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Hitchin Components, Riemannian Metrics and Asymptotics

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ABSTRACT

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Higher Teichmüller spaces are deformation spaces arising from subsets of the space of representations of a surface group into a general Lie group, e.g., $PSL(n, \mathbb{R})$, which share some of the properties of classical Teichmüller space. By the non-abelian Hodge theory, such representation spaces correspond to the space of Higgs bundles. We focus on two aspects on the Higher Teichmüller space: Riemannian geometry and dynamics. First, we construct a new Riemannian metric on the deformation space for $PSL(3, \mathbb{R})$, and then prove Teichmüller space endowed with Weil-Petersson metric is totally geodesic in deformation space for $PSL(3, \mathbb{R})$ with the new metric. Secondly, in a joint work with Brian Collier, we are able to obtain asymptotic behaviors and related properties of representations for certain families of Higgs bundles of rank n.

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Contents

Chapter 1

Preliminaries

We recall some of the basic theory of Hitchin component in this chapter.

1.1 Teichmüller space

We start with talking about Teichmüller theory and define Hitchin component in next section as a natural generalization of Teichmüller space.

Given a topological closed surface S of genus $g > 1$, we consider the Teichmüller space $\mathcal{T}(S)$.

Definition 1.1.1. Teichmüller space is the space of equivalence classes of hyperbolic metrics h on the surface S, where $h \sim h'$ means that there exists a diffeomorphism f of S to itself such that $f^*h = h'$ and f is isotopic to identity map.

Equivalently, Teichmüller space is the space of isomorphism classes of Riemann surface structures on the surfaces up to isotopy.

The cotangent space of Teichmüller space at point Σ can be identified with the space of holomorphic quardratic differentials on Σ [Ahl61]. The Weil-Petersson metric is the Riemannian metric on Teichmüller space $\mathcal{T}(S)$ defined as the dual of the L^2 pairing on the cotangent space defined as follows: for any two holomorphic quardratic differentials ϕ, ψ on Σ,

$$< \phi, \psi >= \int_\Sigma \frac{\phi \wedge \overline{\psi}}{\sigma} \, dz d\bar{z},$$

where $\sigma dzd\bar{z}$ is the hyperbolic metric in the conformal class of Σ. The metric has many intriguing properties; for example, it is Kähler [Ahl61], negatively curved [Chu76, Wol86], incomplete [Tro87, Wol75] and not uniformly bounded away from 0 [Hua07].

1.2 Higher Teichmüller Theory

In this section, we define Hitchin component. Consider the space $Hom(\pi_1(S), G)$ of group homomorphisms $\rho : \pi_1(S) \to G$ from the fundamental group $\pi_1(S)$ to a reductive Lie group G, and define the representation variety

$$\mathcal{R}(\pi, G) = Hom(\pi_1(S), G)//G.$$

For $G = PSL(2, \mathbb{R})$, there are two isomorphic connected components of the space $\mathcal{R}(\pi, PSL(2, \mathbb{R}))$ that are open cells of complex dimension $3(g-1)$. The representations in these components are called Fuchsian, and both components can be identified with Teichmüller space [Gol88].

There is a unique irreducible representation $\tau : SL(2, \mathbb{R}) \to SL(n, \mathbb{R})$, using the induced action on the homogeneous polynomials. More explicitly, for $A \in SL(2, \mathbb{R})$, $\tau(A)$ is defined via the action on the basis $\{X^{n-1}, X^{n-2}Y, ..., Y^{n-1}\}$ under the action of A on $\{X, Y\}$. By post-composing this irreducible irreducible representation τ, there is a natural embedding

$$\begin{aligned} \mathcal{R}(\pi, PSL(2, \mathbb{R})) &\hookrightarrow \mathcal{R}(\pi, PSL(n, \mathbb{R})) \\ \rho &\mapsto \tau \circ \rho. \end{aligned}$$

We can single out a component of $\mathcal{R}(\pi_1, PSL(n, \mathbb{R}))$, namely the component containing representations which factor through Fuchsian representations. Hitchin [Hit92] used Higgs bundles to show that this component is an open cell of complex dimension $(n^2-1)(g-1)$. This component has since been called the Hitchin component or Hitchin-Teichmüller component and will be denoted $Hit_n(S)$.

Such a construction suggests that these representations are geometrically interesting; however, it does not give any information on the structure of this component. Hitchin [Hit92] asked about the geometric interpretation of the special components for general Lie groups. In $PSL(3,\mathbb{R})$, this is answered by Choi-Goldman [CG93]: they showed that the Hitchin compoent for $PSL(3,\mathbb{R})$ is isomorphic to the space of convex $\mathbb{R}P^2$-structures on the surface S. The latter space is also called Goldman space. In $PSL(4,\mathbb{R})$, Wienhard-Guichard [GW08] showed that the Hitchin component is isomorphic to the space of convex foliated structures on the unit tangent bundle of the surface S. In fact, they [GW12] construct the correspondence between space of Anosov representations with some geometric structures on a bundle of the surface S.

There are also studies of the Hitchin component from topological and algebraic aspects. Labourie [Lab06] and Guichard [Gui08] construct an isomorphism between Hitchin component for $PSL(n,\mathbb{R})$ and the space of hyperconvex curves on a topological circle. Also, Labourie [Lab06] showed that every representation in a Hitchin component is discrete, faithful and purely loxodromic, which is a generalization of a classic result [**?**] of Teichmüller space. Labourie [Lab07a] develops the correspondence between Hitchin components for $PSL(n,\mathbb{R})$ with the space of cross-ratios of rank n (an algebraic condition).

1.3 Higgs Bundles

Through the nonabelian Hodge correspondence developed by Hitchin [Hit87], Simpson [Sim92], Donaldson [Don87] and Corlette [Cor88], the representation variety $\mathcal{R}(\pi, G)$

is diffeomorphic to the moduli space of polystable G-Higgs bundles:

$$\mathcal{R}(\pi, G) \cong \mathcal{M}_{Higgs}(G).$$

We now briefly recall some of the theory of Higgs bundles and the non-abelian Hodge correspondence for $G = SL(n, \mathbb{C})$. (Readers can see [Hit92].)

Fix a Riemann surface structure Σ on the surface S, denote the canonical bundle of Σ by $K \to \Sigma$ and fix a square root $K^{\frac{1}{2}}$ of the canonical bundle.

Definition 1.3.1. An $SL(n, \mathbb{C})$-Higgs bundle over Σ is a pair (E, ϕ), where $E \to \Sigma$ is a rank n holomorphic vector bundle with trivial determinant and $\phi \in H^0(\Sigma, End_0(E) \otimes K)$ is a holomorphic traceless K-twisted endomorphism.

To form the moduli space $\mathcal{M}_{Higgs}(SL(n, \mathbb{C}))$ of Higgs bundles, we need a notion of stability.

Definition 1.3.2. (E, ϕ) is **stable**(**semistable**) if for any holomorphic subbundle $F \subset E$ with $\phi|_F : F \to F \otimes K$, we have $deg(F) < (\leq) 0$; it is called **polystable** if it is a direct sum of stable Higgs bundles.

The moduli space $\mathcal{M}_{Higgs}(SL(n, \mathbb{C}))$ consists of isomorphism classes of polystable $SL(n, \mathbb{C})$-Higgs bundles, where two Higgs bundles (E_1, ϕ_1) and (E_2, ϕ_2) are equivalent if there exists a smooth $SL(n, \mathbb{C})$-Higgs bundle isomorphism $g : (E_1, \phi_1) \to (E_2, \phi_2)$, which means a smooth bundle morphism from E_1 to E_2 pulling back both the holomorphic structure and the Higgs field.

One key ingredient in the nonabelian Hodge correspondence is the following theorem, proven by Hitchin [Hit87] in the rank 2 case and Simpson in the general case [Sim92]. Recall on a holomorphic bundle E, given a hermitian metric, there exists a unique connection such that it is compatible with both holomorphic structure on E and the hermitian metric, called Chern connection.

Theorem 1.3.3. *Let (E,ϕ) be a stable $SL(n,\mathbb{C})$-Higgs bundle, then there exists a unique hermitian metric h, with associated Chern connection A_h, solving the Higgs bundle equations*

$$F_{A_h} + [\phi, \phi^{*_h}] = 0 \qquad (1.3.1)$$

where F_{A_h} is the curvature of A_h and ϕ^{*_h} is the hermitian adjoint. Conversely, if (A_h,ϕ) is a solution then the corresponding Higgs bundle is polystable Higgs bundle.

In conclusion, given a stable $SL(n,\mathbb{C})$-Higgs bundle (R,ϕ), we have a unique hermitian metric satisfying the Higgs bundle equation 1.3.1 and hence gives rise to a flat $SL(n,\mathbb{C})$-connection $A+\phi+\phi^{*_h}$. The holonomy of the flat $SL(n,\mathbb{C})$ connection gives a representation of $\pi_1(S)$ to $SL(n,\mathbb{C})$. Hence we obtain a map to the representation variety

$$\mathcal{M}_{Higgs}(SL(n,\mathbb{C}))\longrightarrow\mathcal{R}(\pi_1(S),SL(n,\mathbb{C})).$$

A generalization of the following example will be our main object of study. It was studied in detail in Hitchin's original paper [Hit87].

Example 1.3.4. Consider the following family of stable $SL(2,\mathbb{C})$-Higgs bundles

$$E = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \qquad \phi = \begin{pmatrix} 0 & q_2 \\ 1 & 0 \end{pmatrix}$$

where $q_2 \in H^0(\Sigma, K^2)$ is a holomorphic quadratic differential. When $q_2 = 0$, a solution to (1.3.1) is equivalent to finding a hyperbolic metric on Σ in the conformal class of the complex structure. Furthermore, using q_2, all hyperbolic metrics can be found this way; this gives a parametrization of Teichmüller space by holomorphic quadratic differentials using Higgs bundles.

Remark 1.3.5. For any real reductive Lie group G, there are corresponding definitions and theorems for G-Higgs bundles. We will only need to consider $SL(n,\mathbb{R})$-

Higgs bundles, for which we give a definition below. For the general construction see [BGPG03, GPGiR13]. The more complicated construction for real G ensures that the flat connection, which arises from solving the Higgs bundle equations, has holonomy in the real group G. Before defining the $SL(n,\mathbb{R})$-Higgs bundles, we first define an orthogonal structure on a bundle E.

Definition 1.3.6. An orthogonal structure Q on a bundle E is a non degenerate quadratic form on the bundle E.

Definition 1.3.7. An $SL(n,\mathbb{R})$-Higgs bundle over Σ is a triple (E,Q,ϕ), where (E,ϕ) is an $SL(n,\mathbb{C})$-Higgs bundle and Q is an orthogonal structure on E with the property that ϕ is Q-symmetric, i.e., $\phi^T Q = Q\phi$.

Example 1.3.4 is actually an $SL(2,\mathbb{R})$-Higgs bundle. To see this, consider the orthogonal structure

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \longrightarrow K^{-\frac{1}{2}} \oplus K^{\frac{1}{2}}$$

thought of as a symmetric isomorphism $Q : E \to E^*$.

For $SL(n,\mathbb{R})$-Higgs bundle, the datum of the orthogonal structure Q provides the bundle E with an $SO(n,\mathbb{C})$-structure. By an isomorphism of a $SL(n,\mathbb{R})$-Higgs bundle, we will mean an $SO(n,\mathbb{C})$-gauge transformation, i.e., a bundle isomorphism of E which preserves this additional structure.

Let $p_2,\ldots,p_n$ be a homogeneous basis for the $SL(n,\mathbb{C})$-invariant polynomials $\mathbb{C}[\mathfrak{sl}(n,\mathbb{C})]^{SL(n,\mathbb{C})}$, with $deg(p_j) = j$. Such a choice of basis defines a map (called the Hitchin fibration)

$$h : \mathcal{M}_{Higgs}(SL(n,\mathbb{C})) \longrightarrow \bigoplus_{j=2}^{n} H^0(\Sigma, K^j)$$

given by

$$h([E,\phi]) = (p_2(\phi), \dots, p_n(\phi)).$$

In [Hit92], Hitchin defines a section s_h of this fibration whose image consists of stable Higgs bundles with corresponding flat connections having holonomy in $SL(n,\mathbb{R})$. Furthermore, the section s_h maps surjectively to the connected component of the $SL(n,\mathbb{R})$-Higgs bundle moduli space which naturally contains an embedded copy of Teichmüller space (example 1.3.4). Let

$$(E,Q) = S^{n-1}\left(K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

$$= \left(K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{-\frac{n-3}{2}} \oplus K^{-\frac{n-1}{2}}, \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}\right)$$

be the $(n-1)$'st symmetric power of example 1.3.4, and $(q_2, q_3, \dots, q_n) \in \bigoplus_{j=2}^{n} H^0(\Sigma, K^j)$. The Hitchin section is defined by $s_h(q_2, q_3, \dots, q_n) =$

$$\left\{E, \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2}(n-1)q_2 & q_3 & \cdots & q_{n-1} & q_n \\ 1 & 0 & \frac{1}{2}(n-3)q_2 & \cdots & q_{n-2} & q_{n-1} \\ & \ddots & \ddots & \ddots & & \\ & & & 0 & \frac{1}{2}(n-3)q_2 & q_3 \\ & & & 1 & 0 & \frac{1}{2}(n-1)q_2 \\ & & & & 1 & 0 \end{pmatrix}\right\}.$$

The embedded copy of Teichmüller space results from setting $q_3 = \cdots = q_n = 0$; it arises as the $(n-1)$st symmetric power of example 1.3.4. Through Kostant's work [Kos59] on the principal three-dimensional subalgebra, there exists a homogeneous basis $\{p_2, \dots, p_n\}$ of the invariant polynomials so that $p_j(\phi) = q_j$, verifying that s_h is a section. Because of its link with the principal three-dimensional subalgebra. Let e_j be the matrix where all entries are zeros but the $(k, j+k)$-entries for all $1 \leq k \leq n$ and $\tilde{e}_j$ be the transpose of e_j. Wel denote the Higgs field associated to $s_h(q_2, q_3, \ldots, q_n)$ by

$$\phi = \tilde{e}_1 + q_2 e_1 + q_3 e_2 + \cdots + q_n e_{n-1}.$$

The constants on the q_j's are necessary to make $\langle \tilde{e}_1, e_1, [e_1, \tilde{e}_1] \rangle$ a Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

1.4 Deformation space of convex $\mathbb{R}P^2$-structures

Choi and Goldman [CG93] showed that the Hitchin component for $PSL(3, \mathbb{R})$ is isomorphic to the space of isomorphism classes of convex $\mathbb{R}P^2$-structures on the surface S. The latter space is also called Goldman space. In this section, we briefly recall the definitions of $\mathbb{R}P^2$-structures and Goldman space.

Let M be a smooth 2-manifold.

Definition 1.4.1. A real projective structure on M is an atlas of charts $\{(U_\alpha, \psi_\alpha)\}$, such that

(i) $\{U_\alpha\}$ is an open cover of M;

(ii) For each α, the map $\psi_\alpha : U_\alpha \to \mathbb{R}P^2$ is a diffeomorphism onto its image; and

(iii) The change of coordinates are locally projective: If $\{(U_\alpha, \psi_\alpha)\}$ and $\{(U_\beta, \psi_\beta)\}$ are two such coordinate charts, then the restriction of $\psi_\beta \circ {\psi_\alpha}^{-1}$ to any connected component of ${\psi_\beta}^{-1}(\psi_\alpha(U_\alpha \cap U_\beta))$ is a projective transformation.

A manifold with an $\mathbb{R}P^2$-structure is called an $\mathbb{R}P^2$-manifold.

Definition 1.4.2. An $\mathbb{R}P^2$-structure on M is called convex if its developing map is a diffeomorphism of $\widetilde{M}$ onto a convex domain Ω in some affine $\mathbb{R}^2 \subset \mathbb{R}P^2$. In this case, we can realize $M = \Omega/\Gamma$, where Γ is a subgroup of $PGL(3,\mathbb{R})$ which acts discretely and properly discontinuous on Ω. Moreover, a convex $\mathbb{R}P^2$-structure on M is called properly convex if Ω is bounded.

Definition 1.4.3. The Goldman space $\mathcal{B}(S)$ is defined as $\mathcal{B}(S) := \{(f, M) | f : S \to M$ is a diffeomorphism and M is a convex $\mathbb{R}P^2$-manifold$\}/\sim$. The equivalence relation $\sim$ is defined in the following. Two elements $(f, M), (f', M') \in \mathcal{B}(S)$ are equivalent if and only if there exists a projective isomorphism $h : M \to M'$ such that $h \circ f$ is isotopic to f'.

We have that Goldman space $\mathcal{B}(S)$ is open, and the holonomy map is an embedding of $\mathcal{B}(S)$ to $Hom(\pi, PGL(3,\mathbb{R}))/PGL(3,\mathbb{R})$(see [Gol90]). The Zariski tangent space to $Hom(\pi, PGL(3,\mathbb{R}))/PGL(3,\mathbb{R})$ at $[\rho]$ (hence also the tangent space to $\mathcal{B}(S)$ at $[\rho]$) is isomorphic to $H^1(\pi, sl(3,\mathbb{R}))$ which by de Rham's theorem is isomorphic to $H^1(S; sl(3,\mathbb{R})_{Ad\rho})$, where $sl(3,\mathbb{R})_{Ad\rho}$ is the flat $sl(3,\mathbb{R})$-bundle over S with holonomy representation $Ad\rho$ (see [Gol84], pp. 208-209). Explicitly, $sl(3,\mathbb{R})_{Ad\rho}$ is identified with

$$\widetilde{S} \times sl(3,\mathbb{R})/\{(\widetilde{s}, x) \sim (\gamma\widetilde{s}, Ad\rho(\gamma)(x))\}, \tag{1.4.1}$$

for all $\gamma \in \pi, \widetilde{s} \in \widetilde{S}, x \in sl(3,\mathbb{R})$.

1.5 Correspondence of $\mathcal{B}(S)$ and spaces of pairs (Σ, U)

In this section, we explain another description of $\mathcal{B}(S)$ as follows:

Proposition 1.5.1. *(Loftin [Lof01], Labourie [Lab07b]) There exists a natural bijective correspondence between convex $\mathbb{R}P^2$-structures on S and pairs (Σ, U), where Σ is a Riemann surface homeomorphic to S, and U is a holomorphic cubic differential on Σ.*

This correpsondence is essential in the construction of Loftin metric on the Hitchin component for $PSL(3,\mathbb{R})$.

Since we rely heavily on the construction of the bijection in Proposition 1.5.1, we give a version of the arguments here for reader's convenience. The arguments mainly follow [Lof04].

Before explaining the detail of the correspondence, we first state the main idea as follows: To start with, given a convex $\mathbb{R}P^2$- structure on the surface S, we write $M \cong \Omega/\Gamma$, where Ω is a bounded convex domain in $\mathbb{R}^2$, viewed as an affine chart in $\mathbb{R}P^2$. For a bounded convex domain Ω, there is a unique hypersurface asymptotic to the boundary of the open cone $\mathcal{C} \subset \mathbb{R}^3$ above Ω, called the hyperbolic affine sphere (which will be defined later.) This hyperbolic affine sphere $H \subset \mathcal{C}$ is invariant under automorphisms of $\mathcal{C}$ in $SL(3,\mathbb{R})$. The restriction of the projection map $\pi : \mathcal{C} \to \Omega$ induces a diffeomorphism of H onto Ω. Affine differential geometry provides an $SL(3,\mathbb{R})$-invariant structure on the hyperbolic affine sphere H which then descends to $M = \Omega/\Gamma$. Then the affine metric on the surface (which will be defined later) induces a conformal structure, hence gives a Riemann surface structure Σ on the surface. Moreover, the difference of the Levi-Civita connection of the affine metric on H and the Blaschke connection of H (which will be defined later) induces a holomorphic cubic differential on the Riemann surface Σ.

1.5.1 Affine Sphere

Consider a hypersurface $H \subset \mathbb{R}^3$, and consider a transversal vector field ξ on the hypersurface H. We have the equations defining the derivatives of a frame: For any two tangent vectors X, Y on H,

$$D_X Y = \nabla_X Y + h(X,Y)\xi$$

$$D_X \xi = -SX + \beta(X)\xi.$$

Here, the operator D is the canonical flat connection on $\mathbb{R}^3$, the operator ∇ is a torsion-free connection, the form h is a symmetric bilinear form on T_xH, called the second fundamental form, the map S is an endomorphism of T_xH, called the shape operator, and β is a one-form.

An affine normal of H is a transversal vector field which is invariant under affine automorphisms of H.

Definition 1.5.2. An affine sphere is a hypersurface H in $\mathbb{R}^3$ satisfying the condition that all its affine normals point toward a given point, called the center, in $\mathbb{R}^3$. Moreover, if the center lies on the concave side of H, and if the map $S = LI$, where the affine mean curvature L is a constant negative function on H and I is the identity map, we call H is a hyperbolic affine sphere.

Thus by scaling, we can normalize any hyperbolic affine sphere to have $L = -1$. Also, we can translate so that the center is 0, i.e., from now on, we restrict to hyperbolic affine spheres with center at origin and affine mean curvature -1. In this case, the affine normal $\xi = f$, where f is the position vector. The structure equations (see [Lof04]) then become

$$\begin{aligned} D_X Y &= \nabla_X Y + g(X,Y)f \\ D_X f &= X. \end{aligned}$$

the connection ∇ is called the Blaschke connection. The bilinear form g is called the Blaschke metric, or the affine metric.

Proposition 1.5.3. *(Cheng and Yau [CY77],[CY86], Calabi and Nirenberg (with clarifications by Gigena [Gig81], Sasaki [Sas80] and A.M Li [Li90],[Li92])) Consider* *a convex, bounded domain* $\Omega \subset \mathbb{R}^2$*, where* $\mathbb{R}^2$ *is embedded in* $\mathbb{R}^3$ *as the affine space* $\{x_3 = 1\}$*. Then, there is a unique properly embedded hyperbolic affine sphere* $H \subset \mathbb{R}^3$ *of affine mean curvature -1 and center 0 asymptotic to the boundary of the cone* $\mathcal{C} \subset \mathbb{R}^3$*.*

The tautological bundle

We define $\mathbb{R}P^2$ as the space of all lines l passing through 0 in $\mathbb{R}^3$. Then the subset of $\mathbb{R}P^2 \times \mathbb{R}^3$ consisting of all (p, l) with $p \in l$ is the total space for the tautological line bundle $\mathcal{L}$ of $\mathbb{R}P^2$. Given an $\mathbb{R}P^2$-manifold M, the bundle $dev^{-1}\mathcal{L}$ defines the tautological bundle on $\widetilde{M}$. We say $\widetilde{M}$ admits a tautological bundle if this structure descends to M, i.e., if there is a line bundle on M which pulls back to $dev^{-1}\mathcal{L}$ on $\widetilde{M}$ under the universal covering map. For simplicity, we denote this line bundle as $\mathcal{L}$ also. By Proposition 2.2.1 in Loftin [Lof01], a manifold M with convex $\mathbb{R}P^2$- structure admits an oriented tautological bundle.

Affine sphere structure

Let M be an $\mathbb{R}P^2$-manifold with oriented tautological bundle $\mathcal{L}$. Then the total space of the positive part of $\mathcal{L}$ (i.e., the $\mathbb{R}^+$ part of each fiber of the line bundle $\mathcal{L}$) is locally a cone in $\mathbb{R}^3$. We say M admits an affine sphere structure if there is a section s of $\mathcal{L}$ so that for each coordinate chart $\mathcal{U}$ of M, the image $s(\mathcal{U})$ is a hyperbolic affine sphere with center 0 and affine mean curvature -1 in the cone $\mathcal{C}$.

Combining Proposition 2.2.1 and Theorem 4 in Loftin [Lof01] with the fact that any convex $\mathbb{R}P^2$-structure on a compact surface S must be properly convex, (Kuiper [Kui54]), we have the following proposition:

Proposition 1.5.4. *Let M be a convex $\mathbb{R}P^2$-manifold homeomorphic to S, then we have*

1. M admits a negative strictly convex section u of the dual tautological bundle $\mathcal{L}^*$ satisfying $det(u_{ij}) = (\frac{1}{u})^4$ so that the metric $\frac{-u_{ij}}{u}$ is complete, and 2. M admits an affine sphere structure whose metric is complete.

Now suppose we have $M \cong \Omega/\Gamma$; by Proposition 1.5.3, we have a unique hyperbolic affine sphere H in the cone. Consider a local conformal coordinate $z = x + iy$ on the hyperbolic affine sphere H with center the origin and affine curvature -1. Then the affine metric is given by $g = e^{\psi}|dz|^2$ for some function ψ. Parametrize the surface by

$$f : \mathcal{D} \to \mathbb{R}^3, \quad \text{with } \mathcal{D} \text{ a domain in } \mathbb{C}. \tag{1.5.1}$$

Then we have the following structure equations for the affine sphere:

$$\begin{aligned} D_X Y &= \nabla_X Y + g(X,Y)f \\ D_X f &= X \end{aligned} \tag{1.5.2}$$

Here D is the canonical flat connection on $\mathbb{R}^3$, the operator ∇ is a projectively flat connection, and g is the affine metric.

We also consider $\widehat{\nabla}$, the Levi-Civita connection with respect to the affine metric g. To get a holomorphic cubic differential from this construction, we consider the Pick form $C := \widehat{\nabla} - \nabla$ which is a tensor measuring the difference between the Levi-Civita connection and the Blaschke connection. In index notation, we have the following conditions (see Theorem 4.3 in [NS94])

$$\Sigma_i C^i_{ij} = 0 \quad \text{for all} j; \qquad C_{ijk} \quad \text{symmetric in } i,j,k, \tag{1.5.3}$$

here we use g to lower the index. In addition, if C vanishes identically on the hyperbolic affine sphere H with center the origin and affine curvature -1, then H must be the hyperboloid in $\mathbb{R}^3$ (see Theorem 4.5 in [NS94]). The symmetries of the Pick form show that it has only two linearly independent factors, which are realized as the real and complex parts of a holomorphic cubic differential U on S under the complex structure with complex z coordinate.

1.6 Correspondence for the case of Rank 2 Lie Groups

Labourie [Lab14] recently showed that the correspondence between Hitchin component for $PSL(3,\mathbb{R})$ and pairs $\{(\Sigma, U)\}$ can be extended to Hitchin component for rank 2 Lie groups. In this section, we explain this correspondence in more detail. Given a Riemann surface structure Σ, recall that the Hitchin [Hit92] parametrizes the Hitchin component as $s_h(q_2, q_3, ..., q_n)$ for $(q_2, \ldots, q_n) \in \oplus_2^n H^0(K_\Sigma^i)$.

Labourie [Lab08] considers a mapping class group -equivariant construction. Let $\mathcal{E}^n$ be the vector bundle over Teichmüller space whose fiber over the Riemann surface is $\mathcal{E}^n_\Sigma = H^0(K^3_\Sigma) \oplus \cdots \oplus H^0(K^n_\Sigma)$ and define the Hitchin map by

$$\begin{aligned} H : \mathcal{E}^n &\to H(n) \\ (\Sigma, q_3, ..., q_n) &\mapsto s_{\Sigma,h}(0, q_3, \cdots, q_n). \end{aligned}$$

In the same paper, Labourie shows that this map is surjective and conjectures that this map is also injective.

Labourie's recent work [Lab14] shows that the conjecture is true for Hitchin component for rank 2 Lie groups G_0.

Theorem 1.6.1. *([Lab14]) Given a Hitchin representation ρ, there exist a unique ρ-equivariant minimal mapping from Σ to the symmetric space $S(G_0)$ associated to G_0.*

Theorem 1.6.2. *([Lab14]) There exists an analytic diffeomorphism, equivariant under the mapping class group action, from the Hitchin component for* G_0*, when* G_0 *is of real rank 2, to the space of pairs* (Σ, q_k) *where* Σ *is a Riemann surface structure on* S *and* q_k *a holomorphic differential with respect on* Σ *of degree* $k = \frac{dim(G_0)-2}{2}$.

Chapter 2

The Loftin Metric

2.1 Introduction

Given S a closed surface S of genus $g > 1$, the equivalence classes of convex RP^2-structures form a moduli space $\mathcal{B}(S)$, called Goldman space, is homeomorphic to an open cell of dimension $16(g-1)$. Teichmüller space $\mathcal{T}(S)$ naturally embeds inside $\mathcal{B}(S)$.

It is of interest to know what of the rich geometric structure on Teichmüller space $\mathcal{T}(S)$ extends to Goldman space $\mathcal{B}(S)$. In [Gol84], a symplectic structure on $\mathcal{B}(S)$ is defined, which extends the symplectic structure on the Teichmüller space $\mathcal{T}(S)$ defined by the Weil-Petersson Kähler form. In terms of a Riemannian metric, it is natural to ask the following questions:

Question. *(i) Does there exist a Riemannian metric on the deformation space $\mathcal{B}(S)$ restricting to be the Weil-Petersson metric on Teichmüller space $\mathcal{T}(S)$?*
(ii) If (i) is satisfied, is the Teichmüller space $\mathcal{T}(S)$ endowed with the Weil-Petersson metric totally geodesic within the deformation space $\mathcal{B}(S)$ endowed with the new metric?

To answer the above questions, we first consider the Riemannian metric on $\mathcal{B}(S)$ constructed by Darvishzadeh and Goldman (see [DG96]), which will be referred to as the DG metric.

We show that the DG metric answers (i) affirmatively in this paper (Theorem

2.4). By the nature of Koszul-Vinberg metric, we are not able to see directly whether the DG metric satisfies (ii). To contruct a metric satisfies both (i) and (ii), we make use of the Cheng-Yau metric, which is closely related to the correspondence between $\mathcal{B}(S)$ and the space of pairs (Σ, U), to construct a new Riemannian metric on the deformation space $\mathcal{B}(S)$, will be referred to as the Loftin metric. In fact, Loftin mentioned in [Lof02] that the construction of the DG metric can be carried out with other invariant affine Kähler metrics instead of the Koszul-Vinberg metric, e.g., the Cheng-Yau metric.

Now we can answer both parts (i) and (ii) of the above question affirmatively with the Loftin metric, namely,

Theorem 2.4.1. *The DG metric and the Loftin metric both restrict to a constant multiple of the Weil-Petersson metric on Teichmüller space.*

Theorem 2.5.1. *Teichmüller space endowed with the Weil-Petersson metric is totally geodesic in $\mathcal{B}(S)$ endowed with the Loftin metric.*

Recently, M. Bridgeman, R. Canary, F. Labourie and A. Sambarino in [BCLS13] constructed a mapping class group invariant Riemannian metric on the Hitchin component $\mathcal{H}(S)$ of $Rep(\pi, PSL(n, \mathbb{R}))$, which is called the pressure metric. They showed using thermodynamical formalism that the pressure metric is an extension of the Weil-Petersson metric on the Fuchsian representations. When restricted to the $SL(3, \mathbb{R})$ case, the Hitchin component coincides with the deformation space $\mathcal{B}(S)$ (see [CG93]), hence the pressure metric also answers part (i) of the question affirmatively. Using the work of Berndtsson [Ber09] on positivity of bundles, Kim-Zhang in [KZ13] constructed a Kähler metric on Goldman space for which Teichmüller space is a totally geodesic complex submanifold. Labourie in [Lab14] then extended their result to

Hitchin components for all rank 2 Lie groups.

Outline of the proof

First, to show Theorem 2.4.1, we begin by showing that the Loftin metric and the DG metric are isometric (up to a constant multiple) when restricted to the Teichmüller locus (mainly because the ingredients in the definition of two metrics coincide for the hyperbolic structure case). Then it is sufficient to show that the Loftin metric restricts to be the Weil-Petersson metric on the Teichmüller locus (Proposition 2.4.3). Here, these two metrics are defined on different descriptions of Teichmüller space: The Weil-Petersson metric is defined on the usual Teichmüller space, and the Loftin metric is defined on the Teichmüller locus in $\mathcal{B}(S)$. Hence we need to identify tangent vectors of Teichmüller space with those of the Teichmüller locus in $\mathcal{B}(S)$, written as 1-forms taking values in flat $sl(3,\mathbb{R})$-bundle (Lemma 2.4.7 (i)). Moreover, we calculate that those 1-forms valued in the flat $sl(3,\mathbb{R})$-bundle are, in fact, harmonic in their cohomology class (harmonicity is essential in the definition of the Loftin metric) (Lemma 2.4.7 (ii)). The explicit expression of the metric on the flat $sl(3,\mathbb{R})$-bundle in Lemma 2.4.5 is extremely helpful for the calculation. Then we finish the proof of Proposition 2.4.3 by comparing the Loftin pairing of the harmonic representatives and the Weil-Petersson pairing of the original tangent vectors.

2.2 The construction of the Loftin metric

Darvishzadeh and Goldman [DG96] construct a Riemannian metric, which will be referred to as the DG metric, on the deformation space $\mathcal{B}(S)$ using the Koszul-Vinberg metric on the cone. In this section, we first give the construction of a new Riemannian metric, will be referred to as the Loftin metric, defined on the deformation space

$\mathcal{B}(S)$ but using the Cheng-Yau metric (defined below) on the cone and then give the construction of the DG metric.

The Cheng-Yau metric on the cone

Let $\mathcal{C} \subset \mathbb{R}^3$ be the open cone over the domain Ω in affine space $E = \mathbb{R}^3$, i.e., the cone $\mathcal{C} = \{(tx, t) \in \mathbb{R}^3 | x \in \Omega, t > 0\}$. Cheng and Yau in [CY82] show that there exists a unique strictly convex function σ on the convex cone $\mathcal{C}$ satisfying

$$\det(\frac{1}{3}(\log \sigma)_{ij}) = \sigma^2, \quad \text{and} \quad \sigma \to \infty \ \text{ at } \ \partial\Omega,$$

and that the metric $h = \mathrm{Hess}(\frac{1}{3}(\log \sigma))$ on the cone is complete and invariant under linear automorphisms of $\mathcal{C}$, will be referred as to Cheng-Yau metric.

Calabi [Cal72], Cheng and Yau [CY86] show that each convex domain Ω has associated to it a unique strictly convex function u satisfying

$$\det(u_{ij}) = (\frac{1}{u})^4, \quad \text{and} \quad u|_{\partial\Omega} = 0.$$

The radial graph of $-\frac{1}{u}$ is a hyperbolic affine sphere H asymptotic to the boundary of the cone $\mathcal{C}$ with center 0 and affine mean curvature -1. Moreover, the metric $-\frac{1}{u}u_{ij}dt^i dt^j$ on Ω and the associated affine metric g on H are isometric under the map $-\frac{1}{u}(t_1, t_2, 1) : \Omega \to H$. (so we may sometimes just write $g = -\frac{1}{u}u_{ij}dt^i dt^j$.)

The relation between the Cheng-Yau metric h on the cone and the affine metric g on the hyperbolic affine sphere inside the cone is as follows:

Proposition 2.2.1. *(Loftin [Lof02]) Let $h = \frac{1}{3}(d^2 \log \sigma)$ be the Cheng-Yau metric on the cone $\mathcal{C}$, then hypersurface $H = \sigma^{-1}(1)$ is a hyperbolic affine sphere with center the origin and affine mean curvature -1. The natural foliation $\mathcal{C} = \cup_{s>0} sH$ gives the metric splitting*

$$(\mathcal{C}, h) = (\mathbb{R}^+, \frac{ds^2}{s^2}) \bigoplus (H, g),$$

where g is the Blaschke metric along H.

By definition, the affine normal (the position vector in the above case) of H is invariant under $Aut(H)$. Combining this with equation (1.5.2), the affine metric g on H is also invariant under $Aut(H)$ and hence the metric $g = -\frac{1}{u}u_{ij}dt^i dt^j$ on Ω descends to a metric on $M = \Omega/\Gamma$ which is also called affine or Blaschke metric on M. The Cheng-Yau metric h is invariant under $Aut(\mathcal{C})$, and hence is also invariant under $Aut(H)$, since $Aut(H) \subset Aut(\mathcal{C})$.

Loftin mentioned in [Lof02] that the construction of the DG metric (will be defined later) can be carried out with other invariant affine Kähler metrics, e.g., Cheng-Yau metric, instead of the Koszul-Vinberg metric on the cone. Hence we define a different Riemannian metric on $\mathcal{B}(S)$ using a construction similar to that of the DG metric but using the Cheng-Yau metric on the cone instead of the Koszul-Vinberg metric. For the above reason, we call the new metric ”the Loftin metric”.

We begin the construction of the Loftin metric:

Suppose that the pair $(f, M) \in \mathcal{B}(S)$ corresponds to a convex $\mathbb{R}P^2$ structure on S. Let $\mathcal{C} \subset R^3$ be the corresponding open cone in affine space $E = \mathbb{R}^3$.

On the one hand, the Cheng-Yau metric h at each point of the hyperbolic affine sphere H is an inner product on $\mathbb{R}^3$, hence induces an inner product on the dual space $\mathbb{R}^3*$ and then on $sl(3,\mathbb{R}) \subset gl(3,\mathbb{R}) \subset Hom(\mathbb{R}^3,\mathbb{R}^3) \cong \mathbb{R}^3 \otimes (\mathbb{R}^3)^*$, and therefore also induces a Riemannian metric $l = h \otimes h^*$ on the trivial bundle $sl(3,\mathbb{R}) \times H$ over H. We then obtain a Riemannian metric l (abusing notation) on the bundle $sl(3,\mathbb{R})_{Ad\rho}$. Explicitly, supposing that ϕ, ϕ' are sections of $sl(3,\mathbb{R})_{Ad\rho}$, $\forall p \in S$, we define

$$l(\phi,\phi')|_p := l_x(\widetilde{\phi}_x, \widetilde{\phi}'_x), \text{for some } x \in \pi^{-1}(p) \subset H, \tag{2.2.1}$$

where π is composition of the projective map from hypersurface H to Ω and the quotient map from Ω to S, and $\widetilde{\phi}, \widetilde{\phi}'$ are the liftings of sections ϕ, ϕ' of $sl(3,\mathbb{R})_{Ad\rho}$ to the trivial bundle $sl(3,\mathbb{R}) \times H$. Then by definition of the bundle $sl(3,\mathbb{R})_{Ad\rho}$ (see equation (1.4.1)), we see that (for $\gamma \in \rho(\pi)$, we have that $\widetilde{\phi}, \widetilde{\phi}'$ satisfy

$$\widetilde{\phi}_{\gamma x} = Ad(\gamma)\widetilde{\phi}_x, \quad \text{and} \quad \widetilde{\phi}'_{\gamma x} = Ad(\gamma)\widetilde{\phi}'_x. \tag{2.2.2}$$

For any $\gamma \in \rho(\pi) = \Gamma < Aut(H)$, we compute

$$\begin{aligned}
& l_{\gamma x}(\widetilde{\phi}_{\gamma x}, \widetilde{\phi}'_{\gamma x}) \\
= \;& l_{\gamma x}(Ad(\gamma)\widetilde{\phi}_x, Ad(\gamma)\widetilde{\phi}'_x) \quad \text{by equation (2.2.2)} \\
& \text{under the identification of } gl(3,\mathbb{R}) \text{ and } \mathbb{R}^3 \otimes \mathbb{R}^{*3} \\
= \;& l_{\gamma x}((Ad(\gamma)\widetilde{\phi}_x)^i_j e_i \otimes e^j, (Ad(\gamma)\widetilde{\phi}'_x)^k_l e_k \otimes e^l) \\
= \;& h_{\gamma x} \otimes h^*_{\gamma x}((\widetilde{\phi}_x)^i_j \gamma e_i \otimes (\gamma e_j)^*, (\widetilde{\phi}'_x)^k_l \gamma e_k \otimes (\gamma e_l)^*) \quad \text{since } l = h \otimes h^* \\
= \;& (\widetilde{\phi}_x)^i_j (\widetilde{\phi}'_x)^k_l h_{\gamma x}(\gamma e_i, \gamma e_k) h^*_{\gamma x}((\gamma e_j)^*, (\gamma e_l)^*) \\
& \text{Since } h \text{ and } h^* \text{ are invariant under affine automorphisms of } H, \\
& h_{\gamma x}(\gamma e_i, \gamma e_k) = h_x(e_i, e_k) \quad \text{and} \quad h^*_{\gamma x}((\gamma e_j)^*, (\gamma e_l)^*) = h^*_x(e^*_j, e^*_l) \\
= \;& (\widetilde{\phi}_x)^i_j (\widetilde{\phi}'_x)^k_l h_x(e_i, e_k) h^*_x(e^j, e^l) \quad \text{noting that } \; e^j = e^*_j, e^l = e^*_l \\
= \;& l_x(\widetilde{\phi}_x, \widetilde{\phi}'_x) \quad \text{since } l = h \otimes h^*.
\end{aligned}$$

Hence we obtain that $l(\phi, \phi')|_p$ does not depend on the choice of x in $\pi^{-1}(p)$ and hence is well-defined.

On the other hand, the affine metric and the orientation on S define a metric on $\mathcal{A}^p(S)$ (the space of 1-forms on S) and hence enable us to define a Hodge star operator

$$* : \mathcal{A}^p(S) \to \mathcal{A}^{2-p}(S)$$

by setting

$$\alpha \wedge *\beta = < \alpha, \beta > dvol. \tag{2.2.3}$$

Combining the action of Hodge star operator with the Riemannian metric l on the bundle $sl(3,\mathbb{R})_{Ad\rho}$, we may define a positive definite inner product $\widetilde{g}_{Loftin}$ on the space $\mathcal{A}^1(S, sl(3,\mathbb{R})_{Ad\rho})$, the space of 1-forms taking values in the bundle $sl(3,\mathbb{R})_{Ad\rho}$ as follows.

Let $\sigma \otimes \phi, \sigma' \otimes \phi' \in \mathcal{A}^1(S, sl(3,\mathbb{R})_{Ad\rho})$, where $\sigma, \sigma\prime \in \mathcal{A}^1(S)$ and ϕ, ϕ' are sections of $sl(3,\mathbb{R})_{Ad\rho}$. We define a pairing $\widetilde{g}_{Loftin}$ as follows:

$$\widetilde{g}_{Loftin}(\sigma \otimes \phi, \sigma' \otimes \phi') = \int_S (\sigma \wedge *\sigma\prime) l(\phi, \phi'). \tag{2.2.4}$$

By linearity, we may extend the definition of $\widetilde{g}_{Loftin}$ to a pair $\Sigma_i \sigma_i \otimes \phi_i$ and $\Sigma_j \sigma'_j \otimes \phi'_j$. Hence we obtain an inner product, which is also denoted $\widetilde{g}_{Loftin}$, defined on the whole space $\mathcal{A}^1(S, sl(3,\mathbb{R})_{Ad\rho})$.

Next we define a metric, which will be referred as to the Loftin metric g_{Loftin}, on the cohomology $H^1(S; sl(3,\mathbb{R})_{Ad\rho})$ as follows (see pp. 108-111 in M.S.Raghunathan [Rag72]): The metric l we defined on the fibers of $sl(3,\mathbb{R})_{Ad\rho}$ gives an isomorphism

$$\sharp : sl(3,\mathbb{R})_{Ad\rho} \to sl(3,\mathbb{R}^*)_{Ad\rho^*},$$

$$\text{where } (\rho^* y)(x) = y(\rho^{-1} x), \text{ for } y \in \mathbb{R}_3, \text{and} x \in \mathbb{R}^3$$

defined by setting

$$(\sharp v)_x(u_x) = l_x(u_x, v_x)$$

for $u_x, v_x \in sl(3,\mathbb{R})$, and $x \in M$. This isomorphism extends naturally to an isomorphism again denoted $\sharp$ of $\mathcal{A}^p(S, sl(3,\mathbb{R})_{Ad\rho})$ and $\mathcal{A}^p(S, sl(3,\mathbb{R}^*)_{Ad\rho^*})$:

$$\sharp : \mathcal{A}^p(S, sl(3,\mathbb{R})_{Ad\rho}) \to \mathcal{A}^p(S, sl(3,\mathbb{R}^*)_{Ad\rho^*}).$$

In addition, the Hodge star operator on $\mathcal{A}^p(S)$ naturally extends to be defined on $\mathcal{A}^p(S, sl(3,\mathbb{R})_{Ad\rho})$. Finally, we define an operator

$$\delta : \mathcal{A}^p(S, sl(3,\mathbb{R})_{Ad\rho}) \to \mathcal{A}^{p-1}(S, sl(3,\mathbb{R})_{Ad\rho})$$

by setting

$$\delta = -(\sharp)^{-1} *^{-1} d * (\sharp) \tag{2.2.5}$$

and then define the Laplacian $\Delta : \mathcal{A}^p(S, sl(3,\mathbb{R})_{Ad\rho}) \to \mathcal{A}^p(S, sl(3,\mathbb{R})_{Ad\rho})$ by setting

$$\Delta = d\delta + \delta d.$$

A form $\xi \in \mathcal{A}^p(S, sl(3,\mathbb{R})_{Ad\rho})$ is harmonic if $\Delta\xi = 0$. In particular, if M is compact, the form ξ is harmonic if and only if

$$d\xi = 0, \quad \delta\xi = 0.$$

The kernel $\mathcal{H}^\infty(S, sl(3,\mathbb{R})_{Ad\rho})$ of Δ and the images of $d : \mathcal{A}^0(S, sl(3,\mathbb{R})_{Ad\rho}) \to \mathcal{A}^1(S, sl(3,\mathbb{R})_{Ad\rho})$ and $\delta : \mathcal{A}^2(S, sl(3,\mathbb{R})_{Ad\rho}) \to \mathcal{A}^1(S, sl(3,\mathbb{R})_{Ad\rho})$ decompose the vector space of 1-forms valued in $sl(3,\mathbb{R})_{Ad\rho}$ into an orthogonal direct sum

$$\mathcal{A}^1(S, sl(3,\mathbb{R})_{Ad\rho}) = \mathcal{H}^\infty(S, sl(3,\mathbb{R})_{Ad\rho}) \oplus Image(d) \oplus Image(\delta)$$

Since each de Rham cohomology class contains a unique harmonic representative from non-abelian Hodge theory (see Proposition 7.10 in [Rag72]), we may define the pairing at $[\rho]$

$$g_{\text{Loftin}} : H^1(S, sl(3,\mathbb{R})_{Ad\rho}) \times H^1(S, sl(3,\mathbb{R})_{Ad\rho}) \to R$$

$$\text{by } g_{\text{Loftin}}([\alpha],[\beta]) := \widetilde{g}(\alpha_{harm}, \beta_{harm}), \quad \text{for}[\alpha],[\beta] \in H^1(S, sl(3,\mathbb{R})_{Ad\rho}),$$

where $\alpha_{harm}, \beta_{harm}$ are the unique harmonic representatives of $[\alpha], [\beta]$ respectively and $\widetilde{g}$ is defined above (see equation (2.2.4)). Hence we have a well-defined Riemannian metric g_{Loftin} on Goldman space $\mathcal{B}(S)$.

Next we introduce the Riemannian metric on the Goldman space $\mathcal{B}(S)$ defined by Darvishzadeh and Goldman (see details in [DG96]). Since this Riemannian metric relies heavily on the Koszul-Vinberg metric on the cone $\mathcal{C}$ in $\mathbb{R}^3$, we first recall the definition of the Koszul-Vinberg metric.

Koszul-Vinberg metric

Let $\mathcal{C} \subset \mathbb{R}^3$ be a cone in affine space $E = \mathbb{R}^3$. The dual cone $\mathcal{C}^*$ is the subset of the dual vector space E^* consisting of linear functionals $\psi : E \to \mathbb{R}$ which are positive on $\mathcal{C}$.

Recall the Koszul-Vinberg characteristic function $k(x)$ on the cone $\mathcal{C}$: for $x \in \mathcal{C}$, define

$$k(x) = \int_{\mathcal{C}^*} e^{-\psi(x)} d\psi.$$

Note that

$$\begin{aligned} k(\gamma x) &= \int_{\mathcal{C}^*} e^{-\psi(\gamma x)} d\psi \\ &= \int_{\mathcal{C}^*} e^{-\gamma^* \psi(x)} d\psi \\ &= \int_{\mathcal{C}^*} e^{-\phi(x)} det(\gamma)^{-1} d\phi, \quad \text{let } \phi = \gamma^* \psi \in \mathcal{C}^*, d\phi = det(\gamma) d\psi \\ &= det(\gamma)^{-1} \int_{\mathcal{C}^*} e^{-\phi(x)} d\phi \\ &= det(\gamma)^{-1} k(x), \text{ for any } \gamma \in \mathrm{Aut}(\mathcal{C}). \end{aligned} \tag{2.2.6}$$

Hence the Hessian $d^2 \log k$ is invariant under $\mathrm{Aut}(\mathcal{C})$. Moreover, the Hessian $d^2 \log k$ is actually a positive definite symmetric bilinear form h (which we call the Koszul-Vinberg metric) on the cone $\mathcal{C}$.

The construction of DG metric

We assume the same notation M, Ω, Γ, and $\mathcal{C}$ as in the construction of the Loftin metric.

On one hand, for every $x \in k^{-1}(1)$, the Koszul-Vinberg metric at each point of the cone gives an inner product on $\mathbb{R}^3$, hence induces an inner product on $sl(3,\mathbb{R}) \subset gl(3,\mathbb{R}) \cong Hom(\mathbb{R}^3,\mathbb{R}^3) \cong \mathbb{R}^3 \otimes \mathbb{R}^{*3}$, and therefore also induces a Riemannian metric on the bundle $sl(3,\mathbb{R})_{\mathrm{Ad}\rho}$, since the Koszul-Vinberg metric is invariant under $\mathrm{Aut}(\mathcal{C})$.

On the other hand, consider the map $m : \Omega \to \mathcal{C}$, which takes $[p] \longmapsto k(p)^{\frac{1}{3}}p$. Setting t a positive constant, we have that tI is an element of $\mathrm{Aut}(\mathcal{C})$. After substituting tI for γ into equation (2.2.6), we obtain that

$$k(tp) = t^{-3}k(p). \tag{2.2.7}$$

Then $k(tp)^{\frac{1}{3}}tp = k(p)^{\frac{1}{3}}p$ and hence m is well-defined (i.e., $k(p)^{\frac{1}{3}}p$ does not depend on the choice of elements in $[p]$). Moreover, after substituting $k(p)^{\frac{1}{3}}$ for t into equation (2.2.7), we obtain that $k(k(p)^{\frac{1}{3}}p) = k(p)^{-1}k(p) = 1$ and hence $m(\Omega) = k^{-1}(1)$.

The Riemannian metric $m^*(d^2 \log k)$ on Ω is invariant under Γ. Hence $m^*(d^2 \log k)$ defines a Riemannian metric on Ω/Γ. Thus corresponding to every convex $\mathbb{R}P^2$-structure on S, there exists an associated Riemannian metric on S. Now the remainder is similar to the definition of the Loftin metric: we first have the induced metric $\widetilde{g}_{DG}$ on the space $\mathcal{A}^1(S; sl(3,\mathbb{R})_{Ad\rho})$ and then define Laplacian operator on the space, hence obtain harmonic representatives as kernel of the Laplacian operator. The DG metric g_{DG} on the cohomologous classes as tangent vectors is actually defined as the metric $\widetilde{g}_{DG}$ on harmonic representatives.

2.3 Embedding of Teichmüller Space

Inside this section, we fix the notation as follows:

(i) the hyperbolic affine sphere(the hyperboloid) $H = \{{x_3}^2 - {x_1}^2 - {x_2}^2 = 1\}$;

(ii) the domain $\Omega = \{{t_1}^2 + {t_2}^2 < 1\} \subset \mathbb{R}^2$.

By definition, Goldman space $\mathcal{B}(S)$ is the space of all convex real projective structures on the surface and we can think of the Teichmüller locus inside $\mathcal{B}(S)$ as the subspace of convex real projective structures which arise from hyperbolic structures. Noting that there are a variety of viewpoints of the deformation space $\mathcal{B}(S)$, in this section we give a detailed description of the embedding of Teichmüller space $\mathcal{T}(S)$ from some different viewpoints and then show their equivalence.

1. When Goldman space $\mathcal{B}(S)$ is identified with the space of affine sphere structures that can be given on the surface S (see Proposition 1.5.4), then the Teichmüller locus consists of points representing surfaces which admits an affine sphere structure whose affine sphere is the hyperboloid H;

2. When Goldman space $\mathcal{B}(S)$ is identified with the space containing pairs (Σ, U) (see Proposition 2.3.5), then the Teichmüller locus is the subspace containing pairs $(\Sigma, 0)$;

3. When Goldman space $\mathcal{B}(S)$ is identified with an open subspace of the space containing conjugate classes of representations $\rho : \pi \to SL(3, \mathbb{R})$, then the Teichmüller locus in $\mathcal{B}(S)$ consists exactly of conjugation classes of representations $\rho' : \pi \to PSL(2, \mathbb{R})$ after composing by the irreducible representation $\Phi : PSL(2, \mathbb{R}) \to SL(3, \mathbb{R})$.

Because of the equivalence of the definitions described above, in following sections we will use the description of the embedding of Teichmüller space from different viewpoints for convenience without explanation.

First note that (2) follows from (1) immediately, because the Pick form C for the hyperboloid vanishes, hence the cubic differential is 0 (see the end of §1.5). Then we continue to describe (1) and (3).

To explain (1): Consider a hyperbolic structure on the surface S, then the hyperbolic surface S has $\mathbb{H}^2$ (the upper half plane in $\mathbb{C}$ with hyperbolic metric $\frac{1}{y^2}|dz|^2$) as its Riemannian cover. The following lemma shows that the hyperbolic metric can be realized as the Blaschke metric on Ω induced from the hyperbolic affine sphere H. Hence the hyperbolic surface S actually admits an affine sphere structure as a quotient of the hyperbolic affine sphere H.

Lemma 2.3.1. *(Kim [Kim06]) Suppose the domain Ω is given with Blaschke metric and $\mathbb{H}^2$ is given with the hyperbolic metric, then the map defined by*

$$F : \Omega \to \mathbb{H}^2$$

$$F(t_1, t_2) = \frac{t_1}{1-t_2} + i\frac{1}{1-t_2}\sqrt{1-{t_1}^2-{t_2}^2}$$

is an isometry.

Proof. Since $x = \frac{t_1}{1-t_2}$, $y = \frac{1}{1-t_2}\sqrt{1-{t_1}^2-{t_2}^2}$, then

$$dx = \frac{1}{1-t_2}dt_1 + \frac{t_1}{(1-t_2)^2}dt_2, \tag{2.3.1}$$

$$dy = \frac{-t_1}{(1-t_2)\sqrt{1-{t_1}^2-{t_2}^2}}dt_1 + \frac{1-{t_1}^2-t_2}{(1-t_2)^2\sqrt{1-{t_1}^2-{t_2}^2}}dt_2. \tag{2.3.2}$$

We note that the function $u = -\sqrt{1-{t_1}^2-{t_2}^2}$ on Ω is the solution to the equation

$$\det(\frac{\partial^2 u}{\partial t^i \partial t^j}) = (\frac{1}{u})^4.$$

Then by applying equations (2.3.1) and (2.3.2), we compute the hyperbolic metric on

$\mathbb{H}^2$:

$$\begin{aligned}
& \frac{1}{y^2}(dx^2+dy^2) \\
= \; & \frac{1-{t_2}^2}{(1-{t_1}^2-{t_2}^2)^2}d{t_1}^2 + \frac{2t_1t_2}{(1-{t_1}^2-{t_2}^2)^2}dt_1dt_2 + \frac{1-{t_1}^2}{(1-{t_1}^2-{t_2}^2)^2}d{t_2}^2 \\
& \text{From the solution } u(t_1,t_2) = -\sqrt{1-{t_1}^2-{t_2}^2}, \\
= \; & -\frac{1}{u}u_{11}d{t^1}^2 - 2\frac{1}{u}u_{12}dt^1dt^2 - \frac{1}{u}u_{11}d{t^2}^2, \\
= \; & -\frac{1}{u}u_{ij}dt^idt^j, \quad \text{the Blaschke metric on } \Omega.
\end{aligned}$$

□

Remark. Instead of (Ω, the Blaschke metric), Kim [Kim06] actually defined the map F in the lemma on (Ω, the Hilbert metric). But in fact the Blaschke metric and the Hilbert metric are the same in this case. Hence our approach in the lemma above is a bit different from his proof in [Kim06]. In the end, we can compute

$$F^{-1}(x,y) = (\frac{2x}{x^2+y^2+1}, 1-\frac{2}{x^2+y^2+1}). \tag{2.3.3}$$

In the following lemma, we give a conformal parametrisation of the hyperbolic affine sphere H on $\mathbb{H}^2$ (see [1.5.1]).

Lemma 2.3.2. *The map* $f : \mathbb{H}^2 \to H \subset \mathbb{R}^3$ *defined as*

$$f(z) = (\frac{x}{y}, \frac{x^2+y^2-1}{2y}, \frac{x^2+y^2+1}{2y}) \tag{2.3.4}$$

is an isometry.

Proof. Once again, set $u = u(t_1,u_2) = -\sqrt{1-{t_1}^2-{t_2}^2}$. Noting that the hypersurface H is exactly the radial graph of the function $-\frac{1}{u}$ (i.e., the image of $-\frac{1}{u}(t_1,t_2,1)$), we have the map $G : (\Omega, \text{the Hilbert metric}) \to (H, \text{the Blaschke metric})$ is an isometry map, where

$$G(t_1,t_2) := -\frac{1}{u}(t_1,t_2,1) = \frac{-1}{\sqrt{1-{t_1}^2-{t_2}^2}}(t_1,t_2,1). \tag{2.3.5}$$

Hence, combining with Lemma 2.3.1, we obtain that the composition map $f := G \circ F^{-1}$ is an isometry from $\mathbb{H}^2 \to H$. Explicitly, we have

$$\begin{aligned} & f(z) = G \circ F^{-1}(z) \\ = \; & G\left(\frac{2x}{x^2+y^2+1}, 1 - \frac{2}{x^2+y^2+1}\right) \text{ by definition (equation (2.3.3)) of } F^{-1}, \\ = \; & \left(\frac{x}{y}, \frac{x^2+y^2-1}{2y}, \frac{x^2+y^2+1}{2y}\right) \text{ by the definition (equation (2.3.5)) of } G. \end{aligned}$$

□

To explain (3): A hyperbolic structure on S determines a holonomy homomorphism $\pi \to PSL(2,\mathbb{R})$. Elements in $PSL(2,\mathbb{R})$ keep the hyperbolic metric invariant. So we hope to find $\Phi : PSL(2,\mathbb{R}) \to SL(3,\mathbb{R})$ such that image of Φ fixes the hyperbolic affine sphere H and the Blaschke metric along H. Equivalently, we wish to show that f in Lemma 2.3.2 is Φ-invariant, since the map f is an isometry from $\mathbb{H}^2$ to $H \subset \mathbb{R}^3$. We eventually realize the hyperbolic structure on S as a convex real projective structure with the map Φ defined in the following proposition:

Proposition 2.3.3. *(Kim [Kim06]) The map $\Phi : PSL(2,\mathbb{R}) \to SL(3,\mathbb{R})$ defined by*

$$\Phi(A) = \begin{pmatrix} ad+bc & ac-bd & ac+bd \\ ab-cd & \frac{a^2-b^2-c^2+d^2}{2} & \frac{a^2+b^2-c^2-d^2}{2} \\ ab+cd & \frac{a^2-b^2+c^2-d^2}{2} & \frac{a^2+b^2+c^2+d^2}{2} \end{pmatrix}$$

for

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an injective homomorphism of $PSL(2,\mathbb{R})$ into $SL(3,\mathbb{R})$ with image $SO(2,1)$ such that the map f in Lemma 2.3.2 is Φ-equivariant.

Remark. The map Φ induces a Lie algebra homomorphism at the identity matrix. Abusing the notation Φ, we obtain that

$$\Phi(A) = \begin{pmatrix} 0 & c-b & c+b \\ b-c & 0 & 2a \\ b+c & 2a & 0 \end{pmatrix}$$

for

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in sl(2, \mathbb{R}),$$

which is a Lie algebra homomorphism of $sl(2, \mathbb{R})$ into $sl(3, \mathbb{R})$ with image $so(2, 1) \subset sl(3, \mathbb{R})$. This map Φ on the Lie algebra will help us connect tangent vectors of Teichmüller space with tangent vectors of the Teichmüller locus in $\mathcal{B}(S)$.

2.4 The restriction of two generalized Weil-Petersson metrics

The goal of this section is to prove

Theorem 2.4.1. *The DG metric and the Loftin metric both restrict to a constant multiple of the Weil-Petersson metric on Teichmüller space.*

It is clear that Theorem 2.4.1 follows from the following two propositions.

Proposition 2.4.2. *The restriction of the DG metric g_{DG} to Teichmüller space is a constant multiple of the restriction of the Loftin metric g_{Loftin} to Teichmüller space.*

Proposition 2.4.3. *The Loftin metric g_{Loftin} restricts to a constant multiple of the Weil-Petersson metric on Teichmüller space.*

Hence the remaining goal of this section is to show the above two propositions. We first finish the proof of Proposition 2.4.2 and then show Proposition 2.4.3.

Proof of Proposition 2.4.2

We start with showing Lemma 2.4.4, which is essential in the proof of Proposition 2.4.2 which compares the Cheng-Yau metric with the Koszul-Vinberg metric on the cone $\mathcal{C} = \{{x_3}^2 > {x_1}^2 + {x_2}^2\}$.

Lemma 2.4.4. *(Sataki [Sas80]) In the case of the cone* $\mathcal{C} = \{{x_3}^2 > {x_1}^2 + {x_2}^2\}$, *the function* $k = ({x_3}^2 - {x_1}^2 - {x_2}^2)^{-\frac{3}{2}}$ *is the characteristic function of the cone* $\mathcal{C}$. *Supposing the function* σ *is the solution to the Cheng-Yau equation*

$$det(\frac{1}{3}(\log\sigma)_{ij}) = \sigma^2 \qquad \text{on the cone } \mathcal{C},$$

then

(i) we have that $k = \sigma$;

(ii) the Koszul-Vinberg metric on the cone is 3 times the Cheng-Yau metric; and

(iii) the hypersurface $k^{-1}(1)$ *coincides with the hypersurface* $\sigma^{-1}(1)$, *the hyperbolic affine sphere which is asymptotic to the boundary of the cone.*

(iv) the metric on $M = \Omega/\Gamma$ *where* $\Omega = \{{t_1}^2 + {t_2}^2 < 1\} \subset\subset \mathbb{R}^2 \subset \mathbb{R}P^2$, *induced from Koszul-Vinberg metric on the cone, is 3 times the affine metric obtained from immersing the hyperbolic affine sphere* $\sigma^{-1}(1)$.

Remark. Part (i), (ii), (iii) of this lemma is already proved in by Sataki in [Sas80] (the statement is a bit different from the original version), here we give a detailed computation to prove it.

Proof. (i): In the cone $\mathcal{C} = \{{x_3}^2 > {x_1}^2 + {x_2}^2\}$, the characteristic function is $k(x) = ({x_3}^2 - {x_1}^2 - {x_2}^2)^{-\frac{3}{2}}$ (see Example 4.2, pp 67 in [Shi07]).

Then we have the following computation

$$
\begin{aligned}
& \det(\frac{1}{3}\frac{\partial^2 \log k}{\partial x^i \partial x^j}) \\
= \;& \det(\frac{1}{3}\frac{\partial^2 \log({x_3}^2 - {x_1}^2 - {x_2}^2)^{-\frac{3}{2}}}{\partial x^i \partial x^j}) \\
= \;& \det(-\frac{1}{2}\frac{\partial^2 \log({x_3}^2 - {x_1}^2 - {x_2}^2)}{\partial x^i \partial x^j}) \\
= \;& \det\begin{pmatrix} \frac{2{x_1}^2+t}{t^2} & \frac{2x_1x_2}{t^2} & \frac{-2x_1x_3}{t^2} \\ \frac{2x_1x_2}{t^2} & \frac{2{x_2}^2+t}{t^2} & \frac{-2x_2x_3}{t^2} \\ \frac{-2x_1x_3}{t^2} & \frac{-2x_2x_3}{t^2} & \frac{2{x_3}^2-t}{t^2} \end{pmatrix} \qquad (2.4.1) \\
= \;& t^{-3}, \text{after setting } t = {x_3}^2 - {x_1}^2 - {x_2}^2 \\
= \;& k^2, \text{noting that } k = t^{-\frac{3}{2}}.
\end{aligned}
$$

Therefore by the uniqueness of solution of the Cheng-Yau equation, the function $k(x) = ({x_3}^2 - {x_1}^2 - {x_2}^2)^{-\frac{3}{2}}$ coincides with the solution σ to the Cheng-Yau equation

$$\det(\frac{1}{3}(\log \sigma)_{ij}) = \sigma^2.$$

This finishes the proof of (i).

(ii): From the definition of the Koszul-Vinberg metric as $d^2(\log k)$ and the Cheng-Yau metric as $\frac{1}{3}(d^2 \log \sigma)$, and combining with the equality $k = \sigma$ in (i), we conclude the proof of (ii).

(iii) immediately follows from the equality $k = \sigma$ in (i).

(iv): Combining the fact that the metric on $M = \Omega/\Gamma$ is the quotient metric of the Koszul-Vinberg metric restricted to the hypersurface $k^{-1}(1)$ and the affine metric on M is the quotient metric of the Cheng-Yau metric restricted to the hypersurface $\sigma^{-1}(1)$ with the the facts in (ii) and (iii), we conclude the proof of (iv). □

For further reference, we collect the concepts and facts associated to the convex $\mathbb{R}P^2$-structure M on the surface arising from the hyperbolic structures.

(I) the domain $\Omega = \{{t_1}^2 + {t_2}^2 < 1\} \subset \mathbb{R}^2 \subset \mathbb{R}P^2$;

(II) the holonomy group $\Gamma < Aut(\mathcal{C}) < SL(3, \mathbb{R})$;

(III) the manifold $M \cong \Omega/\Gamma$;

(IV) the solution $\sigma(x) = ({x_3}^2 - {x_1}^2 - {x_2}^2)^{-\frac{3}{2}}$ to the Cheng-Yau equation

$$\det(\frac{1}{3}(\log \sigma)_{ij}) = \sigma^2,$$

on the cone $\mathcal{C}$;

(V) the solution $u(t_1, t_2) = -\sqrt{1 - {t_1}^2 - {t_2}^2}$ to the Cheng-Yau equation

$$\det(\frac{\partial^2 u}{\partial t^i \partial t^j}) = (\frac{1}{u})^4$$

on the domain Ω;

(VI) the hyperbolic affine sphere H asymptotic to the boundary of the cone $\mathcal{C}$ with center 0 and constant affine mean curvature -1 is exactly the hypersurface $\sigma^{-1}(1)$, and also the radial graph of the function $-\frac{1}{u}$ (i.e., the image of $-\frac{1}{u}(t_1, t_2, 1)$).

Proof of Proposition 2.4.2. We start by noting that the Teichmüller locus in $\mathcal{B}(S)$ exactly contains the convex $\mathbb{R}P^2$ manifolds which are diffeomorphic to Ω/Γ, where $\Omega = \{{t_1}^2 + {t_2}^2 < 1\}$.

Then by comparing the two different pairings $\widetilde{g}_{DG}$ and $\widetilde{g}_{Loftin}$ on the space $\mathcal{A}^1(S, sl(3, \mathbb{R})_{Ad\rho})$, and by (ii),(iii) and (iv) of Lemma 2.4.4, we obtain that they are isometric when restricted to Teichmüller locus (up to a constant mulitiple).

Next, since the pairing $\widetilde{g}_{DG}$ and the pairing $\widetilde{g}_{Loftin}$ are isometric (up to a constant multiple), then the harmonic representatives α^{DG}_{harm} and α^{Loftin}_{harm} in the cohomology class $[\alpha] \in H^1(S; sl(3, \mathbb{R})_{Ad\rho})$, i.e. in the kernel of Laplacian operators for the metrics $\widetilde{g}_{DG}$ and $\widetilde{g}_{Loftin}$ respectively on the space $\mathcal{A}^1(S, sl(3, \mathbb{R})_{Ad\rho})$, are the same.

Finally, recall that

$$\begin{aligned} g_{DG}([\alpha],[\beta]) &= \widetilde{g}_{DG}(\alpha^{DG}_{harm},\beta^{DG}_{harm}) \\ \text{and} \quad g_{Loftin}([\alpha],[\beta]) &= \widetilde{g}_{Loftin}(\alpha^{Loftin}_{harm},\beta^{Loftin}_{harm}), \end{aligned}$$

for $[\alpha],[\beta] \in H^1(S; sl(3,\mathbb{R})_{Ad\rho})$. From above arguments, we then conclude the pairing $g_{DG}([\alpha],[\beta])$ and $g_{Loftin}([\alpha],[\beta])$ give the same result (up to a constant multiple). □

Proof of Proposition 2.4.3

We want to show that the Loftin metric restricts to be the Weil-Petersson metric on Teichmüller space. It requires some preparation to achieve this goal. We need to understand the following two objects:

1. an explicit description of the metric l on the Lie algebra bundle $sl(3,\mathbb{R})_{Ad\rho}$.
2. harmonic representatives in the cohomology class which are tangent vectors on Teichmüller space, since we define the Loftin metric after choosing the harmonic representative in its cohomology class.

(1) The Metric l on the Lie Algebra Bundle

The main goal of this part is to give an explicit formula for the metric l on the bundle $sl(3,\mathbb{R})_{Ad\rho}$ by Lemma 2.4.5 and hence to compute the norm of $\Phi\left(\begin{smallmatrix} -z & z^2 \\ -1 & z \end{smallmatrix}\right)$ (Φ is the Lie algebra homomorphism of $sl(2,\mathbb{R})$ into $sl(3,\mathbb{R})$ defined in the remark in the end of §2.3) in Corollary 2.4.6, which is useful in step (iii) of the proof of Proposition 2.4.3.

Lemma 2.4.5. *Supposing that the metric l is defined on the bundle $sl(3,\mathbb{R})_{Ad\rho}$, and the matrix h is the matrix presentation of the Cheng-Yau metric at a point p of the* *hyperbolic affine sphere H under the standard basis of $\mathbb{R}^3$(i.e., $e_1 = (1,0,0)^T, e_2 = (0,1,0)^T, e_3 = (0,0,1)^T$, and $h(e_i, e_j) = h_{ij}$), we then obtain that*

$$l_p(A,B) = tr(A^T h^{-1} Bh) \qquad \text{for } A, B \in sl(3,\mathbb{R}).$$

Proof. Recall that $l = h \otimes h^*$. Suppose we have the matrix presentation of the Cheng-Yau metric h in the standard basis as (h_{ij}) and the dual basis for $\{e_1, e_2, e_3\}$ is $\{e^1, e^2, e^3\}$ and the matrix presentation of the inverse Cheng-Yau metric h^{-1} is (h^{ij}) with $h^{ij} = h^{-1}(e^i, e^j)$.

Next assuming that the matrix $A = (a_i^j), B = (b_k^l)$ (i, k denote for the row), we obtain that $Ae_i = \sum_{\{j=1,2,3\}}(a_i^j)e_j, Be_k = \sum_{\{l=1,2,3\}}(b_k^l)e_l$. Then we may identify A with $\sum_{\{i,j=1,2,3\}} a_i^j e_j \otimes e^i$ and B with $\sum_{\{i,j=1,2,3\}} b_k^l e_l \otimes e^k$, which is exactly the identification of $Hom(\mathbb{R}^3, \mathbb{R}^3) \cong \mathbb{R}^3 \otimes \mathbb{R}^{*3}$.

$$\begin{aligned}
\text{Hence} \quad & l_p(A,B) \\
= \; & l_p\Big(\sum_{\{i,j=1,2,3\}} a_i^j e_j \otimes e^i, \sum_{\{k,l=1,2,3\}} b_k^l e_l \otimes e^k\Big) \\
= \; & \sum_{\{i,j,k,l=1,2,3\}} a_i^j b_k^l l(e_j \otimes e^i, e_l \otimes e^k) \;\; \text{by the linearility of } l \\
= \; & \sum_{\{i,j,k,l=1,2,3\}} a_i^j b_k^l h(e_j, e_l) h^{-1}(e^i, e^k) \\
& (\text{since } l = h \otimes h^*), \\
= \; & \sum_{\{i,j,k,l=1,2,3\}} a_i^j b_k^l h_{jl} h^{ik} \\
= \; & \sum_{\{i,j,k,l=1,2,3\}} a_i^j h^{ik} b_k^l h_{jl} \\
= \; & tr(A^T h^{-1} Bh) \;\; \text{from } A = (a_i^j), B = (b_k^l). \quad \square
\end{aligned}$$

Corollary 2.4.6. *Let $\mathcal{C} = \{{x_3}^2 > {x_1}^2 + {x_2}^2\}$, and $p = f(z) \in H$ for some $z = x + iy \in \mathbb{H}$, after extending the definition of l in Lemma 2.4.5 by $l_p(A,B) = tr(A^T h^{-1} \overline{B} h)$,* *for* $A, B \in sl(3,\mathbb{C})$, *we obtain*

$$l_p(\Phi\left(\begin{smallmatrix} -z & z^2 \\ -1 & z \end{smallmatrix}\right), \Phi\left(\begin{smallmatrix} -z & z^2 \\ -1 & z \end{smallmatrix}\right)) = 16y^2.$$

Proof. Lemma 2.3.3 implies that

$$\Phi\begin{pmatrix} -z & z^2 \\ -1 & z \end{pmatrix} = \begin{pmatrix} 0 & -z^2-1 & z^2-1 \\ z^2+1 & 0 & -2z \\ z^2-1 & -2z & 0 \end{pmatrix}, \text{denoted as } A.$$

To calculate $l_{f(z)}(A,A) = tr(A^T h^{-1}\overline{A}h)$, we need know the matrix presentation of the Cheng-Yau metric h, and also h^{-1}.

We are in the case of the cone $\mathcal{C} = \{{x_3}^2 > {x_1}^2 + {x_2}^2\}$, hence the Cheng-Yau metric $h = \frac{1}{3}d^2 \log\sigma$, for $\sigma = ({x_3}^2 - {x_1}^2 - {x_2}^2)^{-\frac{3}{2}}$(see Lemma 2.4.4). Since we now restrict ourselves to the hypersurface $\sigma^{-1}(1)$, i.e. $\{{x_3}^2 - {x_1}^2 - {x_2}^2 = 1\}$, hence the function t in the equation (2.4.1) is identically 1, and then we have an explicit formula for the Cheng-Yau metric h and hence h^{-1} as follows,

$$h = \begin{pmatrix} 2{x_1}^2+1 & 2x_1x_2 & -2x_1x_3 \\ 2x_1x_2 & 2{x_2}^2+1 & -2x_2x_3 \\ -2x_1x_3 & -2x_2x_3 & 2{x_3}^2-1 \end{pmatrix}, \; h^{-1} = \begin{pmatrix} 2{x_1}^2+1 & 2x_1x_2 & 2x_1x_3 \\ 2x_1x_2 & 2{x_2}^2+1 & 2x_2x_3 \\ 2x_1x_3 & 2x_2x_3 & 2{x_3}^2-1 \end{pmatrix}.$$

Applying the equation (2.3.4), i.e., $f(z) = (x_1, x_2, x_3) = (\frac{x}{y}, \frac{x^2+y^2-1}{2y}, \frac{x^2+y^2+1}{2y})$ to the above matrices, we obtain

$$h = \begin{pmatrix} \frac{2x^2}{y^2}+1 & \frac{x(x^2+y^2-1)}{y^2} & -\frac{x(x^2+y^2+1)}{y^2} \\ \frac{x(x^2+y^2-1)}{y^2} & \frac{(x^2+y^2-1)^2}{2y^2}+1 & -\frac{(x^2+y^2-1)(x^2+y^2+1)}{2y^2} \\ -\frac{x(x^2+y^2+1)}{y^2} & -\frac{(x^2+y^2-1)(x^2+y^2+1)}{2y^2} & \frac{(x^2+y^2+1)^2}{2y^2}-1 \end{pmatrix} \text{ and}$$

$$h^{-1} = \begin{pmatrix} \frac{2x^2}{y^2}+1 & \frac{x(x^2+y^2-1)}{y^2} & \frac{x(x^2+y^2+1)}{y^2} \\ \frac{x(x^2+y^2-1)}{y^2} & \frac{(x^2+y^2-1)^2}{2y^2}+1 & \frac{(x^2+y^2-1)(x^2+y^2+1)}{2y^2} \\ \frac{x(x^2+y^2+1)}{y^2} & \frac{(x^2+y^2-1)(x^2+y^2+1)}{2y^2} & \frac{(x^2+y^2+1)^2}{2y^2}-1 \end{pmatrix}.$$

Then finally we compute

$$\begin{aligned} l_{f(z)}(A, A) &= tr(A^T h^{-1} \overline{A} h) \\ &= 16y^2, \text{ where } z = x + iy \in \mathbb{H}. \quad \square \end{aligned}$$

(2) Harmonic Representative

The goal of this part is to show

Lemma 2.4.7. *Given a Riemann surface* Σ *and* Φ*:* $sl(2,\mathbb{R}) \to sl(3,\mathbb{R})$ *defined in the Remark of Lemma 2.3.3, then*

(i) the holomorphic tangent space at the point $(\Sigma, 0)$ *of the image of Teichmüller space in* $\mathcal{B}(S)$ *is exactly spanned by the cohomology class of* $\phi(z)dz \otimes \Phi\left(\begin{smallmatrix} -z & z^2 \\ -1 & z \end{smallmatrix}\right)$*, where* $\phi(z)dz^2$ *is a holomorphic quadratic differential on* Σ*; and*

(ii) the globally defined $sl(3,\mathbb{R})_{Ad\rho}$*-valued 1-forms of the form* $\phi(z)dz \otimes \Phi\left(\begin{smallmatrix} -z & z^2 \\ -1 & z \end{smallmatrix}\right)$ *are harmonic representatives (in the sense of the Loftin metric on* $\mathcal{A}^1(S; sl(3,\mathbb{R})_{Ad\rho})$*) in their own cohomology class.*

Proof. (i): First, we note that the image of Teichmüller space in $\mathcal{B}(S)$ exactly contains the representations $\rho : \pi \to SL(3,\mathbb{R})$ which are a composition of $\rho' : \pi \to PSL(2,\mathbb{R})$ with $\Phi : PSL(2,\mathbb{R}) \to SO(2,1) \subset SL(3,\mathbb{R})$. Combining this with the fact that the tangent space of Teichmüller space at Σ contains exactly the cohomology classes of $sl(2,\mathbb{R})_{Ad\rho'}$-valued 1-forms of the form $\phi(z)dz \otimes \left(\begin{smallmatrix} -z & z^2 \\ -1 & z \end{smallmatrix}\right)$ (see [Hej78] for details), we conclude the statement of (i).

(ii): (a) We first note that $\phi(z)dz \otimes \Phi\left(\begin{smallmatrix} -z & z^2 \\ -1 & z \end{smallmatrix}\right) \in \mathcal{A}^1(S; sl(3,\mathbb{R})_{Ad\rho})$ is closed. This follows from the computation

$$\begin{aligned}
& d[\phi(z)dz \otimes \Phi\left(\begin{smallmatrix} -z & z^2 \\ -1 & z \end{smallmatrix}\right)] \\
= \;& d[z^2\phi(z)dz \otimes \Phi\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) - \phi(z)dz \otimes \Phi\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) - 2z\phi(z)dz \otimes \Phi\left(\begin{smallmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{smallmatrix}\right)] \\
= \;& d[z^2\phi(z)dz \otimes \left(\begin{smallmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right) - \phi(z)dz \otimes \left(\begin{smallmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right) - 2z\phi(z)dz \otimes \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)] \\
& \text{by the definition of } \Phi \\
= \;& d(z^2\phi(z)dz) \otimes \left(\begin{smallmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right) - d(\phi(z)dz) \otimes \left(\begin{smallmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right) - d(2z\phi(z)dz) \otimes \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right) \\
= \;& 0, \qquad \text{since } \phi(z) \text{ is holomorphic and hence so are } z^2\phi(z) \text{ and } 2z\phi(z).
\end{aligned}$$

(b) We next prove that $\phi(z)dz \otimes \Phi\left(\begin{smallmatrix} -z & z^2 \\ -1 & z \end{smallmatrix}\right)$ is coclosed, which is the heart of the lemma. From definition of δ (see equation (2.2.5)), it is enough to show $d * (\sharp)(\phi(z)dz \otimes \Phi\left(\begin{smallmatrix} -z & z^2 \\ -1 & z \end{smallmatrix}\right)) = 0$, which follows from

$$\begin{aligned}
& d * (\sharp)[\phi(z)dz \otimes \Phi\left(\begin{smallmatrix} -z & z^2 \\ -1 & z \end{smallmatrix}\right)] \\
= \;& d * \sharp[z^2\phi(z)dz \otimes \Phi\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) - \phi(z)dz \otimes \Phi\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) - 2z\phi(z)dz \otimes \Phi\left(\begin{smallmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{smallmatrix}\right)] \\
= \;& d * [z^2\phi(z)dz \otimes \sharp(\Phi\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)) - \phi(z)dz \otimes \sharp(\Phi\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)) \\
& \qquad - 2z\phi(z)dz \otimes \sharp(\Phi\left(\begin{smallmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{smallmatrix}\right))]
\end{aligned} \tag{2.4.2}$$

We then want to calculate $\sharp(\Phi\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)), \sharp(\Phi\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)), \sharp(\Phi\left(\begin{smallmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{smallmatrix}\right))$.

We choose a basis for $sl(3,\mathbb{R})$ as $\{E_1 = \Phi\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right), E_2 = \Phi\left(\begin{smallmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right), E_3 = \Phi\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right), E_4 = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), E_5 = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), E_6 = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), E_7 = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right), E_8 = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)\}$. The map $\sharp : sl(3,\mathbb{R})_{Ad\rho} \to sl(3,\mathbb{R}^*)_{Ad\rho^*}$ is defined by setting

$$\sharp(v)_x(u_x) = l_x(u_x, v_x), \quad \text{for } u_x, v_x \in sl(3,\mathbb{R}), x \in S,$$

we then have

$$\sharp(A)_x = \sum_{\{1 \le i \le 8\}} l(A, E_i)(E_i)^*, \quad \text{for } A \in sl(3,\mathbb{R}), \tag{2.4.3}$$

where $(E_i)^*$ satisfies

$$(E_i)^*(E_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Applying Lemma 2.4.5 to compute $l(E_i, E_j)$, for all $1 \leq i \leq 3, 1 \leq j \leq 8$, and then substituting the values into Equation (2.4.3), we obtain the following formulas:

$$\begin{aligned}
\sharp(E_1) = & \quad \tfrac{4}{y^2}(E_1^* - x^2 E_3^* + xE_2^*) + \tfrac{-1-x^2}{y^2} E_4^* + \tfrac{1-x^2}{y^2} E_5^* + \tfrac{2x}{y^2} E_8^*. \\
\sharp(E_2) = & \quad \tfrac{4}{y^2}(xE_1^* - x(x^2+y^2)E_3^* + (x^2 + \tfrac{1}{2}y^2)E_2^*) + \tfrac{-x^3-x-xy^2}{y^2} E_4^* \\
& + \tfrac{-x^3+x-xy^2}{y^2} E_5^* + (\tfrac{2x^2}{y^2} + 1)E_8^*. \\
\sharp(E_3) = & \quad \tfrac{4}{y^2}(-x^2 E_1^* + (x^2+y^2)^2 E_3^* - x(x^2+y^2)E_2^*) + \tfrac{x^4+y^4+2x^2y^2+x^2}{y^2} E_4^* \\
& + \tfrac{x^4+y^4+2x^2y^2-x^2}{y^2} E_5^* - \tfrac{2x}{y^2} E_8^*.
\end{aligned}$$

Finally we apply the above formulas to compute Equation (2.4.2)

$$\begin{aligned}
= \;& d * [z^2\phi(z)dz \otimes \sharp(E_1) - \phi(z)dz \otimes \sharp(E_3) - 2z\phi(z)dz \otimes \sharp(E_2)] \\
= \;& d * [-4\phi(z)dz \otimes E_1^* + 4z^2\phi(z)dz \otimes E_3^* - 4z\phi(z)dz \otimes E_2^* \\
& +(z^2+1)\phi(z)dz \otimes E_4^* + (z^2-1)\phi(z)dz \otimes E_5^* - 2z\phi(z)dz \otimes E_8^*]
\end{aligned}$$

Observe that all terms inside the bracket remain holomorphic again.

$$\begin{aligned}
= \;& d[-4i\overline{\phi(z)}d\overline{z} \otimes E_1^* + 4i\overline{\phi(z)}\overline{z}^2 d\overline{z} \otimes E_3^* - 4i\overline{\phi(z)}\overline{z}d\overline{z} \otimes E_2^* \\
& +i(\overline{z}^2+1)\overline{\phi(z)}d\overline{z} \otimes E_4^* + i(\overline{z}^2-1)\overline{\phi(z)}d\overline{z} \otimes E_5^* - 2i\overline{z}\overline{\phi(z)}d\overline{z} \otimes E_8^*] \\
= \;& 0, \qquad \text{since } \phi(z) \text{ is holomorphic.}
\end{aligned}$$

Thus (a) and (b) together imply that $\phi(z)dz \otimes \Phi\left(\begin{smallmatrix} -z & z^2 \\ -1 & z \end{smallmatrix}\right)$ is harmonic. □

We now have the ingredients we need to prove Proposition 2.4.3.

Proof of Proposition 2.4.3. It is sufficient to prove the complexified version, i.e., we compute (with explanations of the steps given at the conclusion of the computation)

$$< [\phi(z)dz \otimes \Phi\left(\begin{smallmatrix} -z & z^2 \\ -1 & z \end{smallmatrix}\right)], [\psi(z)dz \otimes \Phi\left(\begin{smallmatrix} z & z^2 \\ -1 & z \end{smallmatrix}\right)] >_{Loftin}$$

$$\overset{(i)}{=\!=} \quad Re \int_S (\phi(z)dz) \wedge *(\psi(z)dz) l_{f(z)}(\Phi\left(\begin{smallmatrix} -z & z^2 \\ -1 & z \end{smallmatrix}\right), \Phi\left(\begin{smallmatrix} z & z^2 \\ -1 & z \end{smallmatrix}\right))$$

$$\overset{(ii)}{=\!=} \quad Re \int_S \phi(z)dz \wedge (i\overline{\psi(z)}d\overline{z}) l_{f(z)}(\Phi\left(\begin{smallmatrix} -z & z^2 \\ -1 & z \end{smallmatrix}\right), \Phi\left(\begin{smallmatrix} z & z^2 \\ -1 & z \end{smallmatrix}\right))$$

$$\overset{(iii)}{=\!=} \quad Re(16i \int_S \phi(z)\overline{\psi(z)} y^2 dz d\overline{z})$$

$$\overset{(iv)}{=\!=} \quad 32 < \phi dz^2, \psi dz^2 >_{WP}$$

(i): Lemma 2.4.7 implies that $\phi(z)dz \otimes \Phi\left(\begin{smallmatrix} -z & z^2 \\ -1 & z \end{smallmatrix}\right)$ is harmonic. Recall the definition of the Loftin metric, if $\sigma \otimes \phi$ is harmonic, we have

$$g_{\text{Loftin}}([\sigma \otimes \phi], [\sigma' \otimes \phi']) = \int_S (\sigma \wedge *\sigma\prime) l(\phi, \phi').$$

(ii): Since x, y are conformal coordinates for the Blaschke metric on the surface, we extend the action of the Hodge star operator to complex 1-forms by complex-antilinearity, i.e., $*(i\alpha) = -i * \overline{\alpha}$. From the definition of Hodge star (see equation (2.2.3), we see that $*dx = dy, *dy = -dx$, then $*\phi(z)dz = i\overline{\phi(z)}d\overline{z}$.

(iii): By Corollary 2.4.6,

$$l_{f(z)}(\Phi\left(\begin{smallmatrix} -z & z^2 \\ -1 & z \end{smallmatrix}\right), \Phi\left(\begin{smallmatrix} z & z^2 \\ -1 & z \end{smallmatrix}\right)) = 16y^2.$$

(iv): From the definition of the Weil-Petersson co-metric,

$$< \phi dz^2, \psi dz^2 >_{WP} = Re \int_S \phi(z)\overline{\psi(z)} y^2 dx dy$$

(see [Gol84], page 212) and $dz \wedge d\overline{z} = -2i dx \wedge dy$. $\square$

2.5 Teichmüller space is totally geodesic in $\mathcal{B}(S)$

The goal of this section is to prove

Theorem 2.5.1. *Teichmüller space endowed with (a constant multiple of) the Weil-Petersson metric is totally geodesic in $\mathcal{B}(S)$, endowed with the Loftin metric.*

To achieve this goal, we make use of a dual map $\tau : \mathcal{B}(S) \to \mathcal{B}(S)$ which takes $(\Sigma, U) \to (\Sigma, -U)$, where U is a holomorphic cubic differential on Σ. Therefore, the fixed set of this dual map τ is exactly the Teichmüller locus $\cong \{(\Sigma, 0)\}$. We will see it is sufficient to show the following theorem:

Theorem 2.5.2. *The dual map τ is an isometry of $\mathcal{B}(S)$ with respect to the Loftin metric.*

We now show how to derive Theorem 2.5.1 from Theorem 2.5.2.

Proof of Theorem 2.5.1. It is known (see [Kob72]) that the fixed set of an isometry of a Riemannian manifold is a totally geodesic submanifold. Consider the manifold $\mathcal{B}(S)$ endowed with the Loftin metric g_{Loftin}: we first have that the set $\{(\Sigma, 0)\} \cong$ Teichmüller space is the fixed set of the dual map τ on the manifold $\mathcal{B}(S)$, next that the dual map τ is an isometry of $\mathcal{B}(S)$ from Theorem 2.5.2, and moreover that the Loftin metric restricts to be a constant multiple of the Weil-Petersson metric on the Teichmüller locus from Theorem 2.4.1. Combining with the fact stated in the first sentence, we conclude that Teichmüller space endowed with (a constant multiple of) the Weil-Petersson metric is totally geodesic in $\mathcal{B}(S)$ with the Loftin metric. $\square$

Thus, the remaining goal of this section is to show Theorem 2.5.2. We divide the remaining part of this section into three parts.

Part (I): we first define the conormal map of hyperbolic affine spheres and then state Proposition 2.5.4 which tells us that the dual map τ is in fact induced by the conormal map ν of hyperbolic affine spheres. The reason we consider the conormal map ν instead of the dual map τ is that it is closely related to the definition of the Loftin metric, which involves the Cheng-Yau metric of the cone restricted to the hyperbolic affine sphere.

Part (II): we compare the Cheng-Yau metric on the cone restricted to the hyperbolic affine sphere H and the one on the dual cone restricted to the dual hyperbolic affine sphere (image of the conormal map ν) by showing Lemma 2.5.5, which will be essential in the proof of Theorem 2.5.2.

Part (III): we describe the induced tangent map of the dual map τ on the tangent space of Goldman space $\mathcal{B}(S)$. Then we continue to finish the proof of Theorem 2.5.2.

Part (I)

We start with the definition of the conormal map ν. Let $H \subset \mathbb{R}^3$ be a nondegenerate hypersurface transverse to its position vector. Let $\mathbb{R}_3$ be the dual space of $\mathbb{R}^3$. We now define a map $\nu : H \to \mathbb{R}_3$ as follows (see §5 in [NS94]).

For each $p \in H$, let ν_p be the element of $\mathbb{R}_3$ such that

$$\nu_p(\vec{p}) = 1 \qquad \text{and} \qquad \nu_p(X) = 0 \quad \text{for all} \quad X \in T_p(H).$$

We have thus a differentiable map $\nu : H \to \mathbb{R}_3$, called the conormal map. (This construction can be done with any transverse vector field ξ in place of $\vec{p}$.)

Proposition 2.5.3. *(Schirokov-Schirokov [SS62], unpublished work of Calabi, Gigena [Gig78] [Gig81]) The image of the conormal map ν of a hyperbolic affine sphere H with center 0 and affine mean curvature -1 is another such hyperbolic affine sphere*

$\overline{H}$ *in the dual space* $\mathbb{R}_3$.

The conormal map can descend to be defined on the quotient of hyperbolic affine spheres (in other words, affine sphere structures); we obtain the following proposition which says that the dual map τ of Goldman space $\mathcal{B}(S)$ is in fact induced by the conormal map ν.

Proposition 2.5.4. *(Loftin [Lof01], [Lof10]) Given a properly convex* $\mathbb{R}P^2$*-manifold* $M = \Omega/\Gamma$*, the conormal map* ν *with respect to the affine sphere structure induces a map to the dual manifold* $M^* = \Omega^*/\Gamma^*$*, where* $\gamma^* \in \Gamma^*$ *is defined by* $\gamma^* y(x) = y(\gamma^{-1}x)$*, for all* $x \in \mathbb{R}^3, y \in \mathbb{R}_3, \gamma \in \Gamma$*. This map is an isometry of the affine metrics. Moreover the conormal map* ν *induces the dual map* $\tau : \mathcal{B}(S) \to \mathcal{B}(S)$ *takes* $(\Sigma, U) \to (\Sigma, -U)$*, where* U *is a holomorphic cubic differential on* Σ*.*

Remark. If we identify $\mathbb{R}_3$ with $\mathbb{R}^3$ by standard inner product, i.e., identify y with y^T, we obtain an induced identification between $sl(3, \mathbb{R})$ and $sl(3, \mathbb{R}^*)$, and between $SL(3, \mathbb{R})$ and $SL(3, \mathbb{R}^*)$. Then we rephrase the description of γ^* in the above proposition as follows: for any $\gamma \in \Gamma$, we have

$$\gamma^*(y) = (\gamma^T)^{-1} y, \text{for all } y \in \mathbb{R}^3. \tag{2.5.1}$$

Part (II)

The goal of this part is to show the following lemma:

Lemma 2.5.5. *Suppose* h *is the matrix presentation of the Cheng-Yau metric (under the standard basis) on the cone restricted to the hyperbolic affine sphere inside the cone and* h^* *is the matrix presentation of the Cheng-Yau metric on the dual cone restricted* *to the dual hyperbolic affine sphere, then if we identify* $\mathbb{R}_3$ *with* $\mathbb{R}^3$ *by the standard inner product, we have* $h^*_{\nu(p)} = h_p^{-1}$*, for* $p \in H$*.*

Proof of Lemma 2.5.5. Take a conformal parametrisation of the hyperbolic affine sphere H as the map $f : \mathcal{D} \to H$ with the coordinates x, y. Consider the conormal map $\nu : H \to \overline{H} \subset \mathbb{R}_3$, composing with the map f, to obtain a map $\nu \circ f : \mathcal{D} \to \overline{H}$. Abusing notation, we continue to write the new map $\nu \circ f$ as ν, so that we then have

$$\nu(f) = 1, \nu(f_x) = 0, \nu(f_y) = 0. \tag{2.5.2}$$

We next derive some properties of tangent map ν_*:

Lemma 2.5.6. *(Proposition 5.1 in [NS94]) Given a hyperbolic affine sphere H, the conormal map ν on H, and the affine metric $g = e^{\psi}|dz|^2$ along H, then*

$$\nu_*(Y)(\vec{p}) = 0 \quad \textit{and} \quad \nu_*(Y)(X) = -g(Y, X) \quad \textit{for all} \quad X, Y \in T_pH.$$

Applying Lemma 2.5.6, and denote $\nu_x = \nu_*(\frac{\partial}{\partial x}), \nu_y = \nu_*(\frac{\partial}{\partial y}), f_x = f_*(\frac{\partial}{\partial x}), f_y = f_*(\frac{\partial}{\partial y})$, we have

$$\nu_x(f_x) = -g(f_x, f_x) = -e^{\psi}, \quad \nu_x(f_y) = 0, \quad \nu_x(f) = 0; \tag{2.5.3}$$

$$\nu_y(f_x) = 0, \quad \nu_y(f_y) = -g(f_y, f_y) = -e^{\psi}, \quad \nu_y(f) = 0. \tag{2.5.4}$$

Collect equations (2.5.2), (2.5.3) and (2.5.4) together, then we obtain

$$\begin{pmatrix} \nu \\ -e^{-\frac{1}{2}\psi}\nu_x \\ -e^{-\frac{1}{2}\psi}\nu_y \end{pmatrix} \cdot (f, e^{-\frac{1}{2}\psi} f_x, e^{-\frac{1}{2}\psi} f_y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Denote the matrix $A = (f, e^{-\frac{1}{2}\psi} f_x, e^{-\frac{1}{2}\psi} f_y)$, and we identify $\mathbb{R}^3$ with $\mathbb{R}_3$ by standard inner product, i.e., identify v with v^T. Then, we have

$$A^{-1^T} = (\nu^T, -e^{-\frac{1}{2}\psi}\nu_x^T, -e^{-\frac{1}{2}\psi}\nu_y^T).$$

We have the fact that pair $\{e^{-\frac{1}{2}\psi} f_x, e^{-\frac{1}{2}\psi} f_y\}$ is an orthonormal basis for the affine metric on H, and combining with Proposition 2.2.1, we then obtain that $\{f, e^{-\frac{1}{2}\psi} f_x, e^{-\frac{1}{2}\psi} f_y\}$ is an orthonormal basis for the Cheng-Yau metric on the cone restricted to H. Therefore we have

$$A^T h A = I. \tag{2.5.5}$$

Similarly, $\{\nu^T, -e^{-\frac{1}{2}\psi}\nu_x^T, -e^{-\frac{1}{2}\psi}\nu_y^T\}$ is also an orthonormal basis for the Cheng-Yau metric on the cone $\mathcal{C}^* \subset \mathbb{R}_3$ restricted to the dual hyperbolic affine sphere $\overline{H}$, therefore we have

$$A^{-1} h^* A^{-1^T} = I. \tag{2.5.6}$$

Combining equations (2.5.5) and (2.5.6), we obtain that $h^* = h^{-1} = AA^T$. □

Part (III)

We give an explicit description of how the dual map τ affects the tangent vector by showing the following lemma.

Lemma 2.5.7. *The dual map τ on $\mathcal{B}(S)$ induces the map τ_* on the tangent space $T_{[\rho]}\mathcal{B}(S) \cong H^1(S; sl(3,\mathbb{R})_{Ad\rho})$ as follows:*

$$\begin{aligned} \tau_* : \quad H^1(S; sl(3,\mathbb{R})_{Ad\rho}) &\to H^1(S; sl(3,\mathbb{R})_{Ad\rho^*}) \\ [\sigma \otimes \phi] &\longmapsto [\sigma \otimes \phi^*] \end{aligned}$$

where $\forall s \in S$, we have

$$\phi_s^* = -\phi_s^T, \textit{ where } [\sigma \otimes \phi] \in H^1(S; sl(3,\mathbb{R})_{Ad\rho}). \tag{2.5.7}$$

Proof. We carry out the proof in three steps.

Step 1: We first show that the tangent vector is represented by the cohomology class $[u]$ of 1-cocycle $u : \pi \to sl(3,\mathbb{R})$ satisfying $u(xy) - u(x) = Ad(\rho(x))(u(y))$, for $x, y \in \pi$.

Consider a family $\{M_t \cong \Omega_t/\Gamma_t\}$ of convex $\mathbb{R}P^2$-structures on the surface S, with corresponding family of conjugation classes of representations $\rho_t \in Hom(\pi, SL(3,\mathbb{R}))$ (i.e., $\Gamma_t = \rho_t(\pi)$). Taking the derivative of both sides of the equation

$$\rho_t(xy) = \rho_t(x)\rho_t(y), \qquad \text{for all } x, y \in \pi,$$

we obtain that the tangent vector at $\rho_0 = \rho$ is a 1-cocycle $u : \pi \to sl(3,\mathbb{R})$ satisfying (see [Gol84] for details)

$$u(xy) - u(x) = Ad(\rho(x))(u(y)).$$

Step 2: We next want to show that the map

$$\tau_* : H^1(\pi, sl(3,\mathbb{R})) \to H^1(\pi, sl(3,\mathbb{R}))$$

takes the tangent vector $u \in H^1(\pi, sl(3,\mathbb{R}))$ to

$$\tau_*(u)(\gamma) = -(u(\gamma))^T \in sl(3,\mathbb{R}), \text{for all } \gamma \in \pi. \tag{2.5.8}$$

Considering the family $\tau(\rho_t) = \rho_t^* \in Hom(\pi, SL(3,\mathbb{R}))$ and substituting γ for $\rho_t{}^*(\gamma)$ into equation (2.5.1), we obtain

$$\rho_t{}^*(\gamma)y = (\rho_t(\gamma)^T)^{-1}y, \text{for all } y \in \mathbb{R}^3 \tag{2.5.9}$$

Taking the derivative of both sides of the above equation, we find that the tangent vector $\tau_*(u)$ satisfies

$$\tau_*(u)(\gamma)y = -(u(\gamma))^T y, \text{for all } y \in \mathbb{R}^3$$

Therefore, we obtain that

$$\tau_*(u)(\gamma) = -(u(\gamma))^T \in sl(3,\mathbb{R}), \text{ for all } \gamma \in \pi. \tag{2.5.10}$$

Step 3: We want to show that $\tau_*([\sigma \otimes \phi]) = [\sigma \otimes \phi^*]$. Consider the canonical isomorphism from

$$\begin{aligned} H^1(M, sl(3,\mathbb{R}^*)_{Ad\rho}) &\to H^1(\pi, sl(3,\mathbb{R})) \\ [\sigma \otimes \phi] &\longmapsto u_{\sigma\otimes\phi}, \end{aligned} \tag{2.5.11}$$

where $u_{\sigma\otimes\phi}(\gamma) := \int_{\tilde{\gamma}} \sigma \otimes \phi \in sl(3,\mathbb{R}), \forall \gamma \in \pi$ and $\overline{\gamma}$ is an arbitrary closed curve in S representing γ.

For any $\gamma \in \pi$, we compute

$$\begin{aligned} u_{[\sigma\otimes\phi^*]}(\gamma) &= \int_{\overline{\gamma}} \sigma \otimes \phi^* \quad \text{by equation (2.5.11)} \\ &= -\int_{\overline{\gamma}} \sigma \otimes \phi^T \quad \text{by equation (2.5.7)} \\ &= -(\int_{\overline{\gamma}} \sigma \otimes \phi)^T \\ &= -(u_{[\sigma\otimes\phi]}(\gamma))^T \quad \text{by equation (2.5.11)} \\ &= \tau_*(u_{[\sigma\otimes\phi]})(\gamma) \;\; \text{after substituting } u_{[\sigma\otimes\phi]} \text{ for } u \text{ in equation (2.5.8).} \end{aligned}$$

Hence we have that $\tau_*(u_{[\sigma\otimes\phi]}) = u_{[\sigma\otimes\phi^*]}$. Combining with the injectivity of the map between $H^1(S, sl(3,\mathbb{R})_{Ad\rho})$ and $H^1(\pi, sl(3,\mathbb{R}))$, we conclude that $\tau_*([\sigma \otimes \phi]) = [\sigma \otimes \phi^*]$. □

Proof of Theorem 2.5.2. We carry out the proof in three steps.

Step 1: We show that the pairing on the two vector spaces $\mathcal{A}^1(S, sl(3,\mathbb{R})_{Ad\rho})$ and $\mathcal{A}^1(S, sl(3,\mathbb{R})_{Ad\rho^*})$ are isometric under $\mu : \sigma \otimes \phi \to \sigma \otimes \phi^*$.

Consider the Riemannian metric l on the Lie algebra bundle $sl(3,\mathbb{R})_{Ad\rho}$, we compute

$$
\begin{aligned}
& l(\phi,\phi')|_x \quad \text{taking } x\in S \\
= \;& l_{f(\widetilde{x})}(A,B) \quad \text{by equation (2.2.1)} \\
& \text{taking } \widetilde{\phi}_{f(\widetilde{x})}=A, \text{and } \widetilde{\phi'}_{f(\widetilde{x})}=B, \text{ where } \widetilde{x} \text{ is a preimage of } x \\
& \text{ in the domain } \mathcal{D}\subset\mathbb{C} \\
= \;& tr(A^T h^{-1} B h) \quad \text{by Lemma 2.4.5} \\
= \;& tr((A^T h^*)(B h^{*-1})) \quad \text{by Lemma 2.5.5, } h^*=h^{-1} \\
= \;& tr((B h^{*-1})(A^T h^*)) \\
= \;& l^*_{\tau(\widetilde{x})}(B^T,A^T) \quad \text{by substituting } l \text{ with } l^* \text{ into Lemma 2.4.5} \\
= \;& l^*_{\tau(\widetilde{x})}(-A^T,-B^T) \quad \text{by the symmetry and linearility of } l^* \\
= \;& l^*(\phi^*,\phi'^*)|_x \quad \text{by Lemma 2.5.7, } \widetilde{\phi}^*_{\tau(\widetilde{x})}=-A^T, \widetilde{\phi'}^*_{\tau(\widetilde{x})}=-B^T
\end{aligned}
$$

We denote this result as fact (1).

Furthermore, from Proposition 2.5.4, we see that the two Blaschke (or affine) metrics on $M\cong\Omega/\Gamma$ and $M^*\cong\Omega^*/\Gamma^*$ are isometric, hence we have fact (2):The Riemannian metrics on space of 1-forms induced by the two Blaschke metrics are isometric. Hence we compute

$$
\begin{aligned}
& g(\sigma\otimes\phi^*,\sigma'\otimes\phi'^*) \\
= \;& \int_S(\sigma\wedge *\sigma')l^*(\phi^*,\phi'^*) \quad \text{by definition of } g \text{ on } \mathcal{A}^1(S,sl(3,\mathbb{R})_{Ad\rho^*}), \\
= \;& \int_S(\sigma\wedge *\sigma')l(\phi,\phi') \quad \text{by fact (1) and (2)}, \\
= \;& g(\sigma\otimes\phi,\sigma'\otimes\phi') \quad \text{by definition of } g \text{ on } \mathcal{A}^1(S,sl(3,\mathbb{R})_{Ad\rho}).
\end{aligned}
$$

Therefore the Riemannian metrics on the two vector spaces $\mathcal{A}^1(S,sl(3,\mathbb{R})_{Ad\rho})$ and $\mathcal{A}^1(S,sl(3,\mathbb{R})_{Ad\rho^*})$ are isometric under μ.

Step 2: We show that if $\sigma \otimes \phi$ is the harmonic representative in the cohomology class $[\sigma \otimes \phi]$, then $\sigma \otimes \phi^*$ is also the unique harmonic representative in the cohomology class $[\sigma \otimes \phi^*]$.

We begin by noting that, because $\sigma \otimes \phi$ is a harmonic representative, we have equivalently that,

$$d(\sigma \otimes \phi) = 0, \quad \delta(\sigma \otimes \phi) = 0.$$

Note that d is linear, $d(\sigma \otimes \phi) = 0$ implies that $d(-\sigma \otimes \phi^T) = 0$, i.e., $\sigma \otimes \phi^* = -\sigma \otimes \phi^T$ is closed.

Next we show that $\sigma \otimes \phi^* = -\sigma \otimes \phi^T$ is coclosed. Now $\delta(\sigma \otimes \phi) = 0$ and definition δ (see equation (2.2.5)) implies that $d * \sharp(\sigma \otimes \phi) = 0$, and thus $d * (\sigma \otimes \sharp\phi) = 0$. Suppose $\sharp$ and $\sharp^*$ are defined on the two vector spaces $\mathcal{A}^1(S, sl(3,\mathbb{R})_{Ad\rho})$ and $\mathcal{A}^1(S, sl(3,\mathbb{R})_{Ad\rho^*})$ respectively. Assume the basis of $sl(3,\mathbb{R})$ is $\{E_i^j\}$, i.e., where$\{E_i^j\}$ is 3×3 matrix whose entries are all 0 except for the (i,j)-entry which is equal to 1, where $1 \leq i, j \leq 3$ and i, j are not both 3. From the definition of $\sharp$ and $\sharp^*$, we have

$$\sharp(u) = l(u, E_i^j)(E_i^j)^*, \quad \sharp^*(u) = l^*(u, E_i^j)(E_i^j)^*.$$

Hence $d * (\sigma \otimes \sharp\phi) = 0$ implies that $d * (\sigma \otimes l(\phi, E_i^j))(E_i^j)^* = 0$, therefore

$$d * (\sigma \otimes l(\phi, E_i^j)) = 0, \quad \text{where } 1 \leq i, j \leq 3 \text{ and } i, j \text{ are not both 3.} \tag{2.5.12}$$

Then we compute

$$\begin{aligned}
& d * (\sigma \otimes \sharp^*(-\phi^T)) \\
= \; & d * (\sigma \otimes l^*(-\phi^T, E_i^j)(E_i^j)^*) \\
= \; & d * (-\sigma \otimes l(\phi, E_j^i)(E_i^j)^*) \text{ Step 1 implies that } l^*(-A^T, -B^T) = l(A, B) \\
= \; & -d * (\sigma \otimes l(\phi, E_j^i))(E_i^j)^* \\
= \; & 0 \quad \text{by above equation (2.5.12), as all the coefficients of } (E_i^j)^* \text{ vanish.}
\end{aligned}$$

Hence $d * \sharp^*(\sigma \otimes (-\phi^T)) = 0$, and then $\delta(\sigma \otimes (-\phi^T)) = 0$. Therefore $\tau_*(\sigma \otimes \phi) = \sigma \otimes (-\phi^T)$ is also a harmonic representative.

Step 3: Supposing $\sigma \otimes \phi$ is the harmonic representative in its cohomology class, we compute (explanations of each step are given in the end of the computations)

$$
\begin{aligned}
& g_{\text{Loftin}}(\tau_*[\sigma \otimes \phi], \tau_*[\sigma' \otimes \phi']) \\
\overset{(i)}{=\!=} \; & g_{\text{Loftin}}([\sigma \otimes \phi^*], [\sigma' \otimes \phi'^*]) \\
\overset{(ii)}{=\!=} \; & g(\sigma \otimes \phi^*, \sigma' \otimes \phi'^*) \\
\overset{(iii)}{=\!=} \; & g(\sigma \otimes \phi, \sigma' \otimes \phi') \\
\overset{(iv)}{=\!=} \; & g_{\text{Loftin}}([\sigma \otimes \phi], [\sigma' \otimes \phi'])
\end{aligned}
$$

(i): Lemma 2.5.7 implies that $\tau_*([\sigma \otimes \phi]) = [\sigma \otimes \phi^*]$.

(ii): Because $\sigma \otimes \phi$ is the harmonic representative in its cohomology class, Step 2 tells us that $\sigma \otimes \phi^*$ is also the harmonic representative.

(iii): Step 1 implies that the Riemannian metrics on the two vector spaces $\mathcal{A}^1(S, sl(3, \mathbb{R})_{Ad\rho})$ and $\mathcal{A}^1(S, sl(3, \mathbb{R})_{Ad\rho^*})$ are isometric under μ.

(iv): By definition of the Loftin metric by choosing harmonic representatives.

Therefore, we conclude the theorem. $\square$

Chapter 3

Asymptotics of Hitchin Components

In this chapter, we describe the asymptotic behavior of representations for certain families of Higgs bundles in the Hitchin Component. This work will appear in a joint paper with Brian Collier (UIUC).

3.1 Introduction

Given a polystable Higgs bundle (E, ϕ), consider the family of polystable Higgs bundles $(E, t\phi)$, where $t \in \mathbb{C}$. Solving the Higgs bundle equations yields a family of harmonic metrics h_t on E and thus a family of flat connections ∇_t with corresponding representations ρ_t. For $P, P' \in \widetilde{\Sigma}$, let $T_{P,P'}(t)$ be the parallel transport matrices of the family of flat connections. In a recent preprint [KNPS13], Katzarkov, Noll, Pandit, and Simpson asked the following question:

Question. *What is the asymptotic behavior of ρ_t and $T_{P,P'}(t)$ as $t \to \infty$?*

They call this the Hitchin WKB problem. As it involves asymptotically solving the Higgs bundle equations, it seems to be a difficult problem. In this paper we restrict to the following situation

- (E, ϕ) is in the Hitchin component
- $t \in \mathbb{R}$
- $\phi = \tilde{e}_1 + q_n e_{n-1}$ and $\phi = \tilde{e}_1 + q_{n-1} e_{n-2}$.

Instead of $t\phi$, we use tq_n and tq_{n-1}, which are equivalent to $t\phi$ after a gauge transformation.

For the Hitchin component, such asymptotics are related to the compactification of $Hit_n(S)$ due to Parreau [Par12] for general n and Kim [Kim05] for $n = 3$. When $n = 2$, the Hitchin component is the classical Teichmüller space; in [Wol91], Wolf gave a detailed description of the asymptotics for families of harmonic maps associated to rays (Σ, tq_2) inside Teichmüller space. He realized the appropriate limit of the corresponding harmonic maps from the universal cover $\widetilde{\Sigma}$ to the hyperbolic plane $\mathbb{H}^2$ as a harmonic map into the $\mathbb{R}$-tree dual to the horizontal measured foliation determined by q_2 [Wol95].

For $n = 3$, the Hitchin component was identified with the space of convex real projective structures on S by Choi and Goldman [CG93]. Using this identification, Labourie [Lab07b] and Loftin [Lof01] independently showed that the space of convex real projective structures on S is in bijection with the space of pairs (Σ, q_3) where Σ is a Riemann surface structure on S and q_3 is a holomorphic cubic differential on Σ. Away from zeros of q_3, Loftin [Lof07] considered the asymptotic holonomy of the convex real projective structures corresponding to the ray (Σ, tq_3) as $t \to \infty$. Recently, Dumas and Wolf [DW14] have given a correspondence between cubic polynomial differentials and polygons in $\mathbb{RP}^2$. As $t \to \infty$, such polygons serve as local models for neighborhoods around the zeros of the cubic differentials.

In context of Higgs bundles, for $SL(2, \mathbb{C})$-Higgs bundles (E, ϕ) such that the zeros of $det(\phi)$ are simple, Mazzeo, Swoboda, Weiss, and Witt [MSWW14] have recently proven very precise asymptotics for the harmonic metric solving the Higgs bundle equations for the family $(E, t\phi)$. In particular, they are able to describe the limiting metric around the zeros of the $det(\phi)$.

We now describe our main results.

Theorem 3.2.3. *Let (E, ϕ) be in the Hitchin component with $\phi = \tilde{e}_1 + \sum_{j=0 \ mod \ k} q_j e_{j-1}$, for some $k \leq n$. Then the harmonic metric solving the Higgs bundle equation splits as a direct sum metric on $E = E_1 \oplus \cdots \oplus E_k$ where*

$$E_j = K^{\frac{n+1-2j}{2}} \oplus K^{\frac{n+1-2j}{2}-k} \oplus K^{\frac{n+1-2j}{2}-2k} \oplus \cdots$$

Note that the dots for the holomorphic bundle E_j go on as long as it makes sense. For instance, when $k = n$, there is only one summand for each j. In the restricted setting $\phi = \tilde{e}_1 + q_n e_{n-1}$ and $\phi = \tilde{e}_1 + q_{n-1} e_{n-2}$, we obtain

Corollary 3.2.4. *For $k = n$ and $k = n-1$ the Higgs fields are $\phi = \tilde{e}_1 + q_n e_{n-1}$ and $\phi = \tilde{e}_1 + q_{n-1} e_{n-2}$, and the harmonic metric splits as*

$$h_1 \oplus h_2 \oplus \cdots \oplus h_2^{-1} \oplus h_1^{-1}$$

on the line bundles

$$K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{-\frac{n-3}{2}} \oplus K^{-\frac{n-1}{2}}$$

For $k = n$, this was proven by Baraglia [Bar10a, Bar10b], and used to study, amongst other things, the relation between the Higgs bundle equations and the affine Toda equations. Higgs fields of the form $\phi = \tilde{e}_1 + \sum_{j=0 \ mod \ k} q_j e_{j-1}$ are fixed points of an action by the k^{th} roots of unity, this will be crucial in the proof of Theorem 3.2.3.

Corollary 3.2.4 significantly simplifies the Higgs bundle equations from n^2 equations to $\lfloor \frac{n}{2} \rfloor$ equations. We first obtain estimates for the solution metric h_t of the Higgs bundle equations as $t \to \infty$.

Theorem 3.3.1. *For every point $p \in \Sigma$ away from the zeros of q_n or q_{n-1} as $t \to \infty$ we have*

1. *For* $(\Sigma,\ \tilde{e}_1 + tq_n e_{n-1}) \in Hit_n(S)$, *the metric* $h_j(t)$ *on* $K^{\frac{n+1-2j}{2}}$ *admits the expansion*

$$h_j(t) = (t|q_n|)^{-\frac{n+1-2j}{n}}(1 + O(t^{-\frac{2}{n}})) \quad \textit{for all } j$$

2. *For* $(\Sigma,\ \tilde{e}_1 + tq_{n-1} e_{n-2}) \in Hit_n(S)$, *the metric* $h_j(t)$ *on* $K^{\frac{n+1-2j}{2}}$ *admits the expansion*

$$h_j(t) = \begin{cases} (t|q_{n-1}|)^{-\frac{n+1-2j}{n-1}}(1 + O(t^{-\frac{2}{n-1}})) & \textit{for } j = 1 \ \textit{ and } \ j = n \\ (2t|q_{n-1}|)^{-\frac{n+1-2j}{n-1}}(1 + O(t^{-\frac{2}{n-1}})) & \textit{for } 1 < j < n \end{cases}$$

Making use of asymptotic estimates of the solution metric and error estimates, we integrate the ODE defined by the flat connection. This yields an estimate of the parallel transport matrices $T_{P,P'}(t)$ as $t \to \infty$. For $(\Sigma, (0, \ldots, 0, tq_n)) \in \mathcal{H}_n$, let $P \in \widetilde{\Sigma}$ be a point at which q_n does not vanish. Choose a neighborhood $\mathcal{U}_p$ centered at P, with coordinate z, so that $q_n = dz^n$. Any $P' \in \mathcal{U}_P$ may be written in polar coordinate as $P' = Le^{i\theta}$. Suppose $\gamma(s)$ is a $|q_n|^{2/n}-$geodesic from P to P' parametrized by arc length s.

With an extra condition on the path, we obtain the entire set of asymptotic eigenvalues of the parallel transport operator along the path asymptotically. More precisely, we prove

Theorem 3.4.4. *Suppose* P, P' *and the geodesic path* $\gamma(s)$ *are as above and* P' *is located so that for every* s,

$$s < d(\gamma(s)) := min\{d(\gamma(s), z_0), \ \textit{for all zeros } z_0 \textit{ of } q_n\}.$$

There exists a constant unitary matrix S, *not depending on the pair* P *and* P', *so* *that as* $t \to \infty$,

$$T_{P,P'}(t) = (Id + O(t^{-\frac{1}{2n}}))S \begin{pmatrix} e^{-Lt^{\frac{1}{b}}\mu_1} & & & \\ & e^{-Lt^{\frac{1}{b}}\mu_2} & & \\ & & \ddots & \\ & & & e^{-Lt^{\frac{1}{b}}\mu_n} \end{pmatrix} S^{-1}$$

where $\mu_j = 2cos(\theta + \frac{2\pi(j-1)}{n})$.

Remark 1. In the case that the angle $<_{z_0} (P, P')$ satisfies $<_{z_0} (P, P') < \pi/3$, for all zeros z_0 of q_n, the $|q_n|^{2/n}-$geodesic from P to P' will satisfy the condition in the above theorem.

Remark 2. For $(\Sigma, (0, \ldots, 0, tq_{n-1}, 0)) \in \mathcal{H}_n$, we have similar results in Theorem 3.4.7. In particular, in this case, $\mu_1 = 0$ and for $j > 1$, $\mu_j = 2cos(\theta + \frac{2\pi(j-2)}{n-1})$.

When P and P' both project to the same point in Σ, the projected path is a loop. In this case, the above asymptotics correspond to the values of the associated family of representations on the homotopy class of the loop.

Generalizing Loftin's techniques in [Lof], we also obtain an elementary proof of the special case of Theorem 3.4.4 which only involves the highest eigenvalue, or WKB exponent, of $T_{P,P'}$.

The chapter is organized as follows: In section 3.2, we fix notation, collect the necessary Higgs bundle background and prove the metric splitting theorem. In section 3.3, we explicitly write down the Higgs bundle equations and the flat connections for the two types of Higgs fields considered. Using the explicit equations, we prove Theorem 3.3.1 concerning the asymptotics of the metric, along with an important estimate on the first derivative of the metric. Next, we use our results on the metric to set up the ODE given by parallel transport of the corresponding flat connection in section 3.4. To prove Theorem 3.4.4 and 3.4.7, we need very precise bounds on the error terms, in sections 3.5 and 3.6 we prove these precise bounds and use them to prove the main theorem. In section 3.7, we provide a more elementary proof of a special case of Theorem 3.4.4 concerning the highest eigenvalue, or WKB exponent, of the parallel transport operator.

3.2 Metric Splitting

The moduli space $\mathcal{M}_{Higgs}(SL(n,\mathbb{C}))$ consists of isomorphism classes of polystable $SL(n,\mathbb{C})$-Higgs bundles.

There is a $\mathbb{C}^*$ action on $\mathcal{M}_{Higgs}(SL(n,\mathbb{C}))$ given by $\lambda\cdot[(E,\phi)] = [(E,\lambda\phi)]$. A point $[(E,\phi)]$ is a fixed point of this action if for all $\lambda\in\mathbb{C}^*$, there exists an automorphism $g_\lambda : E\to E$ so that $Ad_{g_\lambda}\phi = \lambda\phi$. We will be interested in the restriction of this action to two subgroups of $\mathbb{C}^*$, these are $U(1)$ and the k^{th} roots of unity $\langle\zeta_k\rangle \subset U(1)$.

On a complex vector bundle E, a Hermitian metric h defines the unitary gauge group $\mathcal{G}_h$, consisting of all bundle isomorphisms which preserve the metric. If we denote the Hermitian adjoint of a bundle isomorphism g by g^{*_h}, then

$$\mathcal{G}_h = \{g : E\to E \mid g^{*_h}g = Id\}.$$

It is well-known, for instance see chapter 6 of [Kob87], that any two metrics h and h' on E are related by $h' = hv$, where $v \in \Omega^0(\Sigma, End(E))$ is positive and self-adjoint with respect to h. Furthermore, the bundle endomorphism v can be decomposed as $v = g^{*_h}g$, where g is a $SL(n,\mathbb{C})$-gauge transformation. This decomposition is unique up to a unitary gauge transformation.

Remark 3.2.1. If we denote the holomorphic structure on E by $\bar\partial_E$, then given a stable Higgs bundle $(\bar\partial_E,\phi)$, the above theorem says there is a unique metric h solving the

Higgs bundle equations (1.3.1). For any $SL(n,\mathbb{C})$-gauge transformation g, the pair $(g^{-1}\bar{\partial}_E g, g^{-1}\phi g)$ also has a unique metric h' solving (1.3.1). The metrics h and h' are related by $h' = hg^{*_h}g$. This follows from general gauge theoretic arguments, for example see section 3 of [Bra90]; it will be crucial in the proof of Theorem 3.2.3.

Remark 3.2.2. Note that if h is a solution metric for a Higgs bundle (E,ϕ), then for all $\lambda \in U(1)$, h is also the solution metric for $(E,\lambda\phi)$. This gives a $U(1)$ action on the moduli space of solutions to (1.3.1). This action and its restriction to the k^{th} roots of unity $\langle \zeta_k \rangle$ play a key role in the proof of Theorem 3.2.3.

We now prove the metric splitting theorem which generalizes results of Baraglia [Bar10b].

Theorem 3.2.3. *Let (E,ϕ) be a Higgs bundle in the Hitchin component $Hit_n(S)$ with*

$$\phi = \tilde{e}_1 + \sum_{j=0 \ mod \ k} q_j e_{j-1}.$$

Then the metric solving the Higgs bundle equation (1.3.1) splits as a direct sum metric on $E = E_1 \oplus \cdots \oplus E_k$, where

$$E_j = K^{\frac{n+1-2j}{2}} \oplus K^{\frac{n+1-2j}{2}-k} \oplus K^{\frac{n+1-2j}{2}-2k} \oplus \cdots .$$

Proof. Let $\zeta_k = e^{\frac{2\pi i}{k}}$ and consider the gauge transformation of $E = K^{\frac{n-1}{2}} \oplus \cdots \oplus K^{-\frac{n-1}{2}}$ given by

$$g_k = \begin{pmatrix} \zeta_{2k}^{1-n} & & & \\ & \zeta_{2k}^{3-n} & & \\ & & \ddots & \\ & & & \zeta_{2k}^{n-1} \end{pmatrix} : E \longrightarrow E \qquad (3.2.1)$$

Note that $g_{_k}^T Q g_{_k} = Q$ where Q is the orthogonal structure, so $g_{_k}$ is indeed an $SO(n,\mathbb{C})$-gauge transformation. The action of $g_{_k}$ on the Higgs field is

$$Ad_{g_{_k}}(\tilde{e}_1 + \sum_{j=2}^{n} q_j e_{j-1}) = \zeta_{_k}\tilde{e}_1 + \sum_{j=2}^{n} \zeta_{_k}^{1-j} q_j e_{j-1}.$$

Thus, for

$$\phi = \tilde{e}_1 + \sum_{j=0 \ mod \ k} q_j e_{j-1},$$

we have $Ad_{g_{_k}}\phi = \zeta_{_k}\phi$, so (E, Q, ϕ) is a fixed point of the k^{th} roots of unity action on $\mathcal{M}_{Higgs}(SL(n,\mathbb{R})$.

One checks that, with respect to $g_{_k}$, the eigenbundle decomposition $E = E_1 \oplus \cdots \oplus E_k$ is given by

$$E_j = K^{\frac{n+1-2j}{2}} \oplus K^{\frac{n+1-2j}{2}-k} \oplus K^{\frac{n+1-2j}{2}-2k} \oplus \cdots .$$

To see that the metric h splits, we will show the gauge transformation $g_{_k}$ is unitary, that is $g_{_k}^* g_{_k} = Id$. Since the triple $(\bar{\partial}_E, \phi, h)$ solves the Higgs bundle equations (1.3.1), by remark 3.2.1, the triple $(g_{_k}^{-1}\bar{\partial}_E g_{_k}, g_{_k}^{-1}\phi g_{_k}, h g_{_k}^* g_{_k})$ also solves (1.3.1). We have computed

$$(g_{_k}^{-1}\bar{\partial}_E g_{_k}, g_{_k}^{-1}\phi g_{_k}) = (\bar{\partial}_E, \zeta_{_k}\phi),$$

thus $(\bar{\partial}_E, \zeta_{_k}\phi, h g_{_k}^* g_{_k})$ solves (1.3.1) as well. Now, using the $U(1)$ action and remark 3.2.2, the triple $(\bar{\partial}_E, \phi, h g_{_k}^* g_{_k})$ solves (1.3.1). By uniqueness of the metric,

$$h = h g_{_k}^* g_{_k}$$

proving that $g_{_k}$ is unitary. Since $g_{_k}$ is both unitary and preserves the eigenbundle splitting $E_1 \oplus \cdots \oplus E_k$, the metric h splits as $h_1 \oplus \cdots \oplus h_k$. $\square$

So far, we have only used $SL(n,\mathbb{C})$ properties of the Higgs bundles in the Hitchin component, we now use the $SL(n,\mathbb{R})$ nature of the Hitchin component to further constrain the metric. Since h is a metric on an orthogonal bundle, it gives a reduction of structure from $SO(n,\mathbb{C})$ to $SO(n,\mathbb{R})$. As a result, it must be Q-orthogonal, that is

$$h^T Q h = Q.$$

This leads to the following corollary, which is known for $k = n$ [Bar10b].

Corollary 3.2.4. *For $k = n$ and $k = n-1$, the Higgs fields are $\phi = \tilde{e}_1 + q_n e_{n-1}$ and $\phi = \tilde{e}_1 + q_{n-1}e_{n-2}$, and the harmonic metric splits as*

$$h_1 \oplus h_2 \oplus \cdots \oplus h_2^{-1} \oplus h_1^{-1}$$

on the direct sum of line bundles

$$K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{-\frac{n-3}{2}} \oplus K^{-\frac{n-1}{2}}.$$

Here h_j^{-1} denotes the induced metric on the dual bundle.

Proof. When $k = n$, $\phi = \tilde{e}_1 + q_n e_{n-1}$ is a fixed point of the n^{th} roots of unity action. By Theorem 3.2.3, the original holomorphic decomposition is the eigenbundle decomposition of (3.2.1), and $h = h_1 \oplus \cdots \oplus h_n$. The constraint coming from Q is

$$\begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_n \end{pmatrix} \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_n \end{pmatrix} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix},$$

thus $h_1 h_n = h_2 h_{n-1} = \cdots = 1$ and the metric splits as $h = h_1 \oplus h_2 \oplus \cdots \oplus h_2^{-1} \oplus h_1^{-1}$. When $k = n-1$, $\phi = \tilde{e}_1 + q_{n-1}e_{n-2}$ is a fixed point of the $(n-1)^{st}$ roots of unity action, and the eigenbundle splitting of theorem 3.2.3 is

$$(K^{\frac{n-1}{2}} \oplus K^{-\frac{n-1}{2}}) \oplus K^{\frac{n-3}{2}} \oplus K^{\frac{n-5}{2}} \oplus \cdots \oplus K^{-\frac{n-5}{2}} \oplus K^{-\frac{n-3}{2}}.$$

The condition $h^T Q h = Q$ is

$$\begin{pmatrix} h_1^{11} & h_1^{21} & & \\ h_1^{12} & h_1^{22} & & \\ & & \ddots & \\ & & & h_{n-1} \end{pmatrix} \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & & & 1 \\ & & & \iddots & \\ & & 1 & & \end{pmatrix} \begin{pmatrix} h_1^{11} & h_1^{12} & & \\ h_1^{21} & h_1^{22} & & \\ & & \ddots & \\ & & & h_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & & & 1 \\ & & & \iddots & \\ & & 1 & & \end{pmatrix}$$

thus $h_2 h_{n-1} = h_3 h_{n-2} = \cdots = 1$. The constraint $det(h) = 1$ together with

$$\begin{pmatrix} h_1^{11} & h_1^{21} \\ h_1^{12} & h_1^{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h_1^{11} & h_1^{12} \\ h_1^{21} & h_1^{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

implies

$$\begin{pmatrix} h_1^{11} & h_1^{12} \\ h_1^{21} & h_1^{22} \end{pmatrix} = \begin{pmatrix} h_1^{11} & 0 \\ 0 & (h_1^{11})^{-1} \end{pmatrix}$$

So, in the original splitting $E = K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{-\frac{n-3}{2}} \oplus K^{-\frac{n-1}{2}}$, the metric splits as

$$h = h_1 \oplus h_2 \oplus \cdots \oplus h_2^{-1} \oplus h_1^{-1}.$$

□

Remark 3.2.5. For $\phi = \tilde{e}_1 + q_{n-2} e_{n-3}$, the eigenbundle splitting from Theorem 3.2.3 is

$$(K^{\frac{n-1}{2}} \oplus K^{-\frac{n-3}{2}}) \oplus (K^{\frac{n-3}{2}} \oplus K^{-\frac{n-1}{2}}) \oplus K^{\frac{n-5}{2}} \oplus \cdots \oplus K^{-\frac{n-5}{2}}$$

the metric splits as $h = h_1 \oplus h_2 \oplus h_3 \oplus \cdots \oplus h_{n-2}$ where h_1 and h_2 are metrics on rank 2 bundles. The condition $h^T Q h = Q$ tells us $h_3 h_{n-2} = \cdots = 1$ and $h_1^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} h_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which does not imply the metric splits on line bundles. Thus, looking at fixed points for smaller k gives less information about the metric.

Since $\phi^* = h^{-1}\overline{\phi}^T h$, one checks that with respect to the eigenbundle splitting of Theorem 3.2.3, fixed points of $\langle \zeta_k \rangle$ in the Hitchin component have the following form

$$\phi = \begin{pmatrix} & & & \phi_k \\ \phi_1 & & & \\ & \ddots & & \\ & & \phi_{k-1} & \end{pmatrix} \qquad \text{and} \qquad \phi^{*_h} = \begin{pmatrix} & \phi_1^* & & \\ & & \ddots & \\ & & & \phi_{k-1}^* \\ \phi_k^* & & & \end{pmatrix}$$

with $\phi_j : E_j \to E_{j+1} \otimes K$ and $\phi_j^* : E_{j+1} \to E_j \otimes \overline{K}$, where $j+1$ is taken *mod* k. The adjoint ϕ_j^* is defined by $\phi_j^* = h_j^{-1}\overline{\phi_j}^T h_{j+1}$.

Thus, for fixed points of $\langle \zeta_k \rangle$, the equation (1.3.1) simplifies to the following system of coupled equations

$$F_{A_j} + \phi_{j-1} \wedge \phi_{j-1}^* + \phi_j^* \wedge \phi_j = 0. \tag{3.2.2}$$

These equations are a special case of the twisted quiver bundle equations considered in [ÁCGP03].

The above results have generalizations to fixed points of the k^{th} roots of unity actions on the moduli space of G-Higgs bundles, which we call k-cyclic Higgs bundles. This is also the topic of Collier's thesis [Colon].

3.3 Equations, Flat Connections and Metric Asymptotics

In this section we will use Corollary 3.2.4 to write the Higgs bundle equations as a system of $\lfloor \frac{n}{2} \rfloor$ fully coupled nonlinear elliptic equations and write down the corresponding flat connections. After this we prove the main theorem about the asymptotics of the metric solving the Higgs bundle equations.

There are slight differences when n is even compared to when n is odd. We will always work in the even case and mention what the differences are for the odd case.

One obvious difference in the odd case is the middle line bundle of E is a trivial bundle; for both $\phi = \tilde{e}_1 + q_n e_{n-1}$ and $\phi = \tilde{e}_1 + q_{n-1} e_{n-2}$, the metric on the trivial line bundle is the standard one on $\mathbb{C}$.

3.3.1 Equations

Since the metric splits as $h = h_1 \oplus h_2 \oplus \cdots \oplus h_2^{-1} \oplus h_1^{-1}$, the adjoints of the Higgs fields $\phi = \tilde{e}_1 + q_n e_{n-1}$ and $\phi = \tilde{e}_1 + q_{n-1} e_{n-2}$ are respectively

$$\phi^* = \begin{pmatrix} & h_1^{-1}h_2 & & \\ & & \ddots & \\ & & & h_1^{-1}h_2 \\ h_1^2\bar{q}_n & & & \end{pmatrix}, \qquad \phi^* = \begin{pmatrix} & h_1^{-1}h_2 & & \\ & & \ddots & \\ h_1 h_2 \bar{q}_{n-1} & & & h_1^{-1}h_2 \\ & h_1 h_2 \bar{q}_{n-1} & & \end{pmatrix}$$

We are interested in the corresponding family of flat connections as the differentials q_n and q_{n-1} are scaled by a real parameter t. Using (3.2.2), the Higgs bundle equations for n-cyclic Higgs field $\phi = \tilde{e}_1 + tq_n e_{n-1}$ become:

$$\begin{cases} F_{A_1} + t^2 h_1^2 q_n \wedge \bar{q}_n - h_1^{-1} h_2 = 0 \\ F_{A_j} + h_{j-1}^{-1} h_j - h_j^{-1} h_{j+1} = 0 \qquad 1 < j < \dfrac{n}{2} \\ F_{A_{\frac{n}{2}}} + h_{\frac{n}{2}-1}^{-1} h_{\frac{n}{2}} - h_{\frac{n}{2}}^{-2} = 0 \end{cases} \tag{3.3.1}$$

Here all the metrics, and hence, all the curvature forms depend on t. We will suppress the t dependence from the notation. When n is odd, the last equation is changed to $F_{A_{\frac{n-1}{2}}} + h_{\frac{n-1}{2}-1}^{-1} h_{\frac{n-1}{2}} - h_{\frac{n-1}{2}}^{-1} = 0$.

To understand the flat connection we choose a local coordinate z on Σ. Such a choice gives a local holomorphic frame $(s_1, s_2, \ldots, s_2^*, s_1^*)$ for E, where $s_j = dz^{\frac{n+1-2j}{2}}$ is the local frame of $K^{\frac{n+1-2j}{2}}$ induced by the coordinate z. With respect to this choice of coordinates, the Higgs field is locally given by

$$\phi = \begin{pmatrix} & & & f_n \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix} dz$$

where $q_n = f_n dz^n$, for some function f_n.

With respect to this frame, locally represent the metric h_j by $e^{-\lambda^j}$, here the j is a superscript and **not** an exponent. Recall that in a holomorphic frame, the Chern connection has connection 1-form $A = H^{-1}\partial H$ and curvature 2-form given by $F_A = \bar{\partial}(H^{-1}\partial H)$. Since h_j is a metric on a line bundle, the expressions simplify to

$$A_j = -\lambda^j_z dz \qquad \text{and} \qquad F_{A_j} = \lambda^j_{z\bar{z}} dz \wedge d\bar{z}.$$

The equations may be rewritten as:

$$\begin{cases} \lambda^1_{z\bar{z}} + t^2 e^{-2\lambda^1}|q_n|^2 - e^{\lambda^1 - \lambda^2} = 0 \\ \lambda^j_{z\bar{z}} + e^{\lambda^{j-1}-\lambda^j} - e^{\lambda^j - \lambda^{j+1}} = 0 \qquad 1 < j < \dfrac{n}{2} \\ \lambda^{\frac{n}{2}}_{z\bar{z}} + e^{\lambda^{\frac{n}{2}-1} - \lambda^{\frac{n}{2}}} - e^{2\lambda^{\frac{n}{2}}} = 0 \end{cases} \tag{3.3.2}$$

Similarly for $(n-1)$-cylcic Higgs field $\phi = \tilde{e}_1 + t q_{n-1} e_{n-2}$, we may rewrite the Higgs bundle equations as

$$\begin{cases} \lambda^1_{z\bar{z}} + t^2 e^{-\lambda^1 - \lambda^2}|q_{n-1}|^2 - e^{\lambda^1 - \lambda^2} = 0 \\ \lambda^2_{z\bar{z}} + t^2 e^{-\lambda^1 - \lambda^2}|q_{n-1}|^2 + e^{\lambda^1 - \lambda^2} - e^{\lambda^2 - \lambda^3} = 0 \\ \lambda^j_{z\bar{z}} + e^{\lambda^{j-1}-\lambda^j} - e^{\lambda^j - \lambda^{j+1}} = 0 \qquad 2 < j < \dfrac{n}{2} \\ \lambda^{\frac{n}{2}}_{z\bar{z}} + e^{\lambda^{\frac{n}{2}-1} - \lambda^{\frac{n}{2}}} - e^{2\lambda^{\frac{n}{2}}} = 0 \end{cases} \tag{3.3.3}$$

Again, in the odd case, the last equation is changed to $\lambda^{\frac{n-1}{2}}_{z\bar{z}} + e^{\lambda^{\frac{n-1}{2}-1} - \lambda^{\frac{n-1}{2}}} - e^{\lambda^{\frac{n-1}{2}}} = 0$.

3.3.2 Flat Connections

The flat connection is given by $D = A_h + \phi + \phi^*$. If, in the holomorphic frame $(s_1, \ldots, s_{\frac{n}{2}}, s^*_{\frac{n}{2}}, \ldots, s^*_1)$, we have

$$q_n = f_n dz^n \quad \text{and} \quad q_{n-1} = f_{n-1} dz^{n-1}, \tag{3.3.4}$$

then the flat connection for the n-cyclic $\phi = \tilde{e}_1 + tq_n e_{n-1}$ is given by

$$D = \begin{pmatrix} -\lambda^1_z dz & & & tf_n \\ 1 & -\lambda^2_z dz & & \\ & \ddots & \ddots & \\ & & 1 & \lambda^1_z dz \end{pmatrix} + \begin{pmatrix} & e^{\lambda^1-\lambda^2} & & & \\ & & e^{\lambda^2-\lambda^3} & & \\ & & & \ddots & \\ & & & & e^{\lambda^1-\lambda^2} \\ te^{-2\lambda^1}\bar{f}_n & & & & \end{pmatrix}, \tag{3.3.5}$$

and the flat connection for the $(n-1)$-cyclic $\phi = \tilde{e}_1 + tq_{n-1}e_{n-2}$, is

$$D = \begin{pmatrix} -\lambda^1_z dz & & tf_{n-1} & 0 \\ 1 & -\lambda^2_z dz & & tf_{n-1} \\ & & \ddots & \\ & & 1 & \lambda^1_z dz \end{pmatrix} + \begin{pmatrix} & e^{\lambda^1-\lambda^2} & & \\ & & \ddots & \\ te^{-\lambda^1-\lambda^2}\bar{f}_{n-1} & & & e^{\lambda^1-\lambda^2} \\ & te^{-\lambda^1-\lambda^2}\bar{f}_{n-1} & & \end{pmatrix}. \tag{3.3.6}$$

We want to calculate the behavior of the flat connection in the limit $t \to \infty$. To do so, we need to understand the asymptotics of the λ^j's and the asymptotics of their first derivatives λ^j_z. In order to use the maximum principle, we will make a change of variables. Let $\Omega_n \subset \Sigma$ be a compact set away from the zeros of q_n and fix a background metric g_n on Σ with the following properties:

$$\begin{cases} g_n = |q_n|^{\frac{2}{n}} & \text{on } \Omega_n \\ \dfrac{|q_n|^2}{(g_n)^n} \leq 1 & \text{on } \Sigma \end{cases} \tag{3.3.7}$$

Using this metric, we make the following change of variables:

$$u^j = \lambda^j - \frac{n+1-2j}{2}\ln(g_n).$$

For $\phi = \tilde{e}_1 + q_{n-1}e_{n-2}$, we define the analogous compact set Ω_{n-1} and background metric g_{n-1} with the property

$$\begin{cases} g_{n-1} = |q_{n-1}|^{\frac{2}{n-1}} & \text{on } \Omega_{n-1} \\ \dfrac{|q_{n-1}|^2}{(g_{n-1})^{n-1}} \leq 1 & \text{on } \Sigma \end{cases} \tag{3.3.8}$$

Using g_{n-1}, we make the change of variables

$$v^j = \lambda^j - \frac{n+1-2j}{2}\ln(g_{n-1}).$$

Recall that the Laplace-Beltrami operator of a conformal metric g on a Riemann surface is given by $\Delta_g = \frac{4}{g}\partial_{z\bar{z}}$ and the scalar curvature is

$$K_g = -\frac{1}{2}\Delta_g \ln(g) = -\frac{2}{g}\partial_{z\bar{z}}\ln(g).$$

Because q_n and q_{n-1} are holomorphic, $K_{g_n} = 0 = K_{g_{n-1}}$ on Ω_n and Ω_{n-1}.

With respect to u^j, the equations for $\phi = \tilde{e}_1 + tq_n e_{n-1}$ become

$$\begin{cases} (u^1 + \frac{n-1}{2}\ln(g_n))_{z\bar{z}} + t^2 e^{-2u^1-(n-1)\ln(g_n)}|q_n|^2 - e^{u^1-u^2+\ln(g_n)} = 0 \\ (u^j + \frac{n+1-2j}{2}\ln(g_n))_{z\bar{z}} + e^{u^{j-1}-u^j+\ln(g_n)} - e^{u^j-u^{j+1}+\ln(g_n)} = 0 & 1 < j < \frac{n}{2} \\ (u^{\frac{n}{2}} + \frac{1}{2}\ln(g_n))_{z\bar{z}} + e^{u^{\frac{n}{2}-1}-u^{\frac{n}{2}}+\ln(g_n)} - e^{2u^{\frac{n}{2}}+\ln(g_n)} = 0 \end{cases} \tag{3.3.9}$$

Using our knowledge of K_{g_n} and Δ_{g_n}, we rewrite the equations as

$$\begin{cases} -\frac{1}{4}\Delta_{g_n}u^1 = -\frac{n-1}{4}K_{g_n} + \frac{t^2|q_n|^2}{g_n^n}e^{-2u^1} - e^{u^1-u^2} \\ -\frac{1}{4}\Delta_{g_n}u^j = -\frac{n+1-2j}{4}K_{g_n} + e^{u^{j-1}-u^j} - e^{u^j-u^{j+1}} & 1 < j < \frac{n}{2} \\ -\frac{1}{4}\Delta_{g_n}u^{\frac{n}{2}} = -\frac{1}{4}K_{g_n} + e^{u^{\frac{n}{2}-1}-u^{\frac{n}{2}}} - e^{2u^{\frac{n}{2}}} \end{cases} \tag{3.3.10}$$

We will show

$$\lim_{t\to\infty} e^{u^j} = t^{\frac{n+1-2j}{n}} \qquad 1 \le j \le \frac{n}{2}.$$

Similarly, in terms of the v^j's, the equations for $\phi = \tilde{e}_1 + tq_{n-1}e_{n-2}$ become

$$\begin{cases} -\frac{1}{4}\Delta_{g_{n-1}} v^1 = -\frac{n-1}{4}K_{g_{n-1}} + \frac{t^2|q_{n-1}|^2}{g_{n-1}^{n-1}} e^{-v^1-v^2} - e^{v^1-v^2} \\ -\frac{1}{4}\Delta_{g_{n-1}} v^2 = -\frac{n-1}{4}K_{g_{n-1}} + \frac{t^2|q_{n-1}|^2}{g_{n-1}^{n-1}} e^{-v^1-v^2} + e^{v^1-v^2} - e^{v^2-v^3} \\ -\frac{1}{4}\Delta_{g_{n-1}} v^j = -\frac{n+1-2j}{4}K_{g_{n-1}} + e^{v^{j-1}-v^j} - e^{v^j-v^{j+1}} & 2 < j < \frac{n}{2} \\ -\frac{1}{4}\Delta_{g_{n-1}} v^{\frac{n}{2}} = -\frac{1}{4}K_{g_{n-1}} + e^{v^{\frac{n}{2}-1}-v^{\frac{n}{2}}} - e^{2v^{\frac{n}{2}}} \end{cases} \tag{3.3.11}$$

In this case, it will be shown that

$$\lim_{t\to\infty} e^{v^1} = t$$

$$\lim_{t\to\infty} e^{v^j} = (2t)^{\frac{n+1-2j}{n-1}} \qquad 1 < j \le \frac{n}{2}$$

3.3.3 Estimates on Asymptotics of λ^j and λ^j_z

In order to understand the asymptotics of the family of flat connections above, we need to understand the asymptotics of the metric and its first derivative. For the metric, we have the following theorem.

Theorem 3.3.1. *For every point $p \in \Sigma$ away from the zeros of q_n or q_{n-1}, as $t\to\infty$ we have*

1. *For $(\Sigma,\ \tilde{e}_1 + tq_ne_{n-1}) \in Hit_n(S)$, the metric $h_j(t)$ on $K^{\frac{n+1-2j}{2}}$ admits the expansion*

$$h_j(t) = (t|q_n|)^{-\frac{n+1-2j}{n}}(1 + O(t^{-\frac{2}{n}})) \quad \textit{for all } j$$

2. For $(\Sigma,\ \tilde{e}_1 + tq_{n-1}e_{n-2}) \in Hit_n(S)$*, the metric* $h_j(t)$ *on* $K^{\frac{n+1-2j}{2}}$ *admits the expansion*

$$h_j(t) = \begin{cases} (t|q_{n-1}|)^{-\frac{n+1-2j}{n-1}}(1 + O(t^{-\frac{2}{n-1}})) & \text{for } j = 1 \text{ and } j = n \\ (2t|q_{n-1}|)^{-\frac{n+1-2j}{n-1}}(1 + O(t^{-\frac{2}{n-1}})) & \text{for } 1 < j < n \end{cases}$$

Unfortunately, the proof is long and technical, and so may be skipped on first reading. The idea is to use maximum principle on different equations many times.

Proof. While the statement of the theorem for our two cases is similar, due to differences in the equations, the details of the proofs are different. The following notation will be used:

$$M_n = \frac{1}{4}\max_{\Sigma} |K_{g_n}| \qquad A_j = \max_{\Sigma} e^{u^j} \qquad B_j = \max_{\Sigma} e^{u^j - u^{j+1}}$$

$$M_{n-1} = \frac{1}{4}\max_{\Sigma} |K_{g_{n-1}}| \qquad D_j = \max_{\Sigma} e^{v^j} \qquad E_j = \max_{\Sigma} e^{v^j - v^{j+1}},$$

note that $\dfrac{A_j}{A_{j+1}} \leq B_j$ and $\dfrac{D_j}{D_{j+1}} \leq E_j$. We start with a series of lemmas. Lemma 3.3.3 gives a lower bound of u^1 and v^1. Lemma 3.3.4 gives the upper bound of $A_{\frac{n}{2}}$ and shows that $\frac{D_j}{D_{j+1}}$ bounds $D_{\frac{n}{2}}$. Making use of Lemma 3.3.4, Lemma 3.3.5 shows that A_j is bounded by $A_{\frac{n}{2}}$ and $D_{\frac{n}{2}}$ bounds all other D_j's. With these three lemmas, we are then ready to show the proof of Theorem 3.3.1.

The signs in our equations are set up for applications of the maximum principle. At the maxima of e^{u^j}, we have $0 \leq -\frac{1}{4}\Delta_g u^j$. After rearranging the equations (3.3.10) we get the following estimates at the maxima of e^{u^j} respectively

$$\begin{cases} 0 \leq -\dfrac{1}{4}\Delta_{g_n} u^1 \leq (n-1)M_n + t^2 A_1^{-2} - \dfrac{A_1}{A_2} \\ 0 \leq -\dfrac{1}{4}\Delta_{g_n} u^j \leq (n+1-2j)M_n + \dfrac{A_{j-1}}{A_j} - \dfrac{A_j}{A_{j+1}} \quad 1 < j < \dfrac{n}{2} \\ 0 \leq -\dfrac{1}{4}\Delta_{g_n} u^{\frac{n}{2}} \leq M_n + \dfrac{A_{\frac{n-1}{2}}}{A_{\frac{n}{2}}} - A_{\frac{n}{2}}^2. \end{cases} \tag{3.3.12}$$

Similarly, using equations (3.3.11), at the maxima of e^{v^j} respectively

$$\begin{cases} 0 \leq -\frac{1}{4}\Delta_{g_{n-1}} v^{j} \leq (n+1-2j)M_{n-1} + \frac{D_{j-1}}{D_j} - \frac{D_j}{D_{j+1}} \quad 2 < j < \frac{n}{2} \\ 0 \leq -\frac{1}{4}\Delta_{g_{n-1}} v^{\frac{n}{2}} \leq (n+1-2j)M_{n-1} + \frac{D_{\frac{n}{2}-1}}{D_{\frac{n}{2}}} - D_{\frac{n}{2}}^2. \end{cases} \tag{3.3.13}$$

Remark 3.3.2. We *cannot* write

$$0 \leq -\frac{1}{4}\Delta_{g_{n-1}} v^{1} \leq (n-1)M_{n-1} + \frac{t^2}{D_1 D_2} - \frac{D_1}{D_2},$$

nor the corresponding inequality for $\Delta_{g_{n-1}} v^{2}$ because at the maximum of e^{v^1}, it may be the case that $e^{-v^1 - v^2}$ is larger than $(D_1 D_2)^{-1}$.

In what follows, C will be a constant, and C's on different lines should not be assumed to be related.

Lemma 3.3.3. *Set* $f_n = \ln(\frac{|q_n|^2}{g_n^n e^{2u^1}})$ *and* $f_{n-1} = \ln(\frac{|q_{n-1}|^2}{g_{n-1}^{n-1} e^{2v^1}})$ *then*

$$max(\ t^2 e^{f_n}) \leq B_1 + 2M_n \qquad and \qquad e^{f_{n-1}} \leq 1$$

Proof. For the first inequality, we compute

$$\frac{1}{4}\Delta_{g_n}(f_n) = \frac{1}{4}\Delta_{g_n} \ln(\frac{|q_n|^2}{g_n^n e^{2u^1}}) = \frac{1}{g_n}\partial_z\partial_{\bar{z}} \ln|q_n|^2 - \frac{1}{g_n}\partial_z\partial_{\bar{z}} \ln(g_n^n) - \frac{1}{g_n}\partial_z\partial_{\bar{z}} \ln(e^{2u^1}).$$

The first term vanishes since q_n is holomorphic, the second term involves the curvature of g_n and the third term involves the Laplacian of u^1. We have

$$\frac{1}{4}\Delta_{g_n}(f_n) = \frac{1}{4}\Delta_{g_n} \ln(\frac{|q_n|^2}{g^n e^{2u^1}}) = \frac{n}{2}K_{g_n} - \frac{1}{4}\Delta_{g_n}(2u^1).$$

The equation for $\Delta_{g_n}(u^1)$ yields

$$\frac{1}{4}\Delta_{g_n}(f_n) = 2t^2 \frac{|q_n|^2}{g_n^n e^{2u^1}} - 2e^{u^1 - u^2} + \frac{1}{2}K_{g_n}.$$

By the maximum principle, at the maximum of f_n we have

$$0 \geq \frac{1}{4}\Delta_{g_n}(f_n) = 2t^2 e^{f_n} - 2e^{u^1 - u^2}$$

which gives

$$t^2 e^{f_n} \leq e^{u^1 - u^2} \leq B_1 + 2M_n.$$

Similarly, for the second inequality we compute

$$\Delta_{g_{n-1}} f_{n-1} = \Delta_{g_{n-1}} \ln(t^2|q_{n-1}|) - (n-1)\Delta_{g_{n-1}} \ln(g_{n-1}) - 2\Delta_{g_{n-1}} v^1$$

$$= 2(n-1)K_{g_{n-1}} - 2\Delta_{g_{n-1}} v^1$$

since q_{n-1} is holomorphic. Now using the first equation

$$= 2(n-1)K_{g_{n-1}} + (e^{-v^1 - v^2} \frac{|q_{n-1}|t^2}{g_{n-1}^{n-1}} - e^{v^1 - v^2} - \frac{n-1}{4} K_{g_{n-1}})$$

$$= 8e^{v^1 - v^2}(e^{f_{n-1}} - 1).$$

At the maximal point of f_{n-1} we get

$$0 \geq \Delta_{g_{n-1}} f_{n-1} = 8e^{v^1 - v^2}(e^{f_{n-1}} - 1),$$

thus $e^{f_{n-1}} \leq 1$ on Σ. □

Lemma 3.3.4. *We have the following upper bound on* $A^2_{\frac{n}{2}}$ *and* $D^2_{\frac{n}{2}}$*,*

$$\begin{cases} A^2_{\frac{n}{2}} \leq C + t^{\frac{2}{n}} \\ D^2_{\frac{n}{2}} \leq \dfrac{D_j}{D_{j+1}} + C \leq 2\dfrac{D_1}{D_2} + C \quad 1 < j < \dfrac{n}{2}. \end{cases} \tag{3.3.14}$$

Proof. For the first inequality we use two telescoping sums, the first is

$$\frac{A_j}{A_{j+1}} = (\frac{A_j}{A_{j+1}} - \frac{A_{j+1}}{A_{j+2}}) + \cdots + (\frac{A_{\frac{n}{2}-1}}{A_{\frac{n}{2}}} - A^2_{\frac{n}{2}}) + A^2_{\frac{n}{2}} \geq -C(M_n) + A^2_{\frac{n}{2}}$$

by equations (3.3.12). Thus

$$A_1 = \frac{A_1}{A_2}\frac{A_2}{A_3} \cdots \frac{A_{\frac{n}{2}-1}}{A_{\frac{n}{2}}} A_{\frac{n}{2}} \geq (-C + A^2_{\frac{n}{2}})^{\frac{n}{2}-1}(A^2_{\frac{n}{2}})^{\frac{1}{2}} \geq (-C + A^2_{\frac{n}{2}})^{\frac{n-1}{2}}.$$

The second telescoping sum is

$$\frac{t^2}{A_1^2} = (\frac{t^2}{A_1^2} - \frac{A_1}{A_2}) + (\frac{A_1}{A_2} - \frac{A_2}{A_3}) + \cdots + (\frac{A_{\frac{n}{2}-1}}{A_{\frac{n}{2}}} - A_{\frac{n}{2}}^2) + A_{\frac{n}{2}}^2 \geq -C(M_n) + A_{\frac{n}{2}}^2,$$

putting them together we obtain $t^2 \geq (-C + A_{\frac{n}{2}}^2)^n$, giving the desired upper bound.

Now for the second inequality, at the maximum of e^{v^2}

$$0 \leq -\frac{1}{4}\Delta_{g_{n-1}} v^2 = \frac{1}{D_2 e^{v^1}} \frac{|q_{n-1}|^2 t^2}{g_{n-1}^{n-1}} + \frac{e^{v^1}}{D_2} - \frac{D_2}{e^{v^3}} - \frac{n-3}{4} K_{g_{n-1}}$$

$$\leq e^{v^1 - v^2} \left(\frac{|q_{n-1}|^2 t^2}{g_{n-1}^{n-1} e^{2v^1}} + 1 \right) - e^{v^2 - v^3} - (n-3) M_{n-1}.$$

By Lemma 3.3.3

$$0 \leq 2\frac{D_1}{D_2} - \frac{D_2}{D_3} + (n-3)M_{n-1},$$

thus $\frac{D_2}{D_3} \leq 2\frac{D_1}{D_2} + (n-3)M_{n-1}$. Using the inequalities (3.3.13) we have

$$\begin{cases} \frac{D_j}{D_{j+1}} \leq \frac{D_{j-1}}{D_j} + C \quad 2 < j < \frac{n}{2} \\ D_{\frac{n}{2}}^2 \leq \frac{D_{\frac{n}{2}-1}}{D_{\frac{n}{2}}} + C. \end{cases} \tag{3.3.15}$$

Putting the inequalities together gives

$$D_{\frac{n}{2}}^2 \leq \frac{D_j}{D_{j+1}} + C \leq 2\frac{D_1}{D_2} + C \qquad \textit{for } 1 < j < \frac{n}{2}$$

as desired. □

Lemma 3.3.5. *The following upper bound on A_j and D_j hold,*

$$\begin{cases} A_j \leq (A_{\frac{n}{2}}^2 + C)^{\frac{n+1-2j}{2}} \quad \textit{for all } j \\ D_1 \leq \frac{1}{2}(D_{\frac{n}{2}}^2 + C)^{\frac{n-1}{2}} \\ D_j \leq (D_{\frac{n}{2}}^2 + C)^{\frac{n+1-2j}{2}} \quad j > 1 \end{cases} \tag{3.3.16}$$

Proof. In both cases, we first prove an inequality on the B_j's and the E_j's, and then use Lemma 3.3.4. To do this we subtract the equation $j+1$ from equations j.

For the first inequality, when $j=1$ at the maximum of $u^{1}-u^{2}$ we have

$$0 \leq -\frac{1}{4}\Delta_{g_n}(u^{1}-u^{2}) = -2e^{u^{1}-u^{2}} + e^{u^{2}-u^{3}} + e^{-2u^{1}}\frac{t^2|q_n|^2}{g_n^n} - \frac{1}{2}K_{g_n}$$

$$\leq -2B_1 + B_2 + e^{-2u^{1}}\frac{t^2|q_n|^2}{g_n^n} + 2M_n \leq -B_1 + B_2 + 4M_n$$

by Lemma 3.3.3. For $1 < j < \frac{n}{2}-1$ at the maximum of $u^{j}-u^{j+1}$,

$$0 \leq -\frac{1}{4}\Delta_{g_n}(u^{j}-u^{j+1}) = -\frac{1}{2}K_{g_n} + e^{u^{j-1}-u^{j}} - 2e^{u^{j}-u^{j+1}} + e^{u^{j+1}-u^{j+2}}$$

$$\leq 2M_n + B_{j-1} - 2B_j + B_{j+1}.$$

At the maximum of $u^{\frac{n}{2}-1}-u^{\frac{n}{2}}$,

$$0 \leq -\frac{1}{4}\Delta_{g_n}(u^{\frac{n}{2}-1}-u^{\frac{n}{2}}) = -\frac{1}{2}K_{g_g} + e^{u^{\frac{n}{2}-2}-u^{\frac{n}{2}-1}} - 2e^{u^{\frac{n}{2}-1}-u^{\frac{n}{2}}} + e^{2u^{\frac{n}{2}}}$$

$$\leq 2M_n + B_{\frac{n}{2}-2} - 2B_{\frac{n}{2}-1} + A_{\frac{n}{2}}^2.$$

Rewriting the B_j inequalities slightly gives:

$$\begin{cases} B_1 - B_2 \leq 4M_n & \\ B_j - B_{j+1} \leq 2M_n + B_{j-1} - B_j & 1 < j < \dfrac{n}{2}-1 \\ B_{\frac{n}{2}-1} - A_{\frac{n}{2}}^2 \leq 2M_n + B_{\frac{n}{2}-2} - B_{\frac{n}{2}-1}. & \end{cases} \tag{3.3.17}$$

The first two inequalities give $B_j - B_{j+1} \leq C$ for $1 \leq j < \frac{n}{2}-1$, and the third gives $B_{\frac{n}{2}-1} - A_{\frac{n}{2}}^2 \leq C$. Putting these together we have

$$B_j \leq A_{\frac{n}{2}}^2 + C \qquad \text{for all } j. \tag{3.3.18}$$

Since $A_j = \dfrac{A_j}{A_{j+1}} \cdots \dfrac{A_{\frac{n}{2}-1}}{A_{\frac{n}{2}}} \cdot A_{\frac{n}{2}} \leq B_j B_{j+1} \cdots B_{\frac{n}{2}-1} \cdot A_{\frac{n}{2}}$ we have the desired inequality

$$A_j \leq (C + A_{\frac{n}{2}}^2)^{\frac{n}{2}-j} A_{\frac{n}{2}} \leq (C + A_{\frac{n}{2}}^2)^{\frac{n+1-2j}{2}}.$$

The second inequality involves the v^j's; at the maximum of $v^1 - v^2$,

$$0 \le -\frac{1}{4}\Delta_{g_{n-1}}(v^1 - v^2) = -2e^{v^1-v^2} + e^{v^2-v^3} - \frac{1}{2}K_{g_{n-1}} \le -2E_1 + E_2 + 2M_{n-1}$$

thus $2E_1 - E_2 \le C$. At the maximum of $v^2 - v^3$

$$0 \le -\frac{1}{4}\Delta_{g_{n-1}}(v^2 - v^3) = e^{-v^1-v^2}\frac{|q_{n-1}|^2 t^2}{g_{n-1}^{n-1}} + e^{v^1-v^2} - 2e^{v^2-v^3} + e^{v^3-v^4} - \frac{1}{2}K_{g_{n-1}}.$$

By Lemma 3.3.3 and the definition of E_j,

$$0 \le e^{v^1-v^2} + E_1 - 2E_2 + E_3 + M_{n-1} \le 2E_1 - 2E_2 + E_3 + 2M_{n-1}$$

thus $E_2 - E_3 \le C + 2E_1 - E_2$.

For $2 < j < \frac{n}{2} - 1$, at the maximum of $v^j - v^{j+1}$, we have

$$0 \le -\frac{1}{4}\Delta_{g_{n-1}}(v^j - v_{j+1}) = e^{v^{j-1}-v^j} - 2e^{v^j-v^{j+1}} + e^{v^{j+1}-v^{j+2}} - \frac{1}{2}K_{g_{n-1}}$$

thus $E_j - E_{j+1} \le E_{j-1} - E_j + C$.

Finally, at the maximum of $v^{\frac{n}{2}-1} - v^{\frac{n}{2}}$, we obtain

$$0 \le -\frac{1}{4}\Delta_{g_{n-1}}(v^{\frac{n}{2}-1} - v^{\frac{n}{2}}) = e^{v^{\frac{n}{2}-2}-v^{\frac{n}{2}-1}} - 2e^{v^{\frac{n}{2}-1}-v^{\frac{n}{2}}} + e^{2v^{\frac{n}{2}}} - \frac{1}{2}K_{g_{n-1}}$$

thus $E_{\frac{n}{2}-1} - D^2_{\frac{n}{2}} \le E_{\frac{n}{2}-2} - E_{\frac{n}{2}-1} + C$.

Combining the inequalities for the E_j's yields $E_{\frac{n}{2}-1} - D^2_{\frac{n}{2}} \le C + E_{j-1} - E_j \le 2E_1 - E_2 + C$, and hence

$$\begin{cases} 2E_1 \le D^2_{\frac{n}{2}} + C \\ E_j \le D^2_{\frac{n}{2}} + C. \end{cases} \tag{3.3.19}$$

Since $D_j = \frac{D_j}{D_{j+1}} \cdots \frac{D_{\frac{n}{2}-1}}{D_{\frac{n}{2}}} \cdot D_{\frac{n}{2}} \le E_j E_{j+1} \cdots E_{\frac{n}{2}-1} \cdot D_{\frac{n}{2}}$, the desired inequalities are proved

$$\begin{cases} D_1 \le \frac{1}{2}(D^2_{\frac{n}{2}} + C)^{\frac{n-1}{2}} \\ D_j \le (D^2_{\frac{n}{2}} + C)^{\frac{n+1-2j}{2}} \quad j > 1. \end{cases} \tag{3.3.20}$$

□

We are now ready to prove the theorem.

Proof. (*Of Theorem 3.3.1*) Recall that we are proving that on Ω_n and Ω_{n-1}

$$\begin{cases} e^{u^j} = t^{\frac{n+1-2j}{n}}(1+O(t^{-\frac{2}{n}})) & \text{for all } j \\ e^{v^j} = (2t)^{\frac{n+1-2j}{n-1}}(1+O(t^{-\frac{2}{n-1}})) & 1 < j \leq \dfrac{n}{2} \\ e^{v^1} = t(1+O(t^{-\frac{2}{n-1}})) \end{cases} \tag{3.3.21}$$

First the u^j's, by Lemma 3.3.3 we have $\dfrac{t^2|q_n|^2}{g_n^n e^{2u^1}} \leq B_1$. Thus on Ω_n, the choice of metric g_n yields $t^2 e^{-2u^1} \leq B_1$. Using (3.3.18), on Ω_n we have

$$t^2 e^{-2u^1} \leq B_1 \leq C + A_{\frac{n}{2}}^2$$

which, together with Lemma 3.3.5, gives the inequality

$$\left(\frac{t^2}{A_{\frac{n}{2}}^2 + C}\right)^{\frac{1}{2}} \leq e^{u^1} \leq A_1 \leq (C + A_{\frac{n}{2}}^2)^{\frac{n-1}{2}}.$$

We may rewrite this inequality as $t^{\frac{2}{n}} \leq A_{\frac{n}{2}}^2 + C$.

Since $e^{u^1} = e^{u^1-u^2}e^{u^2-u^3}\cdots e^{u^{j-1}-u^j}e^{u^j}$, by (3.3.18) it follows that

$$\left(\frac{t^2}{A_{\frac{n}{2}}^2 + C}\right)^{\frac{1}{2}} \leq e^{u^1} \leq B_1 \cdots B_{j-1} e^{u^j}.$$

Lemma 3.3.4 gives $B_j \leq A_{\frac{n}{2}}^2 + C \leq t^{\frac{2}{n}} + C$, and by Lemma 3.3.5

$$e^{u^j} \leq A_j \leq (t^{\frac{2}{n}} + C)^{\frac{n+1-2j}{2}}.$$

Hence on Ω_n,

$$\frac{t}{(t^{\frac{2}{n}} + C)^{\frac{2j-1}{2}}} \leq e^{u^j} \leq (t^{\frac{2}{n}} + C)^{\frac{n+1-2j}{2}}.$$

Thus, on Ω_n, we have the desired

$$e^{u^j} = t^{\frac{n+1-2j}{n}}(1+O(t^{-\frac{2}{n}})).$$

Now for the second two equations, recall that Lemma 3.3.3 says $\frac{t^2|q_{n-1}|^2}{g_{n-1}^{n-1}e^{2v^1}} \leq 1$. By definition of g_{n-1}, on Ω_{n-1},

$$\frac{t^2}{e^{2v^1}} \leq 1.$$

So on Ω_{n-1}, $t \leq e^{v^1} \leq D_1$. By Lemma 3.3.5

$$(2t)^{\frac{2}{n-1}} - C \leq D_{\frac{n}{2}}^2$$

on Ω_{n-1}, thus by Lemma 3.3.4

$$2^{\frac{3-n}{n-1}} t^{\frac{2}{n-1}} - C \leq \frac{1}{2} D_{\frac{n}{2}}^2 - C \leq \frac{D_1}{D_2}.$$

Rearranging yields

$$\frac{D_2}{D_1} \leq \frac{1}{2^{\frac{3-n}{n-1}} t^{\frac{2}{n-1}} - C}$$

on Ω_{n-1}. Using the first equation and the definition of g_{n-1}, at the maximum of e^{v^1} we have

$$0 \leq -\frac{1}{4}\Delta_{g_{n-1}} v^1 = e^{-v^1 - v^2} \frac{|q_{n-1}|^2 t^2}{g_{n-1}^{n-1}} - e^{v^1 - v^2} - C$$
$$\leq \frac{t^2}{D_1 e^{v^2}} - \frac{D_1}{e^{v^2}} + C \leq \frac{t^2 - D_1^2 + C D_2 D_1}{D_1 e^{v^2}}.$$

After rearranging the inequality it becomes

$$1 \leq \frac{t^2}{D_1^2} + C\frac{D_2}{D_1}.$$

Thus on Ω_{n-1}, we have $1 \leq \frac{t^2}{D_1^2} + Ct^{-\frac{2}{n-1}}$, simplifying yields

$$D_1 \leq \frac{t}{1 - Ct^{-\frac{2}{n-1}}} \leq t(1 + Ct^{-\frac{2}{n-1}}).$$

By Lemma 3.3.5, $D_{\frac{n}{2}}^2 \leq (2D_1)^{\frac{2}{n-1}} + C$, thus on Ω_{n-1}

$$D_{\frac{n}{2}}^2 \leq 2^{\frac{2}{n-1}} (t(1 + Ct^{-\frac{2}{n-1}}))^{\frac{2}{n-1}}$$

Using Lemma 3.3.5 once more, for $1 < j < \frac{n}{2}$

$$D_j \leq 2^{\frac{2}{n+1-2j}} (t(1 + Ct^{-\frac{2}{n-1}}))^{\frac{2}{n+1-2j}}.$$

Using (3.3.19) we have

$$\begin{cases} E_j \leq 2^{\frac{2}{n-1}} (t(1 + Ct^{-\frac{2}{n-1}}))^{\frac{2}{n-1}} & 1 < j \\ 2E_1 \leq 2^{\frac{2}{n-1}} (t(1 + Ct^{-\frac{2}{n-1}}))^{\frac{2}{n-1}}. \end{cases} \tag{3.3.22}$$

As in the proof of the first equation,

$$e^{v^1} = e^{v^1 - v^2} e^{v^2 - v^3} \cdots e^{v^{j-1} - v^j} \cdot e^{v^j} \leq E_1 E_2 \cdots E_{j-1} \cdot e^{v^j},$$

thus restricting to Ω_{n-1}

$$\frac{t}{E_1 \cdots E_{j-1}} \leq e^{v^j}.$$

Applying (3.3.22) yields

$$\frac{2t}{(t(1 + Ct^{-\frac{2}{n-1}}))^{\frac{2(j-1)}{n-1}}} \leq e^{v^j}.$$

Finally, since $e^{v^j} \leq D_j$ we have

$$\begin{cases} t \leq e^{v^1} \leq t(1 + Ct^{-\frac{2}{n-1}}) \\ \dfrac{2t}{(t(1 + Ct^{-\frac{2}{n-1}}))^{\frac{2(j-1)}{n-1}}} \leq e^{v^j} \leq 2^{\frac{2}{n+1-2j}} (t(1 + Ct^{-\frac{2}{n-1}}))^{\frac{2}{n+1-2j}}. \end{cases} \tag{3.3.23}$$

Hence we have shown that on Ω_{n-1},

$$e^{v^1} = t(1 + O(t^{-\frac{2}{n-1}}))$$

$$e^{v^j} = (2t)^{\frac{n+1-2j}{n-1}} (1 + O(t^{-\frac{2}{n-1}})) \qquad for\ 1 < j \leq \frac{n}{2}.$$

□

□

In terms of the u^j's and v^j's, Theorem 3.3.1 says the asymptotics of the metric solving the Higgs bundle equations on Ω_n are

$$e^{u^j} = t^{\frac{n+1-2j}{n}}(1 + O(t^{-\frac{2}{n}})) \qquad 1 \leq j \leq \frac{n}{2},$$

and for $\phi = \tilde{e}_1 + tq_{n-1}e_{n-2}$, the asymptotics of the metric solving the Higgs bundle equations on Ω_{n-1} are

$$e^{v^j} = (2t)^{\frac{n+1-2j}{n-1}}(1 + O(t^{-\frac{2}{n-1}})) \qquad 1 < j \leq \frac{n}{2}$$

$$e^{v^1} = t(1 + O(t^{-\frac{2}{n-1}})).$$

Using our understanding of the u^j's, the v^j's, and their Laplacians, we gain control of their first derivatives.

Proposition 3.3.6. *Let z be a local coordinate so that $q_n = dz^n$, then there is a constant $C_n = C_n(\Sigma, q_n, \Omega_n)$ so that*

$$|u^j_z| \leq C_n t^{-\frac{1}{n}} .$$

Similarly, let z be a local coordinate so that $q_{n-1} = dz^{n-1}$, then there is a constant $C_{n-1} = C_{n-1}(\Sigma, q_{n-1}, \Omega_{n-1})$ so that

$$|v^j_z| \leq C_{n-1} t^{-\frac{1}{n-1}} .$$

Proof. Recall that the u^j's are functions of t, we continue to suppress this from the notation. Theorem 3.3.1 implies

$$\begin{cases} e^{u^j - u^{j+1}} = t^{\frac{2}{n}}(1 + O(t^{-\frac{2}{n}})) \\ \dfrac{t^2}{e^{2u^1}} = t^{\frac{2}{n}}(1 + O(t^{-\frac{2}{n}})) \end{cases} \tag{3.3.24}$$

For $p \in \Omega_n$, choose a local coordinate z centered at p with $q_n = dz^n$. Consider the functions α^j defined by

$$\alpha^j(z) = u^j(t^{-\frac{1}{n}}z) - \frac{n+1-2j}{n}\ln t \quad for\ 1 \leq j \leq \frac{n}{2},$$

where $u^j(t^{-\frac{1}{n}}z)$ is just a rescaling of u^j. The α^j's then satisfy the following two properties

$$\begin{cases} |\alpha^j| = |u^j - \frac{n+1-2j}{n}\ln t| \leq Ct^{-\frac{2}{n}} \\ \alpha^j_{z\bar{z}} = t^{-\frac{2}{n}}u^j_{z\bar{z}}. \end{cases} \tag{3.3.25}$$

By choice of the background metric g_n and our coordinate system, on Ω_n we have $q_n = dz^n$, $K_{g_n} = 0$, and $\frac{|q_n|^2}{(g_n)^n} = 1$. Hence for $j = 1$, using the equations (3.3.9) we have

$$|\alpha^1_{z\bar{z}}| = |t^{-\frac{2}{n}}u^1_{z\bar{z}}| = |t^{-\frac{2}{n}}(\frac{t^2}{e^{2u^1}} - e^{u^1-u^2})|$$
$$= |t^{-\frac{2}{n}}(t^{\frac{2}{n}}O(t^{-\frac{2}{n}}))| \leq Ct^{-\frac{2}{n}}.$$

For $1 < j < \frac{n}{2}$, using the equations (3.3.9) we have

$$|\alpha^j_{z\bar{z}}| = |t^{-\frac{2}{n}}u^j_{z\bar{z}}| = |t^{-\frac{2}{n}}(e^{u^{j-1}-u^j} - e^{u^j-u^{j+1}})|$$
$$= |t^{-\frac{2}{n}}(t^{\frac{2}{n}}O(t^{-\frac{2}{n}}))| \leq Ct^{-\frac{2}{n}}.$$

For $j = \frac{n}{2}$,

$$|\alpha^{\frac{n}{2}}_{z\bar{z}}| = |t^{-\frac{2}{n}}u^{\frac{n}{2}}_{z\bar{z}}| = |t^{-\frac{2}{n}}(e^{u^{\frac{n}{2}-1}-u^{\frac{n}{2}}} - e^{2u^{\frac{n}{2}}})|$$
$$= |t^{-\frac{2}{n}}(t^{\frac{2}{n}}O(t^{-\frac{2}{n}}))| \leq Ct^{-\frac{2}{n}}.$$

Hence for all j, we have both $|\alpha^j| \leq Ct^{-\frac{2}{n}}$ and $|\alpha^j_{z\bar{z}}| \leq Ct^{-\frac{2}{n}}$.

Applying Schauder's estimate gives $|\alpha^j|_{C^1} \leq Ct^{-\frac{2}{n}}$. Thus $|\alpha^j_z| \leq Ct^{-\frac{2}{n}}$, proving

$$|u^j_z| = t^{\frac{1}{n}}|\alpha^j_z| \leq Ct^{-\frac{1}{n}}.$$

The $\phi = \tilde{e}_1 + q_{n-1}e_{n-2}$ case works similarly. For $p \in \Omega_{n-1}$, choose the local coordinate z centered at p, and consider the functions β^j defined by

$$\beta^j(z) = v^j(t^{-\frac{1}{n-1}}z) - \frac{n+1-2j}{n}\ln(t) \quad \text{for} \quad 1 \leq j \leq \frac{n}{2}.$$

The β^j's also satisfy (3.3.25), thus as above, our equations simplify to

$$|\beta^j_{z\bar{z}}| = |t^{-\frac{2}{n-1}}v^j_{z\bar{z}}| = \begin{cases} |t^{-\frac{2}{n-1}}(t^2e^{-v^1-v^2} - e^{v^1-v^2})| \leq Ct^{-\frac{2}{n-1}} & j=1 \\ |t^{-\frac{2}{n-1}}(t^2e^{-v^1-v^2} + e^{v^1-v^2} - e^{v^2-v^3})| \leq Ct^{-\frac{2}{n-1}} & j=2 \\ |t^{-\frac{2}{n-1}}(e^{v^{j-1}-v^j} - e^{v^j-v^{j+1}})| \leq Ct^{-\frac{2}{n-1}} & 2<j<\frac{n}{2} \\ |t^{-\frac{2}{n-1}}(e^{v^{\frac{n}{2}-1}-v^{\frac{n}{2}}} - e^{2v^{\frac{n}{2}}})| \leq Ct^{-\frac{2}{n-1}} & j=\frac{n}{2} \end{cases}$$

Thus, with the same argument as the $\phi = \tilde{e}_1 + q_ne_{n-1}$ case, we get

$$|v^j_z| = t^{\frac{1}{n-1}}|\beta^j_z| \leq Ct^{-\frac{1}{n-1}}$$

□

3.4 Parallel Transport Asymptotics

In this section we set up the parallel transport ODE we wish to integrate. To avoid some redundancy, in this section we will sometimes use a subscript or superscript b to denote objects corresponding to the b-cyclic Higgs field $\phi_b = \tilde{e}_1 + tq_be_{b-1}$ for $b = n, n-1$. We will also work in the universal cover $\widetilde{\Sigma}$ of Σ, all objects should be pulled back to the universal cover.

Let $P \in \widetilde{\Sigma}$ be away from the zeros of the differential q_b. Choose a neighborhood $\mathcal{U}_P$ centered at P, with coordinate z, so that $q_b = dz^b$. Note that for this to make sense, $\mathcal{U}_P$ must be disjoint from the zero set of q_b. In this neighborhood, $u^j = \lambda^j$ for $b = n$ and $v^j = \lambda^j$ for $b = n-1$.

As before, the choice of local coordinate z defines a local holomorphic frame $(s_1, \dots, s_{\frac{n}{2}}, s^*_{\frac{n}{2}}, \dots, s^*_1)$ for

$$E = K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{-\frac{n-1}{2}},$$

where $s_j = dz^{\frac{n+1-2j}{2}}$. In this frame, the connection 1-forms of the corresponding flat connection are given by (3.3.5) and (3.3.6). By our choice of coordinates, the f_b in (3.3.4) is identically 1.

Using our estimates from Theorem 3.3.1 and Proposition 3.3.6, we will solve for the transport matrix $T_{P,P'}(t)$ along paths starting at P and ending at a point P' in the neighborhood $\mathcal{U}_P$. In fact, we calculate $T_{P,P'}(t)$ along geodesics of the background metric $g_b = |dz|^{\frac{2}{b}}$ which start at P and end at P'. Since the connection is flat, the value of $T_{P,P'}(t)$ is path independent in $\mathcal{U}_P$.

We rescale the holomorphic frame so that it stays bounded away from 0 and ∞ as $t \to \infty$. For $\phi = \tilde{e}_1 + tq_n e_{n-1}$, the rescaled frame is given by $F_n = (\sigma_1, \dots, \sigma^*_1)$ where

$$\sigma_j = t^{\frac{n+1-2j}{2n}} s_j \qquad \sigma^*_j = t^{-\frac{n+1-2j}{2n}} s^*_j.$$

Remark 3.4.1. By Theorem 3.3.1, in the rescaled frame, the metric $h = Id(1 + O(t^{-\frac{2}{n}}))$. To see this, consider

$$h(s_i, s_j) = \delta_{ij} t^{\frac{i+j-n-1}{n}} (1 + O(t^{-\frac{2}{n}}))$$

thus

$$h(\sigma_i, \sigma_j) = h(t^{\frac{n+1-2i}{2n}} s_i, t^{\frac{n+1-2j}{2n}} s_j)$$

$$= t^{\frac{n+1-(i+j)}{n}} h(s_i, s_j) = \delta_{ij}(1 + O(t^{-\frac{2}{n}})).$$

For $\phi = \tilde{e}_1 + tq_{n-1}e_{n-2}$, we denote the rescaled frame by $F_{n-1} = (\sigma_1, \dots, \sigma^*_1)$, it is given by

$$\sigma_1 = t^{\frac{1}{2}} s_1 \qquad \sigma^*_1 = t^{-\frac{1}{2}} s^*_1$$

$$\sigma_j = (2t)^{\frac{n+1-2j}{2(n-1)}} s_j \qquad \sigma_j^* = (2t)^{-\frac{n+1-2j}{2(n-1)}} s_j^* \qquad j = 2 \dots \frac{n}{2}$$

As in the previous case, the harmonic metric in this frame is $h = Id(1 + O(t^{-\frac{2}{n-1}}))$.

If we denote the flat connection by $D_b = U_b dz + V_b d\bar{z}$, then by the estimates from Theorem 3.3.1 and Proposition 3.3.6, the matrices in the connection 1-form are given by:

1. For $\phi = \tilde{e}_1 + q_n e_{n-1}$,

$$U_n = \begin{pmatrix} -u_z^1 & & & t^{\frac{1}{n}} \\ t^{\frac{1}{n}} & -u_z^2 & & \\ & \ddots & \ddots & \\ & & t^{\frac{1}{n}} & u_z^1 \end{pmatrix} = t^{\frac{1}{n}} \begin{pmatrix} & & 1 \\ 1 & & \\ & \ddots & \\ & 1 & \end{pmatrix} + O(t^{-\frac{1}{n}})$$

$$V_n = \begin{pmatrix} & t^{-\frac{1}{n}} e^{u^1 - u^2} & & \\ & & \ddots & \\ & & & t^{-\frac{1}{n}} e^{u^1 - u^2} \\ t^{\frac{1-2n}{n}} e^{-2u^1} & & & \end{pmatrix} = t^{\frac{1}{n}} \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & & & \end{pmatrix} + O(t^{-\frac{1}{n}})$$

where $O(t^{-\frac{1}{n}})$ is uniform as $t \to \infty$ for all points in Ω_n.

2. For $\phi = \tilde{e}_1 + q_{n-1} e_{n-2}$,

$$U_{n-1} = \begin{pmatrix} -v_z^1 & & & 2^{-\frac{n-3}{2(n-1)}} t^{\frac{1}{n-1}} & \\ 2^{-\frac{n-3}{2(n-1)}} t^{\frac{1}{n-1}} & -v_z^2 & & & 2^{-\frac{n-3}{2(n-1)}} t^{\frac{1}{n-1}} \\ & 2^{\frac{1}{n-1}} t^{\frac{1}{n-1}} & & & \\ & & \ddots & & \\ & & & 2^{-\frac{n-3}{2(n-1)}} t^{\frac{1}{n-1}} & v_z^1 \end{pmatrix}$$

$$= (2t)^{\frac{1}{n-1}} \begin{pmatrix} & & & \frac{1}{\sqrt{2}} & \\ \frac{1}{\sqrt{2}} & & & & \frac{1}{\sqrt{2}} \\ & 1 & & & \\ & & \ddots & & \\ & & & \frac{1}{\sqrt{2}} & \end{pmatrix} + O(t^{-\frac{1}{n-1}})$$

$$V_{n-1} = \begin{pmatrix} & e^{v^1 - v^2} 2^{\frac{n-3}{2(n-1)}} t^{-\frac{1}{n-1}} & & \\ & & \ddots & \\ e^{-v^1 - v^2} 2^{\frac{n-3}{2(n-1)}} t^{\frac{2n-3}{n-1}} & & & e^{v^1 - v^2} 2^{\frac{n-3}{2(n-1)}} t^{-\frac{1}{n-1}} \\ & e^{-v^1 - v^2} 2^{\frac{n-3}{2(n-1)}} t^{\frac{2n-3}{n-1}} & & \end{pmatrix}$$

$$= (2t)^{\frac{1}{n-1}} \begin{pmatrix} & \frac{1}{\sqrt{2}} & & & \\ & & 1 & & \\ & & & \ddots & \\ \frac{1}{\sqrt{2}} & & & & \frac{1}{\sqrt{2}} \\ & \frac{1}{\sqrt{2}} & & & \end{pmatrix} + O(t^{-\frac{1}{n-1}})$$

where $O(t^{-\frac{1}{n-1}})$ is uniform as $t \to \infty$ for all points in Ω_{n-1}.

As noted above, we will integrate the initial value problem along geodesics of the metric $|q_b|^{\frac{2}{b}}$ which avoid the zeros of q_b. For any $P' \in \mathcal{U}_P$, we express P' in polar coordinates $P' = Le^{i\theta}$, the geodesic γ of the metric $|q_b|^{\frac{2}{b}}$ which starts at P and ends at P' is the straight line

$$\gamma(s) = se^{i\theta} \qquad \text{for } s \in [0, L].$$

We start at P with the initial rescaled holomorphic frame $F_b(P)$. For a fixed t, parallel transportation along the geodesic $\gamma(s) : [0, L] \to \widetilde{\Sigma}$ with respect to the flat connection yields a family of frames $G_b(\gamma(s))(t)$ along γ given by

$$G_b(\gamma(s))(t) = T^b_{P,\gamma(s)}(t)(F_b(P)) \qquad \text{with} \qquad T^b_{P,\gamma(0)}(t) = Id.$$

For each t, consider the family of matrices $\Psi^b_t(s)$ satisfying

$$\Psi^b_t(0) = Id \quad \text{and} \quad \Psi^b_t(s)G_b(\gamma(s))(t) = F_b(\gamma(s)).$$

Since $G_b(\gamma(s))(t)$ is parallel along γ, we can rewrite $\nabla_{\frac{\partial}{\partial s}} F_b(\gamma(s))$ in terms of $G_b(\gamma(s))(t)$

$$\nabla_{\frac{\partial}{\partial s}} F_b(\gamma(s)) = \frac{d\Psi^b_t}{ds} G_b(\gamma(s))(t).$$

Also,

$$\nabla_{\frac{\partial}{\partial s}} F_b(\gamma(s)) = (e^{i\theta}U_b + e^{-i\theta}V_b)F_b(\gamma(s)) = (e^{i\theta}U_b + e^{-i\theta}V_b)\Psi^b_t G_b(\gamma(s))(t),$$

hence,

$$\frac{d\Psi^b_t}{ds} = (e^{i\theta}U_b + e^{-i\theta}V_b)\Psi^b_t.$$

Rewriting $T_{P,\gamma(s)}(t)$ in terms of Ψ^b_t gives

$$T_{P,\gamma(s)}(t)(F_b(P)) = G_b(\gamma(s))(t) = \Psi^b_t(\gamma(s))^{-1}F_b(\gamma(s)). \tag{3.4.1}$$

From Theorem 3.3.1, the metric at P and the metric at $\gamma(s)$ are related by $h^b_{\gamma(s)} = h^b_P(1 + O(t^{-\frac{2}{b}}))$. As a result of rescaling the holomorphic frame, Theorem 3.3.1 also implies

$$\overline{F_b(P)}^T \cdot h^b_P \cdot F_b(P) = Id(1+O(t^{-\frac{2}{b}})) \quad \text{and} \quad \overline{F_b(\gamma(s))}^T \cdot h^b_{\gamma(s)} \cdot F_b(\gamma(s)) = Id(1+O(t^{-\frac{2}{b}}))$$

This means that for large t, the frames F_b are close to unitary, thus

$$F_b(\gamma(s)) = F_b(P)(1 + O(t^{-\frac{2}{b}})).$$

Note here we are being careful to write $Id(1+O(t^{-\frac{2}{b}}))$, which is stronger than $Id + O(t^{-\frac{2}{b}})$, since the former is necessarily diagonal while latter is not.

Applying this to (3.4.1) gives the following proposition.

Proposition 3.4.2. *With respect to the frame $(\sigma_1, \ldots, \sigma_{\frac{n}{2}}, \sigma^*_{\frac{n}{2}}, \ldots, \sigma^*_1)$, parallel transport along the geodesic from P to P' for the flat connection is given by $\Psi^b_t(L)^{-1}(1+O(t^{-\frac{2}{b}}))$, where Ψ^b_t solves the initial value problem*

$$\Psi^b_t(0) = I \qquad \frac{d\Psi^b_t}{ds} = (e^{i\theta}U_b + e^{-i\theta}V_b)\Psi^b_t$$

Explicitly, we have

(1) For $\phi = \tilde{e}_1 + q_n e_{n-1}$,

$$\frac{d\Psi}{ds} = \left[t^{\frac{1}{n}} \begin{pmatrix} 0 & e^{-i\theta} & & & 0 & e^{i\theta} \\ e^{i\theta} & 0 & e^{-i\theta} & & & \\ & \ddots & & \ddots & & \\ & & & & & \\ 0 & & & e^{i\theta} & 0 & e^{-i\theta} \\ e^{-i\theta} & & & & e^{i\theta} & 0 \end{pmatrix} + O(t^{-\frac{1}{n}}) \right] \Psi$$

(2) For $\phi = \tilde{e}_1 + q_{n-1}e_{n-2}$,

$$\frac{d\Psi}{ds} = \left[(2t)^{\frac{1}{n-1}} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}e^{-i\theta} & & & \frac{1}{\sqrt{2}}e^{i\theta} & 0 \\ \frac{1}{\sqrt{2}}e^{i\theta} & 0 & e^{-i\theta} & & & \frac{1}{\sqrt{2}}e^{i\theta} \\ & \ddots & & \ddots & & \\ & & & & & \\ \frac{1}{\sqrt{2}}e^{-i\theta} & & & e^{i\theta} & 0 & \frac{1}{\sqrt{2}}e^{-i\theta} \\ 0 & \frac{1}{\sqrt{2}}e^{-i\theta} & & & \frac{1}{\sqrt{2}}e^{i\theta} & 0 \end{pmatrix} + O(t^{-\frac{1}{n-1}}) \right] \Psi$$

In the above expressions, the matrix inside the bracket may be diagonalized by a constant unitary matrix S_b, and thus can be written as

$$S_b \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} S_b^{-1},$$

where

$$S_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta_n & \zeta_n^2 & \cdots & \zeta_n^{n-1} \\ \vdots & & \vdots & & \\ 1 & \zeta_n^{n-1} & \zeta_n^{2(n-1)} & \cdots & \zeta_n^{(n-1)^2} \end{pmatrix}, S_{n-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ 0 & 1 & \zeta_{n-1} & \cdots & \zeta_{n-1}^{n-2} \\ \vdots & \vdots & & \vdots & \\ 0 & \zeta_{n-1}^{n-2} & \zeta_{n-1}^{2(n-2)} & \cdots & \zeta_{n-1}^{(n-2)^2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{3.4.2}$$

and the set $\{\mu_j\}$ is given by the roots of the characteristic polynomial $det(\mu I - M_b(\theta))$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. More precisely,

(1) For the case $\phi = \tilde{e}_1 + q_n e_{n-1}$, $\mu_j = 2\cos(\theta + \frac{2\pi j}{n})$.

(2) For the case $\phi = \tilde{e}_1 + q_{n-1} e_{n-2}$, $\mu_1 = 0$, and for $j \geq 2$, $\mu_j = 2\cos(\theta + \frac{2\pi j}{n-1})$.

To compute $\Psi^b(L)$, we compute $\Phi^b = S_b^{-1} \Psi^b S_b$

$$\Phi^b(0) = I, \qquad \frac{d\Phi^b}{ds} = \left[t^{\frac{1}{b}} M_b(\theta) + R \right] \Phi^b \tag{3.4.3}$$

where $M_b(\theta) = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix}$, and $S_b^{-1} R S_b$ is the error term in Proposition 3.4.2. There is a constant C so that the matrix elements $R_{ij} = R_{ij}(s,t)$ satisfies $|R_{ij}| \leq Ct^{-\frac{1}{b}}$.

To integrate this initial value problem, we employ the following strategy:

Consider the solution Φ_0^b to the initial value problem

$$\Phi_0^b(0) = I, \qquad \frac{d\Phi_0^b}{ds} = t^{\frac{1}{n}} M_b(\theta)\Phi_0^b. \tag{3.4.4}$$

Hence $\Phi_0^b(s) = \begin{pmatrix} e^{st^{\frac{1}{b}}\mu_1} & & & \\ & e^{st^{\frac{1}{b}}\mu_2} & & \\ & & \ddots & \\ & & & e^{st^{\frac{1}{b}}\mu_n} \end{pmatrix}$.

Instead of solving for Φ^b asymptotically, we solve for $(\Phi_0^b)^{-1}\Phi^b$ which solves the initial value problem

$$(\Phi_0^b)^{-1}\Phi^b(0) = I, \qquad \frac{d((\Phi_0^b)^{-1}\Phi^b)}{ds} = (\Phi_0^b)^{-1}R\Phi_0^b \cdot (\Phi_0^b)^{-1}\Phi^b. \tag{3.4.5}$$

This can be seen by using the product rule

$$\begin{aligned} \frac{d((\Phi_0^b)^{-1}\Phi^b)}{ds} &= \frac{d\Phi_0^b}{ds}\Phi^b + (\Phi_0^b)^{-1}\frac{d\Phi^b}{ds} && (3.4.6) \\ &= -(\Phi_0^b)^{-1}\frac{d\Phi_0^b}{ds}(\Phi_0^b)^{-1}\Phi^b + (\Phi_0^b)^{-1}\frac{d\Phi^b}{ds} && (3.4.7) \\ &= -(\Phi_0^b)^{-1}t^{\frac{1}{b}}M_b(\theta)\Phi^b + (\Phi_0^b)^{-1}(t^{\frac{1}{b}}M_b(\theta) + R)\Phi^b && (3.4.8) \\ &= (\Phi_0^b)^{-1}R\Phi^b && (3.4.9) \\ &= (\Phi_0^b)^{-1}R\Phi_0^b \cdot (\Phi_0^b)^{-1}\Phi^b. && (3.4.10) \end{aligned}$$

For the initial value problem (3.4.5) we will show $(\Phi_0^b)^{-1}R\Phi_0^b$ is $o(1)$, and that $(\Phi_0^b)^{-1}\Phi^b$ is $Id + o(1)$. Hence,

$$\Phi^b = \Phi_0^b(Id + o(1)).$$

Before doing this, we need a more in-depth understanding of the error term.

The estimate of the error term for the ODE relies mainly on the error estimate of the u^j's and v^j's. For the n-cyclic case, we introduce the following notation for the error term for u^j coming from Theorem 3.3.1

$$\tilde{u}^j = u^j - ln|tq_n|^{\frac{n+1-2j}{n}}.$$

Similarly for the $(n-1)$-cyclic case we set

$$\tilde{v}^j = \begin{cases} v^j - ln|tq_{n-1}| & j = 1 \\ v^j - ln|2tq_{n-1}|^{\frac{n+1-2j}{n-1}} & otherwise \end{cases}$$

In the frame we are working in, $\tilde{u}^j = u^j - ln|t|^{\frac{n+1-2j}{n}}$, and similarly for the $\tilde{v}^j$'s. For the n-cyclic case, writing the error term R for the ODE (3.4.3) in terms of $\tilde{u}^j$ gives

$$S_n^{-1}\left(e^{i\theta}\begin{pmatrix} \tilde{u}^1_z & & & \\ & \tilde{u}^2_z & & \\ & & \ddots & \\ & & & -\tilde{u}^1_z \end{pmatrix} + t^{\frac{1}{n}}e^{-i\theta}\begin{pmatrix} 0 & e^{\tilde{u}^1-\tilde{u}^2}-1 & & \\ & \ddots & \ddots & \\ & & 0 & e^{\tilde{u}^1-\tilde{u}^2}-1 \\ e^{-2\tilde{u}^1}-1 & & & 0 \end{pmatrix}\right)S_n \tag{3.4.11}$$

which we will write as $R = B^1_n + t^{\frac{1}{n}}B^2_n$. In a similar fashion, the error term for the $(n-1)$-cyclic case is

$$S_{n-1}^{-1}(e^{i\theta}\begin{pmatrix} \tilde{v}^1_z & & & & \\ & \tilde{v}^2_z & & & \\ & & \ddots & & \\ & & & -\tilde{v}^2_z & \\ & & & & -\tilde{v}^1_z \end{pmatrix} \tag{3.4.12}$$

$$+(2t)^{\frac{1}{n-1}}e^{-i\theta}\begin{pmatrix} & \frac{1}{\sqrt{2}}(e^{\tilde{v}^1-\tilde{v}^2}-1) & & & \\ & & e^{\tilde{v}^2-\tilde{v}^3}-1 & & \\ & & & \ddots & \\ & & & & e^{\tilde{v}^2-\tilde{v}^3}-1 & \\ \frac{1}{\sqrt{2}}(e^{-\tilde{v}^1-\tilde{v}^2}-1) & & & & & \frac{1}{\sqrt{2}}(e^{\tilde{v}^1-\tilde{v}^2}-1) \\ & \frac{1}{\sqrt{2}}(e^{-\tilde{v}^1-\tilde{v}^2}-1) & & & & \end{pmatrix})S_{n-1} \tag{3.4.13}$$

3.4.1 The n-cyclic case

We need the following crucial theorem on estimates on errors.

Theorem 3.4.3. *For any $d < d(p) = min\{p, zeros\ of\ q_n\}$, as $t \to +\infty$, the (k,l)-entry R_{kl} of R satisfies*

$$R_{kl}(p) = O(t^{-\frac{1}{2n}} e^{-2|1-\zeta_n^{k-l}|t^{\frac{1}{n}} d}).$$

Due to the length and level of technicality of the proof, we will spend the next two sections proving Theorem 3.4.3. In section 3.5 we relate the error term with solutions to the $\mathfrak{sl}(n,\mathbb{C})$-cyclic Toda lattice, and prove a recursive formula on the Toda lattice. Then in section 3.6 we apply the recursive formula to prove the desired estimate.

Assuming Theorem 3.4.3, we prove the following Theorem on the asymptotic of the parallel transport operator with an extra condition on the path.

Theorem 3.4.4. *Suppose P, P' and the geodesic path $\gamma(s)$ are as above and P' is located so that for every s,*

$$s < d(\gamma(s)) := min\{d(\gamma(s), z_0),\ for\ all\ zeros\ z_0\ of\ q_n\}.$$

Then as $t \to \infty$,

$$T_{P,P'}(t) = (Id + O(t^{-\frac{1}{2n}}))S^{-1} \begin{pmatrix} e^{-Lt^{\frac{1}{n}}\mu_1} & & & \\ & e^{-Lt^{\frac{1}{n}}\mu_2} & & \\ & & \ddots & \\ & & & e^{-Lt^{\frac{1}{n}}\mu_n} \end{pmatrix} S$$

where $\mu_j = 2cos(\theta + \frac{2\pi(j-1)}{n})$ and S is S_n in equation (3.4.2).

Remark. In the case that the angle $<_{z_0} (P, P')$ satisfies $<_{z_0} (P, P') < \pi/3$, for all zeros z_0 of q_n, the $|q_n|^{2/n}-$geodesic from P to P' satisfies the condition in the above theorem.

When P and P' both project to the same point in Σ, the projected path is a loop. In this case, the above asymptotics correspond to the values of the associated family of representations on the homotopy class of the loop.

Proof. (*of Theorem* 3.4.4) Assuming Theorem 3.4.3, we have the (k,l)-entry of the error term $(\Phi_0^n)^{-1}R\Phi_0^n$ is $R_{k,l}(\gamma(s))e^{(\mu_k-\mu_l)st^{\frac{1}{n}}} = O(t^{-\frac{1}{2n}}e^{-2|1-\zeta_n^{k-l}|t^{\frac{1}{n}}d(\gamma(s))}e^{(\mu_k-\mu_l)st^{\frac{1}{n}}})$. We make use of

$$|\mu_k - \mu_l| = |2\cos(\theta + \frac{2\pi(k-1)}{n}) - 2\cos(\theta + \frac{2\pi(l-1)}{n})| \quad (3.4.14)$$

$$= |4\sin\frac{\pi(k-l)}{n}\sin(\theta + \frac{\pi(k+l-2)}{n})| \quad (3.4.15)$$

$$\leq |4\sin\frac{\pi(k-l)}{n}| = 2|1-\zeta_n^{k-l}| \quad (3.4.16)$$

Hence the (k,l)-entry of $(\Phi_0^n)^{-1}R\Phi_0^n$ is $O(t^{-\frac{1}{2n}}e^{2|1-\zeta_n^{k-l}|t^{\frac{1}{n}}(s-d(\gamma(s)))})$. Since $\gamma(s)$ satisfies the condition that for every s, $s < d(\gamma(s))$, we obtain that $(\Phi_0^n)^{-1}R\Phi_0^n = O(t^{-\frac{1}{2n}})$.

We make use of the following classical theorem in ODE theory.

Lemma 3.4.5. *For the equation $F'(s) = F(s)A(s)$ on an interval $[a,b] \subset \mathbb{R}$, where the coefficient $A : J \to gl_n(\mathbb{R})$ is a continuous function. There exists $C, \delta_0 > 0$ such that if $||A(t)|| < \delta < \delta_0$ for all $s \in [a,b]$, then the solution F with $F(a) = I$ satisfies $|F(s) - I| < C\delta$ for all $s \in [a,b]$.*

For a nice proof of the above lemma, also see the appendix B of [DW14]. Applying Lemma 3.4.5 and $(\Phi_0^n)^{-1}R\Phi_0^n = O(t^{-\frac{1}{2n}})$ into the ODE 3.4.5, we obtain that $(\Phi_0^n)^{-1}\Phi_0^n = Id + O(t^{-\frac{1}{2n}})$. Therefore $\Phi^n = \Phi_0^n(Id + O(t^{-\frac{1}{2n}}))$. □

3.4.2 The $(n-1)$-cyclic case

We need the following crucial theorem on estimates on errors.

Theorem 3.4.6. *For any $d < d(p) = min\{p, zeros\ of\ q_{n-1}\}$, as $t \to +\infty$, the (k,l)-entry R_{kl} of R satisfies*

$$R_{kl}(p) = \begin{cases} O(t^{-\frac{1}{2(n-1)}} e^{-2|1-\zeta_{n-1}^{k-i}|(2t)^{\frac{1}{n-1}} d}) & k,l \geq 2 \\ 0 & k=l=1 \\ O(t^{-\frac{1}{2(n-1)}} e^{-2(2t)^{\frac{1}{n-1}} d}) & otherwise \end{cases}$$

We will spend the next two sections proving Theorem 3.4.6. Assuming Theorem 3.4.6, we prove the following Theorem on the asymptotic of the parallel transport operator with an extra condition on the path.

Theorem 3.4.7. *Suppose P, P' and the geodesic path $\gamma(s)$ are as above and P' is located so that for every s,*

$$s < d(\gamma(s)) := min\{d(\gamma(s), z_0),\ for\ all\ zeros\ z_0\ of\ q_{n-1}\}.$$

Then as $t \to \infty$,

$$T_{P,P'}(t) = (Id + O(t^{-\frac{1}{2(n-1)}}))S^{-1} \begin{pmatrix} e^{-Lt^{\frac{1}{n-1}}\mu_1} & & & \\ & e^{-Lt^{\frac{1}{n-1}}\mu_2} & & \\ & & \ddots & \\ & & & e^{-Lt^{\frac{1}{n-1}}\mu_n} \end{pmatrix} S$$

where S is S_{n-1} in equation (3.4.2) and $\mu_1 = 0$, for $j \geq 2, \mu_j = 2cos(\theta + \frac{2\pi(j-2)}{n-1})$.

Remark. In the case that the angle $<_{z_0} (P, P')$ satisfies $<_{z_0} (P, P') < \pi/3$, for all zeros z_0 of q_{n-1}, the $|q_{n-1}|^{\frac{2}{n-1}}$–geodesic from P to P' will satisfy the condition in the above theorem.

When P and P' both project to the same point in Σ, the projected path is a loop. In this case, the above asymptotics correspond to the values of the associated family of representations on the homotopy class of the loop.

Proof. (*of Theorem* 3.4.7) Assuming Theorem 3.4.6, we have the (k,l)-entry of the error term $(\Phi_0^{n-1})^{-1}R\Phi_0^{n-1}$ is $R_{k,l}(\gamma(s))e^{(\mu_k-\mu_l)st^{\frac{1}{n}}}$. For $k,l \geq 2$, we make use of

$$\begin{aligned} |\mu_k - \mu_l| &= |2\cos(\theta + \frac{2\pi(k-2)}{n-1}) - 2\cos(\theta + \frac{2\pi(l-2)}{n-1})| && (3.4.17) \\ &= |4\sin\frac{\pi(k-l)}{n-1}\sin(\theta + \frac{\pi(k+l-4)}{n-1})| && (3.4.18) \\ &\leq |4\sin\frac{\pi(k-l)}{n-1}| = 2|1-\zeta_{n-1}^{k-l}| && (3.4.19) \end{aligned}$$

Hence for $k,l \geq 2$, (k,l)-entry of $(\Phi_0^{n-1})^{-1}R\Phi_0^{n-1}$ is $O(t^{-\frac{1}{2(n-1)}}e^{2|1-\zeta_n^{k-l}|(2t)^{\frac{1}{n-1}}(s-d(\gamma(s))})$.

For $k = l = 1$, we have $\mu_1 = 0$, hence the $(1,1)$-entry of $(\Phi_0^{n-1})^{-1}R\Phi_0^{n-1}$ is $O(t^{-\frac{1}{2(n-1)}})$.

For $k = 1, l \neq 1$, we have

$$|\mu_k - \mu_l| = |0 - 2\cos(\theta + \frac{2\pi(l-1)}{n-1})| \leq 2.$$

For $l = 1, k \neq 1$, we have

$$|\mu_k - \mu_l| = |2\cos(\theta + \frac{2\pi(k-1)}{n-1}) - 0| \leq 2.$$

Hence for $k = 1, l \neq 1$ or $l = 1, k \neq 1$, the (k,l)-entry of $(\Phi_0^{n-1})^{-1}R\Phi_0^{n-1}$ is

$$O(t^{-\frac{1}{2(n-1)}}e^{2(2t)^{\frac{1}{n-1}}(s-d(\gamma(s)))}).$$

Since $\gamma(s)$ satisfies the condition that for every s, $s < d(\gamma(s))$, we obtain that $(\Phi_0^{n-1})^{-1}R\Phi_0^{n-1} = O(t^{-\frac{1}{2(n-1)}})$.

We make use of Lemma 3.4.5 again and obtain $(\Phi_0^{n-1})^{-1}\Phi_0^{n-1} = Id + O(t^{-\frac{1}{2(n-1)}})$. Therefore $\Phi^{n-1} = \Phi_0^{n-1}(Id + O(t^{-\frac{1}{2(n-1)}}))$. ∎

3.5 The Error terms and the Toda Lattice

To understand the asymptotics of the error terms we reinterpret them in terms of the Toda Lattice. In fact, the Toda lattice we consider here are usually called Toda lattice with opposite sign as oppose to the usual Toda lattice. The Toda equations have been extensively studied from the point of view of integrable systems and the relation to minimal surfaces and harmonic maps. For instance, the readers can see [Gue97, BW94, MOP81].

3.5.1 The Toda Lattice

The formula from the cyclic $\mathfrak{sl}(n, \mathbb{C})$ Toda lattice, or the affine $\mathfrak{sl}(n, \mathbb{C})$ Toda equations, is the following

$$\begin{cases} \Delta d^1 = e^{d^1 - d^2} - e^{d^n - d^1} \\ \Delta d^2 = e^{d^2 - d^3} - e^{d^1 - d^2} \\ \quad \cdots \\ \Delta d^n = e^{d^n - d^1} - e^{d^{n-1} - d^n} \end{cases} \tag{3.5.1}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{4}\frac{\partial^2}{\partial z \partial \bar{z}}$.

Define

$$w_k = \frac{1}{\sqrt{n}} \sum_{i \in \mathbb{Z}_n} \zeta_n^{ik}(d^i - d^{i+1}),$$

and note that since the sum is over $\mathbb{Z}_n$, we have $w_0 = 0$.

Proposition 3.5.1. *The term w_k satisfies the follow properties.*

1.

$$w_k = \frac{1 - \zeta_n^{-k}}{\sqrt{n}} \sum_i \zeta_n^{ki} d^i$$

2.

$$\Delta w_k = \frac{|1-\zeta_n^k|^2}{\sqrt{n}} \sum_i \zeta_n^{(i-1)k} e^{d^i - d^{i+1}}$$

3.

$$\Delta w_k = |1-\zeta_n^k|^2 \sum_{\substack{r \equiv k \\ mod\ n}} \sum_{r_1+\cdots+r_s=r} \frac{1}{s! n^{\frac{s-1}{2}}} \binom{r}{r_1, \cdots, r_s} w_{r_1} w_{r_2} \cdots w_{r_s}$$

Proof. Items 1 and 2 follow immediately from reordering the summation. Proving 3 takes a little more work, we start by linearizing the cyclic Toda system (3.5.1) at zero as

$$\begin{cases} \Delta d^1 = -d^n + 2d^1 - d^2 \\ \Delta d^2 = -d^1 + 2d^2 - d^3 \\ \quad \cdots \\ \Delta d^n = -d^{n+1} + 2d^n - d^1 \end{cases} \tag{3.5.2}$$

Subtracting the $(i+1)^{th}$ equations from the i^{th} equations in the linearization gives

$$\begin{cases} \Delta(d^n - d^1) = -(d^{n-1} - d^n) + 2(d^n - d^1) - (d^1 - d^2) \\ \Delta(d^1 - d^2) = -(d^n - d^1) + 2(d^1 - d^2) - (d^2 - d^3) \\ \quad \cdots \\ \Delta(d^{n-1} - d^n) = -(d^{n-2} - d^{n-1}) + 2(d^{n-1} - d^n) - (d^n - d^1) \end{cases} \tag{3.5.3}$$

Note that $w_k = \frac{1}{\sqrt{n}} \sum_{i \in \mathbb{Z}_n} \zeta_n^{ik}(d^i - d^{i+1})$ is an eigenfunction of equations (3.5.3).

$$\begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{n-1} \end{pmatrix} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta_n & \zeta_n^2 & \cdots & \zeta_n^{n-1} \\ \vdots & & \vdots & & \\ 1 & \zeta_n^{n-1} & \zeta_n^{2(n-1)} & \cdots & \zeta_n^{(n-1)^2} \end{pmatrix} \begin{pmatrix} d^n - d^1 \\ d^1 - d^2 \\ \vdots \\ d^{n-1} - d^n \end{pmatrix}$$

Denote the above matrix by S and note that it is unitary. We have

$$\begin{pmatrix} d^n - d^1 \\ \vdots \\ d^{n-1} - d^n \end{pmatrix} = \bar{S}^T \begin{pmatrix} w_0 \\ \vdots \\ w_{n-1} \end{pmatrix}. \tag{3.5.4}$$

Write S in terms of row vectors $S = \begin{pmatrix} S_0 \\ \vdots \\ S_{n-1} \end{pmatrix}$ then,

$$S^{-1} = \bar{S}^T = \begin{pmatrix} \bar{S}_0^T & \bar{S}_1^T & \cdots & \bar{S}_{n-1}^T \end{pmatrix}$$

with $\bar{S}_i^T = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \zeta_n^{-i} \\ \vdots \\ \zeta_n^{(n-1)i} \end{pmatrix}$. Exploiting the cyclicity of the cyclic Toda lattice, we have

$$\begin{pmatrix} d^1 - d^2 \\ d^2 - d^3 \\ \vdots \\ d^n - d^1 \end{pmatrix} = \begin{pmatrix} \bar{S}_0^T & \zeta_n^{-1}\bar{S}_1^T & \zeta_n^{-2}\bar{S}_2^T & \cdots & \zeta_n^{-(n-1)}\bar{S}_{n-1}^T \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{n-1} \end{pmatrix} \tag{3.5.5}$$

and

$$\begin{pmatrix} d^{n-1} - d^n \\ d^n - d^1 \\ \vdots \\ d^{n-2} - d^{n-1} \end{pmatrix} = \begin{pmatrix} \bar{S}_0^T & \zeta_n^{1}\bar{S}_1^T & \zeta_n^{2}\bar{S}_2^T & \cdots & \zeta_n^{(n-1)}\bar{S}_{n-1}^T \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{n-1} \end{pmatrix}. \tag{3.5.6}$$

Since $\Delta w_k = S_k \Delta \begin{pmatrix} d^n - d^1 \\ d^1 - d^2 \\ \vdots \\ d^{n-1} - d^n \end{pmatrix}$, using the cyclic Toda equations (3.5.1), we have

$$\Delta w_k = S_k \left(2 \begin{pmatrix} e^{d^n - d^1} \\ e^{d^1 - d^2} \\ \vdots \\ e^{d^{n-1} - d^n} \end{pmatrix} - \begin{pmatrix} e^{d^1 - d^2} \\ e^{d^2 - d^3} \\ \vdots \\ e^{d^n - d^1} \end{pmatrix} - \begin{pmatrix} e^{d^{n-1} - d^n} \\ e^{d^n - d^1} \\ \vdots \\ e^{d^{n-2} - d^{n-1}} \end{pmatrix} \right)$$

The next step is to expand all the exponentials, we make use of the Hadamard product $*$ on vectors, which is defined as $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} * \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \\ \vdots \\ x_n y_n \end{pmatrix}$.

$$\Delta w_k = S_k \sum_{l \geq 0} \frac{1}{l!} \left(2 \begin{pmatrix} d^n - d^1 \\ d^1 - d^2 \\ \vdots \\ d^{n-1} - d^n \end{pmatrix}^{*l} - \begin{pmatrix} d^1 - d^2 \\ d^2 - d^3 \\ \vdots \\ d^n - d^1 \end{pmatrix}^{*l} - \begin{pmatrix} d^{n-1} - d^n \\ d^n - d^1 \\ \vdots \\ d^{n-2} - d^{n-1} \end{pmatrix}^{*l} \right)$$

Plugging in Equation (3.5.4),(3.5.5), and (3.5.6), we can rewrite this as

$$\Delta w_k = S_k \left(\sum_{l \geq 0} \frac{1}{l!} \left(2 (\sum_{i \in \mathbb{Z}_n} \bar{S}_i^T w_i)^{*l} - (\sum_{i \in \mathbb{Z}_n} \bar{S}_i^T w_i \zeta_n^i)^{*l} - (\sum_{i \in \mathbb{Z}_n} \bar{S}_i^T w_i \zeta_n^{-i})^{*l} \right) \right).$$

Reindexing the sum by the index of the w_j's we have

$$\Delta w_k = S_k \sum_{r \geq 0} (2 \sum_{i_1 + \cdots + i_s = r} \frac{1}{s!} \binom{r}{i_1, \cdots, i_s} \bar{S}_{i_1}^T * \cdots * \bar{S}_{i_s}^T w_{i_1} w_{i_2} \cdots w_{i_s} \tag{3.5.7}$$

$$- \sum_{i_1 + \cdots + i_s = r} \frac{1}{s!} \zeta_n^r \binom{r}{i_1, \cdots, i_s} \bar{S}_{i_1}^T * \cdots * \bar{S}_{i_s}^T w_{i_1} \cdots w_{i_s} \tag{3.5.8}$$

$$- \sum_{i_1+\cdots+i_s=r} \frac{1}{s!}\zeta_n^r \binom{r}{i_1,\cdots,i_s} \bar{S}_{i_1}^T * \cdots * \bar{S}_{i_s}^T w_{i_1}\cdots w_{i_s}) \tag{3.5.9}$$

$$= S_k \left(\sum_{r\geq 0} (2-\zeta_n^r+\zeta_n^{-r}) \sum_{i_1+\cdots+i_s=r} \frac{1}{s!} \binom{r}{i_1,\cdots,i_s} \bar{S}_{i_1}^T * \cdots * \bar{S}_{i_s}^T w_{i_1}\cdots w_{i_s} \right).$$

Observe that $\bar{S}_i^T * \bar{S}_j^T = \frac{1}{\sqrt{n}}\bar{S}_{i+j \mod n}^T$, so we may rewrite the above equation as

$$\Delta w_k = S_k \left(\sum_{r\geq 0} |1-\zeta_n^r|^2 \sum_{i_1+\cdots+i_s=r} \frac{1}{s!\sqrt{n^{s-1}}} \binom{r}{i_1,\cdots,i_s} \bar{S}_{r(\mathrm{mod}\ n)}^T w_{i_1}\cdots w_{i_s} \right).$$

Finally, using the fact that S is unitary, we have $S_i\bar{S}_j^T = \delta_{ij}$, and thus

$$\Delta w_k = |1-\zeta_n^k|^2 \sum_{\substack{r\equiv k \\ \mathrm{mod}\ n}} \sum_{i_1+\cdots+i_s=r} \frac{1}{s!\sqrt{n^{s-1}}} \binom{r}{i_1,\cdots,i_s} w_{i_1}\cdots w_{i_s}$$

as desired. □

In applications to the Error term we will sometimes need the following perturbed version of the Toda system

$$\begin{cases} \Delta d^1 = a(e^{d^1-d^2} - e^{d^n-d^1} + f_1) \\ \Delta d^2 = a(e^{d^2-d^3} - e^{d^1-d^2} + f_2) \\ \quad \cdots \\ \Delta d^n = a(e^{d^n-d^1} - e^{d^{n-1}-d^n} + f_n) \end{cases} \tag{3.5.10}$$

where a is a constant and $f_j = f_j(d^1,\ldots,d^n)$. Again we define

$$w_k = \frac{1}{\sqrt{n}} \sum_{i\in\mathbb{Z}_n} \zeta_n^{ik}(d^i - d^{i+1}),$$

and have the following proposition analogous to Proposition 3.5.1.

Proposition 3.5.2. *If w_k is as above, then*

1.

$$w_k = \frac{1-\zeta_n^{-k}}{\sqrt{n}} \sum_i \zeta_n^{ki} d^i$$

2.

$$\Delta w_k = \frac{a|1-\zeta_n^k|^2}{\sqrt{n}} \sum_i \zeta_n^{(i-1)k} e^{d^i - d^{i+1}} + \frac{a}{\sqrt{n}} \sum_i \zeta_n^{ik}(f_i - f_{i+1})$$

3.

$$\Delta w_k = a \sum_{\substack{r \equiv k \\ mod\ n}} \sum_{r_1+\cdots+r_s=r} \frac{1}{s! n^{\frac{s-1}{2}}} \binom{r}{r_1, \cdots, r_s} |1-\zeta_n^k|^2 w_{r_1} w_{r_2} \cdots w_{r_s}$$
$$+ \frac{a}{\sqrt{n}} \sum_i \zeta_n^{ik}(f_i - f_{i+1})$$

The proof of the following proposition is straightforwardly induced from the proof of Proposition 3.5.1.

We now examine the error term and relate it to the w_k's from the Toda lattice. Using the results in the previous section, we show that the matrix elements have the desired decay.

3.5.2 The n-cyclic case

In this section we write the error term (3.4.11) for the n-cyclic Higgs field in terms of the w_k's of the previous section. Rewriting the Hitchin equations in terms of the $\tilde{u}^j$ yields,

$$\begin{cases} \Delta \tilde{u}^1 = 4t^{\frac{2}{n}}(e^{\tilde{u}^1 - \tilde{u}^2} - e^{-2\tilde{u}^1}) \\ \Delta \tilde{u}^2 = 4t^{\frac{2}{n}}(e^{\tilde{u}^2 - \tilde{u}^3} - e^{\tilde{u}^1 - \tilde{u}^2}) \\ \qquad \cdots \\ \Delta(-\tilde{u}^1) = 4t^{\frac{2}{n}}(e^{-2\tilde{u}^1} - e^{\tilde{u}^1 - \tilde{u}^2}) \end{cases} \tag{3.5.11}$$

This system is a special case of the cyclic Toda lattice, in fact it is a real form of the $\mathfrak{sl}(n, \mathbb{C})$ cyclic Toda lattice. Our techniques do not rely on this extra symmetry and we will think of $(\tilde{u}^1, \cdots, -\tilde{u}^1)$ as $(d^1, \cdots, d^n)$ satisfying $d^{n+1-i} = -d^i$. Recall from (3.4.11), the error term R is written as $B_n^1 + t^{\frac{1}{n}} B_n^2$.

For ease of notation, we will write the diagonalizing matrix in terms of column vectors, that is,

$$S^{-1} = \bar{S}^T = \begin{pmatrix} \bar{S}_0^T & \bar{S}_1^T & \cdots & \bar{S}_{n-1}^T \end{pmatrix}.$$

Thus, $B^1_{_n}$ is given by

$$\begin{pmatrix} \bar{S}_0^T & \bar{S}_1^T & \cdots & \bar{S}_{n-1}^T \end{pmatrix} \begin{pmatrix} d_z^1 & & & \\ & d_z^2 & & \\ & & \ddots & \\ & & & d_z^n \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ \vdots \\ S_{n-1} \end{pmatrix}$$

$$= \sum_i \bar{S}_i^T S_i d_z^{i+1}$$

Using the definition of S_i

$$\bar{S}_i^T S_i = \frac{1}{n} \begin{pmatrix} 1 \\ \zeta_{_n}^i \\ \vdots \\ \zeta_{_n}^{(n-1)i} \end{pmatrix} \begin{pmatrix} 1 & \zeta_{_n}^{-i} & \zeta_{_n}^{-2i} & \cdots & \zeta_{_n}^{-(n-1)i} \end{pmatrix}$$

thus $(\bar{S}_i^T S_i)_{kl} = \frac{1}{n}\zeta_{_n}^{(k-1)i}\zeta_{_n}^{-(l-1)i} = \frac{1}{n}\zeta_{_n}^{(k-l)i}$. This gives

$$(B^1_{_n})_{kl} = \frac{1}{n}\sum_i \zeta_{_n}^{(k-l)i} d_z^{i+1}.$$

By the lemma $w_k = \dfrac{1-\zeta_{_n}^{-k}}{\sqrt{n}} \sum_i \zeta_{_n}^{ik} d^i$, so we may rewrite $B^1_{_n}$ as

$$(B^1_{_n})_{kl} = \frac{\zeta_{_n}^{-(k-l)}}{(1-\zeta_{_n}^{-(k-l)})\sqrt{n}} (w_{k-l})_z$$

For $B^2_{\frac{}{n}}$, we have

$$\begin{pmatrix} \bar{S}_0^T & \cdots & \bar{S}_{n-1}^T \end{pmatrix} \begin{pmatrix} 0 & e^{d^1-d^2}-1 & & & \\ & 0 & e^{d^2-d^3}-1 & & \\ & & \ddots & \ddots & \\ & & & 0 & e^{d^{n-1}-d^n}-1 \\ e^{d^n-d^1}-1 & & & & 0 \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ \vdots \\ S_{n-1} \end{pmatrix}$$

$$= \frac{1}{n}\sum_i \bar{S}_i^T S_i (e^{d^i-d^{i+1}}-1).$$

Since

$$\bar{S}_i^T S_{i+1} = \frac{1}{n}\begin{pmatrix} 1 \\ \zeta_n^i \\ \vdots \\ \zeta_n^{(n-1)i} \end{pmatrix} \begin{pmatrix} 1 & \zeta_n^{-i} & \zeta_n^{-2i} & \cdots & \zeta_n^{-(n-1)i} \end{pmatrix}$$

we have

$$(\bar{S}_i^T S_{i+1})_{kl} = \frac{1}{n}\zeta_n^{(k-1)i}\zeta_n^{-(l-1)(i+1)} = \zeta_n^{(k-l)i-(l-1)}.$$

This yields

$$(B^2_{\frac{}{n}})_{kl} = \frac{1}{n}\sum_i \zeta_n^{(k-l)i-(l-1)}(e^{d^i-d^{i+1}}-1)$$

since we are summing of $\mathbb{Z}_n$, the constant terms sum to 0 giving

$$(B^2_{\frac{}{n}})_{kl} = \frac{\zeta_n^{1-l}}{n}\sum_i \zeta_n^{(k-l)i} e^{d^i-d^{i+1}}$$

Now using second part of Proposition 3.5.2, we have

$$(B^2_{\frac{}{n}})_{kl} = \frac{\zeta_n^{1-l}}{n}\sum_i \zeta_n^{(k-l)i} e^{d^i-d^{i+1}} = \frac{C}{t^{\frac{2}{n}}}\Delta w_{k-l}.$$

In conclusion, we have the (k,l)-entry of the error term

$$R_{kl} = (B^1_{\frac{}{n}})_{kl} + t^{\frac{1}{n}}(B^2_{\frac{}{n}})_{kl} = C(w_{k-l})_z + Ct^{-\frac{1}{n}}\Delta w_{k-l}. \tag{3.5.12}$$

Lemma 3.5.3. *For the n-cyclic case, the eigensolutions w_k's satisfy*

1, $w_k = w_{n-k}$;

2, w_k is real.

Proof. 1,

$$\begin{aligned}
w_k &= \frac{1}{\sqrt{n}} \sum_{i\in\mathbb{Z}_n} \zeta_n^{ik}(d^i - d^{i+1}) \\
&= \frac{1}{\sqrt{n}} \sum_{i\in\mathbb{Z}_n} \zeta_n^{(n-i)(n-k)}(d^i - d^{i+1}) \\
&\quad \text{Since } d^i\text{'s satisfy that } d^{n+1-i} = -d^i, \\
&= \frac{1}{\sqrt{n}} \sum_{i\in\mathbb{Z}_n} \zeta_n^{(n-i)(n-k)}(d^{n-i} - d^{n-i+1}) \\
&= \frac{1}{\sqrt{n}} \sum_{i\in\mathbb{Z}_n} \zeta_n^{i(n-k)}(d^i - d^{i+1}) = w_{n-k}.
\end{aligned}$$

2,

$$\begin{aligned}
\overline{w_k} &= \frac{1}{\sqrt{n}} \sum_{i\in\mathbb{Z}_n} \zeta_n^{-ik}(d^i - d^{i+1}) \\
&= \frac{1}{\sqrt{n}} \sum_{i\in\mathbb{Z}_n} \zeta_n^{(n-i)k}(d^i - d^{i+1}) \\
&\quad \text{Since } d^i\text{'s satisfy that } d^{n+1-i} = -d^i, \\
&= \frac{1}{\sqrt{n}} \sum_{i\in\mathbb{Z}_n} \zeta_n^{(n-i)k}(d^{n-i} - d^{n-i+1}) \\
&= \frac{1}{\sqrt{n}} \sum_{i\in\mathbb{Z}_n} \zeta_n^{ik}(d^i - d^{i+1}) = w_k.
\end{aligned}$$

□

Remark 3.5.4. It is important to remember that the system we are interested in comes from the error terms $\tilde{u}^j$, and $w_k = \frac{1}{\sqrt{n}} \sum_i \zeta_n^{ki}(\tilde{u}^i - \tilde{u}^{i+1})$. By Theorem 3.3.1, $\tilde{u}^j \sim O(t^{-\frac{2}{n}})$, thus the w_k are also small for large t. This will be important in the proof of Theorem 3.4.4.

3.5.3 The $(n-1)$-cyclic case

Recall from (3.4.13) the error term for the $(n-1)$-cyclic case is written as $B^1_{n-1} + (2t)^{\frac{1}{n-1}} B^2_{n-1}$. In this case, we will reduce the system to a perturbed version of a cylic Toda lattice of rank $(n-1)$. Writing the Hitchin equations in terms of the $\tilde{v}^j$'s gives

$$\begin{cases} \Delta\tilde{v}^1 = 4(2t)^{\frac{2}{n-1}}(\frac{1}{2}e^{\tilde{v}^1-\tilde{v}^2} - \frac{1}{2}e^{-\tilde{v}^1-\tilde{v}^2}) \\ \Delta\tilde{v}^2 = 4(2t)^{\frac{2}{n-1}}(e^{\tilde{v}^2-\tilde{v}^3} - \frac{1}{2}e^{\tilde{v}^1-\tilde{v}^2} - \frac{1}{2}e^{-\tilde{v}^1-\tilde{v}^2}) \\ \Delta\tilde{v}^3 = 4(2t)^{\frac{2}{n-1}}(e^{\tilde{v}^3-\tilde{v}^4} - e^{\tilde{v}^2-\tilde{v}^3}) \\ \quad\cdots \\ \Delta(-\tilde{v}^1) = -\Delta\tilde{v}^1 = 4(2t)^{\frac{2}{n-1}}(-\frac{1}{2}e^{\tilde{v}^1-\tilde{v}^2} + \frac{1}{2}e^{-\tilde{v}^1-\tilde{v}^2}) \end{cases} \tag{3.5.13}$$

We first focus on the equations 2 through $n-1$, if we set $f = \frac{1}{2}(e^{-\tilde{v}^1} + e^{\tilde{v}^1} - 2)e^{-\tilde{v}^2}$. Then the equations 2 through $n-1$ are

$$\begin{cases} \Delta\tilde{v}^2 = 4(2t)^{\frac{2}{n-1}}(e^{\tilde{v}^2-\tilde{v}^3} - e^{-\tilde{v}^2}) - f \\ \Delta\tilde{v}^3 = 4(2t)^{\frac{2}{n-1}}(e^{\tilde{v}^3-\tilde{v}^4} - e^{\tilde{v}^2-\tilde{v}^3}) \\ \quad\cdots \\ \Delta(-\tilde{v}^2) = 4(2t)^{\frac{2}{n-1}}(-e^{\tilde{v}^2-\tilde{v}^3} + e^{-\tilde{v}^2}) + f \\ \Delta 0 = 4(2t)^{\frac{2}{n-1}}(e^{-\tilde{v}^2} - e^{-\tilde{v}^2}) \end{cases} \tag{3.5.14}$$

This system is a special case of the perturbed cyclic Toda lattice of rank $n-1$. As in the n-cyclic case, set $(\tilde{v}^2, \tilde{v}^3, \cdots, -\tilde{v}^2, 0) = (d^1, d^2, \cdots, d^{n-1})$ satisfying $d^{n-1} = 0$ and $d^{n-1-i} = -d^i$ for $1 \leq i \leq n-2$. If we define $w_k = \frac{1}{\sqrt{n-1}}\sum_i \zeta^{ik}_{n-1}(d^i - d^{i+1})$ then

$$\Delta w_k = 4(2t)^{\frac{2}{n-1}}\frac{|1-\zeta^k_{n-1}|^2}{\sqrt{n-1}}\sum_{i\in\mathbb{Z}_{n-1}}\zeta^{(i-1)k}_{n-1}e^{d^i-d^{i+1}} + C(2t)^{\frac{2}{n-1}}f.$$

First for the term $B^1_{_n}$, recall, that $S^{-1} = \bar{S}^T$. A simple calculations shows that

$$\bar{S}^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \cdots & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\bar{S}^T_0 & \bar{S}^T_1 & \cdots & \bar{S}^T_{n-2} & \frac{1}{\sqrt{2}}\bar{S}^T_0 \end{pmatrix}$$

where, as in the $(n-1)$-cyclic case, the column vector $\bar{S}^T_j = \frac{1}{\sqrt{n-1}} \begin{pmatrix} 1 \\ \zeta^j_{_{n-1}} \\ \vdots \\ \zeta^{(n-2)j}_{_{n-1}} \end{pmatrix}$. By definition of $B^1_{_{n-1}}$,

$$B^1_{_{n-1}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \cdots & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\bar{S}^T_0 & \bar{S}^T_1 & \cdots & \bar{S}^T_{n-2} & \frac{1}{\sqrt{2}}\bar{S}^T_0 \end{pmatrix} \begin{pmatrix} \tilde{v}^{\,1}_z & & & & \\ & d^1_z & & & \\ & & \ddots & & \\ & & & d^{n-2}_z & \\ & & & & -\tilde{v}^{\,1}_z \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}S_0 \\ 0 & S_1 \\ \vdots & \vdots \\ 0 & S_{n-2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}S_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}}\tilde{v}^{\,1}_z & 0 & \cdots & 0 & \frac{1}{\sqrt{2}}\tilde{v}^{\,1}_z \\ \frac{1}{\sqrt{2}}\bar{S}^T_0\tilde{v}^{\,1}_z & \bar{S}^T_1 d^1_z & \cdots & \bar{S}^T_{n-2}d^{n-2}_z & -\frac{1}{\sqrt{2}}\bar{S}^T_0\tilde{v}^{\,1}_z \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}S_0 \\ 0 & S_1 \\ \vdots & \vdots \\ 0 & S_{n-2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}S_0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \tilde{v}^{\,1}_z S_0 \\ \tilde{v}^{\,1}_z\bar{S}^T_0 & \bar{S}^T_1 S_1 d^1_z + \cdots + \bar{S}^T_{n-2}S_{n-2}d^{n-2}_z \end{pmatrix} = \begin{pmatrix} 0 & \tilde{v}^{\,1}_z S_0 \\ \tilde{v}^{\,1}_z\bar{S}^T_0 & \sum\limits_{i\in\mathbb{Z}_{n-1}} \bar{S}^T_i S_i d^i_z \end{pmatrix}$$

where, in the matrix above, the $(1,1)$ entry is a 1×1 matrix, the $(1,2)$ entry is a row vector of length $(n-1)$, the $(2,1)$-entry is a column vector of length $(n-1)$ and the

$(2,2)$-entry is a $(n-1)\times(n-1)$-matrix. Thus

$$(B^1_{n-1})_{kl} = \begin{cases} C(w_{k-l})_z & k \geq 2 \text{ and } l \geq 2 \\ 0 & k = l = 1 \\ c\tilde{v}^1_z & \textit{otherwise} \end{cases}$$

Now for B^2_{n-1}, as above, we have

$$B^2_{n-1} = \bar{S}^T \begin{pmatrix} & \frac{1}{\sqrt{2}}(e^{\tilde{v}^1-d^1}-1) & & & \\ & & e^{d^1-d^2}-1 & & \\ & & & \ddots & \\ \frac{1}{\sqrt{2}}(e^{-\tilde{v}^1-d^1}-1) & & & & \frac{1}{\sqrt{2}}(e^{\tilde{v}^1-d^1}-1) \\ & \frac{1}{\sqrt{2}}(e^{-\tilde{v}^1-d^1}-1) & & & \end{pmatrix} S$$

$$= \begin{pmatrix} 0 & \frac{1}{2}(e^{\tilde{v}^1-d^1}-e^{-\tilde{v}^1-d^1}) & 0 & \cdots & 0 \\ \frac{1}{\sqrt{2}}\bar{S}^T_{n-2}(e^{-\tilde{v}^1-d^1}-1) & \frac{1}{2}\bar{S}^T_0(e^{\tilde{v}^1-d^1}+e^{-\tilde{v}^1-d^1}-2) & \bar{S}_1(e^{d^1-d^2}-1) & \cdots & \frac{1}{\sqrt{2}}\bar{S}^T_{n-2}(e^{\tilde{v}^1-d^1}-1) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}S_0 \\ 0 & S_1 \\ \vdots & \vdots \\ 0 & S_{n-2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}S_0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{2}S_1(e^{\tilde{v}^1-d^1}-e^{-\tilde{v}^1-d^1}) \\ \frac{1}{2}\bar{S}^T_{n-2}(-e^{\tilde{v}^1-d^1}+e^{-\tilde{v}^1-d^1}) & * \end{pmatrix}$$

where $*$ is the $(n-1)\times(n-1)$-matrix given by

$$\frac{1}{2}\bar{S}^T_{n-2}S_0(e^{-\tilde{v}^1-d^1}-1) + \frac{1}{2}\bar{S}^T_0 S_1(e^{\tilde{v}^1-d^1}+e^{\tilde{v}^1-d^1}-2) + \frac{1}{2}\bar{S}^T_{n-2}S_0(e^{\tilde{v}^1-d^1}-1)$$
$$+\sum_{i=1}^{n-3}\bar{S}^T_i S_{i+1}(e^{d^i-d^{i+1}}-1)$$

Rewrite $*$ as $I-II$ where I contains all exponential terms and II contains all constant terms. Then

$$II = 2\times\frac{1}{2}\bar{S}^T_{n-2}S_0 + \frac{1}{2}\bar{S}^T_0 S_1\cdot 2 + \bar{S}^T_1 S_2 + \cdots + \bar{S}^T_{n-3}S_{n-2}$$

$$= \sum_{i \in \mathbb{Z}_{n-1}} \bar{S}_i^T S_{i+1} = 0.$$

Writing $\frac{1}{2}\bar{S}_{n-2}^T S_0(e^{-\tilde{v}^1 - d^1} + e^{\tilde{v}^1 - d^1}) = \frac{1}{2}\bar{S}_{n-2}^T S_0(2f + 2e^{-d^1})$ as above, we have

$$\begin{aligned} * &= \bar{S}_{n-2}^T S_0(f + e^{-d^1}) + \bar{S}_0^T S_1(f + e^{-d^1}) + \sum_{i=1}^{n-3} \bar{S}_i^T S_{i+1} e^{d^i - d^{i+1}} \\ &= \sum \bar{S}_i^T S_{i+1} e^{d^i - d^{i+1}} + (\bar{S}_{n-2}^T S_0 + \bar{S}_0^T S_1) f. \end{aligned}$$

Thus similarly as n-cyclic case,

$$(B_{n-1}^2)_{kl} = \begin{cases} ct^{-\frac{2}{n-1}} \Delta w_{k-l} + Cf & k, l \geq 2 \\ 0 & k = l = 1 \\ c(e^{\tilde{v}^1} - e^{-\tilde{v}^1}) e^{-d^1} & otherwise \end{cases}$$

In conclusion, we have the (k, l)-entry of the error term

$$R_{kl} = (B_{n-1}^1)_{kl} + (2t)^{\frac{1}{n-1}} (B_{n-1}^2)_{kl} = \begin{cases} c(w_{k-l})_z + ct^{-\frac{1}{n-1}} \Delta w_{k-l} + cft^{\frac{1}{n-1}} & k, l \geq 2 \\ 0 & k = l = 1 \\ c\tilde{v}_z^1 + ct^{\frac{1}{n-1}} (e^{\tilde{v}^1} - e^{-\tilde{v}^1}) e^{-d^1} & otherwise \end{cases}$$

Similarly as in n-cyclic case, we have

Lemma 3.5.5. *For the $(n-1)$-cyclic case, the eigensolutions w_k's satisfy*

1, $w_{n-1-k} = \zeta_{n-1}^k w_k$;

2, $\frac{w_k}{i(1-\zeta_{n-1}^{-k})}$ is real.

Remark 3.5.6. As in the n-cyclic case, it is important to remember that the system we are interested in comes from the error terms $\tilde{v}^j$. By Theorem 3.3.1, $\tilde{v}^1$ and w_k are also small for large t. This will be important in the proof of Theorem 3.4.4.

3.6 Error Estimate

In this section, let D be the disk of radius R centered at the origin be at least ϵ-bounded away from zero set of q_b.

3.6.1 The n-cyclic case

Proposition 3.6.1. *Let d^i be the error functions $\tilde{u}^i$ in the Hitchin equations for the n-cyclic case, and define $w_k = \frac{1}{\sqrt{n}} \sum_{i\in\mathbb{Z}_n} \zeta_n^{ik}(d^i - d^{i+1})$ on the disk D as in section 3.5, then*

$$w_k(z) = O(t^{-\frac{3}{2n}} e^{-2|1-\zeta_n^k| t^{\frac{1}{n}}(R-|z|)}),$$

where the O is depending on ϵ.

Corollary 3.6.2. *Under the same conditions as above,*

$$\Delta w_k(z) = O(t^{\frac{1}{2n}} e^{-2|1-\zeta_n^k| t^{\frac{1}{n}}(R-|z|)}).$$

and

$$|w_z^k| = O(t^{-\frac{1}{2n}} e^{-2|1-\zeta_n^k| t^{\frac{1}{n}}(R-|z|)}).$$

Proof. (of Corollary 3.6.2) Using part (iii) of Porposition 3.5.2,

$$\Delta w_k = 4|1-\zeta_n^k|^2 t^{\frac{2}{n}} \sum_{\substack{r\equiv k \\ \mathrm{mod}\ n}} \sum_{r_1+\cdots+r_s=r} \frac{1}{s! n^{\frac{s-1}{2}}} \binom{r}{r_1,\cdots,r_s} w_{r_1} w_{r_2} \cdots w_{r_s}$$

Applying Proposition 3.6.1 and the triangle inequality, for $s \geq 2$,

$$|1-\zeta_n^{r_1}| + \cdots + |1-\zeta_n^{r_s}| > |1-\zeta_n^{r_1+\cdots+r_s}|,$$

we obtain

$$\Delta w_k(z) = O(t^{\frac{1}{2n}} e^{-2|1-\zeta_n^k| t^{\frac{1}{n}}(R-|z|)}).$$

Consider the functions α^k defined by

$$\alpha^k(z) = w^k(t^{-\frac{1}{n}} z) \; for \; 1 \leq k \leq n,$$

where $w_k(t^{-\frac{1}{n}} z)$ is just a rescaling of w_k. The α^k's then satisfy the following two properties

$$|\alpha^k| = |w_k| \leq C t^{-\frac{3}{2n}} e^{-2|1-\zeta_n^k| t^{\frac{1}{n}}(R-|z|)}$$

$$|\alpha^k_{z\bar{z}}| = |t^{-\frac{2}{n}} w^k_{z\bar{z}}| \le C t^{-\frac{3}{2n}} e^{-2|1-\zeta_n^k| t^{\frac{1}{n}}(R-|z|)}.$$

Applying Schauder's estimate gives

$$|\alpha^k|_{C^1} \le C t^{-\frac{3}{2n}} e^{-2|1-\zeta_n^k| t^{\frac{1}{n}}(R-|z|)}.$$

Thus $|\alpha^k_z|, |\alpha^k_{\bar{z}}| \le C t^{-\frac{3}{2n}} e^{-2|1-\zeta_n^k| t^{\frac{1}{n}}(R-|z|)}$, proving

$$|w^k_z|, |w^k_{\bar{z}}| \le C t^{-\frac{1}{2n}} e^{-2|1-\zeta_n^k| t^{\frac{1}{n}}(R-|z|)}.$$

□

Now we are ready to show Theorem 3.4.3.

Proof. (of Theorem 3.4.3) For any $d < min\{p, \text{zeros of } q_n\}$, choose a disk D of radius d centered at p. Denote the (k,l)-entry of R as R_{kl}. From section 3.5, we have $R_{kl} = C w_{k-l}(z)_z + C t^{-\frac{1}{n}} \Delta w_{k-l}(z)$. Apply Corollary 3.6.2, as $t \to +\infty$, the (k,l)-entry R_{kl} of R satisfies

$$R_{kl}(p) = O(t^{-\frac{1}{2n}} e^{-2|1-\zeta_n^{k-l}| t^{\frac{1}{n}} d}).$$

□

Denote the function y_k is the unique solution to the system

$$\Delta \eta = k\eta, \quad \text{in } D \tag{3.6.1}$$

$$\eta = 1, \quad \text{on } \partial D. \tag{3.6.2}$$

For $k > 0$, the function y_k satisfies the following important property.

Lemma 3.6.3. *There exists a constant $C > 0$ not depending on k and z such that*

$$e^{-\sqrt{k}(R-|z|)} \le y_k(z) \le C k^{\frac{1}{4}} e^{-\sqrt{k}(R-|z|)}.$$

Proof. Left Inequality: By simple calculation,

$$\Delta e^{-\sqrt{k}(R-|z|)} \geq k e^{-\sqrt{k}(R-|z|)}$$

and $e^{-\sqrt{k}(R-|z|)}|_{\partial D} = 1$. By the maximum principle, we have

$$e^{-\sqrt{k}(R-|z|)} \leq y_k(z).$$

Right Inequality: We first consider radial functions η satisfying

$$\Delta\eta = k\eta, \quad \text{in } D \tag{3.6.3}$$

$$\eta = 1, \quad \text{on } \partial D. \tag{3.6.4}$$

Then $\eta(r)$ satisfies

$$\eta'' + \frac{1}{r}\eta' - k\eta = 0, \quad \text{in } D$$
$$\text{and} \quad \eta(R) = 1.$$

Hence $\eta(r) = \frac{I_0(\sqrt{k}r)}{I_0(\sqrt{k}R)}$, where I_0 is the modified Bessel function of second kind.

Lemma 3.6.4. *There exists $C > 1, M > 0$ such that $\forall x > M$,*

$$\frac{1}{C}\frac{e^x}{\sqrt{x}} \leq I_0(x) \leq C\frac{e^x}{\sqrt{x}}.$$

Lemma 3.6.4 follows from that the asymptotic expansion of $I_0(r)$ [AS92] is $I_0(r) \sim \frac{e^x}{\sqrt{2\pi x}}$ as $x \to \infty$.

Lemma 3.6.5. *[AS92] I_0 is an increasing function.*

So by Lemma 3.6.5 $y_k = \frac{I_0(\sqrt{k}|z|)}{I_0(\sqrt{k}R)} \leq \frac{max\{I_0(M), I_0(\sqrt{k}|z|)\}}{I_0(\sqrt{k}R)}$.

For $|z| \leq \frac{M}{\sqrt{k}}$, as $k \to +\infty$,

$$y_k \leq C\frac{I_0(M)}{I_0(\sqrt{k}R)} \quad \text{by Lemma 3.6.4} \tag{3.6.5}$$

$$\leq C\frac{e^M}{\sqrt{M}}\frac{k^{\frac{1}{4}}R^{\frac{1}{2}}}{e^{\sqrt{k}R}} \tag{3.6.6}$$

$$\leq Ck^{\frac{1}{4}}e^{\sqrt{k}(|z|-R)}. \tag{3.6.7}$$

For $|z| > \frac{M}{\sqrt{k}}$,

$$y_k \leq C\frac{I_0(\sqrt{k}|z|)}{I_0(\sqrt{k}R)} \quad \text{by Lemma 3.6.4} \tag{3.6.8}$$

$$\leq Ck^{\frac{1}{4}}e^{\sqrt{k}(|z|-R)}. \tag{3.6.9}$$

□

Our goal is to use such functions y_k's to bound the eigensolutions w_j's by choosing the right k. We start by proving a lemma about a general system.

Lemma 3.6.6. *Suppose $\eta_1, \cdots, \eta_n$ are functions on D satisfying*

$$\Delta\eta_1 = \lambda_1 t^{\frac{2}{n}}(\eta_1 + f_1) \tag{3.6.10}$$

$$\cdots \tag{3.6.11}$$

$$\Delta\eta_n = \lambda_n t^{\frac{2}{n}}(\eta_n + f_n) \tag{3.6.12}$$

with $\lambda_j > 0$, $\lambda_1 = min\{\lambda_j\}$, and $f_j = f_j(\eta_1, \cdots, \eta_n)$. Suppose further that there exists $C > 0$ such that

$$\begin{cases} |f_j| \leq C(\eta_1^2 + \cdots + \eta_n^2 + y_{kt^{\frac{2}{n}}}) & \text{for some } k > \lambda_1 \\ |\eta_j| \leq Ct^{-\frac{2}{n}}. \end{cases}$$

Then

$$|\eta_1(z)| \leq Ct^{-\frac{3}{2n}}e^{-\sqrt{\lambda_1}t^{\frac{1}{n}}(R-|z|)},$$

Proof. In order to obtain an upper and lower bound for η_1, we first need an upper bound on $\sum_{j=1}^{n}\eta_j^2$

$$\Delta(\sum_{j=1}^{n}\eta_j^2) = 2\sum_j \eta_j\Delta\eta_j + 2\sum_j |\nabla\eta_j|^2 \geq 2\sum_j \eta_j\Delta\eta_j.$$

Using our assumptions on $\Delta\eta_j$ and λ_1 gives

$$\Delta(\sum_{j=1}^{n}\eta_j^2) \geq 2t^{\frac{2}{n}}\sum_j \eta_j\lambda_j(\eta_j + f_j)$$

$$\geq 2t^{\frac{2}{n}}\lambda_1\sum_j \eta_j^2 - 2t^{\frac{2}{n}}\sum_j |\eta_j||f_j|\lambda_j$$

Now with our assumptions on $|f_j|$ and $|\eta_j|$, we have

$$\Delta(\sum_{j=1}^{n}\eta_j^2) \geq (2\lambda_1 t^{\frac{2}{n}} - C)\sum_j \eta_j^2 - Cy_{kt^{\frac{2}{n}}}$$

Let $\lambda = min\{k, \frac{3}{2}\lambda_1\}$, then

$$\Delta(\sum_{j=1}^{n}\eta_j^2) \geq (2\lambda_1 t^{\frac{2}{n}} - C)\sum_j \eta_j^2 - Cy_{\lambda t^{\frac{2}{n}}}$$

Let $C \geq \sum_{j=1}^{n}\eta_j^2|_{\partial D}$ and consider $C \cdot y_{\lambda t^{\frac{2}{n}}}$, for t large enough,

$$\Delta Cy_{\lambda t^{\frac{2}{n}}} \leq (2\lambda_1 t^{\frac{2}{n}} - C)Cy_{\lambda t^{\frac{2}{n}}} - Cy_{\lambda t^{\frac{2}{n}}}$$

Hence by the maximum principle, in D,

$$\sum_{j=1}^{n}\eta_j^2 \leq Cy_{\lambda t^{\frac{2}{n}}}.$$

To obtain an upper bound for η_1, consider

$$\Delta\eta_1 = \lambda_1 t^{\frac{2}{n}}(\eta_1 + f_1) \geq \lambda_1 t^{\frac{2}{n}}\eta_1 - Ct^{\frac{2}{n}}y_{\lambda t^{\frac{2}{n}}}.$$

Let $b = Ct^{-\frac{2}{n}} \geq \eta_1|_{\partial D}$ and consider $by_{\lambda_1 t^{\frac{2}{n}} - C}$, we obtain

$$\Delta by_{\lambda_1 t^{\frac{2}{n}} - C} \leq \lambda_1 t^{\frac{2}{n}} by_{\lambda_1 t^{\frac{2}{n}} - C} - Cy_{\lambda t^{\frac{2}{n}}}$$

Similarly, we have by the maximum principle, on D,

$$\eta_1 \leq by_{\lambda_1 t^{\frac{2}{n}} - C}.$$

To obtain a lower bound, consider $\Delta(-\eta_1) = \lambda_1(-\eta_1) - f_1$. By the same argument as above, we obtain $-\eta_1 \leq by_{\lambda_1 t^{\frac{2}{n}} - C}$.

Finally, by the estimate on y_k from lemma 3.6.3, we obtain the desired

$$|\eta_1(z)| \leq Ct^{-\frac{3}{2n}} e^{-\sqrt{\lambda_1} t^{\frac{1}{n}}(R-|z|)}.$$

□

Proof. (of Proposition 3.6.1) By Lemma 3.5.3, we have that w_k is real, which we will automatically use in the following proof without mentioning. The idea of proof is by induction on k, we first show the base case, which is essentially the conclusion of Lemma 3.6.6.

By part (iii) of Proposition 3.5.2,

$$\Delta w_k = 4t^{\frac{2}{n}} |1 - \zeta_n^k|^2 \sum_{\substack{r \equiv k \\ \text{mod } n}} \sum_{r_1 + \cdots + r_s = r} \frac{1}{s! n^{\frac{s-1}{2}}} \binom{r}{r_1, \cdots, r_s} w_{r_1} w_{r_2} \cdots w_{r_s}$$

Taking the $r = k$ term out of the sum gives

$$\Delta w_k = 4t^{\frac{2}{n}} |1 - \zeta_n^k|^2 w_k + t^{\frac{2}{n}} f_k$$

where f_k is the remainder of the sum. Since we removed the first term, each term in f_k is a polynomial of the w_j's with no linear term or constant term. By remark 3.5.4, each $w_j \sim O(t^{-\frac{2}{n}})$. To show $|f_k| \leq C(\sum w_j^2)$, note that for the $s = 2$ summand of f_k is

$$4t^{\frac{2}{n}} \sum_{\substack{r \equiv k \\ \text{mod } n}} \sum_{r_1 + r_2 = r} \frac{1}{2! \sqrt{n}} \binom{r}{r_1, r_2} |1 - \zeta_n^k|^2 w_{r_1} w_{r_2} \leq C \sum {w_j}^2.$$

For $s > 2$ we use remark 3.5.4 to conclude

$$|f_k| \leq C \sum_j {w_j}^2 (1 + O(t^{-\frac{2}{n}}) + O(t^{-\frac{4}{n}}) + \dots) \leq C \sum_j {w_j}^2.$$

Furthermore, $|1-\zeta_n|^2 \leq |1-\zeta_n^k|^2$ for all $k < n$, thus the system of w_k's satisfies the hypothesis of Lemma 3.6.6 with $\lambda_k = |1-\zeta_n^k|^2$.

$$w_k(z) = O(t^{-\frac{3}{2n}} e^{-2|1-\zeta_n|t^{\frac{1}{n}}(R-|z|)}).$$

Hence in particular, let $k = 1$, this proves the base case.

For the induction step, suppose $w_l = O(t^{-\frac{3}{2n}} e^{-2|1-\zeta_n^l|t^{\frac{1}{n}}(R-|z|)})$ for all $l \leq k$. By Lemma 3.5.3, we have $w_l = w_{n-l}$, and if $l \leq k$ the induction hypothesis implies $w_{n-l}(z) = O(t^{-\frac{3}{2n}} e^{-2|1-\zeta_n^{n-l}|t^{\frac{1}{n}}(R-|z|)})$.

To obtain the estimate for w_{k+1}, the assumptions are not enough to directly estimate the terms containing w_j's for $k+1 \leq j \leq n-k-1$. Therefore we need use the rest system to do the estimate. Expanding the expression for Δw_{k+i}, we have

$$\frac{\Delta w_{k+i}}{4t^{\frac{2}{n}}} = |1-\zeta_n^{k+i}|^2 w_{k+i} + \sum_{r_1+\cdots+r_s=k+i} \frac{1}{s!n^{\frac{s-1}{2}}} \binom{k+i}{r_1,\ldots,r_s} w_{r_1}\cdots w_{r_s}$$

$$+ \sum_{\substack{r_1+\cdots+r_s \\ =k+i+n}} \frac{1}{s!n^{\frac{s-1}{2}}} \binom{k+i+n}{r_1,\ldots,r_s} w_{r_1}\ldots w_{r_s} + \ldots$$

Let $A \subset \{1,\ldots,n-1\}$ be the set $A = \{k+1,\ldots,n-k-1\}$ and $B = \{1,\ldots,n-1\} \setminus A$. Rewrite the above equation as $\dfrac{\Delta w_{k+i}}{4t^{\frac{2}{n}}} = |1-\zeta_n^{k+i}|^2 w_{k+i} + E + F + G$, where

$$\begin{cases} E = \displaystyle\sum_{\substack{r\equiv k+i \\ \bmod n}} \sum_{\substack{r_1+\cdots+r_s=r \\ r_j\in B}} \frac{1}{s!n^{\frac{n-1}{2}}} \binom{r}{r_1,\cdots,r_s} |1-\zeta_n^{k+i}|^2 w_{r_1}w_{r_2}\cdots w_{r_s} \\ F = \displaystyle\sum_{\substack{r\equiv k+i \\ \bmod n}} \sum_{\substack{r_1+\cdots+r_s=r \\ \text{at most one of } r_j\text{'s in } A}} \frac{1}{s!n^{\frac{n-1}{2}}} \binom{r}{r_1,\cdots,r_s} |1-\zeta_n^{k+i}|^2 w_{r_1}w_{r_2}\cdots w_{r_s} \\ G = \displaystyle\sum_{\substack{r\equiv k+i \\ \bmod n}} \sum_{\substack{r_1+\cdots+r_s=r \\ \text{at least two of } r_j\text{'s in } A}} \frac{1}{s!n^{\frac{n-1}{2}}} \binom{r}{r_1,\cdots,r_s} |1-\zeta_n^{k+i}|^2 w_{r_1}w_{r_2}\cdots w_{r_s} \end{cases} \tag{3.6.13}$$

For E, by the induction hypothesis, for $l \in B$,

$$w_l = O(t^{-\frac{3}{2n}} e^{-2|1-\zeta_n^l|t^{\frac{1}{n}}(R-|z|)}).$$

Also, by the triangle inequality, for $s \geq 2$,

$$|1-\zeta_n^{r_1}|+\cdots+|1-\zeta_n^{r_s}|>|1-\zeta_n^{r_1+\cdots+r_s}|,$$

thus

$$\sum_{r_1+\cdots+r_s=k+1}\frac{1}{s!n^{\frac{n-1}{2}}}\binom{k+1}{r_1,\ldots,r_s}w_{r_1}\cdots w_{r_s}\sim O(t^{-\frac{3}{n}}e^{-2|1-\zeta_n^{k+1}|t^{\frac{1}{n}}(R-|z|)}).$$

Hence we have

$$E\sim O(t^{-\frac{3}{n}}e^{-2|1-\zeta_n^{k+1}|t^{\frac{1}{n}}(R-|z|)}).$$

For F, similarly, we have

$$F=\sum_{\substack{r\equiv k+i\\ \mathrm{mod}\ n}}\ \sum_{\substack{r_0+r_1=r\\ r_1\in A}}O(t^{-\frac{3}{2n}}e^{-2|1-\zeta_n^{r_0}|t^{\frac{1}{n}}(R-|z|)})w_{r_1}.$$

We will prove a lemma for a coarse estimate of w_j's for $j\in A$.

Lemma 3.6.7.

$$|w_{k+i}|\leq Ct^{-\frac{3}{2n}}e^{-\sqrt{2}|1-\zeta_n^{k+1}|t^{\frac{1}{n}}(R-|z|)}$$

Proof. We are going to estimate the sum $\sum_{k+1\leq i\leq n-k-1}w_{k+i}^2$. We have

$$\begin{aligned}\frac{\Delta w_{k+i}^2}{4t^{\frac{2}{n}}}&\geq 2w_{k+i}\frac{\Delta w_{k+i}}{4t^{\frac{2}{n}}}=2|1-\zeta_n^{k+i}|^2w_{k+i}^2+2w_{k+i}E+2w_{k+i}F+2w_{k+i}G\\&\geq 2|1-\zeta_n^{k+i}|^2w_{k+i}^2-O(t^{-\frac{3}{n}}e^{-2|1-\zeta_n^{k+i}|t^{\frac{1}{n}}(R-|z|)})(\sum_i w_{k+i}^2)^{\frac{1}{2}}\\&\quad-(O(t^{-\frac{3}{2n}}e^{-2|1-\zeta_n^{k+1}|t^{\frac{1}{n}}(R-|z|)})+O(t^{-\frac{2}{n}}))\sum_i w_{k+i}^2\end{aligned}$$

Thus summing over i for $1\leq i\leq n-2k-1$ yields

$$\frac{1}{4}t^{-\frac{2}{n}}\Delta(\sum_i w_{k+i}^2)\geq(2|1-\zeta_n^{k+1}|^2-Ct^{-\frac{2}{n}})\sum_i w_{k+i}^2-Ct^{-\frac{3}{n}}e^{-2|1-\zeta_n^{k+1}|t^{\frac{1}{n}}(R-|z|)}(\sum_i w_{k+i}^2)^{\frac{1}{2}}$$

Let $b = Ct^{-\frac{4}{n}}(n-2k-1)$ such that we have

$$(\sum_i w_{k+i}^2)|_{\partial D} \leq Ct^{-\frac{4}{n}} \leq b_{\partial D},$$

since $w_{k+i} \leq Ct^{-\frac{2}{n}}$.

Consider $b \cdot y_{8|1-\zeta_n^{k+1}|^2 t^{\frac{2}{n}}-8C}$, in D

$$\frac{1}{4}t^{-\frac{2}{n}}\Delta(b \cdot y_{8|1-\zeta_n^{k+1}|^2 t^{\frac{2}{n}}-8C}) = (2|1-\zeta_n^{k+1}|^2 - 2Ct^{-\frac{2}{n}})b \cdot y_{8|1-\zeta_n^{k+1}|^2 t^{\frac{2}{n}}-8C}$$

Using Lemma 3.6.3, for t sufficiently large,

$$\leq (2|1-\zeta_n^{k+1}|^2 - Ct^{-\frac{2}{n}})b \cdot y_{8|1-\zeta_n^{k+1}|^2 t^{\frac{2}{n}}-8C} - Ct^{-\frac{3}{n}}e^{-|1-\zeta_n^{k+1}|t^{\frac{1}{n}}(R-|z|)}\sqrt{b}(y_{8|1-\zeta_n^{k+1}|^2 t^{\frac{2}{n}}-8C})^{\frac{1}{2}}.$$

By the maximum principle,

$$\sum_i w_{k+i}^2 \leq b \cdot y_{8|1-\zeta_n^{k+1}|^2 t^{\frac{2}{n}}-8C}$$

thus

$$|w_{k+i}(z)| \leq C\sqrt{b}t^{\frac{1}{2n}}e^{-\sqrt{2}|1-\zeta_n^{k+1}|t^{\frac{1}{n}}(R-|z|)}.$$

Therefore

$$|w_{k+i}(z)| \leq Ct^{-\frac{3}{2n}}e^{-\sqrt{2}|1-\zeta_n^{k+1}|t^{\frac{1}{n}}(R-|z|)}.$$

□

We will now use the results from the above lemma to prove the desired estimate for w_{k+1}. Recall that

$$\frac{1}{4}t^{-\frac{2}{n}}\Delta w_{k+1} = |1-\zeta_n^{k+1}|^2 w_{k+1} + E + F + G.$$

and $E = O(t^{-\frac{3}{n}}e^{-2|1-\zeta_n^{k+1}|t^{\frac{1}{n}}(R-|z|)})$, we have

$$\frac{1}{4}t^{-\frac{2}{n}}\Delta w_{k+1} \geq |1-\zeta_n^{k+1}|^2 w_{k+1} - O(t^{-\frac{3}{n}}e^{-2|1-\zeta_n^{k+1}|)^{\frac{1}{n}}(R-|z|)})$$

$$-O(t^{-\frac{3}{2n}}e^{-2|1-\zeta_n|t^{\frac{1}{n}}(R-|z|)})(\sum_i w_{k+i}^2)^{\frac{1}{2}} - C\sum_i w_{k+i}^2$$

Applying Lemma 3.6.7 yields

$$\frac{1}{4}t^{-\frac{2}{n}}\Delta w_{k+1} \geq |1-\zeta_n^{k+1}|^2 w_{k+1} - O(t^{-\frac{3}{n}}e^{-2|1-\zeta_n^{k+1}|t^{\frac{1}{n}}(R-|z|)})$$

$$-O(t^{-\frac{2}{n}}e^{-(2|1-\zeta_n|+\sqrt{2}|1-\zeta_n^{k+1}|)t^{\frac{1}{n}}(R-|z|)}) - O(t^{-\frac{1}{n}}e^{-2\sqrt{2}|1-\zeta_n^{k+1}|t^{\frac{1}{n}}(R-|z|)})$$

On the right hand side of the inequality, the second term dominates the last term,

$$\frac{1}{4t^{\frac{2}{n}}}\Delta w_{k+1} \geq |1-\zeta_n^{k+1}|^2 w_{k+1} - O(t^{-\frac{3}{n}}e^{-2|1-\zeta_n^{k+1}|t^{\frac{1}{n}}(R-|z|)})$$

$$-O(t^{-\frac{2}{n}}e^{-(2|1-\zeta_n|+\sqrt{2}|1-\zeta_n^{k+1}|)t^{\frac{1}{n}}(R-|z|)})$$

For the competition of the second and third term on the right hand side of the inequality, we must consider in two cases:

Case I: $|1-\zeta_n^{k+1}| \leq |1-\zeta_n| + \frac{1}{\sqrt{2}}|1-\zeta_n^{k+1}|$

In this case we have

$$\frac{1}{4}t^{-\frac{2}{n}}\Delta w_{k+1} \geq |1-\zeta_n^{k+1}|^2 w_{k+1} - O(t^{-\frac{3}{n}}e^{-2|1-\zeta_n^{k+1}|t^{\frac{1}{n}}(R-|z|)}).$$

Let $Ct^{-\frac{2}{n}} \geq w_{k+1}|_{\partial D}$ and consider $C \cdot y_\lambda$, where $\lambda = 4t^{\frac{2}{n}}|1-\zeta_n^{k+1}|^2 - 4C$.

$$\frac{1}{4}t^{-\frac{2}{n}}\Delta(Ct^{-\frac{2}{n}} \cdot y_\lambda) = (|1-\zeta_n^{k+1}|^2 - Ct^{-\frac{2}{n}})(Ct^{-\frac{2}{n}}y_\lambda)$$

$$\leq |1-\zeta_n^{k+1}|^2(Ct^{-\frac{2}{n}}y_\lambda) - O(t^{-\frac{3}{n}}e^{-(2|1-\zeta_n|+\sqrt{2}|1-\zeta_n^{k+1}|)t^{\frac{1}{n}}(R-|z|)})$$

By the maximum principle, in D

$$w_{k+1} \leq Ct^{-\frac{2}{n}}y_\lambda.$$

then we have

$$w_{k+1} = O(t^{-\frac{3}{2n}}e^{-2|1-\zeta_n^{k+1}|t^{\frac{1}{n}}(R-|z|)})$$

as desired.

Case II: $|1-\zeta_n^{k+1}| > |1-\zeta_n| + \frac{1}{\sqrt{2}}|1-\zeta_n^{k+1}|$

Then

$$\frac{1}{4}t^{-\frac{2}{n}}\Delta w_{k+1} \geq |1-\zeta_n^{k+1}|^2 w_{k+1} - O(t^{-\frac{2}{n}}e^{-(2|1-\zeta_n|+\sqrt{2}|1-\zeta_n^{k+1}|)t^{\frac{1}{n}}(R-|z|)}).$$

Let $Ct^{-\frac{2}{n}} \geq w_{k+1}|_{\partial D}$ and consider $C\cdot y_k$, where $k = 4t^{\frac{2}{n}}(|1-\zeta_n| + \frac{1}{\sqrt{2}}|1-\zeta_n^{k+1}|)^2$.

$$\begin{aligned}\frac{1}{4}t^{-\frac{2}{n}}\Delta(Ct^{-\frac{2}{n}}\cdot y_k) &= (|1-\zeta_n| + \frac{1}{\sqrt{2}}|1-\zeta_n^{k+1}|)^2(Ct^{-\frac{2}{n}}y_k)\\ &\leq |1-\zeta_n^{k+1}|^2(Ct^{-\frac{2}{n}}y_k) - O(t^{-\frac{2}{n}}e^{-(2|1-\zeta_n|+\sqrt{2}|1-\zeta_n^{k+1}|)t^{\frac{1}{n}}(R-|z|)})\end{aligned}$$

By the maximum principle, on D

$$w_{k+1} \leq Ct^{-\frac{2}{n}}y_k.$$

Similarly applying the same process as above, the same bound holds for $-w_{k+i}$, hence

$$|w_{k+i}| \leq Ct^{-\frac{2}{n}}y_k.$$

Since $|1-\zeta_n^{k+i}| \geq |1-\zeta_n^{k+1}|$, we have the same bound for $|w_{k+i}|$,

$$|w_{k+i}| \leq Ct^{-\frac{2}{n}}y_k. \tag{3.6.14}$$

Therefore by Lemma 3.6.3 and $k = 4t^{\frac{2}{n}}(|1-\zeta_n| + \frac{1}{\sqrt{2}}|1-\zeta_n^{k+1}|)^2$,

$$|w_{k+i}(z)| \leq O(t^{-\frac{3}{2n}}e^{-(2|1-\zeta_n|+\sqrt{2}|1-\zeta_n^{k+1}|)t^{\frac{1}{n}}(R-|z|)})$$

Recall that,

$$\frac{1}{4}t^{-\frac{2}{n}}\Delta w_{k+1} = |1-\zeta_n^{k+1}|^2 w_{k+1} + E + F + G.$$

Using our sharper bounds of w_{k+i} in (3.6.1) yields

$$\geq |1-\zeta_n^{k+1}|^2 w_{k+1} - O(t^{-\frac{3}{n}}e^{-2|1-\zeta_n^{k+1}|t^{\frac{1}{n}}(R-|z|)}) - O(t^{-\frac{2}{n}}e^{-(2|1-\zeta_n|+2\sqrt{2}|1-\zeta_n^{k+1}|)t^{\frac{1}{n}}(R-|z|)}).$$

we again need to consider two cases:

Case i. $|1-\zeta_n^{k+1}| \leq 2|1-\zeta_n| + \sqrt{2}|1-\zeta_n^{k+1}|$

Case ii. $|1-\zeta_n^{k+1}| > 2|1-\zeta_n| + \sqrt{2}|1-\zeta_n^{k+1}|$

For case i. then we can apply similar argument and we will be done. For case ii. we will obtain even more precise bounds on the asymptotics of w_{k+i}, in which case we will have two more cases to consider. Fortunately this process will terminate, since we will eventually be in the first case. Thus we obtain

$$w_{k+1}(z) = O(t^{-\frac{3}{2n}} e^{-2|1-\zeta_n^{k+1}|t^{\frac{1}{n}}(R-|z|)})$$

as desired. □

3.6.2 The $(n-1)$-cyclic case

The main difference between n-cyclic case and $(n-1)$-cyclic case is the latter one contains $\tilde{v}^1$ not in the Toda Lattice. We need estimate $\tilde{v}^1$ separately. Recall

$$\Delta\tilde{v}^1 = 4(2t)^{\frac{2}{n-1}}(\frac{1}{2}e^{\tilde{v}^1-\tilde{v}^2} - \frac{1}{2}e^{-\tilde{v}^1-\tilde{v}^2})$$

Lemma 3.6.8. *In a disc D of radius R,*

$$\tilde{v}^1(z) = O(t^{-\frac{3}{2(n-1)}} e^{-2(2t)^{\frac{1}{n-1}}(R-|z|)}).$$

Proof.

$$\Delta\tilde{v}^1 = 4(2t)^{\frac{2}{n-1}}(\frac{1}{2}e^{\tilde{v}^1-\tilde{v}^2} - \frac{1}{2}e^{-\tilde{v}^1-\tilde{v}^2})$$
$$= 2(2t)^{\frac{2}{n-1}}(e^{\tilde{v}^1} - e^{-\tilde{v}^1})e^{-\tilde{v}^2} \geq 2(2t)^{\frac{2}{n-1}}(2v^1)(1-Ct^{-\frac{2}{n-1}})$$

Since $\tilde{v}^1 \leq Ct^{-\frac{2}{n-1}}$, by the maximum principle, in D, we have

$$\tilde{v}^1(z) \leq Ct^{-\frac{2}{n-1}} y_{4(2t)^{\frac{2}{n-1}}(1-Ct^{-\frac{2}{n-1}})} \leq Ct^{-\frac{3}{2(n-1)}} e^{-2(2t)^{\frac{1}{n-1}}(R-|z|)}.$$

We also consider $-\tilde{v}^1$ and similarly obtain that

$$-\tilde{v}^1(z) \leq Ct^{-\frac{2}{n-1}} y_{4(2t)^{\frac{2}{n-1}}(1-Ct^{-\frac{1}{n-1}})} \leq Ct^{-\frac{3}{2(n-1)}} e^{-2(2t)^{\frac{1}{n-1}}(R-|z|)}.$$

Hence we obtain the lemma. □

Now we want to estimate w_k. In fact first estimate the real function $\frac{w_k}{i(1-\zeta_{n-1}^{-k})}$ by Lemma 3.5.5 and then w_k.

Lemma 3.6.9.

$$w_k(z) = O(t^{-\frac{3}{2(n-1)}} e^{-2|1-\zeta_{n-1}^k|(2t)^{\frac{1}{n-1}}(R-|z|)})$$

Proof. We plan to show the above lemma in a similar manner to the n-cyclic case. The difference between the Toda lattices in the n-cyclic case and in the $(n-1)$-cyclic case is the latter one is perturbed by f. We need to show that f has enough decay such that it never affects the estimate of w_k's. From the estimate of $\tilde{v}^1(z)$, then

$$\begin{aligned} f &= \frac{1}{2}(e^{-\tilde{v}^1} + e^{\tilde{v}^1} - 2)e^{-\tilde{v}^2} \\ &\leq \frac{1}{2}\tilde{v}^1(z)^2(1 + Ct^{-\frac{2}{n-1}}) \\ &\leq Ct^{-\frac{3}{n-1}} e^{-4(2t)^{\frac{1}{n-1}}(R-|z|)}. \end{aligned}$$

Also, from the expression of f, it is clear that $f \geq 0$. Hence we have $|f| = O(t^{-\frac{3}{n-1}} e^{-4(2t)^{\frac{1}{n-1}}(R-|z|)})$. Since $4 \geq 2|1-\zeta_{n-1}^k|$ for all k, we can see that f has the desired decay rate which is enough not to affect estimates of w_k's. From now on, we can repeat the process of estimates in n-cyclic case. Since it is very similar, we don't include the proof here. □

Similarly, as Corollary 3.6.2, we have

Corollary 3.6.10. *In the same condition as above,*

$$\Delta\tilde{v}^1(z) = O(t^{\frac{1}{2(n-1)}} e^{-2(2t)^{\frac{1}{n-1}}(R-|z|)}).$$

$$|\tilde{v}^1_z| = O(t^{-\frac{1}{2(n-1)}} e^{-2(2t)^{\frac{1}{n-1}}(R-|z|)}).$$

and

$$\Delta w_k(z) = O(t^{\frac{1}{2(n-1)}} e^{-2|1-\zeta^k_{n-1}|(2t)^{\frac{1}{n-1}}(R-|z|)}).$$

$$|w^k_z| = O(t^{-\frac{1}{2(n-1)}} e^{-2|1-\zeta^k_{n-1}|(2t)^{\frac{1}{n-1}}(R-|z|)}).$$

Now we are ready to show Theorem 3.4.6.

Proof. (of Theorem 3.4.6) For any $d < min\{p, \text{zeros of } q_n\}$, choose a disk D of radius d centered at p. Denote the (k,l)-entry of R as R_{kl}. From section 3.5, we have

$$R_{kl} = (B^1_{n-1})_{kl} + (2t)^{\frac{1}{n-1}}(B^2_{n-1})_{kl} = \begin{cases} c(w_{k-l})_z + ct^{-\frac{1}{n-1}}\Delta w_{k-l} + cft^{\frac{1}{n-1}} & k,l \geq 2 \\ 0 & k=l=1 \\ c\tilde{v}^1_z + ct^{\frac{1}{n-1}}(e^{\tilde{v}^1} - e^{-\tilde{v}^1})e^{-d^1} & otherwise \end{cases}$$

Apply Corollary 3.6.10, as $t \to +\infty$, the (k,l)-entry R_{kl} of R satisfies

$$R_{kl}(p) = \begin{cases} O(t^{-\frac{1}{2(n-1)}} e^{-2|1-\zeta^{k-l}_{n-1}|(2t)^{\frac{1}{n-1}}d}) & k,l \geq 2 \\ 0 & k=l=1 \\ O(t^{-\frac{1}{2(n-1)}} e^{-2(2t)^{\frac{1}{n-1}}d}) & otherwise \end{cases}$$

□

3.7 Elementary Proof of WKB Exponent

In this section we prove a special case of Theorem 3.4.4 which only concerns the largest eigenvalue, or WKB exponent, of the transport operator. By considering the inverse path, we in fact also obtain the smallest eigenvalue of the transport operatot. We include it here because the proof does not require the precise error estimates obtained in Theorem 3.6.1, and hence is more elementary.

Theorem 3.7.1. *Let $P \in \widetilde{\Sigma}$ be disjoint from the zeros of q_b, and choose a neighborhood $\mathcal{U}_p$ centered at P, with coordinate z, so that $q_b = dz^b$. Any $P' \in \mathcal{U}_P$ can be written in polar coordinate $P' = Le^{i\theta}$. Then as $t \to \infty$ there exists a constant $K > 1$ such that*

$$\frac{1}{K} e^{Lt^{\frac{1}{b}}\mu} \leq ||T^{-1}_{P,P'}(t)|| \leq K e^{Lt^{\frac{1}{b}}\mu}$$

with the following notation:

1. *For the n-cyclic case $b = n$, and $\mu = max\{2cos(\theta + \frac{2\pi j}{n})\}$.*

2. *For the $(n-1)$-cyclic case $b = n-1$, and $\mu = max\{2cos(\theta + \frac{2\pi j}{n-1})\}$.*

Remark 3.7.2. When P and P' both project to the same point in Σ, the projected path is a loop. In this case, the above asymptotics correspond to the smallest eigenvalue of the associated family of representations on the homotopy class of the loop. Note, considering the inverse path from P' to P, we obtain asymptotics of the largest eigenvalue of $T_{P,P'}$.

The proof is a generalization of arguments of Loftin [Lof07, Lof] in which he deals with $n = 3$ situation. To remain self-contained, we include the proof for the case $\phi = \tilde{e}_1 + q_n e_{n-1}$ here.

Proof. Since we are only proving the n-cyclic case, we drop all b subscipts. Using the notation of section 3.4, we need to estimate Φ solving the initial value problem

$$\Phi(0) = I \qquad \frac{d\Phi}{ds} = \left[t^{\frac{1}{n}} \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} + (b_{ij}) \right] \Phi$$

where $|b_{ij}| = O(t^{-\frac{1}{n}})$. We will assume $\mu_1 = \mu$.

Write $\Phi = (\phi_{ij})$, and consider the first column $(\phi_{11}, \phi_{21}, \ldots, \phi_{n1})$, which satisfies the linear system

$$\begin{aligned}
\phi_{11}(0) = 1, &\quad \tfrac{d}{ds}\phi_{11} = (t^{\frac{1}{n}}\mu_1 + b_{11})\phi_{11} + b_{12}\phi_{21} + \cdots + b_{1n}\phi_{n1},\\
\phi_{21}(0) = 0, &\quad \tfrac{d}{ds}\phi_{21} = b_{21}\phi_{11} + (t^{\frac{1}{n}}\mu_2 + b_{22})\phi_{21} + \cdots + b_{2n}\phi_{n1},\\
&\vdots\\
\phi_{n1}(0) = 0, &\quad \tfrac{d}{ds}\phi_{n1} = b_{n1}\phi_{11} + b_{n2}\phi_{21} + \cdots + (t^{\frac{1}{n}}\mu_n + b_{nn})\phi_{n1}
\end{aligned}$$

Each of the above differential equations is first-order linear, and so we must have

$$\begin{aligned}
\phi_{11} &= e^{t^{\frac{1}{n}}\mu_1 s} e^{\int_0^s b_{11}} \left[1 + \int_0^s e^{-t^{\frac{1}{n}}\mu_1\tau - \int_0^\tau b_{11}} (b_{12}\phi_{21} + \cdots + b_{1n}\phi_{n1})d\tau\right]\\
\phi_{21} &= e^{t^{\frac{1}{n}}\mu_2 s} e^{\int_0^s b_{22}} \int_0^s e^{-t^{\frac{1}{n}}\mu_2\tau - \int_0^\tau b_{22}} (b_{21}\phi_{11} + b_{23}\phi_{31} + \cdots + b_{2n}\phi_{n1})d\tau\\
&\vdots\\
\phi_{n1} &= e^{t^{\frac{1}{n}}\mu_n s} e^{\int_0^s b_{nn}} \int_0^s e^{-t^{\frac{1}{n}}\mu_n\tau - \int_0^\tau b_{nn}} (b_{n1}\phi_{11} + \cdots + b_{n,n-1}\phi_{n-1,1})d\tau
\end{aligned}$$

The above n equations can be see as a map $\mathcal{F}$ from the R^n-valued function $(\phi_{11}, \phi_{21}, \ldots, \phi_{n1})$ to the right-hand side.

Now let $N \gg 1$ be a constant independent of t, and consider the Banach space $\mathcal{B}_t$ of continuous R^n-valued functions with norm

$$||(f_1, f_2, \ldots, f_n)||_{\mathcal{B}_t} = \sup_i \sup_{s\in[0,L]} |f_i(s)| e^{-t^{\frac{1}{n}}\mu_1 s}.$$

Let $\mathcal{B}_t(N)$ be the closed ball of radius N centered at the origin in $\mathcal{B}_t$.

Claim 3.7.3. *For t large enough, the solution $(\phi_{1j}, \phi_{2j}, \ldots, \phi_{nj})$ to the ODE system, must lie in $\mathcal{B}_t(N)$ for all $1 \leq j \leq n$.*

Thus, for sufficiently large t, $|\phi_{ij}| \leq N e^{t^{\frac{1}{n}}\mu_1 L}$. Applying this claim to the above equation system yields

$$\phi_{11} = e^{t^{\frac{1}{n}}\mu_1 L}(1 + O(t^{-\frac{1}{n}})), \quad \phi_{21} = e^{t^{\frac{1}{n}}\mu_1 L} O(t^{-\frac{1}{n}}), \quad \cdots, \quad \phi_{n1} = e^{t^{\frac{1}{n}}\mu_1 L} O(t^{-\frac{1}{n}}).$$

For the second column of Φ, we have the equations

$$\begin{aligned}
\phi_{12} &= e^{t^{\frac{1}{n}}\mu_1 s} e^{\int_0^s b_{11}} \int_0^s e^{-t^{\frac{1}{n}}\mu_1 \tau - \int_0^\tau b_{11}} (b_{12}\phi_{22} + \cdots + b_{1n}\phi_{n2}) d\tau \\
\phi_{22} &= e^{t^{\frac{1}{n}}\mu_2 s} e^{\int_0^s b_{22}} \left[1 + \int_0^s e^{-t^{\frac{1}{n}}\mu_2 \tau - \int_0^\tau b_{22}} (b_{21}\phi_{12} + b_{23}\phi_{32} + \cdots + b_{2n}\phi_{n2}) d\tau \right] \\
&\vdots \\
\phi_{n2} &= e^{t^{\frac{1}{n}}\mu_n s} e^{\int_0^s b_{nn}} \int_0^s e^{-t^{\frac{1}{n}}\mu_n \tau - \int_0^\tau b_{nn}} (b_{n1}\phi_{12} + \cdots + b_{n,n-1}\phi_{n-1,2}) d\tau
\end{aligned}$$

Thus, applying $|\phi_{ij}| \leq N e^{t^{\frac{1}{n}}\mu_1 s}$ yields

$$\begin{aligned}
|\phi_{12}| &\leq e^{t^{\frac{1}{n}}\mu_1 s} O(t^{-\frac{1}{n}}) \\
|\phi_{22} - e^{t^{\frac{1}{n}}\mu_2 s}| &\leq e^{t^{\frac{1}{n}}\mu_1 s} O(t^{-\frac{1}{n}}) \\
&\vdots \\
|\phi_{n2}| &\leq e^{t^{\frac{1}{n}}\mu_1 s} O(t^{-\frac{1}{n}}),
\end{aligned}$$

Similarly, $|\phi_{ij} - \delta_{ij} e^{t^{\frac{1}{n}}\mu_i s}| \leq e^{t^{\frac{1}{n}}\mu_1 s}$ for all i, j. To determine the largest eigenvalue of (ϕ_{ij}) asymptotically, we make use of the trace. Suppose $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$ are the eigenvalues, then since $|\phi_{ij}| \leq N e^{t^{\frac{1}{n}}\mu_i s}$, there is a constant C so that

$$max(\xi_j) \leq C e^{t^{\frac{1}{n}}\mu_1 s}.$$

Also,

$$\xi_1 + \cdots + \xi_n = Tr(\Phi) = e^{t^{\frac{1}{n}}\mu_1 s}(1 + O(t^{-\frac{1}{n}})) + e^{t^{\frac{1}{n}}\mu_2 s} + \cdots + e^{t^{\frac{1}{n}}\mu_n s},$$

thus

$$\xi_1 \geq \frac{1}{n}(\xi_1 + \cdots + \xi_n) \geq \frac{1}{n} e^{t^{\frac{1}{n}}\mu_1 s}(1 + O(t^{-\frac{1}{n}})).$$

Hence

$$\frac{1}{n} e^{t^{\frac{1}{n}}\mu_1 s}(1 + O(t^{-\frac{1}{n}})) \leq \xi_1 \leq C e^{t^{\frac{1}{n}}\mu_1 s}(1 + O(t^{-\frac{1}{n}})),$$

and for t sufficiently large, there exist constant $K > 0$ so that

$$\frac{1}{K} e^{t^{\frac{1}{n}}\mu_1 s} \leq \xi_1 \leq K e^{t^{\frac{1}{n}}\mu_1 s}.$$

Since the largest eigenvalue $\xi_1 = ||T_{PP'}^{-1}(t)||$, the result follows. □

Proof. (of Claim 3.7.3) We will show that for t large enough, $\mathcal{F}$ is a contraction map from $\mathcal{B}_t(N)$ to itself, and thus the solution $(\phi_{11}, \phi_{21}, \ldots, \phi_{n1})$ to the ODE system, which is the fixed point of $\mathcal{F}$, must lie in $\mathcal{B}_t(N)$. For the rest of columns, the proof is identical. Consider $F = (f_1, f_2, \ldots, f_n), G = (g_1, g_2, \ldots, g_n) \in \mathcal{B}_t(N)$. Then the first component of $\mathcal{F}(F) - \mathcal{F}(G)$ is given by

$$e^{t^{\frac{1}{n}}\mu_1 s} e^{\int_0^s b_{11}} \int_0^s e^{-t^{\frac{1}{n}}\mu_1 \tau - \int_0^\tau b_{11}} [b_{12}(f_2 - g_2) + \cdots + b_{1n}(f_n - g_n) d\tau]$$

Now assume $|b_{ij}| \leq R$ and recall $s \leq L$. A straightforward calculation shows that the first component of $\mathcal{F}(F) - \mathcal{F}(G)$ is pointwise bounded by

$$e^{t^{\frac{1}{n}}\mu_1 s} e^{2RL} 2RL ||F - G||_{\mathcal{B}_t}.$$

For t sufficiently large, since $R \sim t^{-\frac{1}{n}}$, we may assume $e^{2RL}2RL < 1$. Essentially the same calculation shows that $\mathcal{F} : \mathcal{B}_t(N) \to \mathcal{B}_t(N)$ for large t, since $N \gg 1$. The other $n - 1$ components of $\mathcal{F}$ behave the same way, thus $\mathcal{F}$ is a contraction map.

Since $\mathcal{F}$ is a contraction map on the complete metric space $\mathcal{B}_t(N)$, the unique solution $(\phi_{11}, \phi_{21}, \ldots, \phi_{n1})$ to the ODE system is the fixed point, and so must be in $\mathcal{B}_t(N)$ for all t sufficiently large. □

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