subset of $\mathbb{P}^1$ the torus $T_0$ is given by the equation

$$x^2 - (t - a_1)(t - a_2) \ldots (t - a_{r-1})y^2 = 1$$

where $r$ is even and the $a_i$ are distinct. In another chart with local coordinate $s = 1/t$, this becomes

$$x^2 - s(1 - a_1s)(1 - a_2s) \ldots (1 - a_{r-1}s) \left( \frac{y}{s^{r/2}} \right)^2 = 1.$$ 

Therefore, the change of coordinate $y = y/s^{r/2}$ gives a similar equation in the second chart

$$x^2 - s(1 - a_1s)(1 - a_2s) \ldots (1 - a_{r-1}s)y^2 = 1$$

These two glue together to the complete model over $\mathbb{P}^1$, in fact the Néron model:

**Proposition.** *The model $T_0$ is the Néron model over $\mathbb{P}^1$ of its generic fiber.*

**Proof.** It is enough to check this property locally, i.e. that $\mathcal{O}_p$-scheme $T_0 \times_{\mathbb{P}^1} \mathcal{O}_p$ is the Néron model of its generic fiber for every closed point $p \in \mathbb{P}^1$. By Theorem 1, p. 172 in [BLR90] it is enough to show that $T_0(\mathcal{O}_p) \to T_0(\kappa)$ is surjective, where $\kappa \subset k((t))$ is the field of fractions of strictly henselian ring $\mathcal{O}_p \subset k[[t]]$. Locally the model has the form

$$x^2 - ty^2 = 1$$

Given $y$, the expression for $x$ becomes

$$x = \pm \sqrt{1 + ty^2} = \pm \left( 1 + \frac{1}{2}ty^2 - \frac{1}{8}t^2y^4 + \ldots \right)$$

which is a Laurent series if and only if $y$ is a Taylor series. Hence,

$$T_0(\mathcal{O}_p) = T_0(\kappa)$$

and this finishes the proof.

In contrast, the model $T$ is not the Néron model for $T_{NS}$. In particular, it is not the model of finite type. Because $T_{NS}$ contains a copy of $\mathbb{G}_m$ (given by the variable $z$),
Proposition 6 on p. 292 in [BLR90] implies that its Néron model is only locally of finite type.

We compute the group of torsors for $\mathcal{T}_0$ over $\mathbb{P}^1$:

$$\tilde{H}^1(\mathbb{P}^1, \mathcal{T}_0) \simeq H^1(\mathcal{T}_0) \simeq H^1(\mathbb{P}^1, \mathcal{T}_0).$$

Here the first isomorphism comes from the fact that $\mathcal{T}_0$ is an abelian sheaf ([Poo15], Prop.6.4.11) while the second isomorphism comes from the fact that $\mathcal{T}_0$ is a smooth, quasiprojective and commutative group scheme over $\mathbb{P}^1$ ([Mil80], III.3.9).

**Proposition.** Let $\mathcal{T}_0$ be the group scheme over $\mathbb{P}_k^1$ given locally by

$$x^2 - f(t)y^2 = 1,$$

where $f(t)$ is a polynomial with distinct roots that splits over $k$. Then the group of $\mathcal{T}_0$-torsors over $\mathbb{P}^1$ is isomorphic to

$$H^1(\mathbb{P}^1, \mathcal{T}_0) \simeq J(C),$$

where $C$ is the double cover of $\mathbb{P}^1$ corresponding to the field extension splitting the generic fiber.

**Proof.** Denote the set of points on the base corresponding to singular fibers by

$$Z = \{a_1, a_1, \ldots, a_r, a_r = \infty\}$$

and $U = \mathbb{P}^1 - Z$ its complement in $\mathbb{P}^1$. Let $\hat{U}$ be the open subset of $C$ not containing the ramification locus. This is illustrated in the following diagram

$$\hat{U} \hookrightarrow C \quad \text{étale} \quad \text{branched} \quad U \hookrightarrow \mathbb{P}^1$$

The inclusion $j: U \hookrightarrow \mathbb{P}^1$ induces the exact sequence of sheaves on $\mathbb{P}^1$

$$0 \longrightarrow \mathcal{T}_0 \longrightarrow j_*(\mathcal{T}_0|_U) \longrightarrow \mathcal{F} \longrightarrow 0$$
where $\mathcal{F}$ is a skyscraper sheaf supported on $Z$ for which the stalk at every point $a_i$ is isomorphic to the quotient group

$$\mathcal{F}_{a_i} \cong \begin{cases} x^2 - ty^2 = 1 & |x, y \in k((t))| \\ x^2 - ty^2 = 1 & |x, y \in k[[t]]| \end{cases}, \quad i = 1 \ldots r$$

These two groups were computed above in the proof of the Néron property. In particular, they are equal and $\mathcal{F}_{a_i} = 0$, which implies that $\mathcal{F}$ is the zero sheaf. Therefore, the sheaves $\mathcal{T}_0$ and $j_\ast \mathcal{T}_0$ are isomorphic and we have

$$H^1(\mathbb{P}^1, \mathcal{T}_0) = H^1(\mathbb{P}^1, j_\ast \mathcal{T}_0)$$

where from now on we denote by the same letter $\mathcal{T}_0$ the restriction $\mathcal{T}_0|_U$ where appropriate.

The cohomology of the push-forward $j_\ast \mathcal{T}_0$ fits into the Leray spectral sequence

$$0 \to H^1(\mathbb{P}^1, j_\ast \mathcal{T}_0) \to H^1(U, \mathcal{T}_0) \to \Gamma(\mathbb{P}^1, R^1 j_\ast \mathcal{T}_0)$$

where $R^1 j_\ast \mathcal{T}_0$ is a skyscraper sheaf. Its stalks at points $p = a_i$ are isomorphic to the groups

$$H^1(\text{Frac}(\mathcal{O}^\text{sh}_p), \mathcal{T}_0) \cong H^1(k((t)), \mathcal{T}_0)$$

where $\mathcal{O}^\text{sh}_p$ is the strict henselization of a local ring at $p$. The second group in (2.5) is trivial since $k((t))$ has cohomological dimension $\leq 1$ while for fields of cohomological dimension $\leq 1$ the first cohomology with coefficients in tori vanishes. We conclude that the Leray spectral sequence degenerates and

$$H^1(\mathbb{P}^1, j_\ast \mathcal{T}_0) \cong H^1(U, \mathcal{T}_0)$$

In order to compute the group $H^1(U, \mathcal{T}_0)$ we notice that the double cover $\tilde{U} \to U$ is an étale cover with Galois group $\mathcal{G} = \mathbb{Z}/2\mathbb{Z}$. This implies that the terms of the corresponding Hochschild-Serre spectral sequence are Galois cohomology groups ([Mil80]). In particular, we have the following exact sequence

$$0 \to H^1(\mathcal{G}, \mathcal{T}_0(\tilde{U})) \to H^1(\tilde{U}, \mathcal{T}_0) \to \ker \left[ H^0(\mathcal{G}, H^1(\mathcal{T}_0(\tilde{U})) \to H^2(\mathcal{G}, \mathcal{T}_0(\tilde{U})) \right] \to 0$$

(2.6)
Notice that $\mathcal{T}_0$ splits over $\tilde{U}$. This allows us to consider the Galois module $\mathcal{T}_0(\tilde{U})$ as $\mathbb{G}_m(\tilde{U})$ i.e., the group of invertible functions on $\tilde{U}$ but with a twisted Galois action. The hyperelliptic curve $C$ is defined locally by

$$u^2 = (t - a_1)(t - a_2) \ldots (t - a_{r-1})$$

and its field of functions over $k$ is generated by $u$ and $t$ ([Mir95], Ch.VI, Cor.1.23.(iv)). The group of invertible functions on $\tilde{U}$ up to constants is multiplicatively generated by the set $\{u, (t - a_1), \ldots, (t - a_{r-1})\}$. Therefore,

$$\mathcal{T}_0(\tilde{U}) \cong k^\times \times \mathbb{Z}^r/(-2, 1, 1, \ldots, 1) \cong k^\times \times \mathbb{Z}^{r-1}$$

As a $\mathcal{G}$-module it has a twisted action, and we will denote $\mathcal{T}_0(\tilde{U})$ by $M$ when want to stress the module structure. Denote the generator of $\mathcal{G}$ by $\sigma$. The fiber $T_L$ over the generic point of $C$ splits because of the factorization

$$\left( x + \sqrt{f(t)}y \right) \left( x - \sqrt{f(t)}y \right) = 1$$

where $f(t) = (t - a_1) \ldots (t - a_{r-1})$. In particular, the isomorphism to $\mathbb{G}_m$ over $L$ is given by

$$a \mapsto x + \sqrt{f(t)}y$$

where $a \cdot a^{-1} = 1$ is the defining equation for $\mathbb{G}_m$. The non-trivial Galois action changes the map to

$$a \mapsto x - \sqrt{f(t)}y$$

which is the inverse element. On the other hand, the Galois action on points of the curve $C$ is the involution $(u, t) \mapsto (-u, t)$. These two actions together induce the nontrivial twisted action of $\mathcal{G}$:

$$u \mapsto \frac{1}{u}, \quad t - a_i \mapsto \frac{1}{t - a_i}$$

Every cocycle class in $H^1(\mathcal{G}, \mathcal{T}_0(\tilde{U}))$ is defined by values $\varphi_1, \varphi_\sigma$ satisfying the condition

$$1 = \varphi_1 = \varphi_{\sigma^2} = \varphi_\sigma \cdot ^\sigma \varphi_\sigma$$

which can be restated as

$$^\sigma \varphi_\sigma = \frac{1}{\varphi_\sigma}$$
Obviously, every constant multiple of a function of the form \((t - a_i)^k, k \in \mathbb{Z}\), satisfies this condition: the group of 1-cocycles is isomorphic to \(k^\times \times \mathbb{Z}^{r-1}\). On the other hand, two cocycles \(\varphi_\sigma\) and \(\psi_\sigma\) are cohomologous if there exist an element \(f \in \mathcal{T}(\tilde{U})\) such that
\[
\psi_\sigma = \frac{1}{f} \cdot \varphi_\sigma \cdot \sigma f = \pm \frac{1}{f} \cdot \varphi_\sigma
\]
where the plus or minus sign depends on whether the function \(f\) “contains” \(u\). All this indicates that the group \(H^1(G, \mathcal{T}(\tilde{U}))\) can be computed as the quotient of the group of invertible functions on \(\tilde{U}\) modulo squares. We conclude that
\[
H^1(G, \mathcal{T}_0(\tilde{U})) \cong \frac{\mathbb{Z}^{r-1}}{\langle 1, 1, \ldots, 1 \rangle} \cong (\mathbb{Z}/2\mathbb{Z})^{r-2}
\]
where generators for \(\mathbb{Z}^{r-1}\) in the quotient above are \((t - a_i), i = 1, \ldots r - 1\).

The second cohomology group \(H^2(G, M)\) for cyclic Galois groups is known for arbitrary modules ([Ser95]) and can be computed as
\[
H^2(G, M) \cong \frac{M^G}{(\text{id} + \sigma)M}
\]
Only constant functions are invariant under the Galois action, while a direct check shows that \((\text{id} + \sigma)(M) = k^\times\). We conclude that \(H^2(\mathbb{Z}/2, \mathcal{T}_0(\tilde{U})) = 0\).

Because of the torus splitting, the group \(H^0(G, \mathcal{H}^1(\mathcal{T})(\tilde{U}))\) becomes the group of Galois invariants
\[
H^0(G, \mathcal{H}^1(\mathcal{T})(\tilde{U})) = \left(\text{Pic} \tilde{U}\right)^G
\]
with a twisted action of \(G\) on Pic \(C\). This action maps a line bundle \(\mathcal{L} \in \text{Pic} \tilde{U}\) to \(-i(\mathcal{L})\), where \(i\) is the action induced by the involution. In particular, every degree zero line bundle on \(C\) restricted to \(\tilde{U}\) will be invariant under this twisted action.

The Picard group of \(\tilde{U}\) fits into the exact sequence
\[
\mathbb{Z}^r \longrightarrow \text{Pic} C \longrightarrow \text{Pic} \tilde{U} \longrightarrow 0
\]
where \(r\) copies of \(\mathbb{Z}\) correspond to \(r\) ramification points on \(C\). We notice that each function \(t - a_i\) on \(C\) has a zero of order 2 at each ramification point \((u, t) = (0, a_i)\) and a pole of order two at the ramification point at infinity. These are exactly the invertible
functions on $U$ which arise from rational functions on $\mathbb{P}^1$. The function $u$ has a zero of order 1 at every ramification point $(0, a_i)$, $i = 1 \ldots r - 1$ and a pole of order $r - 1$ at the ramification point at infinity. These observations are summarized in the exact sequence

$$0 \rightarrow \mathbb{Z}/(2, 0, \ldots, 0) \rightarrow \text{Pic } C \rightarrow \tilde{U} \rightarrow 0$$

The subgroup that is being mapped to $\text{Pic}^0 C$ via the left inclusion is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{r - 2} = (\mathbb{Z}/2\mathbb{Z})^2$ and corresponds to order 2 elements of the Jacobian of $C$. We conclude that

$$H^0(G, \mathcal{H}^1(\mathcal{T}_0)(\tilde{U})) \simeq \text{Pic}^0 \tilde{U} \simeq \frac{\text{Pic}^0 C}{\text{Pic}^0 C[2]} \simeq \text{Pic}^0 C$$

where the last isomorphism is the isogeny.

The original Hochschild-Serre spectral sequence (2.6) becomes

$$0 \rightarrow \mathbb{Z}^{r - 2} \rightarrow \tilde{H}^1(\mathbb{P}^1, \mathcal{T}_0) \rightarrow \text{Pic}^0 C \rightarrow 0$$

In this extension the group of interest in the middle is isomorphic to the Jacobian $\text{Pic}^0 C = J(C)$ and the second map is multiplication by 2:

$$H^1(\mathbb{P}^1, \mathcal{T}_0) \cong J(C). \quad (2.7)$$

This ends the proof of the Proposition.

The above fact has a more straightforward proof if we interpret $\mathcal{T}_0$ as the kernel of the norm map from the restriction of scalars. Indeed, the constructed model $\mathcal{T}$ is the scheme representing the restriction of scalars $\text{Res}_{\mathbb{P}^1 \mathbb{G}_m}$ functor ([Vos98]). From this point of view the original sequence (2.3) is

$$1 \rightarrow \text{Res}^1 \mathbb{G}_m \rightarrow \text{Res} \mathbb{G}_m \xrightarrow{\text{Nm}} \mathbb{G}_m \rightarrow 1 \quad (2.8)$$

where $\mathcal{T}_0 = \text{Res}^1 \mathbb{G}_m$ and the norm map is given by the $z$ coordinate in (2.2). Moreover, $\text{Res} \mathbb{G}_m = \varphi_* \mathbb{G}_m$, where $\varphi : C \rightarrow \mathbb{P}^1$ is the double cover map. Since $\varphi$ is a finite morphism, the Leray spectral sequence for $\varphi_* \mathbb{G}_m$ degenerates ([Mil80]) which implies that

$$H^1(\mathbb{P}^1, \mathcal{T}) \simeq H^1(\mathbb{P}^1, \varphi_* \mathbb{G}_m) \simeq H^1(C, \mathbb{G}_m) = \text{Pic } C.$$
The exact sequence (2.4) now has a new interpretation

\[ 1 \rightarrow H^1(\mathbb{P}^1, \mathcal{T}_0) \rightarrow \text{Pic } C \xrightarrow{\text{Nm}} \text{Pic } \mathbb{P}^1 \rightarrow 1 \]

The norm map above is the induced norm on line bundles for the double cover. It doubles the degree. This confirms our previous result (2.7) as the kernel of the norm map is isomorphic to the Jacobian of \( C \).
2.2 Construction of the universal torsor

In this section we construct the universal torsor under the torus $\mathcal{T}$ over the threefold $\mathcal{X}$. Our strategy is as follows: using approach from ([CTS87b]) we first obtain the universal torsor $S$ for the torus $T_{NS}$ over the generic fiber and then check that it compactifies to a model $U$ over $\mathbb{P}^1$ respecting the action of $\mathcal{T}$. As the torus $T_{NS}$ for our example is a permutation module, the group $H^1(K, T_{NS})$ is trivial and the exact sequence (1.6) indicates that the constructed universal torsor $S$ is unique. Our main result of this section is summarized in the following Proposition.

**Proposition.** Given a quadric surface fibration $\mathcal{X} \to \mathbb{P}^1$ with square free discriminant and $r$ singular fibers there exists a rank 6 vector bundle $V \simeq \mathcal{O}^4 \oplus \mathcal{O}^2(r/2)$ with a $\mathcal{T}$-action and a $\mathcal{T}$-torsor $U \to \mathcal{X}$ embedded equivariantly in $V$ and restricting over the generic fiber to the universal torsor $S$.

**Proof.** Following the approach described in [CTS87b] (see also the paper by Peyre [Pey14]) we start with an exact sequence of Galois modules (1.2) which has the following form

$$0 \to \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{bmatrix} 1 & 1 & 1 \\ a & b_1 & b_2 \end{bmatrix}} \mathbb{Z}^2 \to 0$$

Below the groups in the this exact sequence are generators and the corresponding Galois actions permuting them – all groups in this example are permutation modules. The matrices defining the maps between the modules are shown above the arrows. The dual torus $T_{NS}$ was computed in the previous sections. In a similar way, by choosing generators of the corresponding invariant rings as

$$x = \frac{1}{2}(u + v) \quad y = \frac{1}{2\sqrt{f(t)}}(u - v) \quad z = uv \quad z^{-1} = 1/uv$$

$$x_1 = \sqrt{f(t)}(a_1 - b_1) \quad y_1 = a_1 + b_1 \quad z_1 = -4f(t)a_1b_1 \quad z_1^{-1} = -1/4f(t)a_1b_1$$

$$x_2 = \sqrt{f(t)}(a_2 - b_2) \quad y_2 = a_2 + b_2 \quad z_2 = -4f(t)a_2b_2 \quad z_2^{-1} = -1/4f(t)a_2b_2$$
we obtain equations for other two tori \( T \) and \( T_\Sigma \):

\[
T: \quad x^2 - f(t)y^2 = z, \quad zz^{-1} = 1 \quad T_\Sigma: \quad x_1^2 - f(t)y_1^2 = z_1, \quad z_1 z_1^{-1} = 1 \\
\quad x_2^2 - f(t)y_2^2 = z_2, \quad z_2 z_2^{-1} = 1
\]

Deriving maps between these tori is more subtle: for a fixed pair of tori we need to write down the images of the generators of the invariant subalgebra of the first torus in terms of generators of the invariant subalgebra of the second torus. For the pair \( T_{NS} \to T_\Sigma \) we have

\[
x_i = \sqrt{f(t)}(a_i - b_i) \mapsto \sqrt{f(t)}(a - b) = x, \quad i = 1, 2
\]

and similarly \( y_i \mapsto y \) and \( z_i \mapsto z, \quad i = 1, 2 \). Therefore, \( T_{NS} \to T_\Sigma \) is the diagonal embedding.

For the pair \( T_\Sigma \to T \) we have

\[
x \mapsto \frac{1}{2} \left( \frac{a_1}{a_2} + \frac{b_1}{b_2} \right) = \frac{1}{2a_2b_2} \left( a_1b_2 + a_2b_1 \right) = \frac{-1}{4a_2b_2} \left( (a_1 - b_1)(a_2 - b_2) - (a_1 + b_1)(a_2 + b_2) \right) = \frac{1}{2} \left( x_1x_2 - f(t)y_1y_2 \right)
\]

\[
y \mapsto \frac{1}{2\sqrt{f(t)}} \left( \frac{a_1}{a_2} - \frac{b_1}{b_2} \right) = \frac{-1}{2\sqrt{f(t)}a_2b_2} \left( a_1b_2 - a_2b_1 \right) = \frac{-1}{4\sqrt{f(t)}a_2b_2} \left( (a_2 - b_2)(a_1 + b_1) - (a_1 - b_1)(a_2 + b_2) \right) = \frac{1}{2} \left( x_2y_1 - x_1y_2 \right)
\]

\[
z \mapsto \frac{a_1b_1}{a_2b_2} = \frac{z_1}{z_2}
\]

The torus \( T \) compactifies to a projective quadric surface \( X \) after removing the condition \( zz^{-1} = 1 \) and homogenizing the defining equation with a new variable \( w \). The torus \( T_\Sigma \)
extends to the universal torsor $S$ embedded as an open subset in the affine space $\mathbb{A}_K^6$, away from the hypersurface $z_1z_2 = 0$. The torus $T_{NS}$ still acts on $S$. Equations of all maps between varieties are outlined in the following diagram.

As all equations are given by the polynomials, this defines the model over an affine part $\mathbb{A}_k^1 \subset \mathbb{P}_k^1$. A straightforward check shows that in the chart on $\mathbb{P}_k^1$ with a local coordinate $s$ where quadric surface is defined by

$$x^2 - g(s)y^2 = zw, \quad g(s) = s(1 - sa_1) \ldots (1 - sa_{r-1}), \quad y = s^{r/2}y$$

the map $S \to X$ is given locally by similar formula

$$[x_1x_2 - g(s)y_1y_2 : x_2y_1 - x_1y_2 : z_1 : z_2]$$

Therefore, the universal torus $S$ compactifies to the complete smooth model $U$ over $\mathbb{P}_k^1$ embedded into rank 6 bundle $\mathcal{O}^4 \oplus \mathcal{O}^2(r/2)$, where the factors $\mathcal{O}(r/2)$ correspond to variables $y_i$, $i = 1, 2$. This ends the proof of the Proposition. \qed
Chapter 3

Degree 6 del Pezzo bundles

In this chapter we discuss threefolds fibered by degree 6 del Pezzo surfaces over the projective line. First, we state the main results about the arithmetic and geometry of del Pezzo surfaces. References for the most of the results for this part can be found in [Man86]. Then we discuss the singularities and the local monodromy for singular fibers in smooth models. Finally, we compute the cohomology groups for the models of Néron-Severi tori.

3.1 Overview of del Pezzo surfaces

A del Pezzo surface over a field $K$ is a smooth projective algebraic variety of dimension 2 whose anticanonical class is ample. Proper multiple of the anticanonical class induces a projective embedding with the image the degree $d$ surface, the value referred to as the degree of corresponding del Pezzo surface.

From the defining properties, it follows that the only possible values for the degree are $1 \leq d \leq 9$. Over an algebraic closure $\overline{K}$ del Pezzo surfaces can be classified by degree $d$ according to the following table.
Let \( X \) be a del Pezzo surface of degree 6 over a field \( K \). It is known ([Blu10]) that up to isomorphism \( X \) can be determined by two separable \( K \)-algebras whose centers are respectively étale quadratic and cubic algebras over \( K \). Over the algebraic closure \( \overline{K} \), by the above classification, \( \overline{X} \) is isomorphic to the blow-up of \( \mathbb{P}^2 \) at three points not lying on the same line. On the other hand, any three points in general position on \( \mathbb{P}^2 \) can be mapped by a projective automorphism to the points \([1 : 0 : 0], [0 : 1 : 0]\) and \([0 : 0 : 1]\). The open subset obtained after removing the three lines containing pairs of these points becomes isomorphic to the algebraic torus \( \mathbb{G}_m^2, \overline{K} \). This corresponds to a principal homogeneous space under some 2-dimensional torus \( T \) over \( K \). If \( K \) is a \( C_1 \)-field this principal homogeneous space is trivial (see, e.g., 3.2 in [Ser01]), hence it is isomorphic to a torus \( T \) itself and the latter is contained in \( X \) as an open subset. The action of \( T \) extends to \( X \) making it a toric variety. If \( X \) appears as the generic fiber of the threefold \( \mathcal{X} \), the result of Voskresenskii about the rationality of 2-dimensional tori ([Vos67]) implies that the total space \( \mathcal{X} \) is rational. If the surface \( X \) becomes isomorphic to a blowup of \( \mathbb{P}^2 \) at three points after passing to a field extension \( L \) we say that the surface \( X \) splits over \( L \). In particular, passing to the extension \( L \) splits the torus \( T_L \subset X_L \); it becomes isomorphic to \( \mathbb{G}_m^2, L \).

The Picard group of \( \overline{X} \) is isomorphic to \( \mathbb{Z}^4 \) generated by the class of the pull-back \( H \) of the hyperplane section from \( \mathbb{P}^2 \) under the blow up and by the classes of the three exceptional divisors \( E_1, E_2 \) and \( E_3 \):

\[
\text{Pic} \overline{X} \simeq \mathbb{Z} H \oplus \mathbb{Z} E_1 \oplus \mathbb{Z} E_2 \oplus \mathbb{Z} E_3
\]

It can be shown that the degree 6 del Pezzo surface over an algebraically closed field contains 6 curves with self-intersection number \(-1\). These are called exceptional curves.
and are the only curves on $\overline{X}$ with a negative self-intersection. Under the anticanonical embedding exceptional curves are mapped to lines. The arrangement of exceptional curves on $\overline{X}$ can be depicted using the hexagon diagram:

On the diagram are shown 6 exceptional curves along with their intersections. All intersections are of multiplicity 1. Next to the curves are denoted the classes they represent in $\text{Pic} \overline{X}$. The intersection form on $\text{Pic} \overline{X}$ (3.1) is given by the matrix

$$
\begin{bmatrix}
1 & -1 \\
-1 & -1 \\
-1 & -1
\end{bmatrix}
$$

and the Galois action preserves this intersection form.

### 3.2 Dihedral group of order 12

The action of the Galois group $\mathcal{G}$ factors through the action of the dihedral group $D_6$ of order 12, the group of symmetries of a regular hexagon. In the context of degree 6 del Pezzo surfaces it permutes the exceptional curves.

The group has two generators. Its representation can be written as

$$D_6 = \langle R, v_1 \mid R^6 = v_1^2 = e, v_1 R v_1 = R^{-1} \rangle$$

The generator $R$ in terms of the hexagon can be considered as clockwise rotation by 60 degrees while the element $v_1$ is reflection with respect to the axis through $E_3$ and $H - E_1 - E_2$. 
The next page contains the list of all 12 elements with their geometric interpretation. It also includes the matrix representation for the elements with respect to the Galois action on the Picard group. We notice that the dihedral group $D_6$ is isomorphic to the direct product

$$D_6 \cong S_2 \times S_3$$

(3.2)

of the symmetric group on two letters $S_2 \cong \mathbb{Z}/2\mathbb{Z}$ which is generated by the rotation by 180 degrees ($R^3$ in our notation)

and also of a symmetric group on three letters $S_3$. On the surface $\overline{X}$ there are two triples of exceptional lines that are obtained from the blowup of the plane $\mathbb{P}^2$. With this interpretation, the subgroup $S_3$ exchanges three blown up points, while the subgroup $S_2$ interchanges the triples. On the diagram, the representation of $S_3$ on $\text{Pic} \overline{X}$ is denoted by shaded $3 \times 3$ blocks.

The multiplication table can be easily reconstructed from the diagram. The last diagram in this section shows all the subgroups of $D_6$. Non-normal subgroups are outlined inside the dashed frames.
Figure 3.1: Elements of the dihedral group $D_6$ with their representation on $\text{Pic} \overline{X}$ and geometric interpretation
Figure 3.2: $D_6$ subgroups diagram
3.3 Local monodromy and singularities

Let \( \pi : \mathcal{X} \to \mathbb{P}^1 \) be a fibration of degree 6 del Pezzo surfaces over the projective line. In the following we will always assume that the total space \( \mathcal{X} \) is smooth and each singular fiber contains one ordinary double point. For this section only we also assume that the ground field \( k \) is algebraically closed.

It can be shown (see, e.g. [Cos06]) that the ordinary double points in a family of degree 6 del Pezzo surfaces appear from blowing up the plane at three points not in general position: three points on the same line or two points colliding. These two cases correspond to two conjugacy classes of subgroups of order 2 in \( D_6 \). We want to construct models for both types of singular fibers.

The first conjugacy class contains one unique normal subgroup of order 2 generated by element \( R^3 \) (with respect to the notation on Fig. 3.1). We will refer to this case as a type 1 singularity and will denote the number of singular fibers of this type in \( \mathcal{X} \) by the integer \( a \). The local monodromy is defined by the conjugacy class of the group \( \langle R^3 \rangle \) and the corresponding singular surface can be constructed from the blowup of three points on \( \mathbb{P}^2 \) on the same line: on the blow up three exceptional curves intersect the unique \( -2 \)-curve. Blowing down this curve produces the required singular surface:

The singular surface contains three exceptional curves, all of multiplicity 2. In particular, this shows that the local monodromy is defined by class of the subgroup generated by the element \( R^3 \) permuting three pairs of lines on the original hexagon.

We proceed with the explicit construction. The blow up of \( \mathbb{P}^2 \) at the 3 points on the same line, \([1 : 0 : 0], [0 : 1 : 0]\) and \([1 : 1 : 0]\), can be realized as a subset of
\[ \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \]

\[ [\eta_0: \eta_1: \eta_2] [\zeta_0: \zeta_1] [\varphi_0: \varphi_1] [\psi_0: \psi_1] \]

defined by the equations

\[
\begin{align*}
\eta_1 \zeta_1 &= \eta_2 \zeta_0 \\
\eta_0 \varphi_1 &= \eta_2 \varphi_0 \\
(\eta_0 - \eta_1) \psi_1 &= -\eta_2 \psi_0
\end{align*}
\]

Projection of this smooth surface to \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) blows down the unique \(-2\)-curve. In terms of the equation, this corresponds to removing variables \( \eta_i \). This results in one equation

\[ f = \zeta_0 \varphi_1 \psi_1 - \varphi_0 \zeta_1 \psi_1 + \psi_0 \zeta_1 \varphi_1 = 0 \]

defining the singular surface in \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \).

Next we check the type of the singularity. Computing the Jacobian matrix for \( f \) we obtain

\[
\begin{bmatrix}
\frac{\partial f}{\partial \zeta_0}, & \frac{\partial f}{\partial \zeta_1}, & \frac{\partial f}{\partial \varphi_0}, & \frac{\partial f}{\partial \varphi_1}, & \frac{\partial f}{\partial \psi_0}, & \frac{\partial f}{\partial \psi_1}
\end{bmatrix} = \begin{bmatrix} \varphi_1 \psi_1, \ -\varphi_0 \psi_1, \ -\zeta_1 \psi_1, \ \zeta_1 \psi_0, \ \zeta_1 \varphi_1, \ (\zeta_0 \varphi_1 - \varphi_0 \zeta_1) \end{bmatrix}
\]

Vanishing of the first entry implies that either \( \varphi_1 = 0 \) or \( \psi_1 = 0 \). In both cases we obtain one singular point \([1 : 0] \times [1 : 0] \times [1 : 0] \). In the local chart where \( \zeta_0 = \varphi_0 = \psi_0 = 1 \) the hyperplane equation is of the form

\[ vw - uw + uv = 0 \]

with a singularity at the origin. Over an algebraically closed field this quadratic form diagonalizes to a sum of squares

\[ x^2 + y^2 + z^2 = 0 \]

which has a du Val singularity \( A_1 \) at the origin.

The second conjugacy class in \( D_6 \) consists of three subgroups of order 2:

\[ \langle v_1 \rangle, \ \langle v_3 \rangle, \ \langle v_5 \rangle \] (3.3)
(with respect to the notation on Fig. 3.1). We will refer to this case as a type 2 singularity and will denote the number of singular fibers of this type in $\mathcal{X}$ by the integer $b$. The local monodromy is defined by the conjugacy class (3.3) and the corresponding singular surface can be constructed from the blowup of $\mathbb{P}^2$ at two colliding points. For the explicit construction we need to blow up $\mathbb{P}^2$ at two points and then blow up the surface obtained at a point on the exceptional divisor. The new surface will contain a unique $(-2)$-curve. Blowing it down produces the singular surface:

This singular surface contains four exceptional curves: two of multiplicity 2 and two of multiplicity 1. This confirms the claim that the conjugacy class (3.3) defines the local monodromy. Indeed, the corresponding three generators are the only elements in the group $D_6$ that fix two lines on the hexagon while exchanging the other four pairwise.

The explicit equations for this type of singularity define the singular surface as a complete intersection in

$$\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$$

defined by

$$x_0y_1 - x_2y_0 = 0$$
$$x_2y_1 - x_1y_2 = 0$$

Denoting the first polynomial as $f_1$ and the second as $f_2$ we compute the Jacobian:

$$\begin{bmatrix}
\frac{\partial f_1}{\partial x_0} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial y_0} & \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\
\frac{\partial f_2}{\partial x_0} & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial y_0} & \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2}
\end{bmatrix}
= \begin{bmatrix}
y_1 & 0 & -y_0 & -x_2 & x_0 & 0 \\
0 & -y_2 & y_1 & 0 & x_2 & -x_1
\end{bmatrix}$$
A direct check shows that the matrix has rank 1 only at the point \([1 : 0 : 0] \times [1 : 0 : 0]\). In its local chart, the surface is a complete intersection

\[ w - v = vw - uz = 0 \]

which reduces to

\[ w^2 - uz = 0 \]

in \(\mathbb{A}^3\). After the change of variables \(u = \alpha + \beta\), \(v = \alpha - \beta\), we obtain the quadratic form

\[ w^2 + \alpha^2 + \beta^2 = 0 \]

with \(A_1\)-type singularity at the origin.

The following diagram shows finite covers of \(\mathbb{P}^1\) corresponding to certain conjugacy classes of subgroups of \(D_6\).

Next to the covers indicated their genera in terms of number of singular fibers \(a\) and \(b\) in \(\mathcal{X}\) of type 1 and type 2 respectively. Genera can be used for computing the dimension of the Intermediate Jacobian of the complex threefold \(\mathcal{X}\) which is discussed in the following section.
3.4 Intermediate Jacobian and Kanev’s construction

Let $k$ be the field of complex numbers $\mathbb{C}$ and consider a fibration of del Pezzo surfaces $X \to \mathbb{P}^1$ with a smooth total space. The Hodge diamond of $X$ has the form

\[
\begin{array}{cccc}
1 & & & \\
0 & 0 & & \\
0 & h^{11} & 0 & \\
0 & h & h & 0 \\
\end{array}
\]

In particular, the middle cohomology $H^3(X, \mathbb{C})$ in the analytic topology admits a Hodge decomposition

\[H^1(X, \mathbb{C}) \cong H^2(X, \Omega^1) \times H^1(X, \Omega^2).\]

On this vector space there exists a natural Hermitian form $H$ given by

\[H(\omega_1, \omega_2) = 2i \int_X \omega_1 \wedge \omega_2\]

It is positive definite on the subspace $V = H^1(X, \Omega^2)$. The image of $H^3(X_{\text{an}}, \mathbb{Z})$ in $V$ forms a full rank lattice on which $H$ takes integer values. This implies that the quotient

\[IJ(X) = \frac{H^1(X, \Omega^2)}{H^3(X, \mathbb{Z})}\]

is a principally polarized torus and hence is an abelian variety, the Intermediate Jacobian of the threefold $X$.

**Proposition.** Let $\mathcal{X}$ be a complex smooth projective threefold fibered over $\mathbb{P}^1$ by degree 6 del Pezzo surfaces. Assume that $\mathcal{X}$ contains $(a+b)$ singular fibers of type 1 and type 2 respectively. Then the dimension of the Intermediate Jacobian is given by the formula

\[\dim IJ(\mathcal{X}) = h^{11} + \frac{a+b}{2} - 5.\]

where $h^{11} = \dim H^1(\mathcal{X}, \Omega^1)$.  

Proof. We compute the dimension using the Euler characteristic. In particular, using the blow up/down construction from the previous section we can compute Euler characteristic of smooth and singular fibers:

\[ \chi(\text{Smooth del Pezzo}) = 6, \]
\[ \chi(\text{Singular del Pezzo}) = 5. \]

Then, given that the model has \( a \) and \( b \) singular fibers of types 1 and 2 respectively, the Euler characteristic of the model becomes

\[ \chi(X) = (\chi(\mathbb{P}^2) - (a + b)) \cdot 6 + (a + b) \cdot 5 = 12 - (a + b). \]

The dimension of the Intermediate Jacobian equals to \( \dim H^1(X, \Omega^2) = h \). Expressing the Euler characteristic via Betti numbers we obtain

\[ 12 - (a + b) = 2h^{11} + 2 - 2h \]

which simplifies to the required formula.

The value \( h^{11} = \dim H^1(X, \Omega^1) \) in the formula depends on the global monodromy coming from the generic fiber \( X \).

The following result allows us to realize \( \text{IJ}(X) \) as a generalized Prym variety – the abelian subvariety of the Jacobian of a curve satisfying the property that the principal polarization of the Jacobian restricts to a multiple of the principal polarization on a Prym variety.

For a del Pezzo surface fibration over a curve, there exists a natural finite cover \( C \) of the base \( \mathbb{P}^1 \) whose points parametrize exceptional curves on fibers of the bundle. The set of points \( (p, q) \in C \times C \), where \( p \) and \( q \) correspond to curves that intersect on the threefold, forms a symmetric divisor \( D \) on \( C \times C - \text{the incidence correspondence} \). If \( p_1 \) and \( p_2 \) are two natural projections from \( C \times C \), then the incidence correspondence induces an endomorphism

\[ i : J(C) \longrightarrow J(C) \]

by mapping \( \mathcal{L} \) to \( p_2^*(p_1^*(\mathcal{L}) \cdot D) \) for \( \mathcal{L} \in \text{Pic}^0 C \).

The following result is essentially due to Kanev ([Kan89]):
Proposition. Let $\mathcal{X} \to \mathbb{P}^1$ be a del Pezzo fibration of degree $\leq 5$. Then, assuming that any singular fiber has a unique singularity which is an ordinary double point, the abelian subvariety of the Jacobian defined by the image of the map $id - i$ is isomorphic to the intermediate Jacobian of the threefold $\mathcal{X}$:

$$(id - i)J(C) \cong IJ(\mathcal{X}).$$

The proof can be found in ([Kan89]). We use this result later in the context of del Pezzo fibrations of degree 6. Blowing up a section allows us to switch to the case of degree 5 fibrations, where the result of Kanev is applicable.
3.5 Cohomology of a model for the Néron-Severi torus

In this section we consider examples of degree 6 del Pezzo fibrations $\pi: X \to \mathbb{P}^1$ with a smooth total space $X$. We start with the case when the generic fiber splits after passing to a degree two extension and construct an explicit example of the family as well as of the universal torsor.

The rest of the section is devoted to computing the group $H^1(\mathbb{P}^1, T)$ for models $T$ of Néron-Severi tori corresponding to different subgroups of $D_6$. For a degree 6 del Pezzo surface $X$ the flasque resolution for the module $\text{Pic} \, \overline{X} \simeq \mathbb{Z}^4$ has the following form

$$0 \to \mathbb{Z}^2 \xrightarrow{M_1} \mathbb{Z}^6 \xrightarrow{M_2} \mathbb{Z}^4 \to 0$$

where the maps can be defined by the matrices

$$M_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 \\ -1 & -1 & -1 \\ -1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}$$

In the end of the section we use Kanev’s construction for the intermediate Jacobian of threefolds fibered in del Pezzo surfaces to show that for the case of maximal monodromy over the complex numbers the intermediate Jacobian is isomorphic to the identity component of the group $H^1(\mathbb{P}^1, T)$.

3.5.1 Degree 2 splitting, the subgroup $S_2$

Assume that the generic fiber $X$ splits after passing to the degree two extension $L = K(\sqrt{f(t)})$ obtained by adjoining the square root of the polynomial $f(t)$ with distinct roots, in particular the fibration $X' \to \mathbb{P}^1$ is with square free discriminant. Denote the corresponding double cover by $C_2$. Then we have the following Proposition.
Proposition. For a degree 6 del Pezzo fibration $X \to \mathbb{P}^1$ whose generic fiber $X$ splits after a degree 2 extension, the group of torsors under the model $\mathcal{T}$ of the Néron-Severi torus is isomorphic to

$$H^1(\mathbb{P}^1, \mathcal{T}) \simeq \text{Pic} C_2 \times \mathbb{Z}^2.$$  

Moreover, there exists a rank 9 vector bundle $V \simeq \mathcal{O}^6 \oplus \mathcal{O}^3(r/2)$ with a $\mathcal{T}$-action and a $\mathcal{T}$-torsor $U \to X$ embedded equivariantly in $V$ and restricting over the generic fiber to the universal torsor $S$.

Proof. Our strategy is as follows. We construct the universal torsor over the generic fiber and this gives the model over an open subset $\mathbb{A}^1 \subset \mathbb{P}^1$. Then we construct the model at the infinity point, the unique point of bad reduction and check that this glues into a global model respecting the action of $\mathcal{T}$.

Let the action of the Galois group $G = \mathcal{G}_2 \simeq \mathbb{Z}/2\mathbb{Z}$ be generated by the element $R^3$ with respect to our notation (see Fig. 3.1). Then the nontrivial action on the sequence of modules (3.4) is as follows

$$
\text{Generators:} \quad a \ b \quad a_1 a_2 a_3 b_1 b_2 b_3 \quad d_0 d_1 d_2 d_3 \quad \quad \quad \\
0 \to \mathbb{Z}^2 \to \mathbb{Z}^6 \to \mathbb{Z}^4 \to 0
$$

Galois action:

$$
a \to 1/a \quad a_1 \to b_1 \quad d_0 \to d_0^2/d_1 d_2 d_3 \\
b \to 1/b \quad a_2 \to b_2 \quad d_1 \to d_0/d_2 d_3 \\
\quad a_3 \to b_3 \quad d_2 \to d_0/d_1 d_3 \\
\quad \quad d_3 \to d_0/d_1 d_2
$$

In particular, the groups of characters for the tori $T$ and $T_{\text{NS}}$ are not permutation modules anymore. It follows that the equations for the torus $T \subset X$ are

$$
u_1^2 - f(t)v_1^2 = 1 \quad \quad u_2^2 - f(t)v_2^2 = 1
$$

These can be compactified to plane conics and hence the torus $T$ can be embedded into the quadric surface in $\mathbb{P}^2_K \times \mathbb{P}^2_K$. The del Pezzo surface $X$ is obtained from it by blowing up two points. Notice that the singularities are ordinary double points. By construction they are of type 1.

For the Néron-Severi torus we can pick generators for the invariant subalgebra as

$$
x = \frac{1}{2} \left( d_1 + \frac{d_0}{d_2 d_3} \right) \quad y = \frac{1}{\sqrt{r}} \left( d_1 - \frac{d_0}{d_2 d_3} \right)
$$
which give torus equations

\[ x^2 - f(t)y^2 = s_1 s_2 s_3, \quad s_1 s_1^{-1} = s_2 s_2^{-1} = s_3 s_3^{-1} = 1 \]

This torus over the field \( K \) obviously compactifies to the model \( T \) over \( \mathbb{P}^1 \) by a similar glueing as used in the construction in the quadric surface example.

The torus corresponding to the permutation module \( \mathbb{Z}^6 \) is given by

\[ x_i^2 - f(t)y_i^2 = z_i, \quad z_i z_i^{-1} = 1 \quad i = 1, 2, 3 \]

After removing conditions \( z_i z_i^{-1} = 1 \) it becomes a complete intersection in \( \mathbb{A}^9_K \). Removing two 2-plane sections \( z_1 = z_2 = 0 \) and \( z_2 = z_3 = 0 \) from it leads to the quasi-affine variety – the universal torsor \( S \).

The maps between varieties can be derived in a similar way to the construction in the previous chapter. The result is shown in the following diagram.

Over the point at infinity on \( \mathbb{P}^1 \) all three varieties can be compactified via the change of variables

\[ \bar{v}_1 = \frac{v_1}{s^{r/2}}, \quad \bar{v}_2 = \frac{v_2}{s^{r/2}}, \quad \bar{y}_i = \frac{y_i}{s^{r/2}}, i = 1, 2, 3, \quad \bar{y} = \frac{y}{s^{r/2}} \]
where \( s = 1/t \) is the local coordinate. Then the equations remain almost the same, with the only change \( f(t) \) for \( g(s) = s^r f(1/s) \). The analysis of the maps indicates that e.g.,

\[
[x_1 x_2 + f(t) y_1 y_2 : x_1 y_2 + x_2 y_1 : z_1 z_2] =
[x_1 x_2 + s^r f(1/s) \frac{y_1}{s^{r/2} s^{r/2}} \frac{y_2}{s^{r/2} s^{r/2}} : s^{r/2} \left( x_1 \frac{1}{s^{r/2}} y_2 + x_2 \frac{1}{s^{r/2}} y_1 \right) : z_1 z_2] =
[x_1 x_2 + g(s) \bar{y}_1 \bar{y}_2 : s^{r/2} (x_1 \bar{y}_2 + x_2 \bar{y}_1) : z_1 z_2]
\]

which shows the compatibility of the map at the infinity point. Therefore, the universal torsor \( S \) compactifies to the complete model \( U \over \mathbb{P}^1 \) respecting the action of the model \( T \). This embeds into a rank 9 vector bundle \( \mathcal{O}^6 \oplus \mathcal{O}^3(r/2) \) where the second factor comes from variables \( y_i, i = 1, 2, 3 \). The 4-dimensional torus \( T \) in this case is isomorphic to

\[
T = \text{Res}_{C_2 | \mathbb{P}^1} \mathbb{G}_m \times \mathbb{G}_m^2
\]

where \( C_2 \) is the double cover defined by the splitting extension \( L \) for the generic fiber. Since

\[
H^1(\mathbb{P}^1, \text{Res}_{C_2 | \mathbb{P}^1}) \cong H^1(C_2, \mathbb{G}_m) = \text{Pic} \, C_2
\]

(see [SGA3], XXIV, 8.5) we conclude that \( H^1(\mathbb{P}^1, T) \cong \text{Pic} \, C_2 \times \mathbb{Z}^2 \) and this ends the proof of the Proposition.

\[\square\]

### 3.5.2 Degree 4 splitting, the Klein four-group

For the case of the Klein four-group, the tori \( T \) and \( T_{NS} \) have a biquadratic splitting field

\[
L = K(\sqrt{f(t)}, \sqrt{g(t)})
\]

obtained by adjoining square roots of the polynomials \( f(t) \) and \( g(t) \) with distinct and disjoint roots. Denote by \( C_2 \) the double cover of \( \mathbb{P}^1 \) corresponding to the extension \( K(\sqrt{f(t)}) \) and by \( C_4 \) the degree 4 cover corresponding to the extension \( L \).

**Proposition.** For a degree 6 del Pezzo fibration \( X \to \mathbb{P}^1 \) whose generic fiber \( X \) has a biquadratic splitting field, the group of torsors under the model \( T \) of the Néron-Severi torus is isomorphic to

\[
H^1(\mathbb{P}^1, T) \cong \text{Pic} \, C_2 \times J(C_4) \times \mathbb{Z}.
\]
Proof. With respect to the notation in Fig.3.1, the generators of the Galois Klein four-group can be chosen as $R^3$ and $v_1$. First we compute the torus $T$. Picking generators $a$ and $b$ for the group ring of characters, we have the following Galois actions

$$
\begin{array}{c}
\sqrt{f} & \rightarrow & -\sqrt{f} \\
\sqrt{g} & \rightarrow & -\sqrt{g}
\end{array}
\quad
\begin{array}{c}
a \rightarrow -b \\
a \rightarrow -1/a \\
b \rightarrow 1/b
\end{array}
$$

Then the generators for the invariant subalgebra can be chosen as

$$
\begin{align*}
x &= \frac{1}{2} \left( a + \frac{b}{a} \right) \\
y &= \frac{1}{2} \sqrt{f} \left( a - \frac{b}{a} \right) \\
u &= \frac{1}{2} \left( ab + \frac{1}{ab} \right) \\
v &= \frac{1}{2} \sqrt{g} \left( ab - \frac{1}{ab} \right)
\end{align*}
$$

which give equations for the torus $T$:

$$
x^2 - f(t)g(t)y^2 = 1, \quad u^2 - g(t)v^2 = 1.
$$

In particular, this shows that each singular fiber contains one ordinary double point. Similarly, analysis of the Galois action shows that the invariant subring for the torus $T_{NS}$ is generated by

$$
\begin{align*}
x &= \frac{1}{2} \left( d_3 + \frac{d_0}{d_1d_2} \right) \\
y &= \frac{1}{\sqrt{f}} \left( d_3 - \frac{d_1}{d_1d_2} \right) \\
s &= \frac{d_0d_3}{d_1d_2} \\
t &= \frac{d_0}{d_3} \\
u &= \frac{1}{2} \left( \frac{d_1}{d_2} + \frac{d_2}{d_1} \right) \\
v &= \frac{1}{\sqrt{g}} \left( \frac{d_1}{d_2} - \frac{d_2}{d_1} \right)
\end{align*}
$$

which gives equations for $T_{NS}$:

$$
x^2 - f(t)y^2 = s, \quad u^2 - g(t)v^2 = 1, \quad ss^{-1} = tt^{-1} = 1
$$

In particular, this shows that the model $T_{NS}$ is isomorphic to the direct product

$$
\mathcal{T} \simeq \text{Res}_{\mathbb{P}^1 \mathbb{P}^1} \mathbb{G}_m \times \mathcal{T}_0 \times \mathbb{G}_m
$$

where the model $\mathcal{T}_0$ was considered in the Chapter 2.1. We conclude that $H^1(\mathbb{P}^1, \mathcal{T}) \simeq \text{Pic} C_2 \times J(C_4) \times \mathbb{Z}$. 

\□
3.5.3 Degree 6 splitting, the subgroup $S_3$

The example of the Galois group $G = S_3$ can be constructed in the following way. Consider the matrix

$$\begin{bmatrix}
-1 & -1 \\
-1 & f(t) \\
1 & 1
\end{bmatrix}$$

where $f(t)$ is the polynomial with coefficients in $k$ with distinct roots. The characteristic polynomial of the above matrix is $\lambda^3 - \lambda + f(t)$. For general $t$, the roots $\lambda_i, i = 1, 2, 3$ are distinct and the corresponding eigenvectors are $(-\lambda_i, \lambda_i^2 + 1, 1), i = 1, 2, 3$. This can be considered as three points on the projective plane $\mathbb{P}^2_k$. Over the field $K = k(t)$ this defines a degree 3 zero-cycle on $\mathbb{P}^2_k$ given by the equations

$$-z_0^3 + z_0 z_2^2 + f(t)z_2^3 = 0 \quad \text{and} \quad z_1 z_2 = -z_0^2 + z_2^2$$

where $[z_0 : z_1 : z_2] \in \mathbb{P}^2_k$. This corresponds to a simply branched trisection $C_3$ to the trivial $\mathbb{P}^2_k$-bundle over $\mathbb{P}^1$. Blowing up this trisection produces a degree 6 del Pezzo fibration $X \to \mathbb{P}^1$ with type 2 singular fibers. The discriminant of the characteristic polynomial is $4 - 27f(t)^2$ and is not a square in $K$; hence, the Galois group for the generic fiber is isomorphic to the full symmetric group $S_3$ which permutes points on $\mathbb{P}^2_k$, the centers of the blow up. We notice that for a general $t$ the points $[-\lambda_i(t) : \lambda_i(t)^2 - 1 : 1]$ are in general position. At the values of $t$ that make the discriminant vanish, two of the roots $\lambda_i(t)$ coincide and the blowup produces the singularity of the type 2 in the corresponding fiber.

Over the complex numbers $k = \mathbb{C}$, the Intermediate Jacobian becomes isomorphic to the Jacobian of the blown-up curve, $J(C_3)$. This follows from the fact that blowing up the threefold with a center at a smooth curve adds the Jacobian of a curve as a direct summand to the Intermediate Jacobian of the original threefold (Proposition 1 in Lecture 1, [Tur72]), while the Intermediate Jacobian of the projective bundle is trivial.

**Proposition.** Let $X \to \mathbb{P}^1$ be a degree 6 del Pezzo fibration whose singular fibers contain only unique type 2 singularities. If the generic fiber $X$ splits at the degree 6 extension with
the Galois group $\mathfrak S_3$, then the group of torsors under the model $\mathcal T$ of the Néron-Severi torus becomes isomorphic to

$$H^1(\mathbb P^1, \mathcal T) \cong \mathbb Z \times \text{Pic } C_3.$$  

Proof. Notice that the Galois group acts trivially on one generator of Pic $\overline X$ while it permutes the other three generators – it is a permutation module and is isomorphic to the direct product of a multiplicative group (coming from invariant character) and the restriction of scalars:

$$\mathcal T \cong \mathbb G_m \times \text{Res}_{\mathbb C^3 \rightarrow \mathbb P^1} \mathbb G_m.$$  

Therefore, we conclude that $H^1(\mathbb P^1, \mathcal T) \cong \mathbb Z \times \text{Pic } C_3$. \hfill $\square$

### 3.5.4 Degree 12 splitting, the Dihedral group $D_6$

The last case corresponds to the maximal monodromy when the action of $D_6$ on Pic $\overline X$ is faithful. Denote by $C_2$ and $C_3$ the double and triple covers of $\mathbb P^1$ corresponding to the subgroup $\mathfrak S_2$ and the conjugacy class (3.3) respectively.

**Proposition.** Let $\mathcal X \to \mathbb P^1$ be a degree 6 del Pezzo fibration such that every singular fiber contains only one singularity which is of type 1 or 2. If the generic fiber $X$ splits at the degree 12 extension with the Galois group $D_6$, the group of torsors under the model $\mathcal T$ of the Néron-Severi torus becomes isomorphic to

$$H^1(\mathbb P^1, \mathcal T) \cong J(C_2) \times J(C_3) \times \mathbb Z.$$  

Proof. The assumption on the mildness of singularities in fibers implies the existence of curves $C_2$ and $C_3$. We construct a resolution of the module Pic $\overline X \cong \mathbb Z^4$ using the idea mentioned in [Gil14]:

$$0 \to \mathbb Z \to \mathbb Z^5 \to \text{Pic } \overline X \to 0$$

with relations:

- $a_1 \to L - E_1$
- $a_2 \to L - E_2$
- $a_3 \to L - E_3$
- $b_1 \to L$
- $b_2 \to 2L - E_1 - E_2 - E_3$
Under this resolution the action $S_3$ permutes $a_i$’s for $i = 1, 2, 3$ and the action of $S_2$ permutes $b_j$’s for $j = 1, 2$. The kernel $\mathbb{Z}$ becomes the trivial Galois module. This induces the corresponding exact sequence of models for dual modules:

$$1 \rightarrow T \rightarrow \text{Res}_{C_2} \mathbb{G}_m \times \text{Res}_{C_3} \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 1$$

where the curves $C_2$ and $C_3$ are double and triple covers corresponding to subgroups $S_2$ and $S_3$, respectively (see Fig. 3.2). The map to $\mathbb{G}_m$ is induced by norm homomorphisms. The corresponding long exact sequence gives

$$0 \rightarrow H^1(\mathbb{P}^1, T) \rightarrow \text{Pic} C_2 \times \text{Pic} C_3 \rightarrow \mathbb{Z} \rightarrow 0$$

where the map to $\mathbb{Z}$ takes the line bundles $(\mathcal{L}_2, \mathcal{L}_3)$ to $2 \deg \mathcal{L}_2 + 3 \deg \mathcal{L}_3$. It follows that $H^1(\mathbb{P}^1, T) \cong J(C_2) \times J(C_3) \times \mathbb{Z}$. 

Over the field of complex numbers we can use Kanev’s construction described in Section 3.4 to compute the Intermediate Jacobian of the threefold. Notice that blowing up a point on the degree 6 del Pezzo lying on the complement of 6 exceptional curves produces a degree 5 del Pezzo with 10 exceptional curves. One of them is the exceptional divisor and three of them are proper transforms of conics passing through opposite pairs of exceptional curves on the blown-up surface.

Take the section $\sigma \in \text{Sect}(\pi)$ corresponding to a rational point on the torus $T$ and blow up the threefold $\mathcal{X}$ with center at $\sigma$. This gives a degree 5 del Pezzo fibration. The curve $\mathcal{C}$ parametrizing exceptional lines is a 10:1 cover of $\mathbb{P}^1$ which is reduced:

$$\mathcal{C} = \mathbb{P}^1 \sqcup C_3 \sqcup C_6$$

where $\mathbb{P}^1$ parametrizes blown up exceptional divisors on fibers, $C_3$ is the triple cover parametrizing three curves connecting opposite pairs on fibers of $\mathcal{X}$ and passing through the section $\sigma$, and $C_6$ is the 6:1 cover parametrizing exceptional curves of fibers of $\mathcal{X} \rightarrow \mathbb{P}^1$. Notice that $C_6 \simeq C_2 \times_{\mathbb{P}^1} C_3$, the fibered product over $\mathbb{P}^1$. Denote the natural projections to $C_2$ and $C_3$ by $p_2$ and $p_3$ respectively. The induced incidence correspondence $i$ in this case is the endomorphism on the Jacobian $J(\mathcal{C}) \cong J(C_3) \times J(C_6)$. We compute the image of the map

$$(\text{id} - i) : J(C_3) \times J(C_6) \rightarrow J(C_3) \times J(C_6).$$
**Proposition.** The image of the endomorphism \( id - i \) is generated by elements of the form

\[
(L, -p_3^*(L)) \quad \text{and} \quad (0, -p_2^*(M)) \in J(C_3) \times J(C_6)
\]

where \( L \in J(C_3) \) and \( M \in J(C_2) \).

**Proof.** We analyze how the incidence \( i \) acts on points. Let \( q \in C_3 \) be a point parametrizing a line in a smooth fiber. It intersects an exceptional divisor as well as two opposite exceptional lines from the hexagon. The two points on \( C \) parametrizing opposite exceptional lines are of the form \( p_3^*(q) \) which shows that \( i(q, 0) = (0, p_3^*(q)) \) where \( q \) defines the corresponding class of a line bundle on \( C_3 \). Therefore, we found the first generator, \((q, -p_3^*(q))\).

Let \( r \in C \) be the point parametrizing a line on the hexagon on a smooth fiber. Then this line intersects adjacent lines and one line parametrized by the point \( q \in C_3 \). Hence \( i(0, r) = (q, r_1 + r_2) \):
Then we have
\[(q, 0) - (q - p_3^*(q)) + (0, r) - (q, r_1 + r_2) = (0, -r_1 - r_2 - r_3) \in J(C_3) \times J(C)\]
which is an element of the form \((0, -p_2^*(s))\) for some \(s \in C_2\). This gives the second generator and finishes the proof of the Proposition. \(\square\)

The generators in the Proposition are obviously independent, while the pull-backs on line bundles are injective. We conclude that
\[I\text{J}(\mathcal{X}) \simeq J(C_2) \times J(C_3)\]
and notice that it is isomorphic to the identity component of the group of torsors \(H^1(\mathbb{P}^1, \mathcal{T})\) for the model of the Néron-Severi torus over the complex numbers.
Bibliography


