RICE UNIVERSITY

Impedance Control Approaches for Series Elastic Actuators

by

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ABSTRACT

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For applications requiring interaction with humans or unstructured environments, robots are increasingly designed to leverage the intentional drivetrain compliance of series elastic actuators (SEAs). Impedance control, likewise, is of particular value in these applications, having long been considered an effective means of addressing dynamic interaction. Although impedance controlled SEAs appear often in the literature, a number of important questions remain unanswered. If, for example, robust contact stability is required, can an SEA render a virtual stiffness greater than its physical spring rate? Previous studies answer no, but this is largely a question of control architecture. It is proven here, as part of a larger study comparing the stability and passivity of five different control approaches, that this is in fact possible if disturbance observer based impedance control is adopted.

The fidelity with which SEAs render desired impedances is important as well. In comparing the impedance rendering accuracy of multiple control approaches, experimental data in both the time and frequency domain point once more to disturbance observer based impedance control. This new SEA control architecture yields demonstrable improvement in actuator transparency, closed loop hysteresis, and the SEA’s dynamic response to both reference commands and external torques.

Two new performance metrics are formulated based on the $H_\infty$ and $H_2$ system
norms to further quantify SEA impedance rendering accuracy across the frequency spectrum. A novel model matching framework is then constructed that leverages these metrics for the optimal synthesis of SEA impedance control. Passive and accurate controllers result that, having been deployed on physical hardware, represent the first application of LMI-based, multi-objective, optimal control synthesis to series elastic actuation.

These results are all confirmed experimentally on high performance SEAs. Three new actuator designs are presented that provide up to 350 Nm in peak torque and torque sensing resolutions as low as 0.006 Nm. This 58,333:1 dynamic range (an order of magnitude improvement over previous SEAs) is achieved in a torque dense, 94.3 Nm/kg package ideally suited for use in humanoid robots. Demonstrated SEA performance reinforces the practical utility of the recommended control approaches and speaks to the broader applicability of impedance controlled SEAs to human-centric robots.
To Erin, Reagan, and Caleb
Acknowledgments

It was December 26, 2008—I was enjoying dinner with my beautiful wife, her sister, and her parents George and Kelly Nield at my favorite restaurant, Clyde’s of Reston, Virginia. Not long after drinks arrived, the conversation turned to the still nascent idea that I might want to pursue a Ph.D. in the future. Of course, this would likely be a part-time endeavor and I really had no idea what the process would look like, but the idea was there. Over a delicious and memorable meal my in-laws helped me see that this was an idea worth pursuing, offering numerous thoughtful suggestions, invaluable advice, and heartfelt support, just as they have every day since. Thank you both for encouraging me down this path, providing wisdom along the way, and praying for my every step.

There have been numerous participants in this almost seven year journey since that initial dinner, and I am appreciative of them all. I know that I certainly have not been a typical graduate student (if there is such a thing), but that has never seemed to faze my advisor, Marcie O’Malley. Whether I was on campus full time or only popping in once every other month, Marcie always made time for me. Thank you Marcie for being the perfect advisor for me, flexible enough to let me chart my own course yet involved enough to help me search for answers, patient when my research took a back seat to the rest of my life for years at a time, and firm enough to push me over the finish line.

One of the primary reasons I decided to go back to school was a strong desire to learn, and in this regard I have not been disappointed. Thank you to my committee members Fathi Ghorbel and Lydia Kavraki, in particular, from whom I have taken multiple classes and learned a great deal. Thank you as well to Karolos Grigoriadis
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Although I have worked with many great people over the course of my career, three stand out as particularly important contributors to what I present here. Adam Parsons was an invaluable collaborator in the design and analysis of the custom torsion springs that give life to the Robonaut and Valkyrie series elastic actuators. Thank you Adam, it was a pleasure. Developing the original Valkyrie embedded control system with Nick Paine and James Holley was also enjoyable, although every bit as challenging. The work we did together during the DRC inspired many of the questions I’ve tried to answer here, and I cannot thank you enough. James in particular, if it was not for you I might not have any experimental data at all—seriously. You have gone above and beyond in making yourself available to assist at anytime and in anything. If and when the time comes to return the favor, sign me up.

While this specific story starts with a dinner almost seven years ago, its true origin lies much earlier. My parents, Steve and Terri Mehling, have always encouraged my
intellectual curiosity and supported all aspects of my education. Thank you for teaching me to think, explore, and ask questions. Thank you for starting all of this with that first Lego kit. And thank you for preparing me to face, with confidence, all the challenges of life.

Having written the final words and plotted the last piece of data, I can but thank God for providing me strength and sustaining me through this entire process. His principle means of grace for this has been my family—a family that is patient and kind, a true blessing, abounding in steadfast love. Sweet Reagan and Caleb, you haven’t known a day in your lives when Daddy wasn’t working on the “Ph.D.” but through it all you’ve been so helpful and understanding even as the hours grew long. Your smiles and hugs mean everything to me, and I look forward to them now more than ever before. I love you both.

And Erin, I could not have done any of this without you. All of the pages in all of the books in all of the world could not contain the many things you’ve done to make these comparatively few pages possible. I’ve been blessed, and I love you. Thank you for everything these past seven years. And thank you for everything in all the years before, and in all those yet to come.

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Chapter 1

Introduction

Robotic systems are fast becoming ubiquitous in everyday life. A 2014 study by the International Federation of Robotics highlights the growing economic impact of industrial robots, citing over 178,000 units in annual sales and an estimated 1,332,000 to 1,600,000 robots in use worldwide [1]. Most industrial robots, however, are still designed to operate in highly structured workcells with well defined interactions. This is not the case in the burgeoning field of service robots. Professional service robots are still vastly outnumbered by their industrial counterparts [1], but they have a broad impact across the economy with applications in fields such as medicine, defense, and agriculture (e.g. [2], [3], and [4]). Personal service robots, of which iRobot’s Roomba is the classic example [5], are typically smaller and less expensive, but they too have had a marked impact on everyday life due to their growing popularity and adoption.

The ability to operate outside of the structured workcell setting distinguishes most service robots from traditional industrial systems. Determining effective methods of managing a robot’s interaction with its environment is by no means a solved problem though, and this places limits on how robots can be used. In fact, most robots designed for truly unstructured interaction with the human-centric world are only in the early stages of research and development. As this technology matures it will drive the growth of the service robot sector, expanding the number of applications in which robots can provide meaningful work and further improving quality of life for the general population.
1.1 Human-Centric Robots

Because achieving robust interaction with the physical world holds such promise for future robotic applications, a great deal of research interest lies in designing systems to meet this objective. Arising from this work is a broad class of human-centric robots, defined here as systems that operate in direct contact with humans or as those robots that perform human-like tasks by explicitly interacting with unstructured, unmodeled environments. These are the systems that will further cement robotics into all aspects of everyday life, impacting the way people use and interact with robots.

Haptic interfaces are aimed at providing touch sensations or force feedback to a human user. The science of haptic feedback and studies on the design of haptic displays have yielded important progress toward understanding contact stability, the psychophysical perception of human-robot interaction, and virtual environment rendering [6], [7], [8]. Commercial devices like the Geomagic Touch X (Fig. 1.1a) are already used to provide force feedback for desktop design, simulation, and training [9], while the research community has leveraged haptic feedback in applications ranging from the tele-operation of unmanned aerial vehicles [10] to robot assisted surgery [11], [12].

Wearable robots, both for rehabilitation and human augmentation, also leverage haptic feedback, but these systems couple the ability to impart force on a human user with an ability to respond to user inputs and the external environment. Powered prosthetics like the BiOM T2 prosthetic ankle (Fig. 1.1b) have given amputees the ability to walk again [13], and lower limb exoskeletons like those from Ekso Bionics (Fig. 1.1c) have done the same for those with spinal cord injuries [14]. Wearable systems for the upper arm and hand are being developed for the rehabilitation of stroke patients (e.g. [15] and [16]), and there has long been an interest in using both
Figure 1.1: A small sampling of human-centric robots: (a) Geomagic Touch X haptic interface, (b) BiOM T2 powered prosthetic ankles, (c) Ekso bionic suit by Ekso Bionics, (d) NASA’s X1 exoskeleton, (e) NASA’s Robonaut 2, (f) Rethink Robotics Baxter robot, (g) NASA’s Valkyrie robot, and (h) Schaft robot.
upper and lower body exoskeletons to augment human strength and endurance [17]. Unique challenges can also be addressed with wearable robots, as NASA has done by adapting their X1 exoskeleton (Fig. 1.1d) to serve as an astronaut exercise device in the confined volume of a spacecraft over long duration missions [18].

The increasing prevalence of humanoid robots makes it readily apparent that robots are expected, now more than ever, to interact with their physical surroundings. These systems are, by design, intended to perform tasks in much the same way humans do, manipulating tools and unmodeled objects, operating in human-centric spaces, and often times working side-by-side with people. Applications for human-like dexterous manipulators vary widely, ranging from the assembly work done by Baxter from Rethink Robotics (Fig. 1.1f) [19], to the astronaut assistant and in-space maintenance capabilities of NASA’s Robonaut 2 (Fig. 1.1e) [20], [21]. Robots designed for bipedal walking, such as NASA’s Valkyrie (Fig. 1.1g) [22] and the Schaft robot (Fig. 1.1h) [23], must also handle interactions with an unstructured environment at every step, particularly if these robots are to ever robustly navigate uneven, rough terrain. There is considerable interest in improving the energy efficiency, dexterity, task performance capability, and contact stability of bipedal humanoids as evidenced by ongoing efforts in the recent DARPA Robotics Challenge [24] and many other current projects (e.g. [25] and [26]).

While human-centric robots vary widely in form, intended purpose, and technological maturity; a common need for stable, robust behavior when in contact with the external world is shared across all of these systems. In applications requiring direct human contact, the “feel” of a robot is often important and the robot’s interactive behavior becomes an explicit control objective. In other circumstances, task performance may be the primary objective, but an ability to control contact
dynamics greatly enhances robot capability. The discussion in subsequent chapters of interaction control and its connection to actuator design, as well as the technologies, designs, and results presented herein, thus promise broad applicability to the challenges of robotics today and the many human-centric robots of the future.

1.2 Impedance Control

Control techniques based largely on regulating specific output variables, like a robot’s position or force, are insufficient for reliably handling contact and interaction tasks. Impedance control differs, however, in that it defines a desired dynamic relationship between a robot’s motion and force, rather than modulating these variables individually. In his seminal work introducing the concept of impedance control, Hogan specifically cites the desire to manipulate and dynamically interact with an environment as the primary motivation for this control technique [27], [28], [29]. In fact, a wide range of robots have since demonstrated the effectiveness of leveraging impedance control to perform various contact tasks (e.g. [30], [31], and [32]). This should come as no surprise, though, in light of human physiology. The human central nervous system itself relies on an impedance control paradigm to manage interaction [33], [34].

The mechanical impedance of a system is formally defined as the mapping of an input (generalized) velocity to an output (generalized) force at a specific interaction port* within the system:

\[ Z(s) = \frac{F(s)}{\dot{x}(s)} \]  

(1.1)

Thus, it becomes the objective of impedance control to enforce a desired dynamic relationship, \( Z_{\text{des}}(s) \). This idea can be conceptualized as deriving a robot’s control

*Interaction ports, or locations at which energy may be exchanged with the environment, are described at length within the context of robot impedance control by Hogan and Buerger [35].
law from the behavior of a desired physical model, as illustrated in the simple example of Fig. 1.2. Here, the robot, represented by a single mass $m$, is subject to an actuator force $F_m$ and an external force from the environment $F_{ext}$. The desired behavior for the robot is defined as that of a mass-spring-damper model, and simply solving for the actuator force that produces this desired dynamic behavior yields an impedance control law:

$$F_m = -k(x - x_0) - b(\dot{x} - \dot{x}_0)$$  \hspace{1cm} (1.2)$$

where $k$ and $b$ are the stiffness and damping of the desired model, $x$ is the position of the mass, and $x_0$ represents a desired virtual equilibrium point.

This example summarizes Hogan’s “simple” impedance control (see [28] and [35])
for a somewhat trivial one degree-of-freedom case. However, it also highlights a num-
ber of key points important in subsequent chapters. First, the resulting impedance
control law (1.2) resembles a proportional-derivative (PD) motion controller with the
desired virtual stiffness and damping corresponding to proportional and derivative
control gains respectively [35]. From a practical perspective this is quite useful. PD
motion control is easy to implement on most robots and control designers can rely
on their intuitive understanding of the gains’ physical interpretations to quickly gen-
erate and easily iterate impedance control designs. Second, the presence of a virtual
equilibrium point, $x_0$, within the impedance control framework allows for traditional
regulation and trajectory tracking tasks. The closed loop, driving point impedance at
the robot’s interaction port with its environment defines how the robot will “feel” to
the world and the dynamic response it will have to external inputs from the environ-
ment. The robot’s response to reference commands, however, is also simultaneously
defined by impedance controller dynamics, and this is important to consider as well.
When the robot of Fig. 1.2 is given a new position reference, for example, it will move
as if “pulled” by a virtual spring and damper stretched between the new $x_0$ reference
and the robot’s current position. Third, external force feedback is not used in this
example, and as a consequence, apparent output inertia (in this case the mass $m$)
cannot be changed by the impedance control law. Adding a direct measure of $F_{\text{ext}}$ in
the feedback loop would allow for arbitrary desired output inertias (Ott [36] provides
an example of this), but this is less common in practice because adding adequate ex-
ternal force sensing to a robot significantly complicates electromechanical design. For
this reason, much of the work to be presented assumes that actuator output inertia is
not to be modified and that external force sensors are unavailable. This assumption
will be revisited in Chapter 7 when optimal impedance control synthesis is examined.
Also noteworthy in the example of Fig. 1.2 is the consideration of causality. The impedance control law (1.2) takes as input motion variables (position and velocity), and it outputs an actuator force, $F_m$. Therefore, it can appropriately be described as enforcing an impedance, per the causality of (1.1). However, from the environment’s perspective an external force, $F_{\text{ext}}$, is applied to the robot, which in turn responds with a prescribed motion. Thus, the closed loop interaction dynamics exhibited by the robot are best described as having admittance\footnote{Admittance, in linear systems, is simply the inverse of impedance.} causality, and can be expressed by the transfer function:

$$Y(s) = \frac{\dot{x}(s)}{F(s)} \quad (1.3)$$

The decision to treat the robot in this example as an admittance, rather than as an impedance, was implicitly made when choosing to model the system as a mass and the environment’s action as a force. Philosophically, this choice departs somewhat from Hogan’s original intent of impedance control [27], but it can prove quite useful in modeling the interaction of a robot with its environment [37]. Throughout the literature, the term “impedance control” has come to generally describe many approaches that modulate a robot’s dynamic behavior rather than its motion and/or force variables alone. The term “admittance control”, on the other hand, is often reserved only for architectures that explicitly act on measured external forces (e.g. [38], [39], and [40]). Therefore, the various control approaches examined in subsequent chapters will all be referred to as “impedance controllers” generally, but care will be taken to explicitly note if it is the system’s driving point impedance or admittance transfer function being considered when that distinction is important.

Further abusing terminology, some sources use the term impedance (or admittance, in linear systems, is simply the inverse of impedance.}
tance) loosely to refer to a dynamic relationship between position and force (i.e. a gen-
eralized dynamic stiffness), rather than a velocity and force relationship (e.g. [35], [41],
and [42]). This description is convenient since most robots rely on position sensors,
and desired robot trajectories are often expressed in terms of position. Confusing a
true impedance transfer function with one that maps position to force has impor-
tant consequences, however, particularly when analyzing stability and passivity as
in Chapter 4. Therefore, while the term “impedance” will be used throughout as
a general description of system dynamics, care will be taken to distinguish transfer functions by their input and output variables, using the terms “impedance” and
“admittance” to describe specific transfer functions only when warranted.

As noted by Hogan and Buerger [35], impedance and admittance are properties of
the robot alone, entirely independent of the environment. Unlike motion and force,
which cannot be described fully without considering interactions with the environ-
ment, a robot’s impedance (or admittance) can be defined regardless of contact state.
This is, in fact, a principle strength of impedance control, allowing a robot to operate
consistently and predictably during free motion, contact, and the transition between
these two states [43]. By leveraging impedance control, human-centric robots, and
their component actuators, do not need a model of the environment to ensure stable,
human-like interaction. Furthermore, as the design and use of impedance control is
better understood in the context of new robotic archetypes and actuator technologies,
robot task performance will only continue to improve.

1.3 Research Objective

Whereas impedance control addresses the difficulty of performing interaction tasks
through appropriate controller design, series elastic actuation provides a distinctly
hardware-based solution to many of the same problems. Introduced in greater detail in Chapter 2, series elastic actuators (SEAs) incorporate an intentionally compliant element in series with their drivetrains [41]. In so doing, the passive output impedance of an SEA is decreased, and contact stability is improved. In fact, the widespread adoption of series elastic actuation throughout the robotics community in recent years can largely be attributed to the ease with which contact stability is achieved using these low-mechanical-output-impedance actuators.

Although much of the SEA literature focuses on actuator electromechanical design (as does a portion of this work), the influence of control approach must certainly be considered in any discussion of series elastic actuator performance. Here, impedance control is a natural solution to couple with SEAs as this actuation technique becomes evermore prevalent in human-centric robots. There are still a number of open questions, however, when considering impedance controlled series elastic actuators.

How, for example, do the various control approaches presented in the literature compare to one another in regard to the accuracy with which they render desired impedances? And what range of impedances can these various approaches render while preserving robust contact stability? The additional dynamics introduced in a series elastic actuator by the inherent compliance of its drivetrain make the SEA control problem, fundamentally, one of overcoming non-colocation. As will be explained, the physical separation of actuator control effort (i.e. motor torque) and the output variables of interest in a series elastic actuator necessitates control approaches beyond the simple impedance control of the previous section. As more feedback measurements and control gains are introduced, however, these more complex control architectures become difficult for the control designer to efficiently construct and quickly iterate. Resulting performance also becomes more difficult to analyze, particularly when con-
tact with the external environment is made. For example, while anecdotal evidence suggests that a series elastic actuator can, in fact, render impedances higher than its own physical, open loop characteristics, a detailed examination of impedance controlled SEAs has never proven this to be the case while also guaranteeing robust contact stability.

The research objective is thus fourfold in regard to these issues. First, a comparison of various SEA impedance control architectures is desired to specifically examine the impedance range and peak stiffness each approach is capable of rendering while preserving robust stability. Second, the performance of a new disturbance observer based impedance control approach needs to be tested, so that its strengths in comparison to other architectures and the practical benefits of adopting such an approach may be determined. Third, a quantitative metric to compare impedance controller performance must be developed to provide a reliable measure of the accuracy with which various control architectures render desired impedances and the bandwidth over which their performance is achievable. Fourth, the development of a framework for the synthesis of series elastic actuator impedance control is desired that leverages this performance metric to ease the burden on the control designer, making the proper selection of controller gains and achieving optimal impedance rendering easier in practice. Finally, it should be noted that an examination of these issues would not be complete without demonstrating results experimentally on high performance series elastic actuators designed specifically for human-centric robots.

1.4 Outline and Contributions

In accomplishing the aforementioned objectives, seven significant contributions are made across the domains of actuator electromechanical design; the design, analysis,
and performance of SEA impedance control; and optimal control synthesis. These contributions are presented in detail throughout the subsequent chapters.

Following the motivation of human-centric robots and the impedance control background presented in Chapter 1, Chapter 2 provides an in-depth introduction to series elastic actuation. This includes a brief history of the control of robots with flexible joints, and examples that illustrate the benefits of leveraging SEAs in human-centric robot design. The series elastic actuator model used for subsequent analysis is also presented here.

The primary contribution of Chapter 2 comes in Section 2.3 with the detailed design of three novel, high performance, series elastic actuators that demonstrate high torque density and specific torque\(^\dagger\), high peak torque as compared to other SEAs, high resolution sensing leading to an unprecedented dynamic range, and a compact form factor suitable for use in humanoid robots. Introduced briefly as components of the Robonaut 2 and Valkyrie robots [20], [22], [44], [45], these actuator designs, features of which are published in four United States patents [46], [47], [48], [49], are presented in complete detail here. Performance of the new Valkyrie series elastic actuators is further demonstrated in their use as experimental platforms throughout subsequent chapters.

Chapter 3 begins an investigation of four candidate SEA impedance control approaches by outlining the important control objectives to be considered and, in turn, presenting system block diagrams and deriving the necessary closed loop transfer functions for analysis. Chapter 4 then continues this investigation with a detailed look at the closed loop stability and passivity of each architecture (along with exam-

\(^\dagger\) Torque density is simply peak output torque per unit volume \((\text{Nm/cm}^3)\), while specific torque is defined as peak output torque per unit mass \((\text{Nm/kg})\).
ining a flexible-joint impedance control approach, albeit not originally intended for SEAs, introduced in Chapter 2). Following the example of Colgate [37], the passivity of an actuator’s driving point impedance (or admittance) can be used to assess robust stability in the presence of unknown interactions with the environment. Three significant contributions result from this study.

First, a derivation of the necessary and sufficient conditions to ensure both stability and passivity for each of the discussed impedance control approaches is performed. The complete, nonconservative constraints placed on the system parameters by these conditions allow for a detailed study in simulation of each architecture’s full theoretical impedance range when robust contact stability is the primary objective. By comparing each impedance control approach in a single study, using the same modeling assumptions, immediate design decisions can be made and conclusions drawn without the need to reconcile disparate, partial studies from the literature.

Of particular note, as part of the aforementioned comparison, a detailed stability and passivity analysis is produced for a novel disturbance observer (DOB) based impedance control architecture. Cascading an impedance control loop with a new DOB based torque controller, developed for NASA’s Valkyrie robot and recently outlined in the Journal of Field Robotics [45], produces this new architecture, described in Section 3.4. The detailed examination of this impedance control approach, subsequently presented in Chapter 4, provides the first look at theoretical stability and passivity limits for disturbance observer based impedance control implemented on series elastic actuators.

Finally, the analysis of Chapter 4 proves that a series elastic actuator can indeed render impedances greater than its own physical open loop impedance while still guaranteeing passivity (i.e. robust contact stability). Empirical evidence is presented
to support this new result using a Valkyrie series elastic actuator and disturbance observer based impedance control.

With stable and passive impedance ranges well understood, Chapter 5 turns its attention to the question of performance, specifically examining the accuracy with which control approaches from the previous chapters render desired impedances. To facilitate this comparison, two new performance metrics are introduced based on the $H_2$ and $H_\infty$ system norms familiar in robust and optimal control theory. It is demonstrated that these metrics accurately represent a controller’s impedance rendering accuracy while also providing valuable insight into the bandwidth of closed loop system performance.

Having seen its promise throughout the preceding studies, Chapter 6 examines the performance of disturbance observer based impedance control in more detail. Here, the principal contribution is an experimental look at some of the factors affecting this controller’s performance and, more importantly, the practical benefits gained when applying this approach to high performance series elastic actuators. For example, actuator hysteresis significantly decreases and SEA transparency when rendering low impedances is enhanced if a disturbance observer is included in the impedance control framework.

Broadening scope slightly, Chapter 7 addresses the larger issue of impedance control design and the difficulties inherent in constructing multi-loop cascaded architectures with numerous control parameters. A brief introduction to linear matrix inequalities (LMIs) and their use in optimal control serves as the foundation for a novel model matching framework presented in Section 7.4. This final contribution leverages the $H_\infty$ based performance metric of Chapter 5 to formulate SEA impedance control design as an optimal control problem. Leveraging this framework for impedance
control synthesis abstracts gain selection away from the designer, easing the control design task significantly. Both simulated and experimental results demonstrate the effectiveness of this, the first application of LMI-based, multi-objective, optimal control synthesis to series elastic actuation.

Chapter 8 outlines again the significant contributions of the previous chapters while discussing the implications these results have for series elastic actuation and impedance control design. Here, the presented research is placed in a broader context with a look toward the future and the promise of continued progress, not just in series elastic actuator impedance control, but in the entire domain of human-robot interaction and human-centric systems.
Chapter 2

Series Elastic Actuation

2.1 Robot Compliance

Compliance has long been understood to aid robot performance during contact tasks. In fact, early work relied on compliance for stability when robots were required to exert forces on their environment (see [50]). Historically, this compliance often appeared as pliable finger pads, an elastic mechanism at the end-effector, or some other padded material mounted distal to the rest of the robot (e.g. the remote center compliance system [51]). These additional elements improved a robot’s stability during interaction, but were largely excluded from the system model used for control. Because they also went unmodeled, structural and drivetrain compliance were seen to negatively impact robot accuracy during free motion and positioning tasks. Sweet and Good were among the first to recognize this drawback and sought to incorporate the inherent flexibility of robot drivetrains into a traditional, rigid robot model [52]. Subsequently, a large body of work has come to address the perceived nonideality of flexible robot joints, and a variety of methods to overcome this challenge are available to the control designer.

Accepting that the predominant source of compliance in robot joints is geartrain flexibility*, the simple model of Fig. 2.1a can be used to represent an idealized actua-

---

*This is a particularly good assumption in high performance robots utilizing compact, torque-dense, Harmonic Drives [54] or other transmissions that rely on flexible components to produce a high gear ratio in a small volume.
Figure 2.1: Modeling a robot with drivetrain compliance. (a) A linear representation of an idealized actuator: the motor mass $m_m$ and link mass $m_L$ are separated by a drivetrain stiffness $k$ while having positions $x_m$ and $x_L$ respectively. Motor force $F_m$ and external force $F_{\text{ext}}$ act on the system as shown. (b) The KUKA-DLR Lightweight Robot, a manipulator to which this idealized model of drivetrain compliance has been applied [30], [36], [53].

ator. Spong’s elastic-joint robot model [55] extends this simple construction to serial chain manipulators and provides a tractable starting point for a number of efforts aimed at understanding, and reducing, the parasitic effects of actuator compliance. Stable regulation in elastic-joint robots has been demonstrated using simple PD feedback of motor position [56], [57]; and various studies have offered feedback linearization as a means to achieve desirable tracking results (albeit while often requiring the undesirable inversion of parameters in the dynamic model, or higher-order differentiation of measurements) [55], [58], [59]. Decoupling-based approaches and backstepping have been applied to the position control of flexible-joint manipulators [60], and adaptive control has also been extensively explored in this context [61], [62]. While earlier work focused largely on the position control problem, more recently, flexible-joint robots have seen applications of hybrid position/force control [63], compliance control [64], and, as will be discussed extensively, impedance control.
2.1.1 Impedance Control of Flexible-Joint Robots

Applying traditional impedance control techniques to the compliant actuator model of Fig. 2.1a poses a number of challenges. If, for example, motor position and velocity are used in a simple PD feedback architecture, the system is indeed provably stable (see [56]), but little can be done to control the dynamics of the link mass. On the other hand, if link position and velocity are measured instead of the motor variables, closed loop PD control improves dynamic performance, but only if there is sufficient damping present in the system to preserve stability (Section 4.1 will illustrate this point in more detail). Given this trade, full state feedback appears to be a natural solution for the impedance control of flexible-joint robots, and indeed this approach has been used to great effect, most notably in the control of the KUKA and DLR lightweight robots (one of which is shown in Fig. 2.1b) [30], [36], [65], [66].

Designing a full state feedback impedance controller is not without difficulties, however. The intuitive physical interpretation of gains, so helpful during the design of simple impedance control in Section 1.2, begins to break down as the number of required control gains increases. Ott et al. [66] get around this problem by developing a cascaded control approach that, in essence, assigns a physical meaning to each of the state feedback gains. The illustration in Fig. 2.2 helps to conceptualize this architecture.

Given that the application of simple impedance control is hindered by the non-colocation of motor force and measured output position, it is desired to alter the fourth order, mass-spring-mass system of Fig. 2.2a such that it appears, to the greatest extent possible, as a simpler second order system. Intuitively, this can be done by either decreasing the motor mass $m_m$ or by increasing the drivetrain stiffness $k$. The control architecture presented by Ott et al. [66] accomplishes the former with an inner torque
Figure 2.2: Conceptual representation of passivity-based impedance control [66] as applied to a linear flexible-joint model. Variables are defined as in Fig. 2.1a with \( u \) representing the impedance loop command, \( K_P \) and \( K_D \) the PD gains, \( m_0 \) the desired apparent motor mass, and \( x_0 \) the desired virtual equilibrium position.

(or force) feedback loop. This leads to a system with new apparent motor mass \( m_0 \) (Fig. 2.2b). With \( m_0 \) sufficiently small, traditional impedance control, as outlined in Section 1.2, can then be applied effectively in spite of the dynamics introduced by drivetrain compliance (Fig. 2.2c). Here, the feedback gains on motor position and velocity maintain their traditional interpretation as a virtual spring and damper, while torque feedback can be physically interpreted as motor inertia scaling. In addition to demonstrating the successful impedance control of robots with drivetrain compliance, this approach also ensures robust contact stability with a rigorous passivity analysis aided by the physical interpretations attached to each gain [66].

Passivity-based impedance control of an actuator with drivetrain compliance is further formalized by the architecture in Fig. 2.3, resulting in a closed loop expression:

1) Force Feedback
\[
F_m = \frac{m_m}{m_0} u + (1 - \frac{m_m}{m_0}) k (x_m - x_L)
\]

2) PD Impedance Control
\[
u = -K_P (x_m - x_0) - K_D (\dot{x}_m - \dot{x}_0)
\]
Figure 2.3: Passivity-based impedance control architecture [66] as applied to a linear actuator with drivetrain compliance. Variables are defined as in Fig. 2.2 with force gain $K_F = \frac{m_m}{m_0}$.

for commanded motor force:

$$F_m = \frac{m_m}{m_0} u + (1 - \frac{m_m}{m_0}) k(x_m - x_L). \quad (2.1)$$

Here, $k(x_m - x_L)$ provides feedback of the actuator force via Hooke’s Law, and $u$ represents the command provided by the outer impedance loop:

$$u = -K_P(x_m - x_0) - K_D(x_m - x_0). \quad (2.2)$$

Additionally, the force feedback gain in (2.1) appears as a ratio of actual motor mass to desired apparent motor mass,

$$K_F = \frac{m_m}{m_0}, \quad (2.3)$$

thus providing a straightforward implementation of the intuitive physical interpretation described above.

Many control approaches for robots with drivetrain compliance follow a design
paradigm similar to that of Fig. 2.3. First, it is desired to eliminate (i.e. mask) an actuator’s compliance as much as possible using control. Then, having done so, a traditional “rigid robot” control approach, well established in the literature, can be selected to produce the closed loop behavior desired of the compliant system. There are many benefits to framing the control problem in this way, and departing from this paradigm is not necessarily being suggested. It is important, however, to recognize implicit assumptions made during the control design process. Most analyses of drivetrain compliance, for example, assume weak elasticity (i.e. relatively stiff systems). This is perfectly reasonable for many systems, and control approaches developed under this assumption will often perform well even as joint compliance increases. At times, however, stability, passivity, and performance can all be hindered if greater compliance is not explicitly addressed. A fresh look at this specific assumption and the control performance possible in systems with significant compliance is particularly appropriate given the widespread adoption and continued popularity of series elastic actuators, which typically have compliance well exceeding that of standard drivetrains.

2.2 Series Elastic Actuators

First described explicitly by Pratt and Williamson in 1995 [41], series elastic actuators (SEAs) are designed to include an intentionally elastic element (i.e. a spring) in their drivetrains. While the introduction of significant compliance lowers the natural frequency of the physical system, potentially impacting closed loop actuator bandwidth, the potential benefits of actuator compliance are considered to outweigh this concern in a variety of applications. Most notably, series elastic actuators have been used with great success across a broad range of human-centric robots: humanoid
dexterous manipulators [19], [20], [22], [67]; haptic devices [68]; exoskeletons for both rehabilitation and human augmentation [18], [69], [70]; powered prostheses [71]; and walking robots [22], [72], [73], [74], [75].

2.2.1 Why Series Elastic Actuation?

Of course, it bears asking: Why is series elastic actuation so appealing for human-centric applications (or any others for that matter)? The literature, as it turns out, offers a number of compelling answers to this question.

Because spring deflection is required to produce actuator force, a small motor motion in an SEA no longer creates a large force as in rigid actuators. Pratt and Williamson describe this as turning “the force control problem into a position control problem” [41]. Due to a variety of nonlinearities (friction, backlash, hysteresis, etc.) geared actuators are more appropriately treated as accurate position sources rather than force sources. Thus, force accuracy is improved and the force control problem simplified with series elastic actuation [76]. Part of this simplification is also due to improved force sensing in series elastic actuators. Because spring deflection provides a measure of actuator force in SEAs, high-resolution, digital position sensors can be used for feedback, rather than relying on analog strain gauges, as is common in rigid robots. If implemented well, a higher signal-to-noise ratio can be achieved with series elastic force sensing, and cleaner sensing will naturally facilitate better force control.

Intuitively, the increased compliance in SEAs provides improved shock tolerance for a robot during high frequency impacts. This point is an oft cited rationale for leveraging series elastic actuation in legged locomotion, where impacts occur with every step (see for example [74] and [77]). Paluska and Herr have also demonstrated the dramatic effect series elasticity can have on actuator work and power output [78].
With the proper selection of spring rate to achieve a catapult-like effect in the actuator, they observe a 400% increase in the energy delivered over a fixed stroke length and a peak power amplification of 40% versus a traditional rigid actuator [78]. This result, achieved in part due to a series elastic actuator’s ability to store energy, suggests the significant promise SEAs hold for robots required to jump, throw, or perform other highly dynamic tasks.

Energy storage in SEAs has also been observed to improve overall efficiency in legged robots [79], and techniques for optimally selecting series compliance to maximize this benefit have been developed for the compliant humanoid, COMAN [72]. The benefit of exploiting the natural dynamics of series elastic actuators is not limited to legged robots, of course. Rhythmic arm motions used during tasks like crank turning, sawing, drumming, and hammering nails have proven efficient and robust when the resonance of series elasticity is specifically leveraged [80], [81]. Exploiting intentional series compliance is also the principle means by which recent variable stiffness actuators achieve reduced energy consumption and explosive motion sequences during dynamic tasks [82], [83].

The low mechanical output impedance offered by series elastic actuators is appealing for robots required to perform sensitive manipulation or interact with humans. A certain level of safety is inherent in the actuator design itself. However, this and other benefits of series elastic actuation are not fully realized apart from control. The ability to effectively render low closed loop output impedances with series elastic actuators is well documented for example [84], [85]; and indeed, this is important in both haptics and human augmentation where safety and device transparency are important. Researchers, however, often forgo comprehensive studies of SEA impedance control in favor of furthering electromechanical actuator design or achieving practi-
cal results in specific applications. A detailed understanding of both hardware and control system design (particularly in regard to impedance rendering performance) is what drives the continued progress of these practical results though, enabling robots to realize in full, the potential benefits of series elastic actuation.

2.2.2 Actuator Model

A single rotary series elastic actuator is represented by the model in Fig. 2.4. Here, motor inertia $J_m$ and load inertia $J_L$ are separated by the series elastic stiffness $k$, and are subject to viscous damping values $b_m$ and $b_L$ respectively. The SEA is treated as a dual input system with motor current $i_m$ as the command input, and external torque $\tau_{ext}$ representing exogenous inputs from the environment (e.g. human interaction forces). With $\theta_m$ and $\theta_L$ representing motor and output position respectively, the system dynamics are expressed as:

$$J_m\ddot{\theta}_m + b_m\dot{\theta}_m + \frac{1}{N}k\left(\frac{1}{N}\theta_m - \theta_L\right) = k_t i_m$$  \hspace{1cm} (2.4)

$$J_L\ddot{\theta}_L + b_L\dot{\theta}_L - k\left(\frac{1}{N}\theta_m - \theta_L\right) = \tau_{ext}$$  \hspace{1cm} (2.5)

where $N$ represents the actuator gear ratio and $k_t$ is the motor’s torque constant.

Aside from the additional details introduced in this model (gear ratio, damping, torque constant, etc.), it is clear that the only real difference between a series elastic actuator and an actuator with only drivetrain compliance is the implicit understanding that the spring stiffness $k$ in an SEA is considerably lower. For this reason, the dynamics introduced by series elasticity are particularly important to address.

For the purpose of control design, the multi-input-multi-output plant of Fig. 2.5
Figure 2.4: The series elastic actuator model. $J_m$ and $J_L$ are motor and load inertia respectively. $b_m$ and $b_L$ are motor and output viscous damping, $\theta_m$ and $\theta_L$ are motor and output positions. $k$ is the series spring stiffness, $N$ is the gear ratio, $\tau_{ext}$ is the external torque from the environment, and $\tau_m$ is the applied motor torque, a product of torque constant $k_t$ and applied motor current $i_m$.

Figure 2.5: The actuator plant model used for control analysis. Motor current $i_m$ and external torque $\tau_{ext}$ are inputs; motor velocity $\dot{\theta}_m$, actuator output position $\theta_L$, and measured spring torque $\tau_k$ represent outputs.
accurately represents the series elastic actuator. From (2.4) and (2.5), the six open loop transfer functions mapping inputs \( i_m \) and \( \tau_{ext} \) to motor velocity \( \dot{\theta}_m \), actuator output position \( \theta_L \), and measured spring torque, 

\[
\tau_k = k\left(\frac{1}{N}\dot{\theta}_m - \theta_L\right),
\]  

(2.6)

are determined to be:

\[
A(s) = \frac{\dot{\theta}_m(s)}{i_m(s)} = \frac{N^2k_iJ_Ls^3 + N^2k_kb_ks + N^2k_ks}{D(s)}
\]  

(2.7)

\[
B(s) = \frac{\dot{\theta}_m(s)}{\tau_{ext}(s)} = \frac{Nks}{D(s)}
\]  

(2.8)

\[
U(s) = \frac{\theta_L(s)}{i_m(s)} = \frac{Nk_ik}{D(s)}
\]  

(2.9)

\[
V(s) = \frac{\theta_L(s)}{\tau_{ext}(s)} = \frac{N^2J_m s^2 + N^2b_ms + k}{D(s)}
\]  

(2.10)

\[
W(s) = \frac{\tau_k(s)}{i_m(s)} = \frac{NJ_lik_ks^2 + Nb_lik_k}{D(s)}
\]  

(2.11)

\[
X(s) = \frac{\tau_k(s)}{\tau_{ext}(s)} = \frac{-N^2J_m s^2 - N^2b_ms}{D(s)}
\]  

(2.12)

where

\[
D(s) = N^2J_mJ_Ls^4 + (N^2J_mb_L + N^2b_mJ_L)s^3
\]

\[
+ (kJ_L + N^2b_mb_L + N^2J_mk)s^2 + (kb_L + N^2b_mk)s.
\]  

(2.13)
Motor velocity, output position, and actuator torque all, at times, serve as primary feedback variables in the control architectures examined throughout Chapters 3 and 4. Thus, the open loop plant of Fig. 2.5 and the transfer functions (2.7)–(2.12) feature prominently in this forthcoming analysis.

2.2.3 A Note on Actuator Output Inertia

It should be noted that inclusion of actuator output inertia $J_L$ in the SEA model of Fig. 2.4 is not universal throughout the literature. In truth, neglecting this term is somewhat common (see for example [41], [69], [84], and [86]). Many devices such as haptic displays and powered orthoses are, in fact, appropriately modeled with negligible robot output inertia. It has also been suggested that omitting output inertia does not directly affect passivity analyses comparable to those performed in Chapter 4 [69].

There are two primary reasons, however, that output inertia is included here. First, many robots of interest do indeed have appreciable actuator output inertia. Thus, including it in the model provides a more accurate representation of the system. This is particularly true for SEAs in serial chain manipulators where a single actuator experiences the inertia of all distal joints and links. Second, although output inertia does not directly influence the conditions for passivity presented in Section 4.2, it will be seen in Section 4.1 that output inertia does appear as a term in the denominator of both an SEA’s closed loop driving point admittance transfer function and its closed loop forward transfer function in response to reference position commands. Thus, from this perspective actuator output inertia has the potential to impact closed loop pole location and hence system stability. This reasoning agrees with Pratt and Williamson’s observation that a minimum output mass is required to ensure stable
series elastic actuation [41]. Although a number of SEA studies omit output inertia, focusing solely on actuator driving point impedance (again, see [69] or [86] for an example), dexterous manipulators, legged systems, and many other human-centric robots require stable reference tracking that must not be sacrificed when introducing series elastic actuation. Hence, actuator output inertia is included here to more fully address the potential factors affecting the stability, passivity, and performance of series elastic robots across their broad range of applications.

2.3 High Performance Actuator Design†

Any investigation of series elastic actuator impedance control (or any control methodology for that matter) must be grounded in the realities of physical hardware. Improvements in actuator performance are only realized when electromechanical design is considered in tandem with potential advances in control, and when these advances are applied, in turn, to real-world systems. To this end, a suite of new rotary series elastic actuators was designed in parallel to the studies of SEA impedance control detailed in Chapters 3 through 7.

Developed for use in NASA’s Robonaut 2 [20] and Valkyrie [22] humanoid robots, these actuator designs represent a significant advancement in SEA performance. Peak actuator torques in excess of 350 Nm allow these SEAs to function as limb joints in humanoid robots designed to perform meaningful, high-force, human-like work. This peak strength is particularly suited for the hip and knee joints of bipedal walking robots, and enables larger, stronger humanoids than have typically relied on series elastic actuation (cf. [67], [72], [87], and [88]). A compact form factor and high specific

†Portions of this work have been previously published in four United States patents [46], [47], [48], [49]. I gratefully acknowledge the work of my collaborators in these efforts.
torque (up to 94.3 \text{Nm/kg}) also ensure that robots leveraging these SEA designs need not sacrifice human-scale volume or mass in exchange for performance.

While 17 unique series elastic actuators have been designed for the Robonaut 2 (R2) and Valkyrie robots, the following two sections highlight three SEA designs in detail. First, a description of the Robonaut 2 hip actuator serves as an introduction to the R2 SEA design paradigm, including the component selection, actuator sensing modalities, and custom electromechanical design and layout that contribute to this unique SEA architecture. While successfully fielded in robots both on Earth and in space, the Robonaut 2 SEA represents only a first generation design. Section 2.3.2 provides a detailed description of the Valkyrie hip and knee actuators, designs that are steeped in R2 heritage but also incorporate a number of important SEA design improvements. These two Valkyrie SEAs subsequently serve as experimental platforms for the forthcoming series elastic actuator impedance control studies.

2.3.1 NASA’s Robonaut 2 Series Elastic Actuator

Robonaut 2 (R2) is a 58 degree-of-freedom humanoid robot designed for dexterous manipulation in human environments (Fig. 2.6). The robot’s seven-degree-of-freedom arms and five-fingered hands allow human-like manipulation of tools, while long, seven-degree-of-freedom legs, each with a gripping end-effector, enable climbing in microgravity aboard the International Space Station (ISS). All upper arm, leg, and waist joints in R2 (25 in total) rely on a series elastic design architecture similar to that of the Robonaut 2 hip actuator (Fig. 2.7) to be outlined in detail.

Robonaut 2 series elastic actuators all include a frameless, brushless DC motor and a Harmonic Drive gear set, packaged together in a custom designed aluminum structure specific to the actuator’s location in the robot. In the R2 hip SEA, a
Figure 2.6: NASA’s Robonaut 2 aboard the International Space Station. Highlighted are the locations of the R2 hip actuators at the base of each climbing leg.

Figure 2.7: The Robonaut 2 hip series elastic actuator. (a) A picture of the assembled actuator. (b) A CAD rendering with overall SEA dimensions.
Kollmorgen RBE-03010 motor and a Harmonic Drive CSG-25 component set are used. In addition to serving as the proximal joint in each of Robonaut 2’s climbing legs, the R2 hip actuator is also used for sagittal plane leg swing in a prototype bipedal walking robot at NASA/Johnson Space Center (Fig. 2.8). Thus, two unique sets of specifications exist for the actuator (as summarized in Table 2.1) depending on the application specific motor winding and gear ratio chosen.

The flight version of the actuator, used in Robonaut 2, leverages a higher torque constant motor ($k_t = 0.412 \text{Nm/A}$) and a higher gear ratio (160:1) to achieve required output torques at significantly lower motor currents. Because high speed is not necessary (or desired) for ISS operations, heavily favoring torque generation in this design, at the expense of speed, does not hinder performance. In fact, it helps. While a

![Figure 2.8: Prototype bipedal walking legs relying on Robonaut 2 SEAs. The R2 hip series elastic actuator, used for sagittal plane leg swing, is located at the base of each leg as shown.](image)
Table 2.1: Robonaut 2 hip series elastic actuator specifications.

<table>
<thead>
<tr>
<th></th>
<th>Flight Unit</th>
<th>Ground Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(Robonaut 2)</td>
<td>(prototype biped)</td>
</tr>
<tr>
<td>Motor</td>
<td>RBE-03010-A</td>
<td>RBE-03010-H</td>
</tr>
<tr>
<td>Torque Constant (motor)</td>
<td>0.412 Nm/A</td>
<td>0.168 Nm/A</td>
</tr>
<tr>
<td>Gear Ratio</td>
<td>160:1</td>
<td>80:1</td>
</tr>
<tr>
<td>Peak Speed (at 96V)</td>
<td>1.25 rad/s</td>
<td>6.32 rad/s</td>
</tr>
<tr>
<td>Peak Torque (torque sensing limit)</td>
<td>271 Nm</td>
<td></td>
</tr>
<tr>
<td>Spring Stiffness</td>
<td>3884 Nm/rad</td>
<td></td>
</tr>
<tr>
<td>Torque Sensing Resolution</td>
<td>0.74 Nm</td>
<td></td>
</tr>
<tr>
<td>Actuator Mass</td>
<td>4.4 kg</td>
<td></td>
</tr>
<tr>
<td>Dimensions</td>
<td>$\varnothing 15.24 \text{ cm} \times 9.78 \text{ cm}$</td>
<td></td>
</tr>
<tr>
<td>Torque Density</td>
<td>0.152 Nm/cm$^3$</td>
<td></td>
</tr>
<tr>
<td>Specific Torque</td>
<td>61.6 Nm/kg</td>
<td></td>
</tr>
</tbody>
</table>

smaller actuator could be used given the overall power requirements, the flight version of the R2 hip SEA provides additional thermal endurance (due to low current operation relative to the thermal mass of the motor), an important benefit when operating on orbit. When a lower torque constant and gear ratio are used, as in the prototype biped, peak actuator speed is correspondingly increased (reaching 6.32 rad/s at 96 V). The combination of this peak speed and the actuator’s high torque capacity represents a broad performance capability uncommon in SEAs.

The high torque density and specific torque of the R2 hip SEA (0.152 Nm/cm$^3$ and 61.6 Nm/kg respectively) are a product of the torque dense motor and Harmonic Drive used as well as the overall mechanical design and layout of the custom actuator structure. When viewed in cross section (Fig. 2.9) the packaging of the actuator’s
Figure 2.9: CAD rendered cross section of the Robonaut 2 hip series elastic actuator.

major components is seen and some important design features can be appreciated. First, of note is the large hollow bore passing through the center of the actuator. When assembled in a serial chain with other actuators, this allows power and data wiring to pass through, rather than around, the SEA to reach distal joints. Thus, all cable harnessing in Robonaut 2 can be kept safely inside the robot’s structure. Hall effect sensors embedded in the motor stator are used for motor commutation and an incremental encoder provides high resolution (3648 counts/rev) motor position feedback. These sensors complement the absolute position sensors used to measure series elastic deflection and the output position of the actuator.

A custom planar torsion spring (Fig. 2.10) provides series elasticity for the Robonaut 2 hip actuator (comparable custom torsion springs are also used in all other R2 SEAs). Unique in its peak torque to deflection ratio, this spring design allows compact series elastic actuation in high torque actuators, while preserving the ability to measure spring deflection, and thus actuator torque, with position sensors rather
Figure 2.10: The custom planar torsion spring used in the Robonaut 2 hip SEA.

than strain gauges. The R2 hip spring, 12.35 cm in diameter and only 1.17 cm thick, has a stiffness of $3884 \text{ Nm/rad}$ and a maximum deflection of $\pm 0.07 \text{ rad}$. This performance is enabled by the use of high yield strength (1.93 GPa), age hardened maraging 300 steel for spring construction, and the design of the two rotationally symmetric splines spanning between the spring’s input and output mounting segments. These splines are designed via an iterative finite element analysis (FEA) process to optimize stress density throughout the material at maximum deflection, and ensure that the spring maintains a linear elastic behavior across its full operating range. Additionally, integrated hardstops prevent the spring from deflecting past its linear range, and so doing, also define the peak operating torque of the series elastic actuator. A similar design process has since been adopted by Sergi et al. [89] and Accoto et al. [90] for the design of planar torsion springs in powered orthoses.

Two Netzer Precision DS-130 19 bit electric encoders are mounted around the Harmonic Drive gear set and provide an absolute measurement of spring input (i.e. Harmonic Drive output) and spring output (i.e. actuator output) position. The difference between these two positions is a measurement of series elastic spring deflection and can
be used to determine actuator torque via Hooke’s Law. Although nominally 19 bit resolution, in practice, after appropriate filtering, these sensors provide a roughly 15 bit position measurement, corresponding to a $1.9 \times 10^{-4}$ rad position resolution and a 0.74 Nm torque resolution.

2.3.2 NASA’s Valkyrie Series Elastic Actuator

NASA’s Valkyrie robot is a 44 degree-of-freedom humanoid that, like Robonaut 2, is designed for dexterous manipulation in human environments (Fig. 2.11). Unlike R2 however, Valkyrie is a bipedal walking robot, having originally been designed for terrestrial disaster response as part of the DARPA Robotics Challenge [24]. While Robonaut 2 SEAs boast impressive performance capabilities and a successful track record deployed aboard the International Space Station, series elastic actuators de-

![Figure 2.11: NASA’s Valkyrie humanoid robot. Highlighted at right are the locations of the Valkyrie hip and knee SEAs in the robot’s leg.](image)
signed for Valkyrie offer a number of significant performance improvements. Because Valkyrie actuators inherit a great deal from the R2 SEA design paradigm, these improvements are best placed in context by examining the design differences between the Valkyrie hip and knee SEAs and the R2 hip actuator of the previous section.

Like Robonaut 2 SEAs, Valkyrie actuators also rely on frameless, brushless DC motors and Harmonic Drive gearing. The Valkyrie hip and knee SEAs (Fig. 2.12 and Fig. 2.13 respectively) differ, however, in that these two components are not packaged coaxially. Instead the motor and Harmonic Drive are placed side-by-side and connected with a timing belt to minimize actuator height and preserve the anthropomorphic form factor of Valkyrie’s upper leg.

Both Valkyrie actuators use a Parker K089050-6Y motor and a CSG-25, 50:1 Harmonic Drive. Thus, they have the same motor torque constant \( k_t = 0.145 \text{Nm/A} \) and the actuators’ overall gear ratios (120:1 for the hip and 122.7:1 for the knee) only differ by a slight variation in timing belt ratio. These similarities, and the low mass of each actuator, lead to comparable torque density and specific torque specifications (cataloged, along with other actuator specifications, in Table 2.2). The 0.120 \text{Nm/cm}^3 torque density of the Valkyrie hip SEA is perhaps somewhat misleading, given that additional volume in the actuator is dedicated to leg structure and not actuation itself. This makes the torque density measurement artificially low. Nevertheless, the specific torques of both actuators, particularly the 94.3 \text{Nm/kg} value achieved in the Valkyrie knee SEA, clearly represent significant advancements beyond the R2 hip actuator previously presented.

Aside from peak torque improvements, Valkyrie SEAs have higher peak speeds than R2 actuators, having been designed to operate at 150 V, and they also incorporate a fundamental change in how series elastic spring deflection, and hence actuator
Figure 2.12: The Valkyrie hip series elastic actuator. (a) Pictures of the assembled actuator. (b) A CAD rendering with overall SEA dimensions.

Figure 2.13: The Valkyrie knee series elastic actuator. (a) Pictures of the assembled actuator. (b) A CAD rendering with overall SEA dimensions.
Table 2.2: Valkyrie series elastic actuator specifications.

<table>
<thead>
<tr>
<th></th>
<th>Hip SEA</th>
<th>Knee SEA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Motor</td>
<td>Parker K089050-6Y</td>
<td></td>
</tr>
<tr>
<td>Torque Constant (motor)</td>
<td>0.145 Nm/A</td>
<td></td>
</tr>
<tr>
<td>Spring Stiffness</td>
<td>4011 Nm/rad</td>
<td></td>
</tr>
<tr>
<td>Torque Sensing Resolution</td>
<td>0.006 Nm</td>
<td></td>
</tr>
<tr>
<td>Gear Ratio</td>
<td>120:1</td>
<td>122.7:1</td>
</tr>
<tr>
<td>Peak Speed (at 150 V)</td>
<td>4.70 rad/s</td>
<td>4.60 rad/s</td>
</tr>
<tr>
<td>Peak Torque (torque sensing limit)</td>
<td>350 Nm</td>
<td>350 Nm</td>
</tr>
<tr>
<td>Actuator Mass</td>
<td>4.31 kg</td>
<td>3.71 kg</td>
</tr>
<tr>
<td>Torque Density (approx.)</td>
<td>0.120 Nm/cm³</td>
<td>0.161 Nm/cm³</td>
</tr>
<tr>
<td>Specific Torque</td>
<td>81.2 Nm/kg</td>
<td>94.3 Nm/kg</td>
</tr>
</tbody>
</table>

torque, is measured. When the Valkyrie hip and knee SEAs are viewed in cross section (Fig. 2.14 and Fig. 2.15) the dense electromechanical packaging of each actuator can be appreciated, but it is also readily apparent that the two Netzer Precision electric encoders used in Robonaut 2 actuators to measure spring deflection are no longer present. They have been replaced, instead, by a single 32 bit optical absolute encoder (a Renishaw RESOLUTE readhead and RESA ring combination) that measures spring deflection directly. The optical readhead of the spring deflection sensor is fixed relative to the actuator’s output (i.e. spring output), and the absolute scale ring moves with Harmonic Drive output (i.e. spring input). By measuring spring deflection with a single sensor, rather than the difference of two, error in torque measurement is reduced and a more compact overall actuator package is achieved.

The theme of eliminating unnecessary or redundant sensors is continued on the
Figure 2.14: CAD rendered cross section of the Valkyrie hip series elastic actuator.

Figure 2.15: CAD rendered cross section of the Valkyrie knee series elastic actuator.
motor side of each actuator where, as in R2 SEAs, an incremental encoder is used for high resolution motor position feedback. Unlike in R2 however, Valkyrie actuators rely on this sensor for motor commutation as well, eliminating the need for motor hall effect sensors. A 13 bit, hall effect based magnetic absolute position sensor is used though, to measure actuator output position on startup. In both the Valkyrie hip and knee SEAs, a shaft, rotating with actuator output, runs through the center bore of the Harmonic Drive. The magnetic field of a small magnet fixed to the end of this shaft is measured by a stationary sensor above the timing belt assembly to provide output absolute position feedback.

Certainly, the most significant factor contributing to improved SEA performance in the Valkyrie actuators is the previously mentioned high resolution spring deflection sensor and its assembly with a further refined planar torsion spring designed specifically for the hip and knee SEAs (Fig. 2.16). This spring, 10.16 cm in diameter and 0.889 cm thick, closely resembles the R2 spring design, having two symmetric splines extending annularly between inner and outer mounting bolt patterns. Unique in the Valkyrie design, however, is the segmentation of the outer mounting pattern. It no longer forms a complete circle as in the R2 design, but instead has only two small segments at the outer end of each spline in the interest of weight savings. Functionally, this change does not affect performance. As in Robonaut 2, the same high yield strength (1.93 GPa), age hardened maraging 300 steel is used for spring construction, but the spline geometry is slightly modified to yield a stiffer spring (4011 Nm/rad) with a larger linear elastic deflection range (±0.087 rad).

Although the Renishaw RESOLUTE optical readhead provides 32 bit digital feedback, in practice, only the first 22 bits are used for spring deflection measurement in each Valkyrie SEA. When coupled with the spring’s 4011 Nm/rad spring rate and
Figure 2.16: The custom planar torsion spring used in the Valkyrie hip and knee series elastic actuators.

±0.087 rad deflection range, this sensor provides a remarkable 0.006 Nm minimum torque resolution and a peak measurable torque of 350 Nm. This equates to a 58,333:1 dynamic range, far exceeding that of the Robonaut 2 hip SEA (366:1) and unprecedented in the series elastic actuation literature (cf. 500:1 in [84], 2038:1 in [90], and 9032:1 in [91]).

The performance capabilities of the Valkyrie hip and knee SEAs make them ideal platforms for an experimental investigation of series elastic actuator impedance control. Further enabling control experiments are the custom embedded motor controllers used for individual actuator control in the Valkyrie robot. Each motor controller consists of a high current (30 A continuous, 60 A peak) motor bridge, and a modular logic card with both a dedicated microprocessor and FPGA (field-programmable gate array). Actuator sensor data is processed, communication is handled, and high-rate closed loop control is implemented locally at each Valkyrie SEA using the onboard logic card. All control architectures investigated in subsequent chapters are run on

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Dynamic range in this context is simply the ratio of peak measurable torque to minimum resolvable torque.
the microprocessor at 5 kHz (as is standard throughout the Valkyrie robot).

Numerous factors affect series elastic actuator performance and this technology’s potential for improving human-centric robots. Perhaps none are more important than the means by which compliance is added to an actuator and the method used to measure said compliance. The Valkyrie SEA design leverages patented approaches developed for Robonaut 2, further refining these techniques to enhance capability. This enables not just the development of a novel dexterous humanoid robot, but also, and more important to the work of subsequent chapters, the Valkyrie hip and knee SEAs allow an in-depth investigation of series elastic actuator impedance control grounded in the realities of high-performance hardware designed specifically for human-centric robots.
Chapter 3

SEA Impedance Control Architectures

There are two basic objectives in the impedance control of series elastic actuators: emulating desired dynamics at the SEA’s interaction port with its environment and responding appropriately to reference commands. The former, in effect, defines how the SEA “feels” to people or objects in contact with the actuator. For the purposes of this study it is assumed that actuator output inertia is not modified via control. Furthermore, desired dynamics are confined to those of a mass-spring-damper second order system. Within the context of the SEA model presented in Chapter 2, the desired driving point impedance is therefore expressed as:

\[ Z_{\text{des}}(s) = \frac{\tau_{\text{ext}}(s)}{\dot{\theta}_L(s)}_{\text{des}} = \frac{J_Ls^2 + K_{\text{di}}s + K_{\pi}}{s} \]  

(3.1)

where \( J_L \) is, again, the actuator’s inherent output inertia, \( K_{\pi} \) is a desired spring rate, and \( K_{\text{di}} \) is a desired virtual damping. Because \( \tau_{\text{ext}} \) is considered the exogenous input from the environment and the comparison studies to follow are predominantly concerned with the resulting actuator output position, \( \theta_L \), it is helpful to construct the desired external torque to output position transfer function (related to the desired driving point admittance) as follows:

\[ \frac{Y_{\text{des}}(s)}{s} = \frac{\dot{\theta}_L(s)}{\tau_{\text{ext}}(s)}_{\text{des}} = \frac{1}{J_Ls^2 + K_{\text{di}}s + K_{\pi}}. \]  

(3.2)
It should be noted that the selection of a parallel spring-damper as the desired dynamic model does not preclude investigating pure spring rendering \((K_{di} = 0)\), pure damping \((K_{pi} = 0)\), or null impedances \((K_{pi} = K_{di} = 0)\). This choice is intentionally made, however, because rendering damped behavior is specifically cited in previous studies as difficult to accomplish passively [86], and damping is critical to the successful realization of impedance control’s second objective: exhibiting appropriate dynamics in response to reference commands.

While the response of an impedance controlled SEA to reference signals is not always critical in a haptic display, for example, it is of utmost importance in robots required to track trajectories and operate effectively both in free space and during contact. Virtual damping is important here because for most robots, and in most applications, a critically damped position response is desired.

Applying the virtual equilibrium point concept to an impedance controlled SEA, the ideal actuator response to a reference position command, \(\theta_{ref}\), is the behavior exhibited by the desired dynamic model if stretched between the actuator’s current position and the new equilibrium point. This is expressed by the desired reference position to actuator output position transfer function:

\[
\frac{\theta_L(s)}{\theta_{ref}(s)}_{des} = \frac{K_{di}s + K_{pi}}{J_Ls^2 + K_{di}s + K_{pi}}. \tag{3.3}
\]

A variety of impedance control architectures can be applied to series elastic actuators in an effort to achieve the impedance rendering and reference response objectives outlined above. The balance of this chapter introduces four candidate control approaches for later comparison in Chapters 4 and 5. Starting with an application of simple impedance control for SEAs, building in complexity, and then culminating in
a new disturbance observer (DOB) based architecture, each impedance control approach is presented with a system block diagram that is used to derive the closed loop transfer functions representing impedance rendering and reference response dynamics.

3.1 Simple Impedance Control

Although seldom used in the current series elastic literature, the straightforward application of simple position-feedback-based impedance control serves as a useful baseline when comparing various SEA impedance control architectures. As illustrated in Fig. 3.1, a PD impedance compensator:

\[ I(s) = K_{pi} + K_{di}s \]  

provides motor current commands to the actuator plant in response to output position feedback. As in (3.2) and (3.3), control gains \( K_{pi} \) and \( K_{di} \) can be interpreted as the stiffness and damping of the desired impedance to be rendered.

From this control architecture, two transfer functions representing the closed loop

\[ \text{Figure 3.1: Simple impedance control of an SEA.} \]
output position response to a reference position and an external torque are derived:

\[
\frac{\theta_L(s)}{\theta_{ref}(s)}_{CL} = \frac{U(s)I(s)}{1 + U(s)I(s)}.
\]  

(3.5)

\[
\frac{\theta_L(s)}{\tau_{ext}(s)}_{CL} = \frac{V(s)}{1 + U(s)I(s)}.
\]  

(3.6)

As is the case throughout this chapter, open loop plant transfer functions (here \(U(s)\) and \(V(s)\)) are defined as in Section 2.2.2.

3.2 Cascaded Impedance and Torque Control

Series elastic actuation was originally presented as a means for achieving high quality actuator force or torque control [41]. Thus, it is natural to approach the SEA impedance control problem from the perspective that high-precision torque control, if not a prerequisite for, certainly facilitates improved impedance control performance. This premise leads to the cascaded impedance and torque control architecture of Fig. 3.2. Like the passivity-based impedance control of compliant actuators presented in Section 2.1.1, an outer impedance loop coupled with an inner torque loop provides full state feedback control with physically meaningful control gains.

There are, of course, a variety of torque control techniques for series elastic actuators. The cascaded impedance and torque control approach presented here relies on a PID torque compensator,

\[
T(s) = K_{pt} + K_{nt} \frac{s}{s} + K_{dt} s,
\]  

(3.7)

or one of various subsets (P, PD, PI, etc.), to govern actuator torque. This is in line
with much of the original SEA force control work leveraging PID feedback architectures (e.g. [41] and [84]), and closely resembles an SEA impedance control architecture examined by Sensinger and Weir [92].

Closed loop expressions for the actuator’s response to a reference command and an external torque can again be derived using the system diagram in Fig. 3.2. Omitting for clarity the explicit dependence on $s$ of each component:

$$\frac{\theta_L(s)}{\theta_{ref}(s)_{CL}} = \frac{UTI}{1 + TW + UTI} \tag{3.8}$$

$$\frac{\theta_L(s)}{\tau_{ext}(s)_{CL}} = \frac{V + VTW - UTX}{1 + TW + UTI} \tag{3.9}$$

### 3.3 Cascaded Impedance, Torque, and Velocity Control

The inner torque compensator of the previous section can itself leverage a cascaded structure to augment performance. Both inner position loops and inner velocity
loops have been used to improve the torque fidelity, robustness, and stability of torque controlled SEAs [93], [94], [95], [96]. Thus, the control approach illustrated in Fig. 3.3 is constructed with an impedance and nested torque controller (both defined as before) cascaded again with an inner PI compensator acting on motor velocity feedback:

\[ \Omega(s) = K_{pv} + \frac{K_{iv}}{s}. \] (3.10)

This cascaded impedance, torque, and velocity control approach has been extensively examined by Vallery et al. [69] and, more recently, by Tagliamonte and Accoto [86]. These studies specifically address the stability and passivity of this impedance control architecture and thus, provide a valuable point of comparison during the investigations in Chapter 4. Both Vallery et al. [69] and Tagliamonte and Accoto [86], however, assume only P or PI torque compensation in their control ar-

![Figure 3.3: Cascaded impedance, torque, and velocity control of an SEA.](image)

\[ \Phi(s) = K_{pv} + \frac{K_{iv}}{s}. \] (3.10)
chitectures, leading each to the same conclusion that impedance controlled SEAs are not passive unless rendering pure spring-like behavior ($K_{di} = 0$). Furthermore, they conclude that a virtual stiffness greater than an SEA’s physical spring rate is not possible if passivity is to be maintained. Chapter 4 expands this analysis to include control approaches that leverage torque compensator derivative gains to demonstrate the conservative nature of these supposed limits on SEA closed loop impedance.

Once again, enabling the forthcoming analysis are the closed loop forward transfer function from reference position to actuator output position,

$$\frac{\theta_L(s)}{\theta_{ref}(s)_{CL}} = \frac{U\Omega TI}{1 + \Omega TW + \Omega A + U\Omega TI}, \quad (3.11)$$

and the closed loop, driving-point-admittance-like transfer function from external torque to output position,

$$\frac{\theta_L(s)}{\tau_{ext}(s)_{CL}} = \frac{V + V\Omega TW + V\Omega A - U\Omega TX - U\Omega B}{1 + \Omega TW + \Omega A + U\Omega TI}. \quad (3.12)$$

Each of these expressions are derived from the system diagram in Fig. 3.3 and omit again each component’s explicit dependence on $s$ for the sake of clarity.

### 3.4 Disturbance Observer Based Impedance Control

Rather than introducing an inner velocity loop as is done in the previous section, torque compensator performance can also be augmented through the addition of a disturbance observer (DOB) in the control architecture. Conceptualized as early as 1983 by Ohishi et al. [97], a disturbance observer compares actual plant output to an expected nominal output, attributes any discrepancy to disturbances, and provides
an augmented command input to mitigate disturbance effects. This approach is described as enabling improved servo loop performance without requiring higher control gains [98], and it has been applied in numerous ways to the control of rigid actuators and robotic systems (e.g. [98], [99], [100], and [101]). An important point of emphasis here is that the “disturbances” addressed by a DOB can be either exogenous inputs from the environment or variations due to model inaccuracies or system parameter changes. Both can result in an off-nominal or unexpected system response and a disturbance observer can account for and correct this discrepancy regardless of cause.

In SEA control, disturbance observers are largely used as a means to improve actuator torque tracking (see [91] and [102]). Hence, a disturbance observer is used to the same effect as the inner velocity loop in Section 3.3. This allows DOB based impedance control and the cascaded impedance, torque, and velocity approach of Section 3.3 to be treated as similarly aimed extensions of the simpler cascaded impedance and torque control architecture presented in Section 3.2. This comparison will be further explored in Chapters 4 and 5.

### 3.4.1 DOB Based Torque Control

The purpose of a disturbance observer is to enforce a nominal closed loop dynamic behavior, making the system response robust to external disturbances and model variations. The DOB based impedance control architecture examined here leverages an inner torque control approach adapted from the work of Paine et al. [91] that has recently been implemented on NASA’s Valkyrie robot.* Here, the plant model used for control development is that of a fixed output (i.e. $J_L = \infty$) series elastic actuator

---

*Additional detail describing the DOB based torque control used in NASA’s Valkyrie robot, introduced here and leveraged throughout subsequent chapters, has been published in the *Journal of Field Robotics* [45].
Figure 3.4: Disturbance observer based torque control. (a) The fixed output SEA plant model with variables defined as in Fig. 2.4. (b) The DOB based torque control architecture delineating the nominal closed loop plant, $P_n$, and the disturbance observer. $\tau_d$ represents the desired actuator torque, $\tau_r$ is the reference torque input to both $P_n$ and the DOB, and $\tau_k$, as before, is the measured spring torque.

as illustrated in Fig. 3.4a. From this model, the open loop transfer function from motor current to measured spring torque† is derived as:

$$P(s) = \frac{\tau_k(s)}{i_m(s)} = \frac{Nk_t k_{t}}{N^2 J_m s^2 + N^2 b_m s + k}.$$  \hspace{1cm} (3.13)

The nominal closed loop plant to be enforced by the disturbance observer then consists of this actuator model; a PD torque compensator,

$$T(s) = K_{pt} + K_{dt}s,$$  \hspace{1cm} (3.14)

dictating the desired closed loop torque response; and a feedforward term incorporating gear ratio and motor torque constant,

$$FF = (Nk_t)^{-1}.$$  \hspace{1cm} (3.15)

†Note the similarity of (3.13) to the transfer function $W(s)$ in (2.11) as $J_L \rightarrow \infty$. 
The interconnection of these components, outlined in Fig. 3.4b as the system $P_n$, leads to the nominal closed loop transfer function from reference torque input to actuator torque output:

$$P_n(s) = \frac{\tau_k(s)}{\tau_l(s)} = \frac{(Nk_t k K_{dt}) s + (k + Nk_t k K_{pt})}{N^2 J_m s^2 + (N^2 b_m + Nk_t k K_{dt}) s + (k + Nk_t k K_{pt})}. \quad (3.16)$$

Knowledge of the control gains $K_{pt}$ and $K_{dt}$, along with an experimental determination of model parameters $J_m$, $b_m$, $k$, and $k_t$ via system identification of the fixed output SEA‡, allows for the construction of a disturbance observer (as shown in Fig. 3.4b) from the inverse of the nominal closed loop torque response:

$$P_n^{-1}(s) = \frac{N^2 J_m s^2 + (N^2 b_m + Nk_t k K_{dt}) s + (k + Nk_t k K_{pt})}{(Nk_t k K_{dt}) s + (k + Nk_t k K_{pt})}. \quad (3.17)$$

A second order low-pass Butterworth filter,

$$Q(s) = \frac{1}{(1/\omega_q)^2 + (1.4142/\omega_q) s + 1}, \quad (3.18)$$

with cutoff frequency $\omega_q$ is included in the disturbance observer as well to ensure proper causality of the inverse plant model. The cutoff frequency of this filter also serves to adjust the bandwidth over which the DOB is effective.

The performance of DOB based torque control is demonstrated experimentally in Fig. 3.5. For this test, the Valkyrie knee actuator of Section 2.3.2 is mounted with its output fixed (as in Fig. 3.6) and a reference torque chirp signal of magnitude 15 Nm is commanded. The actuator is tested open loop, then with only PD control, and finally

‡While torque constant $k_t$ is typically obtained directly from a motor datasheet, the value determined here via system identification incorporates a measure of actuator efficiency as well. Thus, it is appropriately considered the actuator’s experimentally determined effective torque constant.
Figure 3.5: Experimental frequency response of DOB based torque control as applied to a fixed output Valkyrie knee actuator. (a) The magnitude and phase response of the open loop SEA, PD control, and full DOB based torque control, compared to the modeled open loop and nominal closed loop plants. (b) Torque tracking error versus signal frequency for the three experiments.
Figure 3.6: The Valkyrie knee actuator mounted in a fixed output configuration for torque control testing.

Table 3.1: Torque control parameters used for the fixed output test of Fig. 3.5.

<table>
<thead>
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<th>Parameter</th>
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<th>Units</th>
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<td>$\omega_q$</td>
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</table>

Improved torque response bandwidth versus the open loop system is achieved with PD control (increasing bandwidth from about 10 Hz to over 40 Hz), and the disturbance observer mitigates unmodeled effects, driving the empirical response to more closely match the desired theoretical behavior of the nominal closed loop system (Fig. 3.5a). The difference between PD control and DOB based control at first appears slight in Fig. 3.5a. However, a plot of reference torque tracking error (Fig. 3.5b) underscores the benefit of DOB based torque control. PD control obviously reduces tracking error compared to the open loop SEA when within the actuator’s closed loop bandwidth. The introduction of a disturbance observer though, with its ability to mitigate the effects of unmodeled friction and parameter identification errors, further improves this performance. The 35 dB reduction in tracking error observed at low frequencies with the full DOB torque controller (parameters for which are outlined in Table 3.1).
(Fig. 3.5b) also demonstrates the disturbance observer’s ability to reduce steady state error, thus, effectively replacing integral feedback in the torque compensator.

### 3.4.2 Cascaded Impedance and DOB Based Torque Control

To explore the use of DOB based torque control from an impedance control perspective, the controller of the previous section is cascaded with an outer impedance compensator (defined as before) and then applied to the original series elastic actuator model from Section 2.2.2. This results in the DOB based impedance control architecture of Fig. 3.7.

Using this system diagram, closed loop transfer functions from reference position to actuator output position, and from external torque to actuator output position,

Figure 3.7: Disturbance observer based impedance control of an SEA.
can once again be derived. Omitting for clarity the explicit dependence on s of each component yields:

\[
\frac{\theta_L(s)}{\theta_{\text{ref}}(s)_{\text{CL}}} = \frac{UI(FF + T)}{(P_n^{-1}QW + UI)(FF + T) + (TW + 1)(1 - Q)}
\] (3.19)

\[
\frac{\theta_L(s)}{\tau_{\text{ext}}(s)_{\text{CL}}} = \frac{(VW - UX)[P_n^{-1}Q(FF + T)(1 - Q)] + V(1 - Q)}{(P_n^{-1}QW + UI)(FF + T) + (TW + 1)(1 - Q)}.
\] (3.20)

These two transfer functions, along with the comparable expressions derived for each of the other three control architectures introduced in this chapter, serve as the foundation for the upcoming comparison studies in Chapters 4, 5, and 6. Particular attention is paid in empirical studies to comparing the cascaded impedance, torque, and velocity control approach with DOB based impedance control (Chapter 5), and examining in detail the practical benefits associated with leveraging a disturbance observer to control SEA impedance (Chapter 6). Prior to these investigations, however, Chapter 4 focuses squarely on the question of theoretical stability and passivity. Each of the preceding impedance control approaches have a range over which they can be expected to render either stable or passive impedances. Understanding these impedance ranges, and the factors that influence them, greatly enhances a designer’s ability to develop robust, high-performance impedance control for a variety of series elastic actuators.
Chapter 4

Stability and Passivity

Stability across a broad operating range is key to the effective impedance control of series elastic actuators. Without closed loop stability, an actuator is useless. Likewise, an SEA is of extremely limited use if it is stable for only a narrow range of system parameters or desired impedances. With this in mind, examining the impedance control architectures of Chapter 3 can shed new light on when series elasticity, in general, is applicable, and for what conditions each specific control approach is best suited.

While straightforward closed loop stability is of paramount importance, human-centric systems must also ensure coupled stability when in contact with a person or the environment. It is understood that closed loop passivity can provide such an assurance during unmodeled interactions (see [37]). This chapter, therefore, examines both the stability and passivity of the previously presented control architectures, with the goal of establishing, for each control method, system parameter constraints and an allowable range of desired impedances. These limits lead to necessary and sufficient conditions for the robust contact stability of each SEA impedance control approach, allowing for a greater appreciation of the circumstances in which each controller is appropriate.

Symbolic stability and passivity conditions with respect to the many system parameters of an SEA, while indeed enlightening, are complex expressions not necessarily suited for broad, side-by-side comparisons. As such, the constraint conditions
provided in this chapter are meant as tools, rather than answers in and of themselves. The intent here is not to provide a universal claim regarding the “best” control approach in the form of a symbolic expression, but rather, to explore the potential impedance range of each control architecture in context. General conclusions can, and will, be drawn where helpful or particularly insightful, but the overarching concern of this chapter is one of comparing and selecting amongst various impedance control options using realistic test cases. Parameter values of interest are therefore substituted into the derived stability constraints to enable a practical comparison of controller impedance range, and to determine which control approaches merit further consideration during the performance studies of Chapter 5. This specific process for deriving theoretical stability and passivity constraints, and then comparing the disparate control architectures, will be further outlined throughout the stability and passivity analyses of the next two sections.

4.1 Stability

A linear time-invariant (LTI) system is considered asymptotically stable if all poles of its closed loop transfer function lie in the left-half of the complex $s$-plane (i.e. if all poles have negative real parts). Given the SEA impedance control architectures of the previous chapter, stability can therefore be determined by examining the denominators of the presented closed loop transfer functions. Here, the Liénard–Chipart criterion is helpful.

A variant of the more common Routh–Hurwitz criterion, the Liénard–Chipart criterion [103] states that a general polynomial with real coefficients,

$$
g(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \quad (a_0 > 0), \quad (4.1)$$
has roots, all with negative real parts, if any one of the following four conditions is satisfied:

1. \( a_n > 0, a_{n-2} > 0, \ldots; \) and \( \Delta_1 > 0, \Delta_3 > 0, \ldots \)
2. \( a_n > 0, a_{n-2} > 0, \ldots; \) and \( \Delta_2 > 0, \Delta_4 > 0, \ldots \)
3. \( a_n > 0, a_{n-1} > 0, a_{n-3} > 0, \ldots; \) and \( \Delta_1 > 0, \Delta_3 > 0, \ldots \)
4. \( a_n > 0, a_{n-1} > 0, a_{n-3} > 0, \ldots; \) and \( \Delta_2 > 0, \Delta_4 > 0, \ldots \)

Here, \( \Delta_i \) represents the \( i^{th} \) principle minor of the Hurwitz matrix corresponding to the polynomial \( g(z) \) (i.e. the polynomial’s \( i^{th} \) Hurwitz determinant).

Unlike the Routh–Hurwitz criterion which requires the computation of all Hurwitz determinants to confirm stability, satisfying any one of the above conditions in the Liénard–Chipart criterion requires computing only the odd or even determinants. In particular, the highest order determinant can be omitted, greatly simplifying the complexity of the resulting expressions when deriving symbolic stability conditions for transfer functions with numerous system parameters.

A custom MATLAB script has been developed to generate symbolic transfer functions for each control architecture of interest and the Liénard–Chipart criterion is leveraged to derive symbolic stability constraints. The following section outlines this process and the subsequent results for each SEA impedance control approach.

4.1.1 Stability Conditions

In examining the stability (and later, the passivity) of these various SEA impedance control approaches, it should be noted that actuator output damping is assumed negligible (i.e. \( b_L = 0 \)) when deriving the symbolic closed loop transfer functions for each architecture. Considering that a single output bearing is the only source of viscous friction beyond a Valkyrie SEA’s spring, this is a reasonable assumption to
make for the practical cases examined here. Also, as controller complexity increases, 
the resulting stability conditions grow more unwieldy when represented symbolically. 
To aid in the comparison between controllers, these expressions are thus presented 
as functions of impedance control gains alone by substituting in system parameters 
appropriate for the physical actuators to be tested. While well suited for the goals of 
this study, deviating from this approach is, of course, possible if examining stability 
with respect to other parameters is desired. There is nothing inherent in either the 
analysis process or the MATLAB script developed that precludes this.

Simple Impedance Control

Beginning with the simple impedance control of Section 3.1, the system response 
to a reference position command and the system response to an external torque, as 
previously presented in (3.5) and (3.6), are:

\[
\frac{\theta_L(s)}{\theta_{\text{ref}}(s)_{\text{CL}}} = \frac{U(s)I(s)}{1 + U(s)I(s)} \quad (3.5)
\]

\[
\frac{\theta_L(s)}{\tau_{\text{ext}}(s)_{\text{CL}}} = \frac{V(s)}{1 + U(s)I(s)} \quad (3.6)
\]

Substituting in the definitions of \(U(s), V(s),\) and \(I(s)\) given in (2.9), (2.10), and (3.4) 
yields these same transfer functions expressed in terms of system parameters and 
impedance control gains:

\[
\frac{\theta_L(s)}{\theta_{\text{ref}}(s)_{\text{CL}}} = \frac{NK_{\text{di}}kk_t s + NK_{\text{pi}}kk_t}{\text{den}(s)} \quad (4.2)
\]
\[
\frac{\theta_L(s)}{\tau_{\text{ext}}(s)_{\text{CL}}} = \frac{N^2 J_m s^2 + N^2 b_m s + k}{\text{den}(s)}
\] (4.3)

where

\[
\text{den}(s) = N^2 J_L J_m s^4 + N^2 J_L b_m s^3 + k(J_L + N^2 J_m) s^2 + N k\left( K_{di} k_t + N b_m \right) s + N K_{pi} k k_t.
\]

Immediately apparent in (4.2) and (4.3) is the common denominator shared by both transfer functions. The stability of one response is therefore indicative of the other’s. As will be seen, this relationship holds true for all of the examined impedance control architectures. Perhaps intuitively expected, this provides the mathematical foundation necessary to claim that factors affecting the stability of an SEA’s response to reference commands also, and in the same way, affect the stability of the actuator’s response to external inputs from the environment.

Applying the Liénard–Chipart criterion to the denominator of (4.2) and (4.3), then simplifying, yields the following condition required for negative, and thus stable, closed loop poles:

\[
K_{pi} < \frac{J_m K_{di} k}{J_L b_m} \left[ \frac{J_L}{N^2 J_m} - \frac{K_{di} k_t}{N b_m} - 1 \right] + \frac{k}{N k_t}.
\] (4.4)

This condition places an upper limit on the proportional gain of the impedance controller (and thus, the closed loop virtual stiffness), and allows for a few additional observations. If, for instance, the objective is to render a pure stiffness \((K_{di} = 0)\), the limit on virtual stiffness from (4.4) is equivalent to the passive physical spring stiffness. From (4.3), the rendered virtual stiffness can be expressed by the output
position to external torque relationship at steady state,

\[ k_{\text{rendered}} = \lim_{s \to 0} \frac{\tau_{\text{ext}}(s)}{\theta_L(s)}_{CL} = NK_{\text{pi}}k_t, \quad (4.5) \]

and with \( K_{\text{di}} = 0 \), (4.4) dictates that:

\[ K_{\text{pi}} < \frac{k}{NK_t}. \quad (4.6) \]

Therefore, in stable systems \( k_{\text{rendered}} \) is always less than \( k \). Here, it should be noted that the physical spring stiffness and its role in limiting closed loop performance will continue to be important considerations throughout the investigation of SEA impedance control.

As suggested by (4.4), simple impedance control requires a nonzero damping gain to render a virtual stiffness greater than the physical spring stiffness. Even so, \( k_{\text{rendered}} \) is still limited below this value unless:

\[ J_L > N^2J_m \left( 1 + \frac{K_{\text{di}}k_t}{Nb_m} \right). \quad (4.7) \]

Hence, including output inertia \( J_L \) in the actuator model is important for a full understanding of closed loop behavior.

It is also observed in (4.4) that for stability (with \( K_{\text{pi}} \geq 0 \)) there exists an upper limit on the virtual damping gain, \( K_{\text{di}} \), such that:

\[ \frac{K_{\text{di}}^2k_t^2}{b_m^2} \left( \frac{J_m}{J_L} \right) + \frac{K_{\text{di}}k_t}{b_m} \left( \frac{N J_m}{J_L} - \frac{1}{N} \right) < 1. \quad (4.8) \]

The practical implications of this limit are best understood within the context of a
realistic physical system. By adopting the system parameters of the Valkyrie hip SEA (outlined in Table 4.1), the stability condition (4.4) can be expressed as a function of the impedance control gains $K_{pi}$ and $K_{di}$ alone:

$$f(K_{pi}, K_{di}) = 616.7 - K_{pi} - 179.4K_{di} - 11.62K_{di}^2 > 0.$$  \hspace{1cm} (4.9)

This expression, and the comparable stability conditions developed over the course of this section, can be used to compare the dynamic range of impedances, or Z-width*, possible with the Valkyrie hip actuator and the candidate SEA impedance control approaches of interest.

To demonstrate further simplification of the stability condition (4.9), critically damped impedance gains can be assumed. For this, the desired dynamic behaviors of (3.2) and (3.3) are defined as critically damped second order systems such that:

$$K_{di} = 2\sqrt{J_L K_{pi}}.$$  \hspace{1cm} (4.10)

*Z-width and Z-width plots, which will be leveraged in Sections 4.1.2 and 4.2.2, were first introduced as a method for representing a system’s impedance range by Colgate and Brown [8].
While not applicable in all circumstances, defining the desired impedance via a damping ratio (in this case $\zeta = 1$) is particularly useful for preventing oscillations both during free motion and when transitioning away from contact. Furthermore, combining (4.9) and (4.10) produces a stability constraint with respect to $K_{pi}$ alone:

$$f(K_{pi}) = 616.7 - 14.94K_{pi} - 196.5\sqrt{K_{pi}} > 0.$$ (4.11)

Solving (4.11) for $K_{pi}$ and then applying that solution to (4.5) reveals a peak critically damped virtual stiffness of $41 \text{ Nm/\text{rad}}$. This represents only 1\% of the system’s $3700 \text{ Nm/\text{rad}}$ open loop spring stiffness, and perhaps provides insight as to why, as previously mentioned, simple impedance control is seldom used in current SEA applications. Stable, critically damped responses are just too severely limited in achievable stiffness.

Although simple impedance control is not an ideal approach for the SEA in this example, the preceding derivation of stability conditions (4.9) and (4.11) outlines a useful process for examining all of the impedance control approaches to come. Purely symbolic expressions like (4.4) do provide useful insight, but they quickly become unwieldy for more complex control architectures, thus, the reliance on a realistic test case to compare control approaches. The parameters of the Valkyrie hip SEA in Table 4.1 are held constant throughout the coming analyses and are only augmented with additional variables when the architecture in question dictates. Point solutions are therefore obtained for a consistent example that can be explored experimentally as well. Ultimately, this leads to meaningful conclusions about the relative performance of each SEA impedance control approach and provides practical guidance for their implementation.
Passivity-Based Impedance Control

The passivity-based impedance control of Ott et al. [66], introduced in Chapter 2, was not originally designed for series elastic actuators. Nevertheless, it is worth considering here as a potential solution to the SEA impedance control problem and as an additional datapoint for comparison.

From the control architecture presented in Fig. 2.3 and the rotary series elastic actuator model of Section 2.2.2, closed loop transfer functions can be derived (as done in Chapter 3 for the other control approaches of interest). This again yields expressions in terms of system parameters and impedance control gains:

\[
\theta_L(s) \over \theta_{\text{ref}}(s)_{\text{CL}} = \frac{J_m K_{\text{di}} k s + J_m K_{\text{pi}} k}{\text{den}(s)} \tag{4.12}
\]

\[
\theta_L(s) \over \tau_{\text{ext}}(s)_{\text{CL}} = \frac{N^2 J_m J_0 s^2 + (J_m K_{\text{di}} + N^2 J_0 b_m) s + J_m (K_{\text{pi}} + k)}{\text{den}(s)} \tag{4.13}
\]

where

\[
\text{den}(s) = N^2 J_L J_m J_0 s^4 + J_L (J_m K_{\text{di}} + N^2 J_0 b_m) s^3 + J_m (J_L K_{\text{pi}} + J_L k + N^2 J_0 k) s^2 + k (J_m K_{\text{di}} + N^2 J_0 b_m) s + J_m K_{\text{pi}} k
\]

and \(J_0\) represents a desired apparent motor inertia (just as \(m_0\) represents the desired apparent motor mass in Chapter 2).

Applying the Liénard–Chipart criterion to the denominator of (4.12) and (4.13) leads to an interesting stability condition:

\[
J_L^2 J_m k^2 (J_m K_{\text{di}} + N^2 J_0 b_m)^2 > 0. \tag{4.14}
\]
Namely, if all parameters are positive, the closed loop system is always stable. This does not, however, imply that an SEA under passivity-based impedance control can render any desired dynamic behavior. From (4.12), the rendered virtual stiffness can again be expressed by the output position to external torque relationship at steady state:

\[ k_{\text{rendered}} = \lim_{s \to 0} \frac{\tau_{\text{ext}}(s)}{\theta_{L}(s)_{\text{CL}}} = \frac{K_{\pi}k}{K_{\pi} + k}. \]  

(4.15)

Thus, while \( K_{\pi} \) can theoretically be increased without bound, \( k_{\text{rendered}} \) will always be limited below the physical spring stiffness \( k \). This limit results from a design that conceptualizes the impedance controller as being in series with the physical compliance, and it highlights that passivity-based impedance control is most useful only in cases of weak elasticity or low desired closed loop stiffness.

**Cascaded Impedance and Torque Control**

When cascading impedance and torque control, as in Section 3.2, the system’s closed loop behavior is influenced by the choice of inner torque compensator. Here, two such cases will be examined. First, if PD torque control is used, the closed loop transfer functions obtained from (3.8) and (3.9) are as follows:

\[
\frac{\theta_{L}(s)}{\theta_{\text{tot}}(s)_{\text{CL}}} = \frac{NK_{\text{di}}K_{\text{dt}}kk_{t}s^2 + Nkk_{t}(K_{\pi}K_{\text{di}} + K_{\text{di}}K_{\pi})s + NK_{\pi}K_{\text{pt}}kk_{t}}{a_0s^4 + a_1s^3 + a_2s^2 + a_3s + a_4} 
\]

(4.16)

\[
\frac{\theta_{L}(s)}{\tau_{\text{ext}}(s)_{\text{CL}}} = \frac{N^2J_{\text{m}}s^2 + (NK_{\text{dt}}kk_{t} + N^2b_{\text{m}})s + k(NK_{\pi}k_{t} + 1)}{a_0s^4 + a_1s^3 + a_2s^2 + a_3s + a_4} 
\]

(4.17)
where

\[ a_0 = N^2 J_L J_m \]
\[ a_1 = J_L (N K_{dt} k k_t + N^2 b_m) \]
\[ a_2 = k (J_L + N^2 J_m + N K_{dt} K_{pt} k_t + N J_l K_{pt} k_t) \]
\[ a_3 = k (N K_{pi} K_{dt} k_t + N K_{di} k_t + N^2 b_m) \]
\[ a_4 = N K_{pi} K_{pt} k k_t. \]

Applying the Liénard–Chipart criterion to this denominator, then introducing the SEA parameters and PD torque control gains in Table 4.2, leads to the following stability condition with respect to the impedance control gains \( K_{pi} \) and \( K_{di} \):

\[
f(K_{pi}, K_{di}) = 42190 + 5268 K_{di} + 72.54 K_{di}^2 + 5.470 K_{pi} + 0.4066 K_{di} K_{pi} - 0.0037 K_{pi}^2 > 0.
\] (4.18)

As in the simple impedance control case, assuming critically damped impedance gains further reduces (4.18) to a function of \( K_{pi} \) alone:

\[
f(K_{pi}) = 42190 + 5770 \sqrt{K_{pi}} + 92.52 K_{pi} + 0.4454 K_{pi}^{3/2} - 0.0037 K_{pi}^2 > 0.
\] (4.19)

The steady-state output position to external torque relationship in (4.17) again determines the actual rendered virtual stiffness,

\[
k_{rendered} = \lim_{s \to 0} \frac{\tau_{ext}(s)}{\theta_L(s)}_{CL} = \frac{N K_{pi} K_{pt} k_t}{N K_{pt} k_t + 1}.
\] (4.20)

and in conjunction with (4.19) a 56578 Nm/rad limit on critically damped virtual stiff-
Table 4.2: System parameters and control gains used for the analysis of cascaded impedance and torque control.

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</tbody>
</table>

...ness is obtained. Although $k_{\text{rendered}}$ is always less than $K_{\text{pi}}$ in this example, the theoretically achievable stiffness here is significantly higher than that of simple impedance control or passivity-based impedance control, highlighting the value of an inner PD torque compensator.

Shifting to the second interesting example of cascaded impedance and torque control, a PI compensator can be used in the inner torque loop rather than PD control. Doing so modifies both closed loop transfer functions such that:

$$\frac{\theta_L(s)}{\theta_{\text{ref}(s)}_{\text{CL}}} = \frac{NK_{\text{di}}K_{\text{pt}}kk_ts^2 + Nkk_t(K_{\text{pi}}K_{\text{pt}} + K_{\text{di}}K_{\text{it}})s + NKK_{\text{pt}}K_{\text{it}}kk_t}{a_0s^5 + a_1s^4 + a_2s^3 + a_3s^2 + a_4s + a_5} \quad (4.21)$$

$$\frac{\theta_L(s)}{\tau_{\text{ext}(s)}_{\text{CL}}} = \frac{N^2J_m^3 + N^2b_ms^2 + k(NK_{\text{pt}}k_t + 1)s + NK_{\text{it}}kk_t}{a_0s^5 + a_1s^4 + a_2s^3 + a_3s^2 + a_4s + a_5} \quad (4.22)$$
where

\[ a_0 = N^2 J_L J_m \]
\[ a_1 = N^2 J_L b_m \]
\[ a_2 = k(J_L + N^2 J_m + NJ_L K_{pt} k_t) \]
\[ a_3 = k(NJ_L K_{pt} k_t + NK_{di} K_{pt} k_t + N^2 b_m) \]
\[ a_4 = Nkk_t(K_{pi} K_{pt} + K_{di} K_{it}) \]
\[ a_5 = NK_{pi} K_{it} k k_t. \]

With a fifth order denominator, applying the Liénard–Chipart criterion leads to a set of two nontrivial conditions, both of which must be satisfied for stability. Again using the parameters of Table 4.2:

\[
\begin{align*}
  f_1(K_{pi}, K_{di}) &= 1.054 K_{di}^2 + 19.04 K_{di} + 8.483 K_{di} K_{pi} + 151.4 K_{pi} \\
                    &\quad - 0.1102 K_{di}^3 - 0.8815 K_{di}^2 K_{pi} - 0.0378 K_{pi}^2 > 0 \tag{4.23}

  f_2(K_{pi}, K_{di}) &= 63.17 - 3.364 K_{di} > 0. \tag{4.24}
\end{align*}
\]

In this case, (4.23) is the more restrictive constraint and, when assuming critically damped impedance gains, it leads to the condition:

\[
  f_1(K_{pi}) = 152.7 K_{pi} + 20.86 \sqrt{K_{pi}} + 9.148 K_{pi}^{3/2} - 1.096 K_{pi}^2 > 0. \tag{4.25}
\]

The presence of an integrator in the torque loop ensures that \( K_{pi} \) corresponds to the actuator’s rendered virtual stiffness. Thus, solving for \( K_{pi} \) in (4.25) yields a 280 Nm/rad limit for stable, critically damped virtual stiffness.
Cascaded Impedance, Torque, and Velocity Control

As discussed in Chapter 3, the inner torque compensator of the previous control architecture can itself be augmented in multiple ways. If, as in Section 3.3, a velocity compensator is added as the innermost control loop in the cascaded structure, the system’s closed loop transfer functions become:

\[
\frac{\theta_L(s)}{\theta_{\text{ref}}(s)_{\text{CL}}} = \frac{d_0 s^3 + (a_4 - NJ_L K_{it} K_{iv} k k_t - N^2 K_{iv} K_{kk})s^2 + a_5 s + a_6}{a_0 s^6 + a_1 s^5 + a_2 s^4 + a_3 s^3 + a_4 s^2 + a_5 s + a_6} \quad (4.26)
\]

\[
\frac{\theta_L(s)}{\tau_{\text{ext}}(s)_{\text{CL}}} = \frac{N^2 J_m s^4 + N^2 (K_{pv} k_t + b_m)s^3 + n_2 s^2 + n_3 s + N K_{it} K_{iv} k k_t}{a_0 s^6 + a_1 s^5 + a_2 s^4 + a_3 s^3 + a_4 s^2 + a_5 s + a_6} \quad (4.27)
\]

where

\[
\begin{align*}
a_0 &= N^2 J_L J_m \\
a_1 &= N^2 J_L (K_{pv} k_t + b_m) \\
a_2 &= N^2 J_L K_{iv} k_t + NJ_L K_{pt} K_{pv} k k_t + J_L k + N^2 J_m k \\
a_3 &= k(NJ_L K_{pt} K_{iv} k_t + NJ_L K_{it} K_{pv} k_t + NK_{di} K_{pt} K_{pv} k_t + N^2 K_{pv} k_t + N^2 b_m) \\
a_4 &= N k k_t (J_L K_{it} K_{iv} + K_{pt} K_{pv} + K_{di} K_{pt} K_{iv} + K_{di} K_{it} K_{pv} + N K_{iv}) \\
a_5 &= N k k_t (K_{pi} K_{pt} K_{iv} + K_{pi} K_{it} K_{pv} + K_{di} K_{it} K_{iv}) \\
a_6 &= N K_{pi} K_{it} K_{iv} k k_t \\
d_0 &= NK_{di} K_{pt} K_{pv} k k_t \\
n_2 &= NK_{pt} K_{pv} k k_t + N^2 K_{iv} k_t + k \\
n_3 &= N k k_t (K_{pt} K_{iv} + K_{it} K_{pv}).
\end{align*}
\]

Here, transfer functions (3.11) and (3.12) are recast with respect to the system pa-
rameters assuming both PI torque control and PI velocity control. This allows for a direct comparison to the results of Vallery et al. [69] and Tagliamonte and Accoto [86]. Both of these studies adopt the cascaded PI architecture above but neglect to consider a gear ratio, torque constant, or output inertia in their system models.

Applying the Liénard–Chipart criterion as before, and adopting the Valkyrie hip SEA parameters and control gains, as outlined in Table 4.3, produces the following set of stability conditions:

\[
f_1(K_{pi}, K_{di}) = 0.4475K_{di}^3 + 13.81K_{di}^2 + 93.20K_{di} + 1230K_{pi}^2 + 19.02K_{di}^2K_{pi} + 75.24K_{di}K_{pi}^2 + 553.5K_{di}K_{pi}^2 + 3482K_{pi} - 0.0042K_{di}^4 - 0.3270K_{pi}^3 - 0.1754K_{di}^3K_{pi} - 0.6347K_{di}^2K_{pi}^2 > 0
\]

\[
f_2(K_{pi}, K_{di}) = 758.3 + 46.08K_{di} - 0.3885K_{di}^2 - 0.2003K_{pi} > 0.
\]

As it turns out, these constraints are nearly identical, and, if assuming critically damped impedance gains, (4.28) leads to the condition:

\[
f_1(K_{pi}) = 3499K_{pi} + 102.1\sqrt{K_{pi}} + 606.9K_{pi}^{3/2} + 82.19K_{pi}^{5/2} + 1253K_{pi}^2 - 1.089K_{pi}^3 > 0.
\]

Once again, the presence of integral action in the controller ensures that the impedance proportional gain \( K_{pi} \) corresponds to the rendered virtual stiffness of the closed loop actuator. Thus, solving (4.30) yields a 7844 \( \text{Nm/rad} \) limit for stable critically damped virtual stiffness.

Introducing an inner velocity loop provides a marked improvement over the per-
Table 4.3: System parameters and control gains used for the analysis of cascaded impedance, torque, and velocity control.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_L$</td>
<td>0.30</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>$J_m$</td>
<td>$1.32 \times 10^{-4}$</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>$b_m$</td>
<td>$7.64 \times 10^{-3}$</td>
<td>N·m·s/rad</td>
</tr>
<tr>
<td>$k$</td>
<td>3700</td>
<td>N·m/rad</td>
</tr>
<tr>
<td>$k_t$</td>
<td>0.05</td>
<td>N·m/A</td>
</tr>
<tr>
<td>$N$</td>
<td>120</td>
<td>–</td>
</tr>
<tr>
<td>$K_{pt}$</td>
<td>9.5</td>
<td>rad/N·m·s</td>
</tr>
<tr>
<td>$K_{it}$</td>
<td>0.25</td>
<td>rad/N·m·s$^2$</td>
</tr>
<tr>
<td>$K_{pv}$</td>
<td>1.0</td>
<td>A·s/N·m-rad</td>
</tr>
<tr>
<td>$K_{iv}$</td>
<td>0.25</td>
<td>A/N·m-rad</td>
</tr>
</tbody>
</table>

formance of simple impedance control and PI torque control alone. It should be noted, however, that the torque compensator’s proportional gain, $K_{pt}$, is increased from 2.0 rad/N·ms to 9.5 rad/N·ms for this test case. The rationale behind this change and support for the validity of this comparison will be offered in Chapter 5. For now, however, it suffices to simply introduce the potential that cascaded impedance, torque, and velocity control has for improving SEA impedance rendering.

**Disturbance Observer Based Impedance Control**

Section 3.4 introduced the disturbance observer (DOB) as a means for augmenting SEA torque control and offered it as an alternative to the inner velocity compensator just discussed. Using (3.19) and (3.20), the closed loop transfer functions for
disturbance observer based impedance control are expressed as:

\[
\frac{\theta_L(s)}{\theta_{ref}(s)}_{CL} = \frac{d_0 s^4 + d_1 s^3 + d_2 s^2 + d_3 s + d_4}{a_0 s^6 + a_1 s^5 + a_2 s^4 + a_3 s^3 + a_4 s^2 + a_5 s + a_6} (4.31)
\]

\[
\frac{\theta_L(s)}{\tau_{ext}(s)}_{CL} = \frac{n_0 s^4 + n_1 s^3 + n_2 s^2 + n_3 s + n_4}{a_0 s^6 + a_1 s^5 + a_2 s^4 + a_3 s^3 + a_4 s^2 + a_5 s + a_6} (4.32)
\]

where

\[a_0 = 5N^2 J_J J_m\]

\[a_1 = J_J (7N^2 J_m \omega_q + 5N K_{di} k k_t + 5N^2 b_m)\]

\[a_2 = 5J_J k + 5N^2 J_m k + 7N^2 J_J b_m \omega_q + 5N^2 J_J J_m \omega_q^2 + 5N K_{di} K_{dt} k k_t + 5N J_J K_{pt} k k_t + 7N J_J K_{pt} k k_t \omega_q\]

\[a_3 = 5K_{di} k + 7J_J k \omega_q + 5N^2 b_m k + 7N^2 J_m k \omega_q + 5N^2 J_J b_m \omega_q^2 + 5N K_{di} K_{pt} k k_t + 5N K_{pi} K_{pt} k k_t + 5N K_{di} K_{dt} k k_t \omega_q + 5N J_J K_{pt} k k_t \omega_q + 7N J_J K_{pt} k k_t \omega_q\]

\[a_4 = 5J_J k \omega_q^2 + 7K_{di} k \omega_q + 7N^2 b_m k \omega_q + 5N K_{pi} K_{pt} k k_t + 5N K_{di} K_{dt} k k_t \omega_q + 5N J_J K_{pt} k k_t \omega_q + 7N K_{di} K_{dt} k k_t \omega_q + 5N K_{pt} k k_t\]

\[a_5 = 5K_{di} k \omega_q^2 + 7N K_{pi} K_{pt} k k_t \omega_q + 5N K_{di} K_{pt} k k_t \omega_q^2 + 5N K_{pi} K_{dt} k k_t \omega_q + 7N K_{di} K_{pt} k k_t \omega_q + 5N K_{dt} k \omega_q\]

\[a_6 = 5K_{pt} k \omega_q^2 (1 + N K_{pt} k_t)\]

\[d_0 = 5N K_{di} K_{dt} k k_t\]

\[d_1 = 5K_{di} k + 5N K_{di} K_{pt} k k_t + 5N K_{pi} K_{dt} k k_t + 7N K_{di} K_{di} k k_t \omega_q\]

\[d_2 = 5N K_{pi} K_{pt} k k_t + 5N K_{di} K_{dt} k k_t \omega_q^2 + 7N K_{di} K_{pt} k k_t \omega_q + 7N K_{pi} K_{pt} k k_t \omega_q + 5K_{pi} k + 7K_{di} k \omega_q\]
Key features of this control architecture, as outlined in Fig. 3.7, include an inner PD torque compensator, a feedforward term, and the disturbance observer itself with its inverse nominal closed loop plant and second order, low-pass Butterworth filter.

Applying the Liénard–Chipart criterion to the denominator of (4.31) and (4.32) produces a set of two nontrivial stability conditions. Substituting into these expressions the system parameters and control gains of Table 4.4 yields:

\[
f_1(K_{pi}, K_{di}) = 2.119K_{di}^4 + 325.6K_{di}^3 + 35080K_{di}^2 + 1.158 \times 10^6K_{di} \\
+ 0.0255K_{di}^3K_{pi} + 2.9 \times 10^{-5}K_{di}^2K_{pi}^2 \\
+ 1.352K_{di}^2K_{pi} + 122.4K_{di}K_{pi}^2 + 3.3 \times 10^{-5}K_{pi}^3 \\
- 2.9 \times 10^{-9}K_{pi}^4 - 0.3698K_{pi}^2 - 2990K_{pi} \\
- 4.9 \times 10^{-7}K_{di}K_{pi}^3 - 0.0062K_{di}K_{pi}^2 > 0
\]

(4.33)

\[
f_2(K_{pi}, K_{di}) = 1.373 \times 10^5 + 6.352K_{di}^2 + 1084K_{di} - 4.106K_{pi} \\
- 8.7 \times 10^{-5}K_{pi}^2 - 0.0147K_{di}K_{pi} > 0.
\]

(4.34)

The more restrictive constraint in this set is (4.33) and, when assuming critically
Table 4.4: System parameters and control gains used for the analysis of disturbance observer based impedance control.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_L$</td>
<td>0.30</td>
<td>kg m$^2$</td>
</tr>
<tr>
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<td>N·m/s/rad</td>
</tr>
<tr>
<td>$k$</td>
<td>3700</td>
<td>N·m/rad</td>
</tr>
<tr>
<td>$k_t$</td>
<td>0.05</td>
<td>N·m/A</td>
</tr>
<tr>
<td>$N$</td>
<td>120</td>
<td>–</td>
</tr>
<tr>
<td>$K_{pt}$</td>
<td>2.0</td>
<td>A/N·m</td>
</tr>
<tr>
<td>$K_{dt}$</td>
<td>0.0209</td>
<td>A·s/N·m</td>
</tr>
<tr>
<td>$\omega_q$</td>
<td>188.5</td>
<td>rad/s</td>
</tr>
</tbody>
</table>

Damped impedance gains, it leads to the condition:

$$f_1(K_{pi}) = 6.8 \times 10^{-5} K_{pi}^3 + 4.304 K_{pi}^2 + 3.911 \times 10^4 K_{pi}$$

$$+ 1.269 \times 10^6 \sqrt{K_{pi}} + 562.1 K_{pi}^{3/2} + 0.0267 K_{pi}^{5/2}$$

$$- 2.9 \times 10^{-9} K_{pi}^4 - 5.37 \times 10^{-7} K_{pi}^{7/2} > 0.$$  \hspace{1cm} (4.35)

Because the disturbance observer serves to provide integral-like action in the controller, the impedance gain $K_{pi}$ again corresponds directly to the rendered virtual stiffness of the actuator. Thus, solving (4.35) yields a $59200 \, \text{Nm/rad}$ limit for stable critically damped virtual stiffness. This value suggests that disturbance observer based impedance control is, at least from a theoretical stability standpoint, a viable, if not preferred, alternative to the other control approaches discussed.
4.1.2 Stable Z-Widths

The peak stiffnesses of the previous section (summarized again in Table 4.5) serve as a starting point for determining which control approaches are best suited for use in series elastic systems. Both simple impedance control and the cascaded impedance and PI torque architecture are significantly limited in comparison to the other options, suggesting that they are not appropriate choices. Likewise, passivity-based impedance control is strictly limited by the actuator’s physical spring stiffness. Although this is not universally disqualifying, it is unduly restrictive for most SEA applications.

Rather than relying solely on a single peak stiffness value to differentiate between control approaches, it is helpful to examine controller Z-width. As introduced by Colgate and Brown [8], closed loop impedance width, or Z-width, quantifies an actuator’s range of achievable impedances, thus providing a better understanding of the full dynamic range of impedances possible with each control approach. Furthermore, Z-width can easily be represented by a virtual stiffness versus virtual damping plot,

Table 4.5: The peak stable, critically damped stiffness for each investigated impedance control approach when applied to the Valkyrie hip SEA model. The passive physical spring stiffness of the actuator is $k = 3700 \text{Nm/rad}$.

<table>
<thead>
<tr>
<th>Impedance Control Approach</th>
<th>Peak Stiffness (Nm/rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple Impedance Control</td>
<td>41</td>
</tr>
<tr>
<td>Passivity-Based Impedance Control</td>
<td>3700</td>
</tr>
<tr>
<td>Cascaded Impedance and PD Torque Control</td>
<td>56578</td>
</tr>
<tr>
<td>Cascaded Impedance and PI Torque Control</td>
<td>280</td>
</tr>
<tr>
<td>Cascaded Impedance, Torque, and Velocity Control</td>
<td>7844</td>
</tr>
<tr>
<td>Disturbance Observer Based Impedance Control</td>
<td>59200</td>
</tr>
</tbody>
</table>
wherein the area encompassed by the Z-width curve represents all combinations of stiffness and damping that an actuator can render. While a high peak stiffness is often desired, some applications might instead call for the broadest impedance range possible, and Z-width plots assist in such design trades.

Using the stability conditions (4.9), (4.18), (4.23), (4.28), and (4.33), the stable Z-width of the Valkyrie hip SEA can be compared for each impedance control approach (Fig. 4.1). As noted previously, the virtual stiffness gain $K_{pi}$ is not equivalent to the actuator’s rendered stiffness in some of these control architectures. Thus, Fig. 4.1 is a comparison of impedance gain Z-width, rather than rendered output Z-width. This

![Figure 4.1: Stable Z-width comparison of various impedance control approaches for the system parameters of the Valkyrie hip SEA. Note that the virtual stiffness gains of the figure underestimate, per (4.5), and overestimate, per (4.20), the rendered output stiffness of simple impedance control and cascaded impedance and PD torque control respectively. $K_{pi} = k_{\text{ rendered}}$ for all other architectures.](image-url)
is an important distinction for simple impedance control and the cascaded impedance and PD torque architecture. Nevertheless, a great deal of insight is gained by examining Fig. 4.1 qualitatively.

The Z-width plot shows that only cascaded impedance and PD torque control; cascaded impedance, torque, and velocity control; and disturbance observer based impedance control significantly exceed the SEA’s 3700 Nm/rad physical spring rate. All three of these controllers provide large regions of stability, although cascaded impedance, torque, and velocity control peaks in stiffness just beyond 10000 Nm/rad before decreasing back toward zero past $K_{di} = 60$ Nms/rad. Conversely, the other two controllers similarly increase in achievable impedance beyond the 20000 Nm/rad and 150 Nms/rad bounds of the figure.

The comparable Z-widths of cascaded impedance and PD torque control and DOB based impedance control are not surprising given that both architectures rely on the same inner PD torque compensator, and only differ due to the presence of a disturbance observer in the latter control approach. It is interesting to note, however, that while the DOB enables a higher peak stiffness, it also prevents the actuator from rendering pure spring-like behavior (where $K_{di} = 0$). This is in contrast to all of the other control approaches. Below virtual damping gains of roughly 15 Nms/rad, DOB based control is outperformed by both PD torque control without the DOB and the PI compensator based cascaded impedance, torque, and velocity architecture. This observation should be tempered somewhat, however, because the performance limitation of DOB based control at low damping gains does not present a significant practical issue for the SEA. For example, if DOB based control is used to render a 1000 Nm/rad stiffness, the lowest virtual damping possible is $K_{di} = 2.44$ Nms/rad (see the enlarged, low gain Z-width plot in Fig. 4.2). Although not a pure stiffness, these gains
Virtual Damping Gain \( K_{d_l} \) (Nm/rad)

Virtual Stiffness Gain \( K_{pi} \) (Nm/rad)

Figure 4.2: Stable Z-width of disturbance observer based impedance control at low impedance gains given the system parameters of the Valkyrie hip SEA.

produce a severely underdamped system with a damping ratio of \( \zeta = 0.07 \), a value well below the likely requirements of most applications.

Quantifying peak stiffness and stable Z-width aids in the selection of an appropriate impedance control approach for series elastic actuators, but the analysis thus far hardly provides a complete picture. Continuing toward the goal of understanding the applicability of each architecture, and seeking to arm the control designer with the insight required to leverage SEAs in human-centric applications, focus will now shift to the question of robust stability and determining actuator closed loop behavior when in contact with the external environment.

4.2 Passivity

The pole locations of a closed loop transfer function dictate the system’s isolated stability (i.e. how the system behaves when considering no external interactions). This analysis, although important, is not sufficient for a system designed to operate in contact with the external environment. In this case, coupled stability is of paramount
concern. Series elastic actuators must maintain robust stability in the presence of uncertain contact dynamics when interacting with the outside world if they are to be of use in human-centric systems.

Colgate [37] provides a detailed examination of dynamically interacting systems and points to passivity as an appropriate means of guaranteeing coupled stability. Specifically, an LTI system is stable when interacting with an arbitrary dynamic physical environment (i.e. a passive environment) if, and only if, the system, at its interaction port with the environment, also behaves as a passive system [37]. In other words, the system’s driving point impedance and/or admittance must be passive to ensure robust coupled stability.

Within the context of the previous analysis, the driving point admittance of an SEA is simply the time derivative of the closed loop external torque to output position transfer function:

$$Y(s) = \frac{\theta_L(s)}{\tau_{ext}(s)} s.$$  

(4.36)

This driving point admittance is guaranteed passive if, and only if, it is positive real, satisfying the following conditions [37]:

1. $Y(s)$ has no poles in the right-half of the complex $s$-plane.
2. Any imaginary poles of $Y(s)$ are simple and have positive real residues.
3. $\text{Re}(Y(j\omega)) \geq 0$.

Having examined in the previous section asymptotic stability, which requires all closed loop poles be strictly confined to the left-half of the complex $s$-plane (i.e. no poles lie on the imaginary axis), and desiring such for the impedance controlled SEAs here, it is sufficient to combine conditions 1 and 2 above by simply requiring $Y(s)$ be asymptotically stable. Thus, the passivity, and therefore coupled stability, of
an impedance controlled SEA hinges on determining the range of parameters and controller gains for which $\text{Re}(Y(j\omega)) \geq 0$ is satisfied.

The positive real constraint is also equivalent to bounding the phase of $Y(s)$ between $+90^\circ$ and $-90^\circ$ [37]. Examining this graphically in the actuator’s closed loop frequency response is an efficient way to compare controller performance in simulation. It also provides further insight on the frequency range of passive behavior and the external input frequencies at which passivity is violated.

4.2.1 Passivity Conditions

A custom MATLAB script is again used, in conjunction with the symbolic transfer functions derived in Section 4.1.1, to determine the real part of $Y(j\omega)$ for each candidate SEA impedance control approach. The resulting symbolic polynomials in even powers of $\omega$ serve as nonconservative constraints, defining the entire range of parameters over which passivity is guaranteed for all external input frequencies. Often times, ensuring that all coefficients in the symbolic polynomial are nonnegative (as is done elsewhere [69], [86], [95]) is equivalent to the condition $\text{Re}(Y(j\omega)) \geq 0$. This is not, however, strictly true and it will be pointed out as such when conservatism is introduced in the following analysis.

It is also important to note that while achieving passivity is ideal, it can be overly conservative as a means for providing robust contact stability (particularly when $Y(j\omega)$ only fails to satisfy the positive real condition at high frequencies, or the interacting environment at hand readily dissipates excess energy). Formal passivity criteria are developed in this chapter in lieu of considering these situations explicitly, but they provide the impetus for including the practical performance of non-passive control architectures in the investigations of Chapter 5.
Simple Impedance Control

If using simple impedance control, the real part of the closed loop SEA’s driving point admittance is found from (4.3) and (4.36):

$$\text{Re}(Y(j\omega)) = \left[ -N^3 J_m K_{di} k k_t \right] \omega^4 + \left[ Nk(K_{di} k k_t + N b_m k - N^2 K_{pi} b_m k_t) \right] \omega^2. \quad (4.37)$$

This leads to a set of two constraints, both of which must be satisfied to ensure that Re$(Y(j\omega)) \geq 0$ for all external input frequencies $\omega$:

$$-N^3 J_m K_{di} k k_t \geq 0 \quad (4.38)$$

$$Nk(K_{di} k k_t + N b_m k - N^2 K_{pi} b_m k_t) \geq 0. \quad (4.39)$$

From (4.38) it is obvious that if all parameters are positive, $Y(s)$ cannot be positive real, and therefore, the closed loop SEA is not passive. This is illustrated in a Bode plot of the driving point admittance for the system parameters of Table 4.1 and impedance control gains $K_{pi} = 1000 \text{Nm/rad}$ and $K_{di} = 24.25 \text{ Nms/rad}$ (Fig. 4.3). At roughly $\omega = 5 \text{ Hz}$ ($31.4 \text{ rad/s}$) the phase exceeds the $+90^\circ$ threshold, indicating a non-passive response.

Interestingly, if a pure stiffness is desired (i.e. $K_{di} = 0$), then reducing (4.39) shows that passivity is achieved if:

$$K_{pi} \leq \frac{k}{Nk_t}. \quad (4.40)$$

This matches the stability limit of simple impedance control when $K_{di} = 0$ (see (4.6)). Although a pure stiffness is only marginally stable and is not frequently used in humanoid robots, there are cases (particularly in haptics and wearable robotics) where
Figure 4.3: Simulated frequency response of the driving point admittance of a Valkyrie hip SEA under simple impedance control ($K_{pi} = 1000 \text{Nm/rad}$ and $K_{di} = 24.25 \text{Nms/rad}$). Phase in excess of $+90^\circ$ indicates a non-passive closed loop response.

rendering spring-like behavior or even a null impedance (i.e. $K_{pi} = K_{di} = 0$) is desired. Thus, it is important to note under which control approaches an SEA can achieve these behaviors passively.

**Passivity-Based Impedance Control**

As its name implies, and as is proven by other means elsewhere (see [36] and [66]), passivity-based impedance control is indeed passive for all parameter values. This is confirmed again with the methodology presented here by examining the real part of $Y(j\omega)$ obtained from (4.13) and (4.36):

$$\text{Re}(Y(j\omega)) = \left[N^2 J_m J_0 b_m k^2 + J_m^2 K_{di} k^2 \right] \omega^2. \quad (4.41)$$

For all nonnegative parameter values, (4.41) is always nonnegative. Thus, the series elastic actuator is always passive.
As noted when discussing stability, this result does not imply that an SEA lever-
aging passivity-based impedance control can render any desired dynamic behavior. 
Although both stable and passive for all system parameters and control gains, this 
control approach is still fundamentally limited to a virtual stiffness below the actua-
tor’s physical spring stiffness, as shown by (4.15).

Cascaded Impedance and Torque Control

The real part of the driving point admittance transfer function for a series elastic 
actuator under cascaded impedance and PD torque control is determined from (4.17) 
and (4.36) to be:

\[ \text{Re}(Y(j\omega)) = c_1\omega^4 + c_2\omega^2 \]  

(4.42)

where

\[
c_1 = N^2 k k_t \left[ N J_m K_{dt} (k - K_{pi}) + K_{di} K_{dt} (K_{pt} k_t + K_{dt} k k_t) - N J_m K_{di} K_{pt} \right] 
\]

\[
c_2 = N k \left[ N^2 K_{pt} b_m k_t (k - K_{pi}) + K_{di} K_{pt} k k_t (N K_{pt} k_t + 1) + K_{pt} K_{dt} k k_t + N b_m k \right]. 
\]

Simplifying slightly, both of these coefficients provide a constraint which must be 
satisfied to ensure \( \text{Re}(Y(j\omega)) \geq 0 \):

\[
N J_m K_{dt} (k - K_{pi}) + K_{di} K_{dt} (N b_m + K_{dt} k k_t) - N J_m K_{di} K_{pt} \geq 0 
\]  

(4.43)

\[
N^2 K_{pt} b_m k_t (k - K_{pi}) + K_{di} K_{pt} k k_t (N K_{pt} k_t + 1) + K_{pt} K_{dt} k k_t + N b_m k \geq 0. 
\]

(4.44)

Both (4.43) and (4.44) can be met for certain nonnegative system parameters. Thus, 
an SEA leveraging cascaded impedance and PD torque control can, at times, render 
passive closed loop impedances.
As was done for stability, it is helpful here to explore the practicality of these constraints using the Valkyrie hip SEA as a physical test case. A Bode plot of the actuator’s driving point admittance for impedance gains $K_{pi} = 1000 \text{Nm/rad}$ and $K_{di} = 24.25 \text{Nms/rad}$ reveals a passive response with phase bounded between $+90^\circ$ and $-90^\circ$ (Fig. 4.4). The system parameters and nominal torque control gains used for this simulation (found in Table 4.2) can also be substituted into (4.43) and (4.44) to yield passivity constraints with respect to the impedance control gains alone:

$$3700 + 206.4K_{di} - K_{pi} \geq 0$$  \hspace{1cm} (4.45)

$$6181 + 674.3K_{di} - K_{pi} \geq 0.$$  \hspace{1cm} (4.46)

With smaller constant and $K_{di}$ terms offsetting a negative $K_{pi}$, (4.45) is the more

![Figure 4.4: Simulated frequency response of the driving point admittance of a Valkyrie hip SEA under cascaded impedance and PD torque control ($K_{pi} = 1000 \text{Nm/rad}$ and $K_{di} = 24.25 \text{Nms/rad}$). Phase bounded between $+90^\circ$ and $-90^\circ$ indicates a passive closed loop response.](image)
restrictive constraint in this case, and thus dictates actuator passivity. This expression will be revisited in Section 4.2.2, along with other passivity constraints yet to come, when comparing passive Z-widths on the Valkyrie hip SEA.

For the case of PI torque control, (4.22) and (4.36) are used to determine the real part of the driving point admittance transfer function:

$$\text{Re}(Y(j\omega)) = c_1 \omega^6 + c_2 \omega^4 + c_3 \omega^2$$  \hspace{1cm} (4.47)

where

$$c_1 = -N^3 J_m K_{di} K_{pt} k_t$$

$$c_2 = N k (N^2 J_m K_{pi} K_{it} k_t + N^2 K_{pt} b_m k_t + N K_{di} K_{pt} ^2 k_t^2 + K_{di} K_{pt} k_t)$$

$$+ N b_m k - N^2 K_{pi} K_{pt} b_m k_t - N^2 K_{di} K_{it} b_m k_t - N^2 J_m K_{it} k_t)$$

$$c_3 = NK_{it} k^2 k_t (NK_{di} K_{it} k_t - K_{pi}).$$

Aside from the null impedance case where $K_{pi}$ and $K_{di}$ are both zero, these three coefficients cannot be simultaneously nonnegative for positive system parameters. Furthermore, because the highest order coefficient in (4.47) is negative for nonzero system parameters, Re$(Y(j\omega))$ cannot remain positive as $\omega \to \infty$. Thus, closed loop passivity of the SEA is not possible for a nonzero desired impedance.

The non-passive nature of cascaded impedance and PI torque control is additionally confirmed by comparing its frequency response to that of the passive cascaded impedance and PD torque control approach previously discussed. When simulating a desired impedance of $K_{pi} = 1000 \text{Nm/rad}$ and $K_{di} = 24.25 \text{Nms/rad}$, the phase of the response exceeds $+90^\circ$, indicating that the system is not passive (Fig. 4.5). The requirement that (4.47) be positive for a passive system also reveals the external input
Figure 4.5: A comparison of the admittance frequency response when using either a PD or PI torque compensator within the cascaded impedance and torque control framework. The simulation assumes Valkyrie hip SEA parameters and impedance gains $K_{pi} = 1000 \text{Nm/rad}$ and $K_{di} = 24.25 \text{Nms/rad}$. PD control is passive while PI control violates passivity both below $\omega_1 = 0.086 \text{Hz}$ and above $\omega_2 = 28.68 \text{Hz}$.

Frequencies at which passivity is violated. Setting (4.47) equal to zero and solving for $\omega$ yields:

$$\omega = \left( \frac{-c_2 \pm \sqrt{c_2^2 - 4c_1c_3}}{2c_1} \right)^{1/2}. \quad (4.48)$$

For the given desired impedance, and the system parameters defined in Table 4.2, (4.48) gives $\omega_1 = 0.086 \text{Hz}$ and $\omega_2 = 28.68 \text{Hz}$. Indeed, in Fig. 4.5 the phase of the cascaded impedance and PI torque control response rises slightly above $+90^\circ$ below $0.086 \text{Hz}$, remains within the passivity bounds between $\omega_1$ and $\omega_2$, then exceeds $+90^\circ$ again beyond $28.68 \text{Hz}$. Because passivity is violated both as $\omega \to 0$ and as $\omega \to \infty$,
and because the departure beyond a $+90^\circ$ phase is so drastic above 28.68 Hz, an SEA relying on cascaded impedance and PI torque control offers little practical use if a passive-like response is desired.

At this point, the enabling effect that a torque compensator derivative gain has on closed loop passivity is apparent. Cascaded impedance and PI torque control cannot produce a passive SEA, but cascaded impedance and PD torque control can. Furthermore, looking back at (4.43) and (4.44), if $K_{dt}$ is set to zero (thus removing the derivative gain and adopting only proportional torque control), a passive system cannot be realized unless only rendering a pure stiffness. Damped impedance behaviors, useful in many control applications, are not possible while also ensuring robust contact stability, unless a torque compensator derivative gain is included. This point is apparent in the next two control architectures as well.

**Cascaded Impedance, Torque, and Velocity Control**

Obtained using (4.27) and (4.36), the real part of the driving point admittance for an SEA under cascaded impedance, torque, and velocity control is:

\[
\text{Re}(Y(j\omega)) = c_1\omega^8 + c_2\omega^6 + c_3\omega^4 + c_4\omega^2
\]

(4.49)

where

\[
c_1 = -N^3J_mK_{di}K_{pt}K_{pv} kk_t
\]

\[
c_2 = Nk \left[ N^2k_t(J_mK_{pi} - J_mk - K_{di}b_m)(K_{it}K_{pv} + K_{pt}K_{iv}) + NK_{pv} kk_t \\
+ N^2K_{pt}K_{pv} b_m kk_t + N^2K_{pt}K_{pv}^2 k_t^2 + N^2J_mK_{di}K_{it}K_{iv}k_t \\
+ NK_{di}K_{pt}^2 K_{pv}^2 k_t^2 + Nb_m k + K_{di}K_{pt}K_{pv} kk_t - N^2K_{di}K_{it}K_{pv}^2 k_t \\
- N^2K_{pt}K_{pv}^2 k_t^2 - N^2 K_{pt}K_{pv} b_m k_t \right]
\]
\[ c_3 = Nkk_t(N^2K_{pi}K_{it}K_{iv}b_m + N^2K_{pt}K_{iv}^2k_k + N K_{di}K_{it}^2 K_{pv}^2k_k + N K_{di}K_{it}^2 K_{iv}^2k_k + N K_{di}K_{pt}K_{iv}^2k_k - N^2K_{di}K_{it}K_{iv}^2k_t - N^2K_{pi}K_{pt}K_{iv}^2k_t - N^2K_{pi}K_{pt}K_{pv}k - K_{pt}K_{pv}k - K_{pi}K_{pt}K_{pv}k - K_{pi}K_{pt}K_{pv}k) \]

\[ c_4 = N^2K_{di}K_{it}^2 K_{iv}^2k^2k_t^2. \]

Strictly speaking, all four of these coefficients are not required to be nonnegative to satisfy \( \text{Re}(Y(j\omega)) \geq 0 \). In this case however, the highest order coefficient \( c_1 \) is always negative for positive system parameters, making it clear that \( \text{Re}(Y(j\omega)) \) does not remain positive as \( \omega \to \infty \). Thus, cascaded impedance, torque, and velocity control (as defined here with PI torque and PI velocity compensators) cannot yield a passive closed loop actuator with nonzero virtual stiffness and damping.

This result confirms the claim of Tagliamonte and Accoto [86] that rendering a virtual spring and damper in parallel cannot be done passively across all external input frequencies using cascaded impedance, torque, and velocity control. Following a straightforward substitution, it also supports their claim (in agreement with Vallery et al. [69]) that a pure spring-like behavior (with \( K_{di} = 0 \)) can indeed be rendered passively. The inner velocity loop of this controller enables such behavior since, from \( c_3 \) in (4.47), it is observed that cascaded impedance and PI torque control alone cannot passively render impedances when \( K_{di} = 0 \).

Although strict passivity is violated regardless of system parameters, the qualitative behavior of the closed loop actuator does not always depart drastically from the passive ideal. This is observed when simulating the Valkyrie hip SEA (using the parameters of Table 4.3) at \( K_{pi} = 1000 \text{Nm/rad} \) and \( K_{di} = 24.25 \text{Nms/rad} \) (Fig. 4.6). Here, the frequency response of the driving point admittance bears a remarkable similarity
Figure 4.6: A comparison of the admittance frequency response when using either cascaded impedance and PD torque control or cascaded impedance, torque, and velocity control. The simulation assumes Valkyrie hip SEA parameters and impedance gains $K_{pi} = 1000 \text{Nm/rad}$ and $K_{di} = 24.25 \text{Nms/rad}$. 

Figure 4.7: Expanded view of the phase in Fig. 4.6 between 0.0028 Hz and 0.06 Hz.
Figure 4.8: Expanded view of the phase in Fig. 4.6 between 50 Hz and 300 Hz.

to the passive cascaded impedance and PD torque control response obtained previously. While difficult to discern given the scale of Fig. 4.6, the cascaded impedance, torque, and velocity response does exhibit a phase exceeding ±90° across two frequency ranges (see Fig. 4.7 and Fig. 4.8). Once again, the frequencies at which these violations occur are obtained by setting (4.49) equal to zero and solving for $\omega$. This yields $\omega_1 = 0.003 \text{ Hz}$, $\omega_2 = 0.044 \text{ Hz}$, and $\omega_3 = 64.2 \text{ Hz}$. Fig. 4.7 confirms that between $\omega_1$ and $\omega_2$ the phase angle is slightly greater than +90°, while in Fig. 4.8, phase is shown to dip slightly below −90° past $\omega_3$, before asymptotically approaching −90° again as $\omega \to \infty$.

Requiring passivity is a conservative means of ensuring coupled stability with the environment. Strictly limiting the closed loop phase to between +90° and −90° prevents instabilities when in contact with “worst case” pure spring- or pure inertia-like environments [37]. If the environment dissipates energy, however, coupled stability might still be possible without strict passivity. The small degree to which cascaded impedance, torque, and velocity control violates passivity in Fig. 4.6 suggests that
this architecture might still be of practical use in a variety of circumstances, and it should not be immediately discarded for being non-passive. Nevertheless, in control design it is important to recognize that this approach does prevent the passive rendering of a virtual spring and damper in parallel, and this carries with it potential consequences.

**Disturbance Observer Based Impedance Control**

As already discussed, a disturbance observer improves inner torque compensator performance in a cascaded impedance control architecture, providing an alternative to an inner velocity loop. The DOB produces integral-like action to remove steady-state error and enforce $k_{\text{rendered}} = K_{\text{pi}}$, but it does so while acting on a PD torque loop that, as previously established, can render a large range of passive desired impedances. As such, it is of interest to determine the DOB’s impact on closed loop passivity.

From (4.32) and (4.36), the real part of the driving point admittance for a series elastic actuator under disturbance observer based impedance control is found to be:

$$\text{Re}(Y(j\omega)) = c_1\omega^8 + c_2\omega^6 + c_3\omega^4 + c_4\omega^2$$  \hspace{1cm} (4.50)

where

$$c_1 = 25N^2k(NK_{\text{di}}K_{\text{dt}}b_mk_t + NJ_mK_{\text{di}}k_kk_t + K_{\text{di}}K_{\text{dt}}^2kk_t^2 - NJ_mK_{\text{di}}K_{\text{pt}}k_t$$

$$- NJ_mK_{\text{dt}}K_{\text{pt}}k_t - J_mK_{\text{di}})$$

$$c_2 = k(35N^4J_m^2\omega_q^3 + 25N^3K_{\text{pt}}b_mkk_t + N^3J_mK_{\text{di}}K_{\text{pt}}k_t\omega_q^2 + N^3J_mK_{\text{pt}}K_{\text{dt}}k_t\omega_q^2$$

$$+ 24N^3J_mK_{\text{dt}}kk_t\omega_q^2 + 25N^2b_mk + N^2J_mK_{\text{di}}k_i\omega_q^2 + 25N^2K_{\text{dt}}K_{\text{pt}}^2kk_t^2$$

$$+ 50NK_{\text{di}}K_{\text{pt}}kk_t + 25K_{\text{di}}k - 25N^3K_{\text{pt}}K_{\text{dt}}b_mk_t - N^3K_{\text{di}}K_{\text{dt}}b_mk_t\omega_q^2$$

$$- N^2K_{\text{di}}K_{\text{dt}}^2kkr_i\omega_q^2 - 25N^2K_{\text{pt}}b_m)$$
\[ c_3 = k_\omega q^2 (35N^4 b_m^2 \omega_q + 24N^3 K_{pt} b_m k k_t + 35N^3 K_{dt} b_m k k_t \omega_q + N^3 K_{pi} K_{pt} b_m k t \\
+ 25N^3 K_{di} K_{dt} b_m k k_t \omega_q^2 + 25N^2 K_{di} K_{dt}^2 k k_t^2 \omega_q^2 + 24N^2 b_m k + N^2 K_{pi} b_m \\
- 35N^3 J_m K_{pt} k k_t \omega_q - 25N^3 J_m K_{di} K_{pt} k k_t \omega_q^2 - 25N^3 J_m K_{pi} K_{dt} k k_t \omega_q^2 \\
- 35N^2 J_m k_\omega q - 25N^2 J_m K_{di} \omega_q^2 - N^2 K_{di} K_{pt}^2 k k_t^2 - 2N K_{di} K_{pt} k k_t - K_{di} k) \]
\[ c_4 = 25k_\omega q^4 (N K_{pt} k t + 1)(N K_{di} K_{pt} k k_t + K_{di} k - N^2 K_{pi} b_m). \]

This expression establishes, for the first time, a necessary and sufficient condition for the closed loop passivity of an SEA under disturbance observer based impedance control. No assumptions or simplifications are made in (4.50) beyond those of the SEA model presented in Chapter 2. As such, the constraint \( \text{Re}(Y(j\omega)) \geq 0 \) represents a nonconservative condition and can be used to fully explore the influence on passivity of all system parameters.

As before, all four of these coefficients need not be greater than or equal to zero to satisfy the positive real constraint. However, from inspection, all can be simultaneously positive given appropriate system parameters, thus confirming that passivity is achievable. The simulated frequency response of the Valkyrie hip SEA’s driving point admittance once again demonstrates this result graphically. Using the system parameters of Table 4.4 and a desired impedance of \( K_{pi} = 1000 \text{ Nm/rad} \) and \( K_{di} = 24.25 \text{ Nms/rad} \), the Bode plot of Fig. 4.9 is generated. The phase of this response is indeed bounded between +90° and −90° over all input frequencies, indicating a passive system.

Further exploration of the passivity range of DOB based impedance control is aided by examining the set of constraints arising from the coefficients in (4.50):

\[ NJ_m K_{di} k_t (k - K_{pi}) + K_{di} \left[ K_{dt} k_t (N b_m + K_{dt} k k_t) - J_m \alpha \right] \geq 0 \quad (4.51) \]
Figure 4.9: Simulated frequency response of the driving point admittance of a Valkyrie hip SEA under disturbance observer based impedance control ($K_{pi} = 1000 \text{Nm/} \text{rad}$ and $K_{di} = 24.25 \text{Nms/} \text{rad}$). Phase bounded between $+90^\circ$ and $-90^\circ$ indicates a passive closed loop response.

\[
25N^2b_m\alpha(k - K_{pi}) + N^3J_mK_{dt}k_t\omega_q^2(24k - K_{pi}) + 35N^4J_m^2\omega_q^3 \\
+ K_{di}\left[\alpha(N^2J_m\omega_q^2 + 25k\alpha) - N^2K_{dt}k_t\omega_q^2(Nb_m\omega_q + K_{dt}k_t)\right] \geq 0
\] 
(4.52)

\[
N^2K_{pi}(b_m\alpha - 25NJ_mK_{dt}k_t\omega_q^2) + N^2k\alpha(24b_m - 35J_m\omega_q) \\
+ K_{di}\left[25N^2K_{dt}k_t\omega_q(Nb_m + K_{dt}k_t) - \alpha(25N^2J_m\omega_q^2 + k\alpha)\right] \geq 0
\] 
(4.53)

\[
K_{di}k\alpha - N^2K_{pi}b_m \geq 0
\] 
(4.54)

where
\[
\alpha = (NK_{pi}k_t + 1).
\]

Satisfying (4.51) – (4.54) is sufficient for ensuring passivity. Although, as mentioned,
it is not strictly necessary and thus leads to conservatism. Nevertheless, these con-
straints provide valuable insight during the control design process.

Using the Valkyrie hip SEA again, the system parameters and nominal control
gains of Table 4.4 can be used with (4.51) – (4.54) to produce constraints with respect
to the impedance control gains alone:

\[
\begin{align*}
3700 + 198.4K_{di} - K_{pi} & \geq 0 \\
63414 + 511.4K_{di} - K_{pi} & \geq 0 \\
196.8K_{di} - K_{pi} - 282.6 & \geq 0 \\
437.3K_{di} - K_{pi} & \geq 0.
\end{align*}
\]

From inspection, (4.57) is the most restrictive constraint in this case, thus, it serves
as the conservative benchmark for actuator passivity.

The analysis thus far shows that using a disturbance observer does not preclude
passive impedance rendering. In fact, a variety of impedances are achievable given the
above constraints (Fig. 4.9 provides one such example). However, an SEA’s achievable
range of passive impedances is restricted somewhat by the disturbance observer. As
already established when examining stability, DOB based impedance control cannot
render pure spring-like behaviors (and this passivity analysis agrees, as satisfying
(4.54) is not possible if \(K_{di} = 0\) and all other parameters are positive). Beyond this,
it is interesting to note that a null impedance is also more difficult to achieve given
the presence of a disturbance observer. From (4.57), if \(K_{pi}\) and \(K_{di}\) are both set to
zero, the Valkyrie hip actuator is not passive (in this case (4.58) is trivially satisfied
and (4.57) drives \(\text{Re}(Y(j\omega))\) negative as \(\omega \to 0\)).
Returning to (4.53) and examining the null impedance case, a general requirement for the passive rendering of such impedances is:

\[ b_m \geq \frac{35}{24} J_m \omega_q. \]  

(4.59)

Here the influence of the disturbance observer is explicit. As the range of frequencies over which the DOB acts is increased (i.e. as the cutoff frequency \( \omega_q \) of the DOB filter increases) it becomes more difficult to satisfy (4.59).

While more restrictive than cascaded impedance and PD torque control, the presence of a disturbance observer does not prevent the passive rendering of closed loop impedances. Furthermore, a passive damped impedance, comprised of a virtual spring and damper in parallel, is still possible with DOB based impedance control. This result again sets apart those control architectures that rely on PD torque compensation, and, as will be illustrated shortly, both low virtual stiffnesses and values exceeding the SEA’s physical spring rate can be passively rendered using these control approaches.

### 4.2.2 Passive Z-Widths

Over the course of the previous section it was established that only passivity-based impedance control, cascaded impedance and PD torque control, and disturbance observer based impedance control are capable of rendering passive damped impedances. The other control architectures examined are limited to pure spring-like behavior or null impedances if passivity is desired.

Z-width plots are once again useful for quantifying the range of passive impedances that each of these controllers can render. Using (4.45), the passive Z-width of cascaded impedance and PD torque control can be compared to its stable Z-width previously
generated from (4.18) in Section 4.1.2 (Fig. 4.10). The actuator’s passive Z-width falls entirely within its stable Z-width, thus verifying that an actuator satisfying (4.45) indeed satisfies all three positive real conditions.

A comparable Z-width plot depicting the stable and passive impedances possible with disturbance observer based impedance control is generated using (4.33) and (4.57) (Fig. 4.11). Here, a similar observation is made; the passive range represents a subset of all stable impedances. Therefore, the control designer need only consult the passivity criteria if both stable and passive performance is desired.

Although the virtual stiffness gains in Fig. 4.10 overestimate, per (4.20), the actual steady-state stiffness of the closed loop actuator, and the passivity bound in Fig. 4.11 represents the conservative limit (4.57), a number of important observations

![Graph](image)

Figure 4.10: Stable and passive Z-widths for cascaded impedance and PD torque control assuming Valkyrie hip SEA parameters. Note again, that the virtual stiffness gains of the figure overestimate, per (4.20), the actual rendered virtual stiffness of this control architecture.
Virtual Damping Gain $K_{di} (\text{Nm/rad})$

Virtual Stiffness Gain $K_{pi} (\text{Nm/rad})$

Figure 4.11: Stable and passive Z-widths for disturbance observer based impedance control assuming Valkyrie hip SEA parameters. Note that the passive Z-width defined here represents a conservative bound on passivity per (4.57).

can nevertheless be made. First, a disturbance observer does adversely affect SEA passive Z-width, particularly at low values of virtual damping. This is not disqualifying, although it should be noted that the passive Z-width boundary of DOB based control fails to intersect the origin, highlighting again that the Valkyrie hip SEA violates (4.59) and cannot render a passive null impedance. Second, as virtual damping increases, both control approaches are able to render virtual stiffnesses far greater than the actuator’s 3700 Nm/rad physical spring rate (the inherent limit of passivity-based impedance control). Guaranteed passivity of a virtual stiffness greater than the actuator’s physical spring rate has never been demonstrated in the SEA literature. Thus, this result for both controllers is new, adding significant weight to the arguments in favor of adopting cascaded impedance and PD torque control or disturbance observer based impedance control.
The veracity of the theoretical analysis and simulations that lead to this new conclusion can be confirmed experimentally. Using the hardware setup and testing protocol detailed in Chapter 5, the frequency response of the Valkyrie hip SEA under DOB based impedance control is measured (Fig. 4.12). Control gains are taken from Table 4.4 with the desired impedance defined as $K_{pi} = 5000 \text{Nm}/\text{rad}$ and $K_{di} = 30 \text{Nms}/\text{rad}$ (roughly 35% stiffer than the physical spring and within the passive Z-width of Fig. 4.11).

Rather than plotting the driving point admittance, Fig. 4.12 examines the actuator’s output position response to external torque as modeled by (4.32). As such, a phase bounded between $0^\circ$ and $-180^\circ$ indicates passivity. Here, the SEA’s response to a 5 Nm external torque chirp is compared to that of the simulated closed loop model.

![Figure 4.12: Experimental and simulated output position frequency responses of the Valkyrie hip SEA under DOB based impedance control. The actuator is excited with a 5 Nm external torque chirp while rendering impedance gains $K_{pi} = 5000 \text{Nm}/\text{rad}$ and $K_{di} = 30 \text{Nms}/\text{rad}$.](image)
Although there are small discrepancies due to nonidealities in the physical system, the linear model used in the preceding analysis provides an accurate representation of the SEA’s behavior. The physical response exhibits the expected closed loop natural frequency,

\[ \omega_n = \sqrt{\frac{K_{pi}}{J_L}} = 129.1 \text{ rad/s } (20.5 \text{ Hz}), \]  

and before measurement noise masks data beyond 40 Hz, a phase angle transitioning from \(0^\circ\) to \(-180^\circ\) is clearly seen.

The -74 dB gain at low frequencies in Fig. 4.12 provides a measure of the steady-state rendered stiffness of the closed loop actuator. This can be seen more clearly by plotting the applied external torque versus the resulting actuator output position over this range (Fig. 4.13). The rendered stiffness of the Valkyrie hip SEA closely

![Graph showing torque versus position](image)

**Figure 4.13:** External torque versus actuator output position when rendering a 5000 \(\text{Nm/rad}\) virtual stiffness with DOB based impedance control. The SEA response closely matches the ideal desired closed loop stiffness.
matches the desired $5000 \text{Nm/rad}$ ideal, thus confirming the actuator’s ability to accurately render a virtual stiffness exceeding its physical spring rate when leveraging disturbance observer based impedance control.

4.3 Implications

The stability and passivity results of this chapter serve as tools for selecting amongst SEA impedance control architectures. Using actuator physical parameters and expected controller gains, the performance of each control approach can be classified across a spectrum of desired driving point impedances. This informs the control designer of potential pitfalls when applying a specific controller to a specific application.

The summary of this analysis in Table 4.6 makes clear that each examined control architecture is capable of stable impedance rendering given the right circumstances, and for most, passivity is possible as well. A few key points must be reiterated, however. While null impedances and spring-like behaviors are readily achievable, only two control approaches succeed in rendering passive damped impedances with stiffnesses greater than the SEA’s physical spring rate. This class of impedance is highly desirable in tasks requiring high strength or accurate positioning (the traditional realm of “rigid” robots), thus SEA control approaches that leverage PD torque compensation hold great promise for realizing the benefits of series elastic actuation in these arenas.

The analysis here provides, for the first time, proof of an SEA’s passivity when rendering a virtual stiffness greater than its own physical spring rate. Aside from this important contribution, the preceding analysis also establishes the necessary and sufficient conditions for the stability and passivity of an SEA leveraging disturbance observer based impedance control, another new result. Presented alongside the analysis of four other control approaches well known in the series elastic actuation or
Table 4.6: A summary of the stability and passivity of each examined impedance control approach. Here, a specific control approach is deemed stable or passive for a given class of impedances if such performance is theoretically possible given any specific set of system parameters within that class. As discussed, stable and passive Z-widths do vary across controllers, providing a finer delineation that is not captured by these classifications.

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<thead>
<tr>
<th>Impedance Control Approach</th>
<th>Null Impedance ($K_{pi} = K_{di} = 0$)</th>
<th>Pure Spring ($k_{rendered} &lt; k$)</th>
<th>Damped Impedances ($k_{rendered} &lt; k$)</th>
<th>Damped Impedances ($k_{rendered} &gt; k$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple Impedance</td>
<td>Stable and Passive</td>
<td>Stable and Passive</td>
<td>Stable</td>
<td>Stable</td>
</tr>
<tr>
<td>Passivity-Based</td>
<td>Stable and Passive</td>
<td>Stable and Passive</td>
<td>Stable and Passive</td>
<td>Not Possible</td>
</tr>
<tr>
<td>Cascaded Impedance and PD Torque</td>
<td>Stable and Passive</td>
<td>Stable and Passive</td>
<td>Stable and Passive</td>
<td>Stable and Passive</td>
</tr>
<tr>
<td>Cascaded Impedance and PI Torque</td>
<td>Stable</td>
<td>Stable</td>
<td>Stable</td>
<td>Stable</td>
</tr>
<tr>
<td>Cascaded Impedance, Torque, and Velocity</td>
<td>Stable and Passive</td>
<td>Stable and Passive</td>
<td>Stable</td>
<td>Stable</td>
</tr>
<tr>
<td>Disturbance Observer Based</td>
<td>Stable and Passive</td>
<td>Unstable$^a$</td>
<td>Stable and Passive</td>
<td>Stable and Passive</td>
</tr>
</tbody>
</table>

$^a$ Not explored in detail here, stable spring-like impedances are theoretically possible with DOB based impedance control if output damping is reintroduced to the model (i.e. $b_L \neq 0$).
flexible-joint robot literature, the relative merit of DOB based control is determined, and the control designer is equipped with a single comparison study that serves to reconcile the conclusions drawn by disparate studies across the literature, with their own unique modeling assumptions and performance objectives.

This work is meant, first and foremost, to aid SEA control design, but it also directly improves upon the few prior studies of SEA passivity. Vallery et al. [69] and Tagliamonte and Accoto [86] investigate the cascaded impedance, torque, and velocity control architecture as it relates to passive impedance rendering, but they both limit themselves to this control approach alone. Neither incorporate a gear ratio or actuator output inertia in their model, nor do they draw conclusions on the passivity of damped impedances or virtual stiffnesses exceeding the SEA’s open loop dynamics (this as a result of leveraging only PI compensators for control). While Colgate and Hogan [104] offered an early look at the coupled stability range of systems with series elasticity, using a fourth order mass-spring-mass model as a test case in their study of robust interaction, passivity is rarely discussed in the SEA literature and much of the intervening work has focused on practical performance.

Ultimately, the objective here is to leverage the insight gained from a rigorous passivity analysis to determine which control approaches merit further investigation in a practical setting. To this end a number of observations can be made. First, adopting either cascaded impedance and PD torque control or disturbance observer based impedance control seems reasonable given their large stable and passive Z-widths. Velocity signals are noisy in practice, however, and derivative gains cannot be increased without bound as theory might suggest. This potentially undercuts the advantage of PD-based control techniques, particularly when remembering that passivity is a conservative means of ensuring coupled stability, not an absolute requirement. Second,
the linear model used in this analysis does not capture nonidealities present in the physical hardware (e.g. friction, hysteresis, etc.). An approach incorporating integral control to minimize steady-state error, or a disturbance observer to account for model inaccuracies, might be preferred for this reason. Third, stability and passivity says little about impedance rendering performance. Indeed, a system can be both stable and passive yet be wholly ineffective at rendering a desired impedance. Attention shifts to this question in the next chapter, specifically examining the accuracy with which various control approaches render desired impedances, and determining how best to quantify this performance.
Chapter 5

Impedance Rendering

The previous chapter sought to differentiate control approaches by examining the range of impedances over which stability and passivity are achieved. However, as is rightly pointed out by Colgate and Hogan [104], “ensur[ing] coupled stability . . . does not necessarily constitute an effective design procedure.” This point is reinforced by the example of passivity-based impedance control. As seen in Chapter 4, this control approach is guaranteed both stable and passive for all rendered stiffnesses less than the actuator’s physical spring rate. Fig. 5.1, which reproduces a simulated step response from Ott et al. [66, Fig. 4] given the system parameters of Table 5.1, illustrates the accurate rendering of a critically damped 1000 $\text{Nm/rad}$ virtual stiffness by an actuator with a physical stiffness of $k = 15000 \text{Nm/rad}$. If, however, the desired virtual stiffness is increased to 10000 $\text{Nm/rad}$ (a value still well below the physical stiffness), the resulting response (Fig. 5.2) no longer adequately matches that of the critically damped desired impedance, in spite of stability and passivity being preserved.

How then should this control approach be classified? It is stable and passive for all $K_{\text{pi}} < k$, but is it effective at rendering that full range of impedances? Clearly not. In this case, performance shortcomings outweigh the benefit of guaranteed passivity if accurate rendering of high stiffnesses is the objective. Of course, the original intent of passivity-based impedance control was to improve the performance of relatively stiff actuators (e.g. $k = 15000 \text{Nm/rad}$) when rendering relatively low desired impedances (e.g. $K_{\text{pi}} = 1000 \text{Nm/rad}$). Thus, when measured by this objective, passivity-based
Figure 5.1: Simulated response of passivity-based impedance control to a 10 Nm step in external torque when rendering $K_{pi} = 1000 \text{Nm/rad}$ and $K_{di} = 100 \text{Nms/rad}$. This plot reproduces the results observed by Ott et al. [66, Fig. 4] for the system parameters outlined in Table 5.1.

Figure 5.2: Simulated response of passivity-based impedance control to a 10 Nm step in external torque when rendering $K_{pi} = 10000 \text{Nm/rad}$ and $K_{di} = 313 \text{Nms/rad}$. Although the response is both stable and passive, the closed loop system (with the parameters of Table 5.1) does not faithfully reproduce the desired impedance behavior at these gains (where the desired virtual stiffness is still only 66% of the actuator’s physical spring rate).
Table 5.1: System parameters used with (4.13) to produce the simulated responses of Fig. 5.1 and Fig. 5.2. These parameters are taken to approximate those used by Ott et al. [66, Fig. 4] in their original investigation of passivity-based impedance control.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_L$</td>
<td>2.45</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>$J_m$</td>
<td>$1.7 \times 10^{-4}$</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>$J_0$</td>
<td>$1.7 \times 10^{-5}$</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>$b_m$</td>
<td>0</td>
<td>N·m·s/ rad</td>
</tr>
<tr>
<td>$k$</td>
<td>15000</td>
<td>N·m/ rad</td>
</tr>
<tr>
<td>$k_t$</td>
<td>1</td>
<td>N·m/ A</td>
</tr>
<tr>
<td>$N$</td>
<td>160</td>
<td>–</td>
</tr>
</tbody>
</table>

impedance control is indeed effective. Yet, it is clearly miscast as an SEA control approach, considering the need for rendered impedances much closer to (and sometimes exceeding) the low spring rates of series elastic actuators.

With this example in mind, the most promising control approaches of the previous chapter (i.e. cascaded impedance and PD torque control; cascaded impedance, torque, and velocity control; and DOB based impedance control) are examined throughout this chapter from the perspective of practical impedance rendering performance in order to classify completely each controller’s fitness for SEA applications. Using these experimental performance comparisons, a set of new metrics is also explored, that serves to quantify impedance rendering performance and better equip the control designer for the task at hand: differentiating between SEA control approaches and their applicable impedance ranges, then implementing an appropriate controller given specific actuator parameters and performance objectives.
5.1 The Experimental Platform

The empirical studies of impedance rendering performance carried out here leverage the Valkyrie series elastic actuators as experimental platforms. Described in Chapter 2, these actuators represent significant advancements in the design and overall capability of SEAs, and using them for these tests serves two purposes. First, the applicability of the discussed control approaches to high performance, real-world hardware can be explored. In tandem with the theoretical analysis of Chapter 4, the practical performance observed in the tests to come provides a complete picture for the control designer of the strengths and weaknesses of each SEA impedance control approach when deployed. Second, the experimental data gathered here serve to quantify the performance capabilities of the Valkyrie actuators themselves, contributing additional empirical details to the hardware description begun in Chapter 2.

All subsequent tests are performed in one of two configurations. Closed loop responses to external torques are obtained using a dual actuator setup where the outputs of the Valkyrie hip and knee actuators are bolted together (Fig. 5.3). The knee actuator, in a torque control mode akin to Section 3.4.1, serves as an external torque source providing either step or sinusoidal chirp inputs. The hip actuator, meanwhile, operates under impedance control, and measuring its response to the external inputs from the other actuator provides data on controller performance and rendered driving point impedance. For testing the closed loop response to reference commands and for experiments involving human interaction, the Valkyrie hip actuator is used alone with an unconstrained output as in Fig. 5.4.

In both test configurations, a single axis torque sensor is mounted to the output of the hip actuator to record ground truth torque input from the environment. Additional output inertia is also added as shown to emulate the inertial load of a Valkyrie
Figure 5.3: Dual actuator experimental test configuration used to examine rendered driving point impedance. A Valkyrie knee SEA in torque control mode provides the external torque inputs used to test a Valkyrie hip SEA in impedance control mode. The outputs of both actuators are bolted together through an external torque sensing loadcell and an additional inertial load.

Figure 5.4: Single actuator experimental test configuration used to examine the SEA’s closed loop response to reference commands. Mounting the Valkyrie hip SEA with an unconstrained output also allows for tests involving human interaction with the actuator.
Table 5.2: System parameters of the Valkyrie hip series elastic actuator for the experimental test configurations of Fig. 5.3 and Fig. 5.4.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_L$</td>
<td>0.30</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>$J_m$</td>
<td>$1.32 \times 10^{-4}$</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>$b_m$</td>
<td>$7.64 \times 10^{-3}$</td>
<td>N·m/s/rad</td>
</tr>
<tr>
<td>$k$</td>
<td>3700</td>
<td>N·m/rad</td>
</tr>
<tr>
<td>$k_t$</td>
<td>0.05</td>
<td>N·m/A</td>
</tr>
<tr>
<td>$N$</td>
<td>120</td>
<td>–</td>
</tr>
</tbody>
</table>

lower leg on the actuators. Configuring the experimental platform in this way leads to a 0.30 kg m$^2$ output inertia. This inertia, the actuator gear ratio, and other pertinent system parameters obtained via experimental system identification (that were previously used throughout Chapter 4) are again summarized in Table 5.2.

5.2 Comparison Methodology

Comparing the performance of disparate control architectures is not straightforward, in the sense that different control approaches have their own unique ways in which they can be tuned. Deciding which knobs to turn, or at what point a controller with numerous control gains is “optimally” tuned can be an arduous process. It is hard to know with any certainty a priori what constitutes a controller’s “best” performance, and comparing one control approach to another with a better (or worse) tuning makes for an unfair comparison. To this end, two steps are taken to provide the most meaningful comparisons possible between the three SEA impedance control
approaches examined in the coming tests.

First, unlike disturbance observer based impedance control or an approach leveraging cascaded PI compensators, cascaded impedance and PD torque control does not enforce a steady-state virtual stiffness equivalent to the impedance compensator’s proportional gain $K_{pt}$ (see (4.20)). Thus, for the same impedance gains this controller’s expected (or desired) behavior is not equal to that of DOB based impedance control or cascaded impedance, torque, and velocity control. To rectify this problem and make graphical comparisons between these three controllers easier, a feedforward term equivalent to that used in DOB based control, $FF = (Nk_t)^{-1}$, is added to the cascaded impedance and PD torque control architecture. In essence, this controller then becomes the disturbance observer based impedance control of Fig. 3.7 with the DOB portion removed.

While this adjustment does not fundamentally change the nature of cascaded impedance and PD torque control, it does augment its closed loop steady-state stiffness to match $K_{pt}$. Thus, it becomes intuitive to compare the performance of all three controllers side-by-side on the same plot, relating them simultaneously to the desired dynamics defined in (3.2) and (3.3) by the impedance control gains. For the balance of this work, all discussion of cascaded impedance and PD torque control refers to this augmented architecture.

The second issue of concern, selecting the appropriate gains to use when comparing these three control approaches, is addressed by looking at torque control bandwidth. Although the effect of torque compensator gains is not consistent across each architecture (primarily due to the inner velocity loop in cascaded impedance, torque, and velocity control), a fair comparison of impedance controllers can nevertheless be made if the inner torque control response of each has the same bandwidth.
The inner torque control of disturbance observer based impedance control and cascaded impedance and PD torque control is represented by the nominal closed loop transfer function from reference torque input to actuator torque output previously presented in (3.16) for a fixed output SEA:

$$P_n(s) = \frac{\tau_k(s)}{\tau_t(s)} = \frac{(Nk_tkK_{dt})s + (k + Nk_tkK_{pt})}{N^2J_m s^2 + (N^2b_m + Nk_tkK_{dt})s + (k + Nk_tkK_{pt})}.$$  \hspace{1cm} (3.16)

Using the same fixed output assumption, a comparable transfer function for the inner torque control of cascaded impedance, torque, and velocity control can be derived:

$$P_n(s) = \frac{\tau_k(s)}{\tau_t(s)} = \frac{(K_{pt}K_{pv}k)s^2 + k\beta s + K_{it}K_{iv}k}{NK_{pv}s^3 + (NK_{iv} + K_{pt}K_{pv}k)s^2 + k\beta s + K_{it}K_{iv}k}.$$  \hspace{1cm} (5.1)

where

$$\beta = (K_{it}K_{pv} + K_{pt}K_{iv}).$$

If gains are selected such that both (3.16) and (5.1) satisfy $|P_n(j\omega)| = 0.707$ (i.e. $-3$ dB) at the same frequency $\omega$, then the torque control bandwidth for all three impedance control architectures is equivalent.

The nominal control gains used throughout Chapter 4 (summarized again in Table 5.3) produce a torque bandwidth of 47 Hz for all three control architectures. This is confirmed experimentally by measuring the torque response of a fixed output Valkyrie hip SEA when subjected to a 5 Nm reference torque chirp signal (Fig. 5.5). Thus, when comparing SEA impedance control approaches, performance differences can be attributed to the fundamental design characteristics of each architecture rather than the range of frequencies over which each regulates torque, as this is held constant across experiments.
Table 5.3: Nominal torque control gains for each experimentally investigated control approach. The chosen values produce an identical 47 Hz inner torque control bandwidth for each architecture, thus ensuring a fair comparison between impedance control approaches.

<table>
<thead>
<tr>
<th>Gain</th>
<th>Value</th>
<th>Units</th>
<th>Gain</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{pt}$</td>
<td>9.5</td>
<td>rad/N-m·s</td>
<td>$K_{pt}$</td>
<td>2.0</td>
<td>A/N·m</td>
</tr>
<tr>
<td>$K_{it}$</td>
<td>0.25</td>
<td>rad/N-m·s²</td>
<td>$K_{dt}$</td>
<td>0.0209</td>
<td>A·s/N·m</td>
</tr>
<tr>
<td>$K_{pv}$</td>
<td>1.0</td>
<td>A·s/N-m·rad</td>
<td>$\omega_q$</td>
<td>188.5</td>
<td>rad/s</td>
</tr>
<tr>
<td>$K_{iv}$</td>
<td>0.25</td>
<td>A/N-m·rad</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Cascaded impedance and PD torque control also adopts the torque gains of DOB based control without the need for a filter cutoff frequency $\omega_q$. 

![Graph](image1)

![Graph](image2)

Figure 5.5: Magnitude of the experimental torque frequency responses for a fixed output Valkyrie hip SEA subjected to a 5 Nm reference torque chirp. Control gains taken from Table 5.3 produce an inner torque bandwidth of 47 Hz for both (a) cascaded impedance, torque, and velocity control, and (b) DOB based impedance control.
5.3 Controller Performance

With the means for making meaningful comparisons established, attention now turns directly to the impedance rendering performance of the three control approaches of interest. Both time domain and frequency domain responses provide valuable insight in this effort.

5.3.1 Time Domain Comparisons

Using the test configuration of Fig. 5.3, a torque controlled Valkyrie knee actuator is used to apply a 4.4 Nm step in external torque to an impedance controlled Valkyrie hip actuator. The output position response of the hip SEA for a desired virtual stiffness and damping of $K_{pi} = 100 \text{ Nm/rad}$ and $K_{di} = 7.67 \text{ Nms/rad}$ is recorded in Fig. 5.6.

All three impedance control approaches (DOB based control; cascaded impedance, torque, and velocity control; and cascaded impedance and PD torque control) are compared to the response of the ideal desired dynamics defined by (3.2):

$$\frac{\theta_L(s)}{\tau_{\text{ext}}(s)}_{\text{des}} = \frac{1}{J_Ls^2 + K_{di}s + K_{pi}} = \frac{1}{0.30s^2 + 7.67s + 100}.$$  \hfill (5.2)

Looking first at the simulated response of each control approach (the dashed lines in Fig. 5.6), it is immediately apparent that DOB based impedance control provides the best theoretical approximation to the ideal desired behavior. While all three simulated responses match the 0.044 rad steady-state deflection of the ideal $100 \text{ Nm/rad}$ virtual stiffness, the transient behaviors (as quantified by the standard step response metrics in Table 5.4) differ significantly. The performance differences between control approaches are even more drastic in the experimental data (the solid lines in Fig. 5.6), highlighting a number of important points.
Figure 5.6: Response of the Valkyrie hip SEA to a 4.4 Nm step in external torque when rendering $K_{pi} = 100 \text{Nm/rad}$ and $K_{di} = 7.67 \text{Nms/rad}$. The desired response defined by (5.2) is compared to that of (1) DOB based impedance control, (2) cascaded impedance, torque, and velocity control, and (3) cascaded impedance and PD torque control.

Table 5.4: Step response characteristics derived from Fig. 5.6 for each impedance control approach. Disturbance observer based impedance control provides a better approximation to the desired impedance response across all metrics, both in simulation (sim) and experimentally (exp).

<table>
<thead>
<tr>
<th>Desired Impedance Response</th>
<th>DOB Based</th>
<th>Cascaded Imp., Torque, &amp; Vel.</th>
<th>Cascaded Imp. &amp; PD Torque</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\text{sim} \quad \text{exp} )</td>
<td>(\text{sim} \quad \text{exp} )</td>
<td>(\text{sim} \quad \text{exp} )</td>
</tr>
<tr>
<td>Rise Time(^a) (s)</td>
<td>0.115</td>
<td>0.116 0.112</td>
<td>0.417 0.266</td>
</tr>
<tr>
<td>Settling Time(^b) (s)</td>
<td>0.328</td>
<td>0.351 0.173</td>
<td>0.671 0.539</td>
</tr>
<tr>
<td>Overshoot (rad)</td>
<td>0.002</td>
<td>0.003 0</td>
<td>0.001 0</td>
</tr>
<tr>
<td>Steady-State Deflection (rad)</td>
<td>0.044</td>
<td>0.044 0.044</td>
<td>0.044 0.035</td>
</tr>
<tr>
<td>Steady-State Error(^c) (rad)</td>
<td>0</td>
<td>0 0</td>
<td>0 0.009</td>
</tr>
</tbody>
</table>

\(^a\) from 10% – 90% of final value \(^b\) to 2% of final value \(^c\) relative to desired response
Disturbance observer based impedance control tracks the desired response closely over the entire step, never varying by more than 0.004 rad (9% of the desired steady-state deflection). Conversely, cascaded impedance and PD torque control has a peak tracking error of 0.0134 rad (31% of the desired steady-state deflection), and cascaded impedance, torque, and velocity control varies by as much as 0.0177 rad, over 39% of the desired steady-state deflection and more than four times the error of DOB based control (Fig. 5.7).

Large-scale errors in these experimental transient responses are a direct result of the control approach chosen and cannot be attributed to physical nonidealities in the actuator. Friction, for instance, is not included in the simulation, yet the simulated responses in Fig. 5.6 provide a remarkably accurate representation of the experimental transients between 0 s and 0.2 s. This reinforces the validity of the system model used,
adding weight to the theoretical analysis of Chapter 4, while also clearly illustrating that DOB based impedance control inherently outperforms the other two approaches both in simulation and in practice.

Although not the exclusive cause of the observed differences, the practical effect of system nonidealities is revealed in the experimental data, specifically in the steady-state responses. Both cascaded impedance, torque, and velocity control, and cascaded impedance and PD torque control exhibit significant experimental steady-state error with respect to the desired impedance response (again see Fig. 5.6 and Table 5.4). Without the presence of an integrator, cascaded impedance and PD torque control does not compensate for unmodeled friction in the test setup, leading to a 0.011 rad steady-state error (25% of the desired deflection). This hinders the faithful rendering of the desired virtual stiffness by masking the commanded external torque to output position relationship observable from the environment.

Cascaded impedance, torque, and velocity control only slightly improves this performance. The integral gains in this architecture serve to decrease steady-state error to 0.009 rad, but this still represents a 20% error relative to the desired closed loop deflection. Clearly the integral gains chosen do not provide enough command authority to effectively compensate for friction over this time scale. Increasing them, however, changes torque control bandwidth, negating the comparison methodology outlined in Section 5.2. That aside, addressing steady-state error here requires undesirable hardware-in-the-loop tuning. A disturbance observer, though, does not suffer from this shortcoming. By enforcing the nominal closed loop plant as described in Chapter 3, the DOB treats physical nonidealities and the effects of nonlinearities like friction as disturbances to counteract. For the purposes of SEA impedance rendering, this makes the integral-like action of the DOB more effective than the integrators
in the PI based cascaded impedance, torque, and velocity approach. Not only does DOB based control render a more accurate steady-state virtual stiffness, but the cycle-by-cycle nature of its compensation (not requiring errors to build up over time for command authority to increase, as integral control does) allows for faster error correction and a closer match to the desired step response transients.

The basic trends observed in the above data are also seen as desired impedance is increased. Fig. 5.8 illustrates the experimental position response of the Valkyrie hip SEA to a 4.7 Nm step in external torque when rendering $K_{pi} = 1000 \text{Nm/rad}$ and $K_{di} = 24.25 \text{Nms/rad}$. In this case, cascaded impedance and PD torque control again misrepresents the closed loop desired stiffness, exhibiting a $1.1 \times 10^{-3}$ rad steady-state error (24% of the desired deflection). Cascaded impedance, torque, and velocity control improves upon this performance but still varies by $3.4 \times 10^{-4}$ rad at steady-state (7% of the desired deflection). DOB based impedance control, however, decreases the steady-state error magnitude to $3.1 \times 10^{-5}$ rad, less than 0.7% of the desired deflection (see Fig. 5.9).

If impedance gains are further increased to $K_{pi} = 5000 \text{Nm/rad}$ and $K_{di} = 30 \text{Nms/rad}$ (a noteworthy impedance in that the virtual stiffness exceeds the physical spring-rate of the SEA) cascaded impedance and PD torque control still exhibits an appreciable $8.8 \times 10^{-4}$ rad (or 12%) steady-state error with respect to the desired deflection (see Fig. 5.10). In this case, however, cascaded impedance, torque, and velocity control outperforms DOB based impedance control. While all three control approaches have comparable transients below 0.025 s, Fig. 5.10 reveals that DOB based control has both the largest overshoot and longest settling time relative to the desired dynamic response. From a practical perspective, controller differences at high desired impedances are less noticeable because actuator deflections are smaller and tran-
Figure 5.8: Response of the Valkyrie hip SEA to a 4.7 Nm step in external torque when rendering $K_{pi} = 1000 \text{Nm/} \text{rad}$ and $K_{di} = 24.25 \text{Nms/} \text{rad}$. The desired response is compared to that of (1) DOB based impedance control, (2) cascaded impedance, torque, and velocity control, and (3) cascaded impedance and PD torque control.

Figure 5.9: Experimental error of the step responses in Fig. 5.8 with respect to the ideal desired response.
Figure 5.10: Response of the Valkyrie hip SEA to a 5 Nm step in external torque when rendering $K_{pi} = 5000 \text{Nm/rad}$ and $K_{di} = 30 \text{Nms/rad}$. The desired response is compared to that of (1) DOB based impedance control, (2) cascaded impedance, torque, and velocity control, and (3) cascaded impedance and PD torque control.

sients settle much faster. This specific example does speak to the merits of cascaded impedance, torque, and velocity control, however, whereas the case for DOB based impedance control is one made by considering a range of performance characteristics (Z-width, passivity, both low and high impedance rendering, etc.).

Selecting a control approach that is best across all potential use cases is not possible; this controller simply does not exist. As the high impedance case suggests, each control approach has its own unique set of strengths and weaknesses. Noteworthy in Fig. 5.10 though, is evidence that none of these three controllers suffer from the same significant degradation in impedance rendering performance observed with passivity-based impedance control when the desired virtual stiffness approaches (or exceeds) the physical stiffness of the SEA.

Using the test configuration of Fig. 5.4, the closed loop SEA’s response to reference
Figure 5.11: Experimental responses of the Valkyrie hip SEA to a 0.15 rad step in commanded position when rendering the desired dynamics of (5.3). The desired response is compared to that of (1) DOB based impedance control, (2) cascaded impedance, torque, and velocity control, and (3) cascaded impedance and PD torque control.

Position commands is also examined experimentally. Step response plots such as these characterize how closely the actuator’s trajectory follows that of the desired virtual dynamics when commanding a new set point. In the examples that follow the Valkyrie hip SEA’s response to a 0.15 rad step in commanded position with impedance gains $K_{pi} = 100 \, Nm/\text{rad}$ and $K_{di} = 7.67 \, Nms/\text{rad}$ is considered.

Fig. 5.11 compares all three impedance control approaches to the response of the ideal desired dynamics defined by the transfer function (3.3):

$$\frac{\theta_L(s)}{\theta_{ref(s)}_{\text{des}}} = \frac{K_{di} s + K_{pi}}{J_L s^2 + K_{di} s + K_{pi}} = \frac{7.67s + 100}{0.30s^2 + 7.67s + 100}. \quad (5.3)$$

In this example disturbance observer based impedance control clearly provides the
best qualitative match to the desired dynamics, but a few discrepancies are worth noting. First, there is a noticeable delay in the transient response of DOB based control relative to the desired behavior. While not nearly as significant as the delay observed in cascaded impedance and PD torque control, it is present nonetheless. It should be noted as well that none of the controllers exhibited this type of delay in the prior external torque response data. Second, although overshoot beyond the steady-state 0.15 rad position is expected in the desired response, DOB based control leads to even further overshoot.

Both of these discrepancies are explained in large part by current saturation in the physical actuator. At time zero the position error acted on by the PD impedance controller experiences a discontinuous jump from zero to 0.15 rad due to the ideal step command. This leads to an infinite error derivative multiplied by $K_{di}$ and, after propagating through the control architecture, an infinite current command to the motor. The Valkyrie SEA is limited to 30 A inputs and this physical saturation of the command produces the tracking discrepancies observed.

In most practical applications the issues associated with ideal step functions need not be considered as robots are typically commanded along continuous trajectories (or at least filtered command inputs) rather than via discontinuous set points. As an alternative to shaping the reference signal, the impedance controller itself can be modified from PD control of position error to individual proportional control of both position and velocity error. This necessitates providing the controller with a desired velocity signal but otherwise does not change its behavior, except in response to discontinuous reference signals. For an ideal position step the desired velocity remains zero throughout, and this slight controller modification eliminates excessive current commands above the actuator’s capacity.
When individually calculating a position and a velocity error, the desired virtual
dynamics in this example change from those expressed in (5.3) to:

\[
\frac{\theta_L(s)}{\theta_{\text{ref}(s)}_{\text{des}}} = \frac{K_{\pi}}{J_L s^2 + K_{\text{di}} s + K_{\pi}} = \frac{100}{0.30 s^2 + 7.67 s + 100}.
\]  

Fig. 5.12 plots this new desired behavior in comparison to the experimental responses
of the Valkyrie hip SEA when the impedance control modification described is imple-
mented in the hardware. Because current saturation no longer affects the responses,
an even closer match between the empirical behavior and the simulated ideal is ob-
served. Once again, DOB based impedance control outperforms the other two archi-
tectures. This is apparent both qualitatively and as quantified by the step response
metrics in Table 5.5.

Interestingly, there is little change in the cascaded impedance, torque, and velocity
control and cascaded impedance and PD torque control responses between Fig. 5.11
and Fig. 5.12. These approaches, on one hand, appear to be less affected by the
saturation of the command signal. On the other hand, however, the desired response
clearly does change and only disturbance observer based impedance control tracks
this difference.

Although not definitive in all cases (particularly at higher desired impedances)
the balance of the time domain response data suggests that DOB based impedance
control is preferable to both cascaded impedance, torque, and velocity control and
cascaded impedance and PD torque control if accurate impedance rendering is of
paramount importance. This is true for rendering driving point impedances in re-
response to external torque inputs from the environment, and for displaying appropriate
virtual dynamics in response to reference commands.
Figure 5.12: Experimental responses of the Valkyrie hip SEA to a 0.15 rad step in commanded position when rendering the desired dynamics of (5.4). The desired response is compared to that of (1) DOB based impedance control, (2) cascaded impedance, torque, and velocity control, and (3) cascaded impedance and PD torque control.

Table 5.5: Step response characteristics derived from Fig. 5.12 for each impedance control approach. DOB based impedance control provides a better approximation of the desired response across all metrics except settling time.

<table>
<thead>
<tr>
<th></th>
<th>Desired Response</th>
<th>DOB Based</th>
<th>Cascaded Imp., Torque, &amp; Vel.</th>
<th>Cascaded Imp. &amp; PD Torque</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rise Time$^a$ (s)</td>
<td>0.116</td>
<td>0.123</td>
<td>0.389</td>
<td>0.148</td>
</tr>
<tr>
<td>Settling Time$^b$ (s)</td>
<td>0.328</td>
<td>0.188</td>
<td>0.657</td>
<td>0.255</td>
</tr>
<tr>
<td>Steady-State Deflection (rad)</td>
<td>0.150</td>
<td>0.148</td>
<td>0.141</td>
<td>0.143</td>
</tr>
<tr>
<td>Steady-State Error$^c$ (rad)</td>
<td>0</td>
<td>0.002</td>
<td>0.009</td>
<td>0.007</td>
</tr>
<tr>
<td>Peak Error$^c$ (rad)</td>
<td>0</td>
<td>0.01</td>
<td>0.07</td>
<td>0.04</td>
</tr>
</tbody>
</table>

$^a$ from 10% – 90% of final value  $^b$ to 2% of final value  $^c$ relative to desired response
5.3.2 Frequency Domain Comparisons

Time domain comparisons of actuator step response provide a useful look at impedance rendering performance, particularly in regard to steady-state deflection (i.e. steady-state rendered stiffness). Developing an intuitive sense for how each of these controllers behaves and how the closed loop actuator might “feel” to the environment or an interacting human user is possible at times from this data, but in many ways each step response represents a single point solution. True interaction with the environment, particularly in human-centric applications, is not as consistent or predictable as a single step function. A frequency domain comparison is therefore helpful in answering questions related to impedance control bandwidth and the range of frequencies over which specific desired impedances are effectively rendered.

In Section 3.4 and Section 5.2 inner torque control bandwidth was discussed, but this does not provide a complete picture of impedance control bandwidth. With the principal objective of impedance control being to emulate a desired dynamic behavior, the bandwidth of an impedance controller can appropriately be defined as the frequency range over which the rendered behavior accurately matches the desired. This is examined in simulation for $K_{pi} = 100 \text{Nm/rad}$ and $K_{di} = 7.67 \text{Nms/rad}$ in Fig. 5.13. Here, a Bode plot of the desired second order dynamics in (5.2) is compared to the closed loop actuator under DOB based impedance control (red line); cascaded impedance, torque, and velocity control (blue line); and cascaded impedance and PD torque control (green line).

Above roughly 50 Hz, all three control approaches converge to closely approximate the desired impedance. This is to be expected because the high frequency roll-off of the desired second order response is defined purely by the physical actuator’s output inertia $J_L$. Impedance control performance is thus a question of how well each
controller renders the desired behavior below this frequency, when active control is required to emulate physical characteristics different than those of the SEA. Impedance control bandwidth is therefore defined as the frequency at which a ±3 dB departure from the desired response occurs, recognizing that at some higher frequency each controller will again converge to within this threshold.

In the example of Fig. 5.13, cascaded impedance, torque, and velocity control is the first architecture to diverge from the desired dynamics of (5.2), at a frequency of 0.8 Hz. Cascaded impedance and PD torque control then follows at 1.2 Hz. Disturbance observer based impedance control, however, never diverges by as much as
Figure 5.14: Experimental frequency response of $\frac{\theta_l(s)}{\tau_{ext}(s)}$ for the Valkyrie hip SEA leveraging either (1) DOB based impedance control, (2) cascaded impedance, torque, and velocity control, or (3) cascaded impedance and PD torque control ($K_{pi} = 100 \text{Nm/rad}$ and $K_{di} = 7.67 \text{Nms/rad}$). The departure of (2) and (3) from the desired magnitude response between 0.8 Hz and 10 Hz closely mirrors that observed in simulation (Fig. 5.13).

3 dB, effectively giving this control approach an infinite bandwidth for rendering this desired impedance.

The relative performance of these control approaches in simulation is confirmed experimentally. Again using the test configuration of Fig. 5.3, the Valkyrie knee actuator is used to apply a 5 Nm exponentially varying sinusoidal chirp in external torque to the impedance controlled Valkyrie hip actuator. The hip SEA’s experimental frequency responses for $K_{pi} = 100 \text{Nm/rad}$ and $K_{di} = 7.67 \text{Nms/rad}$ are given in Fig. 5.14. The responses here closely mirror those observed in simulation with both cascaded impedance, torque, and velocity control and cascaded impedance and PD torque control noticeably departing from the desired response between 0.8 Hz and 10 Hz. DOB
based impedance control, though, provides a close match to the desired dynamics across the full frequency range.

Examining the SEA’s frequency response to external torques at higher desired impedances yields Fig. 5.15 and Fig. 5.16 for $K_{pi} = 1000 \text{Nm/ rad}$ and $K_{di} = 24.25 \text{Nms/ rad}$, and Fig. 5.17 and Fig. 5.18 for $K_{pi} = 5000 \text{Nm/ rad}$ and $K_{di} = 30 \text{Nms/ rad}$. The simulated response for $K_{pi} = 1000 \text{Nm/ rad}$ (Fig. 5.15) shows yet again that DOB based control provides a close match to the desired dynamics while the other two controllers differ between roughly 5 Hz and 15 Hz. This difference is relatively small, though, with only cascaded impedance, torque, and velocity control exceeding the -3 dB threshold at a bandwidth of 7 Hz. The experimental responses of Fig. 5.16 verify this behavior, exhibiting the expected gradation of controllers between 5 Hz and 15 Hz. Additionally, the smaller gain of cascaded impedance and PD torque control at low frequencies is indicative of the larger steady-state error observed in its step response (see Fig. 5.8).

For $K_{pi} = 5000 \text{Nm/ rad}$ and $K_{di} = 30 \text{Nms/ rad}$ the simulated responses in Fig. 5.17 show that all three controllers exhibit a higher resonant peak than the desired response, and each deviates by more than 3 dB, yielding bandwidths of 16.8 Hz (DOB based impedance control), 22.3 Hz (cascaded impedance, torque, and velocity control), and 22.9 Hz (cascaded impedance and PD torque control). Interestingly, DOB based impedance control has the highest resonant peak and lowest bandwidth, but also the closest resonant frequency to the desired response. Again, simulation accurately predicts the experimental behavior, as Fig. 5.18 displays the same response characteristics. In this case, with competing metrics, it is hard to definitively say which controller provides a better match to the desired dynamics. The larger resonant peak of DOB based control, however, does serve to explain the higher amplitude transient oscillations observed in the step response of Fig. 5.10.
Figure 5.15: Simulated frequency response of $\theta_L(s)/\tau_{ext}(s)$ for the Valkyrie hip SEA leveraging either (1) DOB based impedance control, (2) cascaded impedance, torque, and velocity control, or (3) cascaded impedance and PD torque control ($K_{pi} = 1000 \text{Nm/rad}$ and $K_{di} = 24.25 \text{Nms/rad}$). Only (2) deviates from the desired response by more than 3 dB, with $BW_{(2)} = 7$ Hz.

Figure 5.16: Experimental frequency response of $\theta_L(s)/\tau_{ext}(s)$ for the Valkyrie hip SEA leveraging either (1) DOB based impedance control, (2) cascaded impedance, torque, and velocity control, or (3) cascaded impedance and PD torque control ($K_{pi} = 1000 \text{Nm/rad}$ and $K_{di} = 24.25 \text{Nms/rad}$).
Figure 5.17: Simulated frequency response of $\theta_L(s)/\tau_{ext}(s)$ for the Valkyrie hip SEA leveraging either (1) DOB based impedance control, (2) cascaded impedance, torque, and velocity control, or (3) cascaded impedance and PD torque control ($K_{pl} = 5000 \text{Nm/rad}$ and $K_{di} = 30 \text{Nms/rad}$). BW$_{(1)} = 16.8$ Hz, BW$_{(2)} = 22.3$ Hz, and BW$_{(3)} = 22.9$ Hz.

Figure 5.18: Experimental frequency response of $\theta_L(s)/\tau_{ext}(s)$ for the Valkyrie hip SEA leveraging either (1) DOB based impedance control, (2) cascaded impedance, torque, and velocity control, or (3) cascaded impedance and PD torque control ($K_{pl} = 5000 \text{Nm/rad}$ and $K_{di} = 30 \text{Nms/rad}$).
5.4 Performance Metrics

In the previous section both time domain and frequency domain performance were examined in an effort to quantify the accuracy with which each control approach is able to render desired impedances. Various metrics were used throughout for this comparison including standard step response characteristics like rise time, settling time, and steady-state error; as well as a measurement of impedance control bandwidth derived from the SEA’s closed loop frequency response. While the classification of step responses is perhaps more familiar, the actuator’s frequency response provides more information across a broader operating range. In the absence of a perfect match, however, it is difficult to determine what specific traits in a frequency response plot make for better rendering of desired dynamics.

To alleviate this problem, two new frequency response metrics are proposed for use in SEA impedance control design and analysis. Based on the $H_\infty$ and $H_2$ system norms familiar in robust and optimal control theory, these metrics quantify the variation between an SEA’s rendered dynamics and the desired closed loop response. They provide additional insight in relation to more commonly used measurements and ease the comparison of disparate control architectures across applications. Aside from offering a powerful new analysis technique, these metrics also serve as the basis for the optimal SEA impedance control synthesis framework presented in Chapter 7.

5.4.1 A Background on System Norms*

Introducing the $H_\infty$ and $H_2$ system norms first requires a preliminary look at signal norms and their physical interpretations. If given a time varying signal $x(t)$, the $L_\infty$ norm

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*This introduction is intended to be brief. A more detailed treatment of norms (vector, matrix, signal, and system) can be found in a number of texts (e.g. [105], [106], and [107]).
norm of this signal is defined as:

$$\|x\|_{L_\infty} \triangleq \sup_t |x(t)| .$$  \hspace{1cm} (5.5)

This value is often termed the peak norm, as it simply represents the peak value of the signal over time. The $L_2$ norm of the same signal is defined as:

$$\|x\|_{L_2} \triangleq \left( \int_0^\infty x(t)^2 \, dt \right)^{1/2} .$$  \hspace{1cm} (5.6)

Here, $\|x\|_{L_2}$ can be conceptualized as the square root of the total energy contained in the signal $x(t)$. Thus, (5.6) is often termed the energy norm.

Given the above notions of signal size, the $H_\infty$ and $H_2$ system norms can be defined with respect to the system gain between input and output signals. The $H_\infty$ norm is equivalent to the system’s energy-to-energy gain $\Gamma_{ee}$, and is thus defined for the system transfer function $G(s)$ as:

$$\|G(s)\|_\infty = \Gamma_{ee} \triangleq \sup_w \frac{\|y\|_{L_2}}{\|w\|_{L_2}} .$$  \hspace{1cm} (5.7)

where $w$ and $y$ represent the input and output signals of the system respectively. This definition is equivalent to the maximum singular value of the system and leads directly, via Parseval’s theorem, to the physical interpretation of the $H_\infty$ norm as the peak magnitude in a single-input-single-output system’s frequency response plot:

$$\|G(s)\|_\infty = \sup_\omega \sigma(G(j\omega)) = \sup_\omega |G(j\omega)| .$$  \hspace{1cm} (5.8)

Similarly, the $H_2$ norm provides a measure of energy-to-peak gain $\Gamma_{ep}$, relating the energy of the input signal $w$ to the peak value of the output signal $y$. For a
single-input-single-output system $G(s)$:

$$\|G(s)\|_2 = \Gamma_{ep} \triangleq \sup_w \frac{\|y\|_{L^\infty}}{\|w\|_{L^2}}. \quad (5.9)$$

Alternatively, the $H_2$ norm can be defined with respect to the system’s frequency response:

$$\|G(s)\|_2 \triangleq \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega \right)^{1/2}. \quad (5.10)$$

In so doing, it is seen that the $H_2$ norm captures system gain information across all frequencies and can be correlated to the area under the system’s magnitude response plot (providing an RMS-like measurement of gain). Thus, the $H_\infty$ norm can be used to measure a worst case impedance rendering performance (i.e. a peak), and the $H_2$ norm can be used as an overall measurement of performance across the frequency spectrum.

### 5.4.2 An $H_\infty$ Norm Metric

To quantify impedance rendering performance with the $H_\infty$ norm, a closed loop SEA’s $H_\infty$ match\(^\dagger\) is defined as:

$$M_{H_\infty} \triangleq \left\| \frac{\theta_L(s)}{\tau_{ext}(s)_{des}} - \frac{\theta_L(s)}{\tau_{ext}(s)_{CL}} \right\|_{\infty}. \quad (5.11)$$

If the closed loop response perfectly matches the desired dynamics, $M_{H_\infty} = 0$. However, when this is not the case the $H_\infty$ match provides a useful measure of the worst

\(^\dagger\)The $H_\infty$ match as presented here bears some resemblance to an admittance error measurement suggested for use in rigid actuator impedance control by Chapel and Su [108] in 1992. While (5.11) was formulated without this knowledge, it may rightly be considered a follow-up to this original work, of which there is none in the subsequent literature.
case discrepancy between the system’s actual and desired frequency responses.

Although the $H_{\infty}$ norm corresponds to a transfer function’s peak magnitude as in (5.8), (5.11) is not equivalent to the maximum difference between the actual and desired magnitude response plots (though this too is a useful metric in and of itself). For desired dynamics $G_{\text{des}}(s)$ and a closed loop response $G_{\text{CL}}(s)$:

$$M_{H_{\infty}} \neq \max_{\omega} \left( |G_{\text{des}}(j\omega)| - |G_{\text{CL}}(j\omega)| \right).$$

Likewise, it is not equal to the difference of the individual system norms either.

$$M_{H_{\infty}} \neq \|G_{\text{des}}(s)\|_{\infty} - \|G_{\text{CL}}(s)\|_{\infty}.$$  (5.13)

The important distinction in (5.11) is that the system transfer functions are subtracted prior to taking the norm. Thus, the composite error transfer function for which $M_{H_{\infty}}$ is measured accounts for both magnitude and phase variation between the responses.

In addition to quantifying peak impedance rendering error, the $H_{\infty}$ match also provides another means for characterizing SEA impedance control bandwidth. As with all $H_{\infty}$ norms, $M_{H_{\infty}}$ occurs at a specific frequency (see (5.8)). Therefore, both the peak error and the input signal frequency $\omega_{\infty}$ at which this error is expected are available to the control designer.

### 5.4.3 An $H_2$ Norm Metric

Moving beyond a single peak error occurring at a single specific frequency, the $H_2$ norm can be used to characterize an SEA’s impedance rendering performance across a full range of potential input frequencies. Analogous to the previous metric, a closed
loop SEA’s $H_2$ match is defined as:

\[
M_{H_2} \triangleq \left\| \frac{\theta_L(s)}{\tau_{ext}(s)}_{des} - \frac{\theta_L(s)}{\tau_{ext}(s)_{CL}} \right\|_2.
\] (5.14)

Once again, if the closed loop response perfectly matches the desired dynamics, $M_{H_2} = 0$. If, however, there is a discrepancy, $M_{H_2}$ quantifies the cumulative error between the desired dynamic response and the actual closed loop behavior over all input frequencies, integrating as in (5.10).

Rather than serving as an alternative to the aforementioned $M_{H_\infty}$ metric, the $M_{H_2}$ metric is a useful complement. When taken together these two measurements quantify the maximum deviation of a closed loop SEA from the prescribed desired dynamics ($M_{H_\infty}$), the frequency at which this error occurs ($\omega_\infty$), and the extent over which error spans the full frequency range ($M_{H_2}$). They indicate whether impedance rendering accuracy degrades only for specific inputs (high $M_{H_\infty}$, low $M_{H_2}$), if small errors persist globally (low $M_{H_\infty}$, high $M_{H_2}$), or if poor impedance rendering is likely (high $M_{H_\infty}$, high $M_{H_2}$).

Implicit in much of optimal control theory is the concept of system norms as performance measures, and the work here serves to introduce this idea to series elastic actuation for the first time. (5.11) and (5.14) represent novel contributions in this effort, designed to distill broad impedance rendering characteristics across the frequency spectrum down to single easily interpreted accuracy measurements. Although human psychophysical perception is not addressed here to determine what constitutes an accurate “feel” for rendered dynamics, examining $M_{H_\infty}$ and $M_{H_2}$ in relation to the previously presented response data makes clear the explicit connection between these metrics and impedance rendering performance.
5.4.4 Performance Analysis

Revisiting the frequency response data of Section 5.3.2, the performance of each SEA impedance control approach is analyzed using the $M_{H_\infty}$ and $M_{H_2}$ metrics. Fig. 5.19 provides both the simulated and experimental frequency responses of the composite error transfer function,

$$E(s) = \left( \frac{\theta_L(s)}{\tau_{\text{ext}}(s)_{\text{des}}} - \frac{\theta_L(s)}{\tau_{\text{ext}}(s)_{\text{CL}}} \right),$$  \hspace{1cm} (5.15)

given desired dynamics defined by $K_{pi} = 100^{\text{Nm/rad}}$ and $K_{di} = 7.67^{\text{Nms/rad}}$ (the Bode plots of the individual responses for this case are found in Fig. 5.13 and Fig. 5.14).

A lower gain error magnitude in Fig. 5.19 indicates that a control approach provides a better closed loop approximation of the desired dynamics. Both $M_{H_\infty}$ and $M_{H_2}$ capture aspects of this comparison.

Looking at the simulated curves (the dashed lines in Fig. 5.19), the peak magnitude of each is represented by the $M_{H_\infty}$ metric and corresponds to the SEA’s worst case impedance match (revealing, as well, the input frequency at which this worst case behavior occurs). The relative area under the three curves provides a sense of how well each control approach renders impedance across all frequencies. This information is captured by the relative magnitude of $M_{H_2}$ in each case. From these measurements (summarized in Table 5.6) conclusions about the relative merit of each control approach can be drawn.

For the desired impedance presented here, disturbance observer based control performs best, with matching gains an order of magnitude lower than the other two controllers and a higher frequency $\omega_\infty$. Concerning frequency, it should be noted that $\omega_\infty$ loosely correlates to actuator bandwidth as well. Impedance control bandwidth,
Figure 5.19: Frequency responses of the error transfer function (5.15) for the Valkyrie hip SEA leveraging (1) DOB based impedance control, (2) cascaded impedance, torque, and velocity control, and (3) cascaded impedance and PD torque control. Desired dynamics are defined by $K_{\text{pi}} = 100 \text{Nm/rad}$ and $K_{\text{di}} = 7.67 \text{Nms/rad}$.

Table 5.6: $H_2$ and $H_\infty$ based metrics derived from simulation for a Valkyrie hip SEA rendering $K_{\text{pi}} = 100 \text{Nm/rad}$ and $K_{\text{di}} = 7.67 \text{Nms/rad}$. Disturbance observer based impedance control performs best in this case with both lower matching gains and a higher $\omega_\infty$.

<table>
<thead>
<tr>
<th></th>
<th>DOB Based</th>
<th>Cascaded Imp., Torque, &amp; Vel.</th>
<th>Cascaded Imp. &amp; PD Torque</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{H_2}$</td>
<td>0.0021</td>
<td>0.0141</td>
<td>0.0111</td>
</tr>
<tr>
<td>$M_{H_\infty}$</td>
<td>$8.45 \times 10^{-4}$</td>
<td>$5.80 \times 10^{-3}$</td>
<td>$4.70 \times 10^{-3}$</td>
</tr>
<tr>
<td>$\omega_\infty$ (Hz)</td>
<td>2.7</td>
<td>1.5</td>
<td>1.7</td>
</tr>
</tbody>
</table>
per the definition used in Section 5.3.2, indicates that both cascaded impedance, torque, and velocity control and cascaded impedance and PD torque control degrade in performance at relatively low frequencies (0.8 Hz and 1.2 Hz respectively). While not the same frequency values, the $M_{H\infty}$ metric similarly orders these controllers with cascaded impedance, torque, and velocity control experiencing its worst match at the lowest frequency (1.5 Hz), followed by cascaded impedance and PD torque control (1.7 Hz), then DOB based control (2.7 Hz). This correlation is observed throughout the experimental data presented here, and though it is not guaranteed to hold in all cases, $\omega_\infty$ does present a method for classifying the bandwidth of systems that do not exceed the ±3 dB threshold previously used. This measurement also allows control approaches that might otherwise be viewed as equal using other metrics to be distinguished based upon their optimal frequency range.

Examining the experimental responses in Fig. 5.19 reveals the same relative controller performance observed in simulation. The peak error magnitudes of both cascaded impedance, torque, and velocity control and cascaded impedance and PD torque control match those predicted in simulation remarkably well, and while DOB based impedance control underperforms relative to its simulated response, it still offers a clear improvement over the other two control approaches. This is particularly apparent in the dip exhibited by the DOB based response between 0.8 Hz and 5 Hz, the same range over which significant discrepancies were observed between the controllers in the original frequency response plots of Fig. 5.13 and Fig. 5.14.

Two additional points should be noted regarding the experimental data in Fig. 5.19. First, as expected, all three error plots converge at high frequency as the physical output inertia of the actuator comes to dominate the system dynamics. Second, at low frequencies all three controllers exhibit a steady-state error not predicted by the sim-
ulation. This is likely due to parasitic damping and/or friction in the test setup that artificially lowers the torque amplitude exerted on the SEA. This error can be seen at low frequencies in Fig. 5.14 as well, but its presence does not alter the conclusions drawn concerning relative controller performance.

Fig. 5.20 and Table 5.7 summarize the impedance rendering performance of the Valkyrie hip SEA when rendering $K_{pi} = 1000 \text{Nm/}\text{rad}$ and $K_{di} = 24.25 \text{Nms/}\text{rad}$. As in the individual response plots of Fig. 5.15 and Fig. 5.16, the difference between controllers is less obvious than in the lower impedance case. Nevertheless, the $M_{H\infty}$ and $M_{H2}$ metrics clearly regard disturbance observer based control as the preferable approach yet again. This assessment, as it was for the lower impedance case, is consistent with the conclusions drawn from both the time domain and frequency domain data in Section 5.3.

Examining performance when $K_{pi} = 5000 \text{Nm/}\text{rad}$ and $K_{di} = 30 \text{Nms/}\text{rad}$ is interesting in that this case is more difficult to assess from the time domain steps and closed loop Bode plots alone. Fig. 5.21 plots the error of each control approach at this desired impedance, while Table 5.8 summarizes the $H2$ and $H\infty$ based performance metrics. Experimental and simulated responses are in close agreement here, and all three SEA control approaches are largely similar to one another (see the closely matched $M_{H\infty}$ values in Table 5.8 for example). Interestingly, DOB based impedance control again has lower matching gains (albeit not by nearly as much as in the previously examined cases). This implies that it more accurately renders the desired impedance. Qualitatively, however, it is unclear if the responses of the previous section agree.

As was seen in Section 5.3, when rendering $K_{pi} = 5000 \text{Nm/}\text{rad}$ and $K_{di} = 30 \text{Nms/}\text{rad}$ DOB based impedance control resulted in a response with the lowest impedance control bandwidth and highest resonant peak (see Fig. 5.17 and Fig. 5.18). This likely
Figure 5.20: Frequency responses of the error transfer function (5.15) for the Valkyrie hip SEA leveraging (1) DOB based impedance control, (2) cascaded impedance, torque, and velocity control, and (3) cascaded impedance and PD torque control. Desired dynamics are defined by $K_{pi} = 1000 \text{Nm/} \text{rad}$ and $K_{di} = 24.25 \text{Nms/} \text{rad}$.

Table 5.7: $H_2$ and $H_\infty$ based metrics derived from simulation for a Valkyrie hip SEA rendering $K_{pi} = 1000 \text{Nm/} \text{rad}$ and $K_{di} = 24.25 \text{Nms/} \text{rad}$.

<table>
<thead>
<tr>
<th></th>
<th>DOB Based</th>
<th>Cascaded Imp., Torque, &amp; Vel.</th>
<th>Cascaded Imp. &amp; PD Torque</th>
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<tbody>
<tr>
<td>$M_{H_2}$</td>
<td>$3.12 \times 10^{-4}$</td>
<td>$1.20 \times 10^{-3}$</td>
<td>$9.63 \times 10^{-4}$</td>
</tr>
<tr>
<td>$M_{H_\infty}$</td>
<td>$7.84 \times 10^{-5}$</td>
<td>$2.56 \times 10^{-4}$</td>
<td>$2.03 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\omega_\infty$ (Hz)</td>
<td>6.2</td>
<td>6.9</td>
<td>7.6</td>
</tr>
</tbody>
</table>
Figure 5.21: Frequency responses of the error transfer function (5.15) for the Valkyrie hip SEA leveraging (1) DOB based impedance control, (2) cascaded impedance, torque, and velocity control, and (3) cascaded impedance and PD torque control. Desired dynamics are defined by $K_{pi} = 5000 \text{ Nm/rad}$ and $K_{di} = 30 \text{ Nms/rad}$.

Table 5.8: $H_2$ and $H_\infty$ based metrics derived from simulation for a Valkyrie hip SEA rendering $K_{pi} = 5000 \text{ Nm/rad}$ and $K_{di} = 30 \text{ Nms/rad}$.

<table>
<thead>
<tr>
<th></th>
<th>DOB Based</th>
<th>Cascaded Imp., Torque, &amp; Vel.</th>
<th>Cascaded Imp. &amp; PD Torque</th>
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<tr>
<td>$M_{H_2}$</td>
<td>$7.06 \times 10^{-4}$</td>
<td>$1.10 \times 10^{-3}$</td>
<td>$8.21 \times 10^{-4}$</td>
</tr>
<tr>
<td>$M_{H_\infty}$</td>
<td>$1.61 \times 10^{-4}$</td>
<td>$1.99 \times 10^{-4}$</td>
<td>$1.64 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\omega_\infty$ (Hz)</td>
<td>18.6</td>
<td>24.6</td>
<td>23.7</td>
</tr>
</tbody>
</table>
led to the larger amplitude oscillations observed in the step response of Fig. 5.10, and accordingly suggested worse impedance rendering performance. Here, however, DOB based control is deemed slightly better by the $M_{H_{\infty}}$ and $M_{H_2}$ metrics. This discrepancy between qualitative and quantitative performance is perhaps best explained by examining both frequency and phase.

The frequency at which $M_{H_{\infty}}$ occurs for DOB based control is 18.6 Hz. This corresponds to the controller’s resonant frequency in Fig. 5.17, and it also closely matches the calculated resonant frequency of the underdamped desired dynamics (17.2 Hz). The other two controllers exhibit peak errors at higher frequencies (24.6 Hz and 23.7 Hz). This is consistent with their higher frequency resonant peaks in Fig. 5.17. Although the other two controllers maintain a close match to the desired magnitude response over a broader range, $M_{H_{\infty}}$ and $M_{H_2}$ implicitly capture DOB based control’s similarity in resonant frequency and penalize the other control approaches for dissimilar behavior.

Additionally, between 15 Hz and 30 Hz the phase response of DOB based control better approximates that of the desired dynamics (see both Fig. 5.17 and Fig. 5.18). This characteristic is not captured in bandwidth or gain measurements based solely on the magnitude response plot, but it certainly speaks, at least in part, to the fidelity of the rendered behavior. $M_{H_{\infty}}$ and $M_{H_2}$ both implicitly incorporate phase variation in their measurements, contributing again to the slight favoring of DOB based impedance control by these metrics.

Is a matched resonant frequency or phase response critical to impedance rendering performance? More so than high impedance control bandwidth? These are open, and indeed application specific, questions. Importantly, the $M_{H_{\infty}}$ and $M_{H_2}$ metrics do not ignore bandwidth or magnitude error, rather they offer a means for combining this
information with other more difficult to quantify characteristics to better measure various aspects of SEA impedance rendering performance. These metrics are not meant to replace all other comparison methods, but they do provide unique insight in the presented examples and serve to reframe the analysis of series elastic actuator impedance control in a manner conducive to the optimal control synthesis explored in Chapter 7.

5.5 Implications

Empirical step and frequency responses highlight that all three of the examined SEA impedance control approaches (DOB based control; cascaded impedance, torque, and velocity control; and cascaded impedance and PD torque control) effectively render a wide range of desired impedances. This includes desired virtual stiffnesses that approach or exceed the actuator’s physical spring rate (in contrast to the breakdown in performance experienced by passivity-based control as desired impedance increases).

Of note, disturbance observer based impedance control typically outperforms the other two approaches both qualitatively and when quantified using standard step response measurements, impedance control bandwidth, and system norm based metrics that characterize the overall match between rendered closed loop impedance and desired system dynamics. This is particularly true when rendering low impedances. A DOB’s ability to enforce nominal closed loop dynamics in the presence of model inaccuracies and system nonidealities is most apparent (and advantageous) when these discrepancies represent a larger percentage of the desired actuator torque output (as is the case in low torque and low impedance operation). This has significant implications for SEA control design in haptics or wearable robotics, for instance, where actuator transparency is often required.
For high impedance rendering it is more difficult to determine a “best” control approach, as all three offer comparable performance with competing strengths and weaknesses. This too strengthens the case for DOB based impedance control, however, presenting it as a versatile option across many potential applications. Of course, the merits of the other, more well established in the literature, control options should not be overlooked in this case.

As with the stability and passivity analyses of Chapter 4, the simulated and experimental impedance rendering results presented here serve to differentiate SEA control approaches from one another. The various methods and metrics presented for doing this further arm the control designer and allow more informed decision making during the analysis and design process. The performance data presented also serve to quantify the high performance capability of the Valkyrie hip SEA design. The electromechanical hardware, regardless of control approach, accurately renders a variety of impedances and provides a wealth of sensor feedback to assess actuator performance. Thus, empirical evidence supports the claim in Chapter 2 that this newly designed suite of rotary SEAs represents a significant advancement in current series elastic actuator technology.
Chapter 6

Disturbance Observer Based Impedance Control

While disturbance observers have been used to improve the torque tracking of series elastic actuators (see [91] and [102]), little has been done to explicitly examine the use of this control approach within an impedance control framework. Neither its closed loop passivity nor its impedance rendering accuracy has been analyzed prior to the preceding chapters, and the direct comparison to other control approaches more common in the literature provides new insight. Given the relative novelty of disturbance observer based SEA impedance control, various features of this architecture will be briefly discussed and some of the practical benefits of DOB based control will be examined in greater detail to place the positive results established thus far in context.

6.1 Factors Affecting Performance

Looking back at the architecture of Fig. 3.7, there are three primary components of disturbance observer based impedance control: the PD torque compensator itself, the inverse nominal plant model $P_n^{-1}(s)$, and the low pass DOB filter $Q(s)$. The design of each impacts overall impedance rendering performance, and investigating their individual effects aids in understanding the sensitivity to design parameters and robustness of DOB based impedance control.
6.1.1 $Q$ Filter Cutoff Frequency

The low-pass Butterworth filter $Q(s)$ within the DOB based impedance control framework is defined solely by its cutoff frequency $\omega_q$ as in (3.18). This value defines the frequency range over which the disturbance observer actively addresses variation from the nominal closed loop behavior. In the limit, as $\omega_q \to 0$, DOB based impedance control behaves just as cascaded impedance and PD torque control would, whereas increasing $\omega_q$ improves the controller’s rendering of a given desired impedance. This behavior is observed both in simulation and experimentally in Fig. 6.1.

![Figure 6.1: The effect of $Q$ filter cutoff frequency on the impedance rendering accuracy of DOB based impedance control. (a) Simulated magnitude responses of $\theta_L(s)/\tau_{ext}(s)$ at four different cutoff frequencies, compared to the desired dynamics defined by $K_{pi} = 1000 \text{Nm/rad}$ and $K_{di} = 24.25 \text{Nms/rad}$. (b) Experimental magnitude responses of the same.](image-url)
Two things are important to note in these plots. First, the variation in performance across filter cutoff frequencies is not drastic, as even the worst case match at \( \omega_q = 31.4 \text{ rad/s} \) (5 Hz) only departs the desired response by as much as 5 dB. In spite of this, the prediction in simulation of degraded impedance rendering at lower cutoff frequencies (Fig. 6.1a) is clearly supported by the experimental results (Fig. 6.1b). Second, the magnitude responses here are only plotted from 2 Hz to 10 Hz. For lower frequency external torque inputs all four cases respond similarly because the DOB remains active at low frequencies across tests. At higher frequencies, as mentioned in Chapter 5, the SEA’s physical output inertia dominates the closed loop response, thus, there is little difference between tests here as well. This point also explains the similarity between the 30 Hz and 50 Hz test cases. At high frequencies where these values might produce varying responses, the effects are masked by the dominant physical output inertia, and over the frequency range plotted in Fig. 6.1 both \( \omega_q = 188.5 \text{ rad/s} \) (30 Hz) and \( \omega_q = 314.2 \text{ rad/s} \) (50 Hz) produce equally responsive disturbance observers. This observation suggests a point of diminishing returns when selecting \( \omega_q \). If a cutoff frequency is selected above the level of discernible performance differences (in this case 30 Hz), the DOB will not yield further performance benefits. It will, however, be more responsive to noise and other high frequency inputs, increasing power draw and decreasing overall actuator efficiency.

6.1.2 Model Inaccuracies

Disturbance observer based impedance control is the only model based architecture discussed thus far, relying on knowledge of the SEA open loop system parameters to construct \( P_n^{-1}(s) \). As such, the accuracy of the identified SEA open loop model must be considered. If poor system identification leads to poor impedance render-
Figure 6.2: The effect of SEA model parameter variation on the impedance rendering accuracy of DOB based impedance control. The magnitude response of $\theta_L(s)/\tau_{ext}(s)$ sees little change when varying the identified motor inertia $\hat{J}_m$ and motor viscous damping coefficient $\hat{b}_m$ by $\pm 25\%$. Impedance gains here are $K_{p_i} = 1000 \text{Nm/rad}$ and $K_{d_i} = 24.25 \text{Nms/rad}$.

Figure 6.2: The effect of SEA model parameter variation on the impedance rendering accuracy of DOB based impedance control. The magnitude response of $\theta_L(s)/\tau_{ext}(s)$ sees little change when varying the identified motor inertia $\hat{J}_m$ and motor viscous damping coefficient $\hat{b}_m$ by $\pm 25\%$. Impedance gains here are $K_{p_i} = 1000 \text{Nm/rad}$ and $K_{d_i} = 24.25 \text{Nms/rad}$.

First, the nominal experimentally identified motor inertia and motor viscous damping ($\hat{J}_m = 1.32 \times 10^{-4} \text{kgm}^2$ and $\hat{b}_m = 7.64 \times 10^{-3} \text{Nms/rad}$ respectively) are used to construct the inverse nominal plant per (3.17). Then, these values are adjusted $\pm 25\%$ and various combinations are used to test potential model inaccuracies. From the experimental responses it is apparent that varying the identified SEA open loop parameters has little effect on impedance rendering performance. Thus, DOB based impedance control is robust to uncertainty and inaccuracy in the physical actuator model.
6.1.3 Torque Control Gains

Higher gain control typically leads to better performance (assuming, of course, that stability limits are not exceeded). With increased torque compensator gains comes higher torque bandwidth, and an SEA is able to more accurately track desired torque signals. This has a direct impact on the actuator’s ability to render impedances. Looking first at an example with no disturbance observer, this correlation is obvious. In Fig. 6.3 the frequency response of cascaded impedance and PD torque control is plotted for three values of inner torque proportional gain $K_{pt}$. Here, the torque derivative gain $K_{dt}$ is also changed with $K_{pt}$ to ensure a constant torque compensator damping ratio\(^*\) of $\zeta = 0.95$ per:

$$K_{dt} = \frac{2\zeta \sqrt{J_m k (NK_{pt} k_t + 1)} - N b_m}{kk_t}. \quad (6.1)$$

The three experimental responses in Fig. 6.3 clearly show that the rendered closed loop dynamics in response to external torque are a closer match to the desired behavior (in this case one defined by $K_{pi} = 100 \text{ Nm/rad}$ and $K_{di} = 7.67 \text{ Nms/rad}$) when torque gains are increased. The improvement in steady-state error (observed in the magnitude plot at low frequencies) also specifically highlights the ability of higher torque gains to counteract unmodeled effects such as friction to enforce the desired impedance response experimentally. All of this is, of course, as expected.

Interestingly, when a disturbance observer is introduced back into the control architecture the difference between these three cases becomes significantly less obvious. The frequency responses of DOB based impedance control in Fig. 6.4 exhibit little

\(^*\)Additional details describing torque compensator damping ratio and the rationale behind (6.1) have been published in the *Journal of Field Robotics* [45].
Figure 6.3: The effect of increasing torque compensator gains on the frequency response of cascaded impedance and PD torque control. The Valkyrie hip SEA’s experimental response $\frac{\theta_L(s)}{\tau_{ext}(s)}$ more closely matches the desired response (defined by $K_{pi} = 100 \text{Nm/}rad$ and $K_{di} = 7.67 \text{Nms/}rad$) as torque feedback gain $K_{pt}$ increases.

Figure 6.4: The effect of increasing torque compensator gains on the frequency response of DOB based impedance control. There is little discernible change in the Valkyrie hip SEA’s rendering of $\frac{\theta_L(s)}{\tau_{ext}(s)}$ for $K_{pi} = 100 \text{Nm/}rad$ and $K_{di} = 7.67 \text{Nms/}rad$ as torque feedback gain $K_{pt}$ increases.
discernible difference across the same values of $K_{pt}$. This speaks to a disturbance observer’s ability to effectively compensate for unmodeled dynamics, more so than a moderate increase in control gains, and it suggests again a point of diminishing returns at which a control designer leveraging DOB based impedance control would see little benefit to further increasing torque compensator gains.

6.2 The Practical Benefits of a Disturbance Observer†

While choices made in the design of a DOB based impedance controller affect performance as outlined in the previous section, the most significant factors affecting impedance rendering accuracy are unmodeled dynamics not captured in the linear system model. It is in addressing these discrepancies that the practical benefits of leveraging a disturbance observer in SEA impedance control are most apparent.

6.2.1 An Alternative to Higher Gains

As alluded to in the previous section, one of the most common, and perhaps easiest, ways to improve any closed loop performance is to increase feedback compensator gains. In practice, however, this is not always possible because gain magnitude (particularly that of derivative gains) is fundamentally limited by nonidealities like sensor noise and quantization. Fig. 6.4 suggests that a disturbance observer also improves closed loop performance though. Indeed, if it is assumed that torque compensator gains are tuned to acceptable (or perhaps maximum) practical values a priori, the introduction of a disturbance observer serves as a viable (if not preferred) alterna-

†Portions of this work are to be published in the proceedings of the IEEE/RSJ International Conference on Intelligent Robots and Systems [109]. I gratefully acknowledge the work of my collaborators in this effort.
tive to higher gains when seeking improved SEA impedance rendering. This point is explicitly illustrated by the experimental data in both Fig. 6.5 and Fig. 6.6.

In Fig. 6.5 the experimental response of the Valkyrie hip SEA to a 4.6 Nm step in external torque is plotted given a torque proportional gain of $K_{pt} = 0.5 \text{A/Nm}$, no DOB, and impedance control gains $K_{pi} = 100 \text{Nm/rad}$ and $K_{di} = 7.67 \text{Nms/rad}$ (the green line). Significant steady-state error (over 90% relative to the desired steady-state deflection) is observed. To reduce this error, the torque gain is increased to $K_{pt} = 2.0 \text{A/Nm}$ (and a corresponding increase to $K_{di}$ is made to maintain the controller’s $\zeta = 0.95$ damping ratio). While this case (the blue line) sees improvement, it is vastly outperformed by the introduction of a disturbance observer to the control architecture at

![Graph showing experimental responses of the Valkyrie hip SEA to 4.6 Nm steps in external torque, illustrating the disturbance observer as a preferred alternative to increased torque compensator gains. The DOB provides a greater improvement in rendering the desired $K_{pi} = 100 \text{Nm/rad}$ and $K_{di} = 7.67 \text{Nms/rad}$ dynamics than a factor of four increase in torque proportional gain $K_{pt}$.]

Figure 6.5: Experimental responses of the Valkyrie hip SEA to 4.6 Nm steps in external torque, illustrating the disturbance observer as a preferred alternative to increased torque compensator gains. The DOB provides a greater improvement in rendering the desired $K_{pi} = 100 \text{Nm/rad}$ and $K_{di} = 7.67 \text{Nms/rad}$ dynamics than a factor of four increase in torque proportional gain $K_{pt}$. 
Figure 6.6: Experimental frequency responses of $\theta_L(s)/\tau_{ext}(s)$ for the Valkyrie hip SEA, illustrating the disturbance observer as a preferred alternative to increased torque compensator gains. The DOB is shown again to provide a greater improvement in rendering the desired $K_{pi} = 100 \text{Nm/rad}$ and $K_{di} = 7.67 \text{Nms/rad}$ dynamics than a factor of four increase in torque proportional gain $K_{pt}$.

the original torque gain of $K_{pt} = 0.5 \text{A/Nm}$ (the red line). The DOB based impedance control approach does not require an increase in torque compensator gains, yet, in this example, it provides a greater impedance rendering improvement than is realized by increasing $K_{pt}$ by a factor of four.

This practical benefit of DOB based control is observed again, across the frequency spectrum, in the Bode plot of Fig. 6.6. A torque control gain of $K_{pt} = 0.5 \text{A/Nm}$ with a disturbance observer noticeably outperforms $K_{pt} = 2.0 \text{A/Nm}$ without a disturbance observer. For the specific case of the Valkyrie hip SEA, an even better impedance control option is leveraging both the high $K_{pt} = 2.0 \text{A/Nm}$ gain and a DOB. This, in fact, is the nominal controller examined throughout the preceding chapters. Torque compensator gains cannot be increased without bound, however, and in this, and
other applications, a disturbance observer provides an effective means of improving
impedance rendering performance in spite of the inherent practical limits to high gain
control.

6.2.2 Transparency

An important benchmark for impedance controlled devices is the ability to render low,
ear zero impedances. Device transparency is particularly desirable in human-centric
applications. Furthermore, it defines a lower bound on the effective impedance range
an actuator is able to render. Fig. 6.7 displays the results of an experiment in which

![Graph](image)

Figure 6.7: SEA transparency test. (a) The position disturbance introduced through
user interaction with a Valkyrie hip SEA rendering a null impedance ($K_{pi} = K_{di} = 0$).
(b) The resulting torque felt at the actuator output. Leveraging a DOB improves
transparency, decreasing peak torque magnitude from 4.83 Nm to 1.04 Nm.
the Valkyrie hip SEA is commanded to render a null impedance \( (K_{pi} = K_{di} = 0) \).

A human user then disturbs the SEA output via the attached lever arm shown in Fig. 5.4 while both actuator output position (Fig. 6.7a) and measured external torque (Fig. 6.7b) are recorded.

During the roughly 0.6 Hz position sinusoid input by the user, the torque magnitude felt at the output when not using a disturbance observer reaches 4.83 Nm. This value drops to 1.04 Nm with the DOB, making a significant difference in the device dynamics perceived by the user.

### 6.2.3 Actuator Hysteresis

Less commonly explored than device transparency, actuator hysteresis is another important measure of impedance control performance. Due to its reliance on an intentionally compliant element, a series elastic actuator can be particularly susceptible to hysteresis in both torque measurement and controlled response. From an impedance rendering perspective, hysteresis hurts performance by preventing a consistent, repeatable, one-to-one mapping of applied external torque to actuator output position. Qualitatively, it also degrades the haptic sensation experienced when interacting with the SEA.

Careful consideration in design must be given to the structural mounting of both the spring and its deflection sensor to ensure that off-axis loads and unsensed structural deflection do not contribute to erroneous torque measurements in the SEA. This is especially true for actuators, like those in Valkyrie, that are expected to exert large torques while also exhibiting high dynamic range and small torque sensitivity. In addition to the deflection of structural elements, potential sources of closed loop hysteresis in the Valkyrie SEAs include friction, and the nonlinearities and inherent
hysteresis of the harmonic drive gear train. These effects are all difficult to quantify and model, thus, a disturbance observer’s ability to enforce nominal plant dynamics is helpful.

Disturbance observers have been shown to reduce hysteresis in piezoelectric actuators [110], and indeed, as Fig. 6.8 and Fig. 6.9 demonstrate, they also serve this purpose in impedance controlled SEAs. Relying again on human input, external torques ranging from 30 Nm to −30 Nm are applied to a Valkyrie hip actuator that is rendering a virtual stiffness of either $K_{pl} = 100 \text{ Nm/} \text{rad}$ (Fig. 6.8) or $K_{pl} = 1000 \text{ Nm/} \text{rad}$ (Fig. 6.9). The dashed black line in each figure represents the ideal rendered stiffness, while the other curves represent SEA output torque (as measured by an external loadcell) as a function of actuator output position.

Two things are readily apparent in these plots. First, without a disturbance observer the slope of the experimental data does not correspond to the desired virtual stiffness. In both cases, the SEA renders a slightly stiffer response without a DOB. Numerous factors could cause this behavior including a mismatch between the actual physical spring rate and that assumed in the controller for torque calculation, friction in the actuator drivetrain, and misalignment of the spring deflection sensor. Second, without a disturbance observer the width of the observed hysteresis curve is indeed larger.

Leveraging DOB based impedance control addresses both of these issues. The disturbance observer serves to realign the SEA’s rendered stiffness with the desired ideal value in Fig. 6.8 and Fig. 6.9, and it also significantly reduces overall closed loop hysteresis. Around zero position, when rendering $K_{pl} = 100 \text{ Nm/} \text{rad}$, control without a DOB exhibits an 8.24 Nm torque range, whereas control with a DOB reduces this range by 69% to only 2.54 Nm. The same is true at $K_{pl} = 1000 \text{ Nm/} \text{rad}$, where an 84%
Figure 6.8: Hysteresis exhibited by the Valkyrie hip SEA when rendering a virtual stiffness of $K_{pi} = 100 \text{ Nm/rad}$. Leveraging a DOB provides a more accurate match to the ideal desired stiffness and decreases the width of the hysteresis curve by 69%.

Figure 6.9: Hysteresis exhibited by the Valkyrie hip SEA when rendering a virtual stiffness of $K_{pi} = 1000 \text{ Nm/rad}$. Leveraging a DOB provides a more accurate match to the ideal desired stiffness and decreases the width of the hysteresis curve by 84%.
reduction from 5.93 Nm to 0.944 Nm is observed. In light of a disturbance observer’s ability to enforce nominal plant dynamics, this behavior makes sense, and it provides a valuable tool for improving the accuracy with which series elastic actuators render desired impedances.

6.3 Implications

The fidelity with which SEAs render desired impedances is important. Actuators used in haptic displays must “feel” right to the user, wearable robots must appear transparent when commanded to do so, and humanoid robots such as NASA’s Valkyrie must reliably modulate impedance to perform human-like tasks and interactions. Chapter 5 examined the impedance rendering accuracy of three SEA control approaches (DOB based impedance control; cascaded impedance, torque, and velocity control; and cascaded impedance and PD torque control). It was in this study that the relative merit of a disturbance observer based architecture first became apparent, and a further look at DOB performance continues to add additional useful insight.

Although model based, disturbance observer based impedance control is robust to inaccurate system parameter identification, showing no discernible difference in impedance rendering when modeled motor inertia and/or viscous damping are varied by as much as 25%. DOB based control is also straightforward from the perspective of the control designer. Low sensitivity to torque compensator gains and the DOB filter cutoff frequency make tuning this controller easier than other approaches with a comparable number of parameters.

Using a DOB to effectively replace increased compensator gains is also an incredible practical benefit. This alone suggests a number of potential avenues for future work. Feedback control is fundamentally limited by practical bounds on gain magni-
tude. If a DOB can be applied beyond just cascaded impedance and PD torque control to augment other impedance control architectures with their own unique strengths, further improvements in SEA impedance rendering could be realized with little difficulty across a number of systems and applications.

Already, it is clear from the experimental data that disturbance observer based impedance control is capable of addressing one of the most significant obstacles to accurate SEA impedance rendering: the difficult to model nonlinear effects present in physical hardware. Both actuator transparency at low impedances and hysteresis in response to external torques are improved with a disturbance observer. This allows for a more faithful reproduction of a desired virtual impedance by the series elastic actuator and greatly enhances the practical utility of impedance controlled SEAs.
Chapter 7

Optimal Impedance Control Synthesis

7.1 Inherent Difficulties in Controller Design

If the practical utility of impedance controlled SEAs is of paramount importance, straightforward methods for constructing well-performing controllers must be developed. Indeed, ease of design is an important prerequisite in any control application, SEA impedance control included. The preceding chapters examined various control approaches, outlining their unique architectures and performance characteristics, while providing insight into SEA closed loop passivity and the relative merit of disturbance observer based impedance control. In focusing on controller differences however, one common trait, perhaps obvious, went largely unstressed. Each of the SEA impedance control approaches considered thus far rely on some form of cascaded or nested loop structure, and this design choice brings with it both benefits and drawbacks.

Consider cascaded impedance, torque, and velocity control for example. The three nested feedback compensators can each be designed individually, starting with the innermost velocity loop and working outward. The control design problem then simplifies to a series of PI control tuning tasks with (largely) known plants that can be performed either in simulation or on the actual hardware. Compartmentalizing the control of actuator impedance, torque, and velocity also provides the control designer a straightforward means within the architecture to implement limits on these
individual outputs. This approach has been leveraged in the design of Robonaut 2’s human-rated control system on board the International Space Station [111], and it represents a valuable benefit whenever system safety must be considered.

While a disturbance observer was shown in Section 6.1.3 to reduce the need for an optimally tuned torque compensator, the design process is largely unchanged from the previous example; torque PD gains are first tuned to provide a desired response and then impedance control is implemented assuming an ideal torque response from the inner loop. Wrapping the torque compensator with a DOB serves to mitigate the effects of physical nonidealities in the actuator, making the ideal torque response assumption more reasonable.

The PD structure of the outer impedance loop in these architectures also eases the control design burden. As mentioned in Chapter 1, impedance proportional and derivative gains have straightforward physical interpretations that intuitively map to the virtual stiffness and damping of the closed loop system. Herein lies one inherent difficulty related to controller design, however. How should impedance control be structured to render virtual dynamics beyond second order? Furthermore, augmenting these control approaches to simply render a virtual inertia different than the SEA’s physical output inertia, while certainly possible, also requires a fundamental change in architecture. These are not straightforward design tasks.

Perhaps the most significant difficulty associated with SEA control design appears when considering overall impedance rendering performance. As alluded to in Section 5.2, a controller with numerous gains is difficult to tune in practice because it is often unclear which gain, or group of gains, causes a given problem. It is also hard to determine if further tuning will yield positive benefit at all. While Chapter 6 provides guidance regarding the various factors affecting DOB based impedance control,
it is still an open question as to what constitutes the “best” impedance rendering performance possible and how the many parameters in each of the examined control approaches should be tuned to achieve such.

Here, the $H_\infty$ and $H_2$ based performance metrics developed in Section 5.4 present an opportunity. Provided these measures of impedance rendering accuracy, it is possible to frame SEA impedance control as an optimal control problem. If, for example, the $H_\infty$ match from (5.11) is minimized for the closed loop SEA, optimal impedance rendering (as measured by the maximum theoretical discrepancy between desired and actual system dynamics) is guaranteed. Furthermore, it is possible via optimal control synthesis to derive a controller that achieves this performance while abstracting gain selection from the design process. The balance of this chapter focuses on the construction of this novel framework for SEA impedance control synthesis, while offering experimental results to support its efficacy and outlining potential extensions deserving of continued study.

7.2 The General Control Problem

Much of the work in robust and optimal control theory (as presented by Skogestad and Postlethwaite [106], Mackenroth [112], and Zhou et al. [113], [114], among others) considers the general control configuration presented in Fig. 7.1. This simple formulation of plant $P$ and controller $K$ can be used to represent a broad range of closed loop systems with varied plant and controller architectures. As will be seen, it is also useful in the construction of the optimal SEA impedance control problem to come. Exogenous inputs to the system (disturbances, reference signals, etc.) are represented by $w$, while the outputs of interest are expressed as $y$. The controller accepts measurements $z$ from the plant, and in turn produces command signals $u$. A
Figure 7.1: The general control configuration. A closed loop system $P_{CL}$ consists of a plant $P$ and controller $K$ interconnected as shown. Exogenous inputs to the system are represented by $w$, while system outputs are $y$, measurements from the plant are $z$, and command signals generated by the controller are $u$.

A linear time invariant plant is therefore described in state space by:

$$
\begin{align*}
P & \left\{ 
\dot{x}_p &= A_p x_p + B_p u + D_p w \\
y &= C_y x_p + B_y u + D_y w \\
z &= C_z x_p + D_z w 
\right. 
\end{align*}
$$

(7.1)

with $x_p$ representing plant states. In the most general case, controller dynamics are also considered and $K$ is described by:

$$
\begin{align*}
K & \left\{ 
\dot{x}_K &= A_K x_K + B_K z \\
u &= C_K x_K + D_K z 
\right. 
\end{align*}
$$

(7.2)

with $x_K$ representing the internal states of the controller. Combining the plant (7.1)
and controller (7.2) forms the closed loop system:

\[
P_{\text{CL}} \begin{cases} \dot{x}_{\text{CL}} = A_{\text{CL}}x_{\text{CL}} + B_{\text{CL}}w \\ y = C_{\text{CL}}x_{\text{CL}} + D_{\text{CL}}w \end{cases} \tag{7.3}
\]

where

\[
x_{\text{CL}} = \begin{bmatrix} x_p \\ x_K \end{bmatrix} \tag{7.4}
\]

and

\[
A_{\text{CL}} = \begin{bmatrix} A_p + B_pD_kC_z & B_pC_k \\ B_kC_z & A_K \end{bmatrix}, \quad B_{\text{CL}} = \begin{bmatrix} D_p + B_pD_kD_z \\ B_kD_z \end{bmatrix}, \tag{7.5}
\]

\[
C_{\text{CL}} = \begin{bmatrix} C_y + B_yD_kC_z \\ B_yC_K \end{bmatrix}, \quad D_{\text{CL}} = \begin{bmatrix} D_y + B_yD_kD_z \end{bmatrix}.
\]

In the case of static state feedback, which is of particular interest, all plant states are measured \((z = x_p)\) and there are no controller dynamics \((A_K = B_K = C_K = 0)\). Thus, the controller of (7.2) simplifies to:

\[
u = D_kx_p \tag{7.6}
\]

and the closed loop matrices of (7.3) become:

\[
A_{\text{CL}} = \begin{bmatrix} A_p + B_pD_kC_z \end{bmatrix}, \quad B_{\text{CL}} = D_p, \tag{7.7}
\]

\[
C_{\text{CL}} = \begin{bmatrix} C_y + B_yD_kC_z \end{bmatrix}, \quad D_{\text{CL}} = D_y.
\]
7.3 Linear Matrix Inequalities

Numerous tools exist to analyze closed loop performance and synthesize controllers within the generic problem formulation of the previous section. Perhaps none are as powerful as linear matrix inequalities (LMIs). Although only first termed as such by Willems [115] in 1971, linear matrix inequalities find their origin in Lyapunov theory, where it is proven that a system of differential equations:

\[ \dot{x}(t) = Ax(t) \]  

(7.8)

is stable if and only if there exists a positive definite matrix \( P \) such that:

\[ PA + A^\top P < 0. \]  

(7.9)

These two constraints (\( P > 0 \) and \( PA + A^\top P < 0 \)) represent a set of linear matrix inequalities which, in Lyapunov’s case, can be solved analytically.

Progressing from this original result, as Boyd et al. [116] outline, LMIs offer a means to solve numerous control problems via computationally efficient convex optimization. This has given rise to software packages such as CVX [117] and the MATLAB Robust Control Toolbox [118] that are designed specifically to solve such optimization problems. Two specific LMI results related to the \( H_\infty \) and \( H_2 \) system norms have unique relevance when addressing the problem of optimal SEA impedance control synthesis.

7.3.1 The \( H_\infty \) Optimal Control Problem

The Bounded Real Lemma (as presented by Skelton et al. [107] for example) states that a system defined as in (7.3) is stable, with the \( H_\infty \) norm of its closed loop transfer matrix \( P_{\text{cl}}(s) \) bounded such that \( \| P_{\text{cl}}(s) \|_\infty < \gamma \), if and only if there exists a positive...
definite matrix $Y$ satisfying:

$$\begin{bmatrix}
YA_{cl} + A_{cl}^TY & YB_{cl} & C_{cl}^T \\
B_{cl}^TY & -\gamma^2I & D_{cl}^T \\
C_{cl} & D_{cl} & -I
\end{bmatrix} < 0$$

(7.10)

where $I$ represents identity matrices of appropriate dimension.

If the closed loop system is known, (7.10) is an LMI and $\| P_{cl}(s) \|_\infty$ can be determined via convex optimization by minimizing $\gamma^2$ subject to (7.10) and the constraint $Y > 0$. This is $H_\infty$ analysis. The $H_\infty$ synthesis problem is slightly more complex, however, as the presence of unknown controller variables within the closed loop system matrices (for which it is desired to solve) makes (7.10) nonlinear with respect to the unknowns.

Drawing from the proofs presented by Skelton et al. [107], $H_\infty$ controller synthesis is performed in two steps (for the example that follows only static state feedback is considered). First, for a controller $u = D_K x_p$ to exist that satisfies (7.10) it is necessary and sufficient that there exists a positive definite matrix $X$ such that:

$$\begin{bmatrix}
B_p \\
B_y
\end{bmatrix}^\top \begin{bmatrix}
A_p X + X A_p^T + D_p D_p^T & X C_y^T + D_p D_y^T \\
C_y X + D_y D_p^T & D_y D_y^T - \gamma^2 I
\end{bmatrix} \begin{bmatrix}
B_p \\
B_y
\end{bmatrix} < 0$$

(7.11)

Here $[\cdot]^\top$ as is standard, represents the orthogonal complement of the matrix. The condition (7.11) is an LMI with respect to $X$ and $\gamma^2$, relying only on these variables and known plant matrices from (7.1). Thus, both $X$ and a minimum $\gamma$ can be determined via convex optimization.
Once determined, $X$ and $\gamma$ are then used to solve for $D_K$ via a second linear matrix inequality:

$$\Gamma D_K \Lambda + \Lambda^T D_K^T \Phi^T \Phi < 0$$  \hspace{1cm} (7.12)

where

$$\Gamma = \begin{bmatrix} YB_p \\ 0 \\ B_y \end{bmatrix}, \quad \Lambda = \begin{bmatrix} C_z & D_z & 0 \end{bmatrix}, \quad \Phi = \begin{bmatrix} YA_p + A_p^T Y & YD_p & C_y^T \\ D_p^T Y & -\gamma^2 & D_y^T \\ C_y & D_y & -I \end{bmatrix}$$  \hspace{1cm} (7.13)

and

$$Y = \left( \frac{1}{\gamma^2} X \right)^{-1}.$$  \hspace{1cm} (7.14)

This process of minimizing $\gamma$ subject to (7.11) then solving for control gains in (7.12) yields $H_\infty$ optimal control (in the sense that the resulting controller $D_K$ produces a stable closed loop system with the minimum realizable energy to energy gain, or $H_\infty$ norm, from exogenous inputs $w$ to system outputs $y$).

### 7.3.2 The $H_2$ Optimal Control Problem

$H_2$ optimal control, which is a generalization of the linear quadratic regulator (LQR) and linear quadratic gaussian (LQG) control problems [112], can similarly be constructed as a convex optimization problem using LMIs. Given the system defined in (7.3), the closed loop transfer matrix $P_{cl}(s)$ is again stable, with $\|P_{cl}(s)\|_2 < \gamma$, if and only if there exists a positive definite matrix $X$ satisfying the following two constraints [107]:

$$C_{cl}XC_{cl}^T < \gamma^2 I$$  \hspace{1cm} (7.15)
\[ A_{\text{cl}}X + X A_{\text{cl}}^T + B_{\text{cl}} B_{\text{cl}}^T < 0. \] (7.16)

As in the \( H_\infty \) case, (7.15) and (7.16) represent inequality constraints with respect to two unknowns: the solution matrix \( X \) and the unknown controller gains implicit in the closed loop system matrices. Controller synthesis can again be performed using a comparable two step process. Necessary and sufficient solvability conditions are examined and the results are then used in a second LMI to derive \( H_2 \) optimal control gains (further detail is provided, again, by Skelton et al. [107]).

7.4 A Model Matching Framework

Reducing closed loop system gains between exogenous inputs and system outputs within the context of the general control configuration in Fig. 7.1 is conceptually a problem of disturbance rejection. The fundamental goal of impedance control, however, is to prescribe, rather than minimize, the dynamic relationship between external torque and actuator position/velocity. To simplify the development of controllers that accomplish this goal while also providing rigorous performance guarantees using the techniques of the preceding section, SEA impedance control is framed as a model matching problem. Fig. 7.2 outlines this framework for the static state feedback case (Section 7.6 will revisit this architecture for the more general dynamic output feedback case).

While the concept of model matching is not new (see for example [120] and [121]), leveraging this idea for SEA impedance control synthesis is a novel contribution to the field principally achieved by augmenting the generic plant \( P \) of the general con-

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*A preliminary presentation of this framework and its subsequent extension in Section 7.6 was previously published in the proceedings of the 22nd Mediterranean Conference on Control and Automation [119]. I gratefully acknowledge the work of my collaborators in this effort.*
Figure 7.2: Model matching framework for the optimal synthesis of static state feedback SEA impedance control. The plant $P$ of the general control configuration is augmented to include both SEA dynamics $P_{\text{SEA}}$ and desired dynamics $P_{\text{des}}$. The principal output of interest is defined by $C_{\text{error}}$ as the difference between desired and actual output positions, $y_1 = (\theta_{\text{des}} - \theta_{\text{L}})$, while the motor command, $u = i_{\text{m}}$, is also included as output $y_2$, $w = \tau_{\text{ext}}$ and an identity matrix $I$ concatenates $x_{\text{des}}$ and $x_{\text{SEA}}$ to feedback $z = x_p = [\theta_{\text{des}} \dot{\theta}_{\text{des}} \theta_{\text{m}} \dot{\theta}_{\text{m}} \theta_{\text{L}} \dot{\theta}_{\text{L}}]^{\top}$ to the controller $K$.

trol configuration to include both a model of the physical actuator and the desired closed loop dynamics. Within the model matching framework of Fig. 7.2 the physical actuator $P_{\text{SEA}}$ is described by the state-space equations:

$$
P_{\text{SEA}} \begin{cases} 
\dot{x}_{\text{SEA}} = A_{\text{SEA}} x_{\text{SEA}} + B_{\text{SEA}} u + D_{\text{SEA}} w \\
y_{\text{SEA}} = x_{\text{SEA}} 
\end{cases} \tag{7.17}
$$

where

$$
x_{\text{SEA}} = \begin{bmatrix} \theta_{\text{m}} & \theta_{\text{L}} & \dot{\theta}_{\text{m}} & \dot{\theta}_{\text{L}} \end{bmatrix}^{\top}, \quad u = i_{\text{m}}, \quad w = \tau_{\text{ext}}, \quad \tag{7.18}
$$
and

\[
A_{\text{SEA}} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-k/N^2 J_m & k/N J_m & -b_{m}/J_m & 0 \\
k/N J_L & -k/J_L & 0 & 0
\end{bmatrix},
\] (7.19)

\[
B_{\text{SEA}} = \begin{bmatrix}
0 & 0 & k_l/J_m & 0
\end{bmatrix}^\top, \quad D_{\text{SEA}} = \begin{bmatrix}
0 & 0 & 1/J_m
\end{bmatrix}^\top.
\]

This representation is derived from the original SEA model (2.4) – (2.5) in Chapter 2 (assuming \( b_L = 0 \)), and is analogous to the open loop transfer functions (2.9) and (2.10) previously considered.

The desired second order dynamics \( P_{\text{des}} \) are likewise expressed as:

\[
P_{\text{des}} \begin{cases}
\dot{x}_{\text{des}} = A_{\text{des}} x_{\text{des}} + B_{\text{des}} u + D_{\text{des}} w \\
y_{\text{des}} = x_{\text{des}}
\end{cases}
\] (7.20)

where

\[
x_{\text{des}} = \begin{bmatrix}
\theta_{\text{des}} \\
\dot{\theta}_{\text{des}}
\end{bmatrix}, \quad u = i_m, \quad w = \tau_{\text{ext}},
\] (7.21)

and

\[
A_{\text{des}} = \begin{bmatrix}
0 & 1 \\
-k_{\text{des}}/J_{\text{des}} & -b_{\text{des}}/J_{\text{des}}
\end{bmatrix}, \quad B_{\text{des}} = 0, \quad D_{\text{des}} = \begin{bmatrix}
0 \\
1/J_{\text{des}}
\end{bmatrix}.
\] (7.22)
Parameters $k_{\text{des}}$, $b_{\text{des}}$, and $J_{\text{des}}$ are the desired virtual spring rate, damping coefficient, and output inertia of the closed loop actuator respectively. The state vector $x_{\text{des}}$ therefore represents the actuator output position and velocity that is desired in response to external torque input.

To be optimized (i.e. minimized) in Fig. 7.2 is the closed loop system gain between exogenous input $w$ and the outputs of interest $y$. In this case, the principal output of interest $y_1$ is defined by:

$$C_{\text{error}} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

such that

$$y_1 = C_{\text{error}} x_p.$$  \hfill (7.24)

Here, $x_p$ is simply a concatenation of the desired and actual plant states:

$$x_p = \begin{bmatrix} x_{\text{des}} \\ x_{\text{SEA}} \end{bmatrix} = \begin{bmatrix} \theta_{\text{des}} & \dot{\theta}_{\text{des}} & \theta_m & \dot{\theta}_m & \theta_L & \dot{\theta}_L \end{bmatrix}^\top$$ \hfill (7.25)

and (7.24) yields the difference between the desired actuator output position and the SEA’s actual response:

$$y_1 = (\theta_{\text{des}} - \theta_L).$$ \hfill (7.26)

Selecting a controller $K$ that minimizes (7.26) in the presence of external torque inputs will in effect drive the SEA’s closed loop response as close as possible to that of the desired dynamics. This structure formalizes the mapping here between $w$ and $y_1$ as equivalent to the composite error transfer function (5.15) introduced in the performance metric discussion of Chapter 5. Herein lies the utility of this model.
matching framework. The full plant of Fig. 7.2 is a combination of $P_{\text{SEA}}$, $P_{\text{des}}$, and $C_{\text{error}}$ such that when represented as in (7.1):

$$\dot{x}_p = A_p x_p + B_p u + D_p w$$
$$y = C_y x_p + B_y u + D_y w$$
$$z = C_z x_p + D_z w$$

(7.27)

where

$$A_p = \begin{bmatrix} A_{\text{des}} & 0 \\ 0 & A_{\text{SEA}} \end{bmatrix}, \quad B_p = \begin{bmatrix} B_{\text{des}} \\ B_{\text{SEA}} \end{bmatrix}, \quad D_p = \begin{bmatrix} D_{\text{des}} \\ D_{\text{SEA}} \end{bmatrix},$$

(7.28)

$$C_y = \begin{bmatrix} C_{\text{error}} \\ 0 \end{bmatrix}, \quad B_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_y = 0, \quad C_z = I, \quad D_z = 0.$$

Using either $H_\infty$ or $H_2$ synthesis as described in the previous section, a controller $K$ can be generated for the plant (7.27) that when applied alone to $P_{\text{SEA}}$ will render closed loop dynamics optimally matched to $P_{\text{des}}$ (as measured by the corresponding $M_{H_\infty}$ or $M_{H_2}$ metric of Chapter 5).

At this point it should be noted that an additional output, $y_2 = i_m$, is included in the framework of Fig. 7.2 such that:

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

(7.29)

In practice, command authority on the physical hardware is limited, either by the power stage of the embedded motor controller or by the current capacity of the mo-
tor itself. Feedback gains too have practical limits, as high velocity gains coupled with inherently noisy sensor signals can lead to undesirable behavior in even theoretically well-performing systems. Respecting these limits is possible by including the command signal as a second system output and treating the controller synthesis problem as a multi-objective optimization: seeking to optimize the $H_\infty$ or $H_2$ match as described, while also placing a hard cap on the system gain from external torque input to command signal (measured again by either the $H_\infty$ or $H_2$ norm). While only of tangential importance to the preceding discussion of model matching, this feature is of use experimentally (as is seen in the next section), hence its inclusion in the presented framework.

### 7.5 Static State Feedback Synthesis

For an initial experimental look at model matching based synthesis of SEA impedance control, desired closed loop parameters are chosen as in Table 7.1. The virtual stiffness and damping selected, along with an output inertia equal to that of the physical actuator, represent the same dynamics sought in many of the experiments throughout Chapters 5 and 6. Coupled with the parameters of the Valkyrie hip SEA (provided again in Table 7.1 for reference) these values fully define the open loop model matching plant (7.27).

With the plant defined, static state feedback gains can then be generated via multi-objective optimization. The msfsyn function in the MATLAB Robust Control Toolbox [118] is used to generate a solution. Here, a choice is made. Either $M_{H_\infty}$ or $M_{H_2}$ can be used to optimize impedance rendering performance, and a limit on command effort can be enforced using either the $H_\infty$ or $H_2$ norm. Optimizing $M_{H_\infty}$ as the principal performance objective while limiting the $H_2$ norm between external
Table 7.1: Model matching parameters for SEA impedance control synthesis. These desired dynamics correspond to those sought in many of the experiments throughout Chapters 5 and 6. Valkyrie hip SEA parameters are the same as well, repeated here for reference.

<table>
<thead>
<tr>
<th>Desired Dynamics</th>
<th>Valkyrie hip SEA</th>
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<tbody>
<tr>
<td>$k_{\text{des}}$</td>
<td>$J_L$</td>
</tr>
<tr>
<td>$b_{\text{des}}$</td>
<td>$J_m$</td>
</tr>
<tr>
<td>$J_{\text{des}}$</td>
<td>$b_m$</td>
</tr>
<tr>
<td>$k$</td>
<td>$k_t$</td>
</tr>
<tr>
<td>$N$</td>
<td></td>
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</tbody>
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<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{\text{des}}$</td>
<td>100</td>
<td>N·m/rad</td>
</tr>
<tr>
<td>$b_{\text{des}}$</td>
<td>7.67</td>
<td>N·m·s/rad</td>
</tr>
<tr>
<td>$J_{\text{des}}$</td>
<td>0.3</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>$J_L$</td>
<td>0.30</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>$J_m$</td>
<td>$1.32 \times 10^{-4}$</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>$b_m$</td>
<td>$7.64 \times 10^{-3}$</td>
<td>N·m·s/rad</td>
</tr>
<tr>
<td>$k$</td>
<td>3700</td>
<td>N·m/rad</td>
</tr>
<tr>
<td>$k_t$</td>
<td>0.05</td>
<td>N·m/A</td>
</tr>
<tr>
<td>$N$</td>
<td>120</td>
<td>–</td>
</tr>
</tbody>
</table>

torque and command effort to 70 (an arbitrarily tuned value) is found to produce gains of reasonable magnitude for implementation on the Valkyrie SEAs. Thus, this approach is adopted here, resulting in the static state feedback controller:

\[
D_K = \begin{bmatrix}
4174.808 & 79.971 & -74.185 & 4410.554 & -0.498 & -28.607
\end{bmatrix}.
\] (7.30)

The controller (7.30) requires position and velocity feedback from both the motor and the SEA output. These measurements are easily obtained with the sensors already integrated in the Valkyrie hip actuator. Additionally, the controller also requires realtime feedback of the desired output position and velocity in response to external torque. To calculate these values, a direct measurement of external torque is required, as is realtime integration of the desired dynamic model.
In the experimental setup of Fig. 5.4, an external torque measurement can be obtained from the in-line loadcell, although it must be admitted that the requirement for an additional external sensor is a drawback to this control approach. When actuator output inertia is low, however, the SEA’s torque measurement via spring deflection provides a reasonable approximation of applied external torque and could potentially be used instead, circumventing the additional sensing requirement. Further study is needed though, to fully appreciate the potential for, and ramifications of, such a switch in practical applications.

Addressing the need for realtime tracking of the desired dynamic response, a fourth order Runge–Kutta algorithm is implemented on the SEA’s embedded motor controller to numerically integrate the desired dynamic model and output a new desired position and velocity at each timestep (as with the rest of Valkyrie embedded control, this calculation occurs at 5 kHz). With this addition it then becomes possible to run the state feedback controller (7.30) on the Valkyrie hip SEA.

### 7.5.1 Impedance Rendering

Relying on the experimental platform of Fig. 5.4, $M_{H_{\infty}}$ optimal control is tested using the same methodology adopted in Chapters 5 and 6. Fig. 7.3 illustrates both the experimental and simulated responses of the Valkyrie hip SEA to a 4.3 Nm step in external torque when leveraging the optimal controller (7.30). The simulated step confirms the optimal nature of the impedance match as it is nearly identical to the desired response (apart from a 1% steady-state error directly attributable to the limit on command effort imposed during optimization). Experimentally the desired impedance is not as accurately rendered as in simulation, but the step response still closely approximates the desired dynamics. Looking back at the data in Fig. 5.6, this
Figure 7.3: Experimental (exp) and simulated (sim) responses of the Valkyrie hip SEA to a 4.3 Nm step in external torque when leveraging the $M_{H\infty}$ optimized controller (7.30) generated via the model matching framework of Section 7.4.

Experimental impedance rendering performance is comparable to that of disturbance observer based impedance control while clearly outperforming the other two cascaded control approaches.

Model matching based synthesis is seen to produce accurate rendering of the desired dynamics when measured in the frequency domain as well. In Fig. 7.4 the experimental response of the Valkyrie hip SEA to a 5 Nm external torque chirp (solid red line) closely tracks that predicted by the closed loop simulation (dashed blue line). This correlation is evident even when the simulated response dips below the desired by more than $-3$ dB between 15.5 Hz and 20.7 Hz. Experimental reproduction of this behavior serves to validate the 15.5 Hz impedance control bandwidth measured in simulation, which incidentally represents a significant improvement over the cascaded control approaches examined in Chapter 5 (cf. Fig. 5.13).
Figure 7.4: Experimental (exp) and simulated (sim) frequency responses of $\theta_L(s)/\tau_{\text{ext}}(s)$ for the Valkyrie hip SEA when leveraging the $M_{H_{\infty}}$ optimized controller (7.30) generated via the model matching framework of Section 7.4.

Closed loop control with (7.30) also yields a simulated $M_{H_{\infty}} = 1.45 \times 10^{-4}$ and $M_{H_2} = 8.17 \times 10^{-4}$. Thus, when measured by these new performance metrics, $M_{H_{\infty}}$ optimal control appears superior to all of the impedance control approaches considered earlier (as outlined in Table 5.6). This is somewhat expected given the specific optimization of $M_{H_{\infty}}$ in the synthesis process, but nonetheless serves to validate the efficacy of the model matching based design approach.

The stiffness and hysteresis data for the Valkyrie hip SEA when leveraging $M_{H_{\infty}}$ optimal control also illustrate accurate impedance rendering (Fig. 7.5). Performance here is an improvement over the cascaded impedance and PD torque control results reported earlier in Fig. 6.8, though DOB based control still offers more accurate impedance rendering than that now observed in Fig. 7.5. This latter point is interesting. While optimal in theory and in simulation, the practical application of
Figure 7.5: Rendered stiffness and hysteresis exhibited by the Valkyrie hip SEA when leveraging the $M_{H^\infty}$ optimized controller (7.30) generated via the model matching framework of Section 7.4.

$M_{H^\infty}$ optimal control still exhibits a slight error in overall rendered stiffness as compared to the desired value. This suggests that a controller such as (7.30), although well-performing in its own right, would benefit from additional active disturbance attenuation to address parameter errors and unmodeled dynamics as in disturbance observer based impedance control.

### 7.5.2 Augmented Virtual Inertia

A principle benefit of the model matching based approach to impedance control synthesis is that desired impedance definitions are no longer bound by the structure of an outer PD impedance compensator. This makes augmenting virtual inertia in addition to stiffness and damping particularly straightforward. No control architecture changes are necessary. The parameters associated with the desired plant dynam-
Table 7.2: Desired dynamic parameters used to augment the virtual inertia exhibited by the closed loop Valkyrie hip SEA ($J_{\text{des}} = 0.8 \text{kgm}^2$ rather than the SEA’s $0.3 \text{kgm}^2$ physical inertia). These parameters are used to produce the controller (7.31) and the resulting performance data of Fig. 7.6.

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<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
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<tbody>
<tr>
<td>$k_{\text{des}}$</td>
<td>100</td>
<td>N·m/ rad</td>
</tr>
<tr>
<td>$b_{\text{des}}$</td>
<td>7.67</td>
<td>N·m·s/ rad</td>
</tr>
<tr>
<td>$J_{\text{des}}$</td>
<td>0.8</td>
<td>kg·m$^2$</td>
</tr>
</tbody>
</table>

The frequency response of the Valkyrie hip SEA when leveraging controller (7.31) is given in Fig. 7.6. Plotted alongside the experimental data (red solid line) are the desired second order response defined by the parameters of Table 7.2 (black solid line), the simulated closed loop response of the SEA (blue dashed line), and a delineation of the SEA’s physical $0.3 \text{kgm}^2$ output inertia (gray dashed line). The closed loop actuator does indeed accurately render the new augmented output inertia, as evidenced by the closely matched experimental, simulated, and desired curves between 1 Hz and
Figure 7.6: Experimental (exp) and simulated (sim) frequency responses of $\theta_L(s)/\tau_{\text{ext}}(s)$ when augmenting the output inertia (along with stiffness and damping) of the Valkyrie hip SEA using the $M_{H\infty}$ optimized controller (7.31). Note the accurate rendering of the virtual dynamics defined in Table 7.2 rather than the SEA’s physical output inertia until a roughly 13.5 Hz impedance control bandwidth is reached.

10 Hz. Interestingly, at roughly 13.5 Hz the simulated response departs from within 3 dB of the desired dynamics and the closed loop SEA transitions from rendering the desired virtual inertia to rendering the actuator’s physical output inertia. This shift is also perceptible, although not as clear, in the experimental data. Such a transition provides a clear indication of impedance control bandwidth as there is a definitive frequency (in this case 13.5 Hz) at which the closed loop SEA ceases to track the desired response and begins reacting according to the open loop dynamics of the actuator.

7.6 Extensions to the Synthesis Framework

The presented model matching framework is proven to produce optimized state feedback controllers that are of practical utility when applied to high performance series
elastic actuators. This work represents the first example of leveraging LMI-based optimal control synthesis in SEA control design. The study of this problem, though, is by no means complete. There are a number of avenues worthy of further consideration, and two such extensions are introduced here. These concepts, along with preliminary work regarding their implementation, offer a path forward for the future study of $M_{H_\infty}$ and $M_{H_2}$ optimal SEA impedance control.

7.6.1 Passivity

Looking back at the data presented in Fig. 7.4 and Fig. 7.6, the closed loop phase response of $\theta_{L}(s)/\tau_{\text{ext}}(s)$ in both cases (at least in simulation) appears bounded between $0^\circ$ and $-180^\circ$. This, like a $\pm 90^\circ$ bound on the phase of the driving point admittance ($Y(s) = \dot{\theta}_{L}(s)/\tau_{\text{ext}}(s)$), is indicative of a passive system. An LMI condition can be used to confirm this observation.

The Kalman–Yakubovich–Popov Lemma (or Positive Real Lemma) states that a closed loop system defined as in (7.3) is passive (i.e. the transfer matrix $P_{\text{CL}}(s)$ is positive real) if and only if there exists a positive definite matrix $P$ satisfying the following LMI [116]:

\[
\begin{bmatrix}
PA_{\text{CL}} + A_{\text{CL}}^TP & PB_{\text{CL}} - C_{\text{CL}}^T \\
B_{\text{CL}}^TP - C_{\text{CL}} & -D_{\text{CL}}^T - D_{\text{CL}}
\end{bmatrix} \leq 0.
\tag{7.32}
\]

The closed loop plants defined by Fig. 7.2 and the controllers (7.30) and (7.31) are modified slightly to treat actuator output velocity as the sole system output (thus $P_{\text{CL}}(s)$ represents closed loop driving point admittance) and the feasibility of (7.32) is checked using CVX [117]. In both cases a positive definite matrix $P$ is found that
satisfies (7.32) to within the numerical precision of the solver. Thus, the closed loop
driving point admittances are both positive real, and the $M_{H_{\infty}}$ optimal control of the
previous section is confirmed passive.

Nothing inherent in the synthesis process ensures a passive result, however. Rather
than confirming this a posteriori, it would be preferable to explicitly include passivity
within the model matching framework. Because passivity can be represented as an
LMI constraint, this potential exists. An interesting and particularly useful follow-up
to the work presented here would be to create such a multi-objective optimization that
generates guaranteed passive, $M_{H_{\infty}}$ optimal SEA impedance control without added
complexity for the control designer. Scherer et al. [122] provide a starting point for
this by cataloging a number of synthesis LMIs, including one applicable to passivity,
for use in multi-objective optimization. They do this within the context of dynamic
output feedback synthesis, a further generalization of the presented model matching
framework to be discussed next.

7.6.2 Dynamic Output Feedback Synthesis

In the most general case, SEA impedance control need not be static, nor is it required
to rely on full state feedback. Fig. 7.7 outlines modifications to the model matching
framework to extend the synthesis problem to dynamic output feedback controllers.
Here, the physical plant $P_{SEA}$, the desired dynamics $P_{diss}$, and the output vector $C_{error}$
are all defined as before in (7.17), (7.20), and (7.23). New in this architecture is
a second output vector $C_{sense}$ that selects feedback measurements from among the
physical plant states only. The feedback signal $z$ can therefore include full SEA state
(i.e. position and velocity at both the motor and the actuator output), or any subset
of these measurements. Important though, is that $z$ no longer includes the desired
Figure 7.7: Model matching framework for the optimal synthesis of dynamic output feedback SEA impedance control. As in the static state feedback case, the plant $P$ of the general control configuration is augmented to include both SEA dynamics $P_{SEA}$ and desired dynamics $P_{des}$. The principal output of interest is still defined by $C_{error}$ as $e = (\theta_{des} - \theta_L)$, although measurements provided to the controller $K$ are now defined by a second output vector $C_{sense}$. Weighting functions are also introduced on the external torque input ($W_{ext}$), the error output ($W_{error}$), and the command effort output ($W_u$) to further define the optimization problem.

Thus, an external torque sensor is no longer required, nor is the realtime numerical integration of the desired dynamic model. This is of practical utility due to the fewer requirements placed on the physical SEA’s sensor suite.

As stated, the framework of Fig. 7.7 is used to generate dynamic controllers of the form (7.2). This is required because $H_\infty$ synthesis in the static output feedback case presents a nonlinear solvability condition and is thus a nonconvex problem (see [107]). For the synthesis of a dynamic controller (specifically, one of equivalent order to the full plant $P$) the problem is again convex and can be solved via LMI-based optimiza-
tion. Although particularly intriguing because accurate impedance control can be achieved without measuring external torque or the full state of the physical actuator, the additional burden placed on implementation due to realtime numerical integration of the controller (a system typically no less than sixth order) is not negligible.

Adding further to the size of the resulting controller, but also aiding the design process, are the weighting functions incorporated in Fig. 7.7. Introducing a weight on external torque ($W_{\text{ext}}$), the model matching error $e = \theta_{\text{des}} - \theta_L (W_{\text{error}})$, and the command effort ($W_u$) allows normalization of these inputs and outputs while also providing a means for the designer to define the scope of the optimization problem. If, for instance, $W_{\text{ext}}$ and $W_{\text{error}}$ are defined as low pass filters, minimizing the closed loop $H_\infty$ norm between $w$ and $y_1$ will mathematically guarantee an accurate model match only at low frequencies (as dictated by the cutoff frequencies of these filters). Constructing the synthesis problem in this way prevents undo control effort from being spent optimizing impedance rendering performance across frequencies unlikely to be encountered during the specific application at hand. Additional dynamics in these weighting functions adds to the order of the synthesized controller, however, so they should not be introduced indiscriminately.

Using the $H_\infty$ synthesis LMI offered by Scherer et al. [122, eq. (42)] various SEA impedance controllers have been generated for a number of manually tuned weights. In many cases accurate impedance rendering is achieved in simulation with only position feedback from the motor and actuator output (see the step response plot of Fig. 7.8 for example). Eliminating the feedback of velocity signals has important practical implications and this represents one reason why dynamic output feedback is of interest for SEA impedance control. Further work is needed, however, to appropriately constrain the magnitude of the resulting controller parameters before reliable
Figure 7.8: Simulated response of the Valkyrie hip SEA to a 5 Nm step in external torque when leveraging an $M_{H\infty}$ optimized, dynamic output feedback controller. Desired dynamics are defined as in Table 7.1 and measurements provided to the controller consist only of motor position $\theta_m$ and SEA output position $\theta_L$.

performance on physical hardware is possible. Another noteworthy benefit of optimal impedance control synthesis is that it does not require tuning of individual control gains. To truly streamline the design process, though, the optimal selection of weights should be addressed, as should the prevention of numerically stiff, difficult to solve controller dynamics, and the practical implementation issues associated with higher order dynamic controllers on SEA hardware (a ninth order controller, for example, generated the response of Fig. 7.8). These issues are suggested for future study.

7.7 Implications

Straightforward design of SEA impedance control hinges on one’s ability to intuitively relate control parameters to physical actuator performance. A simple proportional-
derivative structure provides this intuition with control gains analogous to the desired virtual stiffness and damping of the closed loop system. Anything more complex though, and this intuition begins to break down. Chapters 4, 5, and 6 examined a number of SEA impedance control approaches, most notably disturbance observer based impedance control, and found accurate and robust performance attainable. The synthesis approach offered here fills a gap in these studies, however, providing an alternative to the ad hoc tuning of controllers with numerous parameters.

Simply by defining desired closed loop dynamics, a designer can generate SEA impedance control via an automated process without addressing gains individually. Successful implementation on the Valkyrie hip SEA proves the efficacy of such a design approach and represents the first application of $H_{\infty}$ optimal control to series elastic actuation. Model matching based synthesis facilitates the rendering of virtual inertias (again demonstrated experimentally), and can provide a passive closed loop response. Additionally, nothing inherent in the framework would prevent the synthesis of controllers that render impedances beyond second order. This point suggests a specific utility in haptics and hardware based simulations where rendering higher order dynamics is often desired. Performing such a task with an SEA is simplified if control gains are derived within the model matching framework.

While many of the concepts and tools discussed in this chapter are well established in the robust and optimal control literature, there is specific value here in adapting and applying them to series elastic actuation. The model matching framework presented represents a novel approach to SEA impedance control design and it serves to introduce optimal synthesis and related methodologies to a field dominated by more traditional approaches. Furthermore, performance metrics are formalized within the design process. The fundamental goal of impedance control is to shape a given sys-
tem’s behavior to match that of a predefined desired dynamic model. Yet too often in practice, controllers are designed and implemented without an appreciation of their true impedance rendering accuracy. Explicitly incorporating the $M_{H_{\infty}}$ and $M_{H_2}$ metrics of Chapter 5 in optimal impedance control synthesis addresses this oversight. No longer must a designer worry about the merits of further tuning because the “best” impedance control possible (as defined by the problem formulation) is found immediately, and quantifiable performance measures produced as part of the design process serve to verify this.

Numerous avenues to explore optimal control synthesis within the context of series elastic actuation still exist and continued research promises to broaden the impact of the ideas presented here. Leveraging dynamic output feedback, for example, potentially eliminates reliance on noisy velocity measurements and the need for external torque sensing when modulating rendered inertia. These higher order controllers, however, are not easily constructed outside of the presented framework. Control synthesis via LMI-based, multi-objective, convex optimization allows for the consideration of numerous characteristics during the design process. These include impedance rendering performance, command effort magnitude, passivity, and robustness among others. Developing a means of leveraging this approach while also realizing some of the practical benefits demonstrated with DOB based impedance control is a task worthy of future study. With further refinement, the framework presented here could lead to a broad, unified approach to SEA impedance control that simplifies the design task, improves performance, and enhances the practical utility of impedance controlled SEAs across a wide range of applications.
Chapter 8

Conclusions

As robots continue to move out of the highly structured industrial workcell, instead finding use in a diverse range of human-centric applications, achieving stable and robust interaction with the physical world has become of even greater importance in both design and control. The advent of series elastic actuation speaks to this point. Introducing significant intentional compliance in an actuator’s drivetrain offers a number of benefits (as outlined in Chapter 2), but it also leads to higher order plant dynamics and the non-colocation of motor effort and measured actuator output, complicating control.

Impedance control, likewise, is of particular value in human-centric applications, and when coupled with series elastic actuation a unique combination of feedback control and passive physical characteristics is brought to bear on the challenges of dynamic interaction. The literature offers a variety of approaches to the impedance control of series elastic actuators, still more have been introduced in this work, and it was in this context that answers were sought to five key questions:

1. What range of driving point impedances can be expected when leveraging various SEA impedance control approaches, and how do these approaches compare to one another in regard to stable and passive Z-width?

2. Can a series elastic actuator render a virtual stiffness greater than its physical spring rate while preserving robust contact stability (i.e. passivity)?
3. How does the accuracy of closed loop impedance rendering vary across architectures, and does this point to a preferred control approach?

4. How best should impedance rendering performance be quantified and compared?

5. Is it possible to ease the burden of SEA impedance control design by leveraging appropriate performance metrics and techniques from optimal control synthesis?

In answering these questions throughout the preceding chapters, seven novel contributions were made to series elastic actuator design and control that are worthy of highlighting once more.

### 8.1 Contributions to SEA Design and Control

Investigating SEA impedance control rightly begins with actuator design, as the control approaches considered are only of practical use insofar as they relate to the realities of physical hardware. Chapter 2 provided the detailed design of three novel, high performance SEAs developed for use in the Robonaut 2 and Valkyrie humanoid robots. Unique design features in these actuators have been published in four United States patents [46], [47], [48], [49]; and the performance specifications outlined in Table 2.1 and Table 2.2 differentiate these designs from others in the literature. The empirical evidence provided in subsequent chapters further demonstrated actuator capability, and a selection of these tests along with a detailed description of the SEA electromechanical design is currently in preparation for submission to the *ASME Journal of Mechanisms and Robotics*.

Chapter 4 addressed the first two questions above with a detailed look at the closed loop stability and passivity of five different SEA impedance control approaches. Here, three significant contributions were made: a derivation of the necessary and sufficient
conditions to ensure stability and passivity of each considered control approach, the first stability and passivity analysis of DOB based impedance control on SEAs, and evidence that SEA impedance control can indeed be structured to render a virtual stiffness greater than the actuator’s physical spring rate while also preserving passivity. Results of this analysis are summarized in Table 4.6, highlighting the importance of torque control derivative gain if passive behavior is to be achieved when $k_{\text{rendered}} > k$.

The comparison study of SEA impedance control approaches continued in Chapter 5 where the question of impedance rendering accuracy was explicitly addressed. Both simulated and experimental data in the time and frequency domains point to DOB based impedance control as the preferred approach amongst those considered. Significant practical benefit is gained from a disturbance observer’s ability to mitigate the effects of model inaccuracies and hard to quantify nonlinear dynamics inherent in the actuator. This was further explored in Chapter 6 as well.

The full breadth of the comparison study presented here and the specific details provided on disturbance observer based impedance control are new to the literature. Drawing from the results of Chapters 4, 5, and 6, two individual manuscripts are in preparation for submission to the IEEE/ASME Transactions on Mechatronics. One work specifically focuses on comparing cascaded impedance, torque, and velocity control (fairly well established in the literature) to DOB based impedance control (a new architecture), while discussing both the presented Z-width results and the practical performance differences between these approaches. The second manuscript is a detailed introduction to and examination of DOB based impedance control, highlighting the important passivity results regarding this approach, discussing its application to the Valkyrie humanoid, and extending work that is currently pending publication in
the proceedings of the *IEEE/RSJ International Conference on Intelligent Robots and Systems* [109].

Two new performance metrics were also proposed in Chapter 5 that quantify impedance rendering accuracy and provide an effective means for comparing disparate control architectures. The $M_{H_\infty}$ and $M_{H_2}$ metrics, defined in (5.11) and (5.14) respectively, distill an array of frequency domain data (including both magnitude and phase information) down to easily interpreted accuracy measurements. Relying on these new analysis tools, a control designer can quickly determine a closed loop SEA’s maximum deviation from prescribed desired dynamics, the input frequency at which this error occurs, and the extent over which errors span the frequency spectrum. The significance of this contribution as it relates to performance analysis was demonstrated both in simulation and experimentally in Section 5.4.4.

Performance metrics based on the $H_\infty$ and $H_2$ system norms have further utility as design tools. In Chapter 7 a novel model matching framework was presented that leverages $M_{H_\infty}$ (or $M_{H_2}$ if desired) to frame SEA impedance control synthesis as an optimal control problem. The benefits of this new approach to SEA impedance control design are vast: abstracting gain selection from the design process virtually eliminates ad hoc tuning while mathematically optimal impedance rendering is ensured through explicit reliance on $M_{H_\infty}$ and/or $M_{H_2}$; the definition of desired dynamics is not limited by the specific architecture of the controller; and control synthesis via LMI-based, multi-objective, convex optimization allows rendering accuracy, command effort magnitude, passivity, and many other factors to be addressed simultaneously.

Having extended the preliminary presentation of this model matching framework published in the proceedings of the 22nd *Mediterranean Conference on Control and Automation* [119], the results presented here regarding optimal impedance control
synthesis are now in preparation for submission to the *IEEE Transactions on Robotics*. The introduction of norm based performance metrics and a framework in which to leverage them for streamlined synthesis of SEA impedance control represents a valuable contribution to the field, as does empirical evidence demonstrating passive and accurate impedance rendering as a result. This is, in fact, the first demonstration of $H_\infty$ based optimal control on a series elastic actuator.

### 8.2 Looking to the Future

Series elastic actuation has had, and is likely to continue having, a broad impact on the robotics field. Haptic devices, exoskeletons, rehabilitation robots, powered prostheses, and dexterous humanoids have all seen applications of this technology. Robust and accurate impedance control is also of vital importance considering the human-centric contexts in which many of these robots are expected to operate — hence the focus of this work.

Each SEA impedance control approach considered here, ranging from the simple impedance control common in traditional, rigid systems to that generated via $H_\infty$ based optimal control synthesis, has its own unique set of strengths and weaknesses. Understanding these differences has been one goal of this work, but future applications are likely to see even greater benefit if a concerted effort is made to blend these control approaches, combining their respective strengths while seeking to counterbalance weaknesses. Robonaut 2, for example, currently relies on joint-level cascaded impedance, torque, and velocity control while Valkyrie leverages DOB based impedance control. Would the integration of these two controllers serve to benefit both robots? It is quite possible.

Optimal SEA impedance control synthesis is perhaps most promising in this re-
gard. The framework described in Chapter 7 already allows for multiple control objectives, and natural extensions exist for guaranteeing closed loop passivity and reducing the number of feedback measurements required. Having demonstrated this design process and a resulting controller on high performance SEA hardware, further study of practical performance should be next. Developing methods to address nonlinearities and other yet unmodeled effects within this framework would be an important step toward a broad, unified approach to SEA impedance control.

At its core, this investigation of impedance control approaches for series elastic actuators is meant, first and foremost, as a practical guide for the control designer. As such, it is equally relevant to the specific applications discussed and many others yet to come. It is hoped, however, that the results presented herein also inspire new questions and encourage new research directions that, in time, advance the study of human-centric robots, series elastic actuation, and impedance control to greater and evermore fruitful ends.
Bibliography


