RICE UNIVERSITY

Essays on Fair Division and Monopoly Pricing

by

Jin Li

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

Doctor of Philosophy

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ABSTRACT

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The first chapter is based on a paper with Jingyi Xue in fair division problems. In this chapter, we consider the problem of fairly dividing a finite number of divisible goods among agents with the generalized Leontief preferences. We propose and characterize the class of generalized egalitarian rules which satisfy efficiency, group strategy-proofness, anonymity, resource monotonicity, population monotonicity, envy-freeness and consistency. On the Leontief domain, our rules generalize the egalitarian-equivalent rules with reference bundles. We also extend our rules to agent-specific and endowment-specific egalitarian rules. The former is a larger class of rules satisfying all the previous properties except anonymity and envy-freeness. The latter is a class of efficient, group strategy-proof, anonymous and individually rational rules when the resources are assumed to be privately owned.

The second chapter is about monopoly pricing with social learning. In this chapter, we consider a two-period monopolistic model in which the consumers who purchase in the first period would reveal the unknown quality of the product through their experiences to the consumers in the second period. Due to this effect, some consumers would strategically choose to delay to the second period in order to take this informational free-ride. We show that there always exists a unique symmetric equilibrium of consumers for each price set by the monopolist. Then we further investigate the seller's optimization pricing problem. In a range of moderate patience, the seller would be likely to induce the consumers to effectively transmit information. We also discuss the impact of information disclosure on the monopolistic profit.

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Chapter 1

Egalitarian Division under Leontief Preferences

1.1 Introduction

In the fair division literature, efficiency and strategy-proofness typically imply totally unfair outcomes. For example, Zhou (1991) shows that an efficient and strategy-proof allocation rule must be dictatorial already in a two-agent economy with continuous, strictly monotonic and strictly convex preferences. Such negative result has been extended to several more restricted domains by many researchers. Serizawa and Weymark (2003) further shows that in a many-agent many-good economy no efficient and strategy-proof rule can guarantee every agent a consumption bundle bounded away from the origin. (Additional discussion of related literature is given at the end of this section.)

However, the picture changes a lot if we assume full complementarity among the goods and consider the domain of Leontief preferences. On the Leontief domain, for most efficient divisions of a given set of resources, some of the resources are redundant*. Thus, it makes sense to give the agents only the least amount of goods to achieve given welfare levels, while transferring the redundant resources to other potential users outside the rule†. We speak in this case of a *non-wasteful rule*. In addition to the normative concern of parsimony, the

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*For example, in a two-agent two-good economy, both agents have the same preference represented by the utility function $u(x) = \min\{\frac{x_1}{2}, x_2\}$, and the endowment vector is (2,2). Then 1 unit of good 2 is redundant in any efficient allocation which divides up all the resources.

†Notice that here withholding the redundant resources does not affect efficiency since they are useless to the agents. It is different from the budget loss in VCG mechanisms which directly reduce the welfare of the agents.

restriction to non-wasteful rules reduces the possibility of strategic manipulation. It turns out that then there exist rules satisfying efficiency, strategy-proofness and many fairness axioms.

The Leontief preferences and the corresponding non-wasteful rules are of natural practical interests, as shown in the computer science literature like Ghodsi et al. (2010), Hindman et al. (2011), Bodwin et al. (2011), Joe-Wong et al. (2011), Dolev et al. (2012), etc. For example, they consider multiple resource sharing problems in cloud computing systems. The users are allocated with computing resources like CPU, memory and I/O resources to do their different jobs with heterogeneous demands. In such circumstance, each user needs the resources in a customized proportion while redundant resources should not be allocated in order to avoid waste.

Two earlier papers inspire our work. Ghodsi et al. (2010) are the first to propose non-wasteful rules for the Leontief domain. They prove that in a many-agent many-good economy the egalitarian-equivalent (EE) rule proposed by Pazner and Schmeidler (1978) (they call it the Dominant Resource Fairness mechanism) is efficient, strategyproof, envy-free and satisfies several other fairness axioms. Prior to them, Nicolò (2004) characterizes in a two-agent two-good economy with generalized Leontief preferences, a class of rules which are efficient, fully implementable in truthful strategies (a requirement stronger than strategy-proofness) and individually rational. However, Nicolò's rules are wasteful, and he finds it difficult to generalize his result to an economy with more agents and more goods.

Our contribution is to bring the existing results to a much more general level. Under Leontief preferences, we propose a class of non-wasteful rules which generalize the EE rules with reference bundles (see Section 3 for the relation of the EE rule and those with reference bundles). They satisfy efficiency, (group) strategy-proofness and almost all the fairness axioms in the literature (see below for further discussion). We also characterize our rules by these axioms. Moreover, the characterization works as well on a much larger preference domain — the generalized Leontief preference domain, which we shall discuss later. Lastly, we provide two natural extensions of our rules.

The rules we propose are called *generalized egalitarian rules* (defined in Section 3). A generalized egalitarian rule assumes that there is a continuous monotonic "benchmark preference" on the commodity space owned by the society. It looks for the non-wasteful efficient allocation where all the agents get the bundles among which the society is indifferent according to its benchmark preference. In another way, we can visualize that in the commodity space, the agents walk on their own "minimum-demand" paths associated with their Leontief preferences at some given speeds which guarantee that at any time they all simultaneously stand on the same indifference curve of the benchmark preference, and then our rule picks the end points where they reach the endowment feasibility constraints. Essentially, egalitarian rules set a standard for society to measure different ordinal preferences of the agents so that they are treated equally by this standard. While a classical EE rule makes the agents feel indifferent between their allocations and the same fraction of the social endowment, our rule gives the agents "equal" bundles according to a utility function of the society. It turns out that when the social endowment is fixed, a classic EE rule on the Leontief domain is one of our rules with a particular benchmark preference. We discuss about it in detail in Example 2 of Section 3.

There is another interpretation of generalized egalitarian rules. Thomson (1994) proposes a concept of equity to capture the notion of equal opportunities. Given a family $\mathcal{C}$ of choice sets, he defines an equal opportunity allocation relative to $\mathcal{C}$ as one giving every agent his optimal bundle from a common choice set in $\mathcal{C}$. Since such an allocation is obtained by having the agents choose in a common choice set, they can be viewed to get equal opportunities. It turns out that a general egalitarian rule always picks the Pareto optimal equal opportunity allocation relative to a corresponding family of nested choice sets.

Our first main result (Theorem 1) shows that a generalized egalitarian rule satisfies efficiency, group strategy-proofness, anonymity, resource monotonicity, population monotonicity, envy-freeness and consistency; and conversely, given an efficient, resource monotonic and consistent rule, if it is either strategy-proof and anonymous, or envy-free, then it must be a generalized egalitarian rule.

All these axioms are very familiar in the fair division literature. Among the incentive compatibility axioms, group strategy-proofness is a very strong one. It allows no group of agents to misreport their preferences together and achieve Pareto improvement within the group (see Pattanaik (1978), Barberà (1979), Moulin and Shenker (2001), Juarez(2008)). For the fairness axioms, anonymity simply rules out the discrimination of the agents by their names; resource monotonicity guarantees that every agent benefits from the growth of the social endowment (see Roemer (1986a,b), Chun and Thomson (1988)); population monotonicity ensures that no agent will get worse off when less agents join in the division (see Thomson (1983)); and envy-freeness makes every agent weakly prefer his own allocation to anybody else's (see Foley (1967), Varian (1974, 1976)). The consistency axiom has also played an important role in the fair division literature, in particular, the rationing (or bankruptcy) problems (see Aumann and Maschler (1985), Young (1987), Thomson (1988)). It requires that when some agents leave first with their allocated bundles, if we apply the rule again to the reduced economy, and the rest of the agents will still be allocated with the same bundles as in the original economy. For a survey of these and some other axioms in the fair division literature, see Thomson (2010).

Many of the axioms above are known to be very demanding and typically incompatible. For example, Moulin and Thomson (1988) show that any efficient and resource monotonic rule must generate envy in an economy with continuous, monotonic, convex and homothetic preferences. However, generalized egalitarian rules under Leontief preferences surprisingly satisfy them all.

Our rules and characterization apply for a much larger preference domain — the domain of generalized Leontief preferences (see Theorem 2). While for a standard Leontief preference, the set of minimum commodity bundles that achieve given utility levels, which we called the *critical set*, is a ray from the origin in the commodity space, the critical set of a generalized Leontief preference can be an arbitrary strictly increasing curve starting from the origin. In real life, generalized Leontief preferences are relevant when the agents are production units and the goods are inputs. For example, a group of people are dividing some cotton, silk and lace to make clothes. They would like to use these materials in different proportions according to their own tastes. Given the precise combination of the materials to make some pieces of clothes, more material of one kind is useless, which captures the essence of a Leontief preference. Moreover, when the amount of all materials increases, one might be able to make a dress instead of a shirt which requires different proportion of materials. There might also exist different types of returns to scale which alter the input proportion. Hence, one's critical set is an increasing curve, as exhibited in the generalized Leontief preferences.

Our results crucially depend on the restriction to non-wasteful rules. We give an example in Section 3 showing that our results do not hold without this restriction. Our characterization is tight with respect to all the axioms.

Our next two results (Theorem 3 and 4) extend the generalized egalitarian rules in two directions. First, instead of using one single benchmark preference to measure all agents' utilities, a rule may assign to each agent a personal welfare index and equalize their utilities according to these agent-specific welfare indices. This family of rules is a much larger and non-anonymous class. Naturally, we do not expect envy-freeness in this case. However, all the other good properties are preserved.

The second extension is motivated when the resources are assumed to be privately, rather than commonly, owned by the agents. A compelling requirement here is the voluntary participation of the agents in the social reallocation. This is ensured by the individual rationality axiom, which requires the allocation to an agent to be no worse (for this agent) than his initial endowment. In this case, we can set the welfare indices such that it is always an "equal treatment" allocation to give every agent the minimum bundle that provides him the same welfare level as his private endowment. The welfare indices then depend on the endowment profile. By slightly modifying the argument in Moulin and Thomson (1988), one can check that efficiency, resource monotonicity and individual rationality are also incompatible in our context. We show that our endowment-specific egalitarian rules are efficient, group strategy-proof, anonymous, consistent and individually rational.

For both agent-specific and endowment-specific rules, our results are one-sided and we leave the characterizations as open questions.

After the literature review below, the paper is organized as follows. Section 2 presents the basic model and the axioms. Section 3 defines the generalized egalitarian rules under Leontief preferences and gives the characterization result. Section 4 introduces the generalized Leontief preference domain, on which the characterization still holds. Section 5 contains the main proofs. Section 6 checks the tightness of our characterization. Section 7 and 8 provide two extensions of the generalized egalitarian rules: agent-specific and endowment-specific egalitarian rules. Section 9 provides concluding remarks. The appendix contains some supporting proofs.

Related Literature

For the incompatibility of efficiency and strategy-proofness with fairness properties in exchange economies, Hurwicz (1972) first proves that any efficient and individually rational rule is manipulable in two-agent, two-good economies where both agents have continuous, strictly convex, and strictly monotonic preferences. Dasgupta et al. (1979) replace individual rationality with non-dictatorship, while allowing discontinuous preferences. Zhou (1991) shows that in two-agent many-good exchange economies with the same preference domain as in Hurwicz (1972), a strategy-proof and efficient rule has to be inverse-dictatorial[‡], and hence dictatorial. From then on, many authors consider various restricted domains, either obtain similar impossibility results or compromise with weakened axioms, such as Schummer (1997, 2004), Ju (2003), Hashimoto (2008), and Momi (2011a) for two-agent cases, Barberà and Jackson (1995), Kato and Ohseto (2002, 2004), Amorós (2002), Serizawa (2002), Serizawa and Weymark (2003), Ju (2004), Morimoto et al. (2010) and Momi (2011b) for many-agent cases. As we mentioned before, both Nicolò (2004) and Ghodsi et al. (2010) study the Leontief preference domain and achieve positive results. The main difference between their works is that Nicolò (2004) studies a two-agent two-good economy with generalized Leontief preferences and gives a characterization, while Ghodsi et al. (2010) study a many-agent many-good economy with standard Leontief preferences and give several one-sided results. In this paper, we consider generalized Leontief preferences and get very positive characterization results for many-agent many-good economy, without weakening any axioms.

1.2 The Model

Throughout this paper, for all $x, y \in \mathbb{R}^m$ where $m \in \mathbb{N}$, $x \geq y$ means that $x_k \geq y_k$, $\forall k = 1, ..., m$; $x > y$ means that $x_k > y_k$, $\forall k = 1, ..., m$. The latter will be the order that we refer to when we consider totally ordered sets in $\mathbb{R}^m$. Let $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m | x \geq 0\}$, $\mathring{\mathbb{R}}^m_+ = \{x \in \mathbb{R}^m | x > 0\}$, and $\partial\mathbb{R}^m_+ = \mathbb{R}^m_+ \setminus \mathring{\mathbb{R}}^m_+$.

Fix the set of perfectly divisible goods $L = \{1, ..., l\}$, $l \in \mathbb{N}$. Let $\mathbb{R}^l_+$ be the commodity space. Up to Section 3, every agent is assumed to have a standard Leontief preference on $\mathbb{R}^l_+$, which can be represented by a utility function $u(x) = \min\limits_{k \in L}\{\frac{x^k}{\lambda_k}\}$, $\forall x \in \mathbb{R}^l_+$, where x^k denotes the amount of the k-th good, $\lambda_k > 0$, $\forall k \in L$, and $\sum\limits_{k \in L} \lambda_k = 1$ for normalization. Let $\mathcal{U}$ denote

[‡]A rule is inverse-dictatorial if there exists some agent who always gets nothing. In a two-agent economy, it is equivalent to a dicatorial rule.

the set of all such utility functions[§]. We will generalize this preference domain in Section 4.

Definition 1 *Let $u \in \mathcal{U}$ with $u(x) = \min\limits_{k \in L}\{\frac{x^k}{\lambda_k}\}$ be given. We call $\gamma = \{(\lambda_1 t, ..., \lambda_l t) \in \mathbb{R}^l_+ | t \in \mathbb{R}_+\}$ the critical set of the preference u.*

A critical set of a preference $u \in \mathcal{U}$ consists of all the minimum commodity bundles required to achieve given utility levels. It is a ray starting from the origin, and thus a connected, totally ordered and closed subset in $\mathbb{R}^l_+$. It is easy to see that γ is uniquely defined for each $u \in \mathcal{U}$. Hence, in the following, we will interchangeably use u and γ as needed.

An economy E is a triple (N, u_N, ω) where $N \subseteq \mathbb{N}$ is a nonempty finite set of agents, $u_N = (u_i)_{i \in N}$ with $u_i \in \mathcal{U}$, $\forall i \in N$, is a preference profile, and $\omega \in \mathbb{R}^l_+$ is the social endowment of the economy. Up to Section 7, the resources are assumed to be collectively owned. In Section 8, we consider the case where every agent has a private endowment and their endowments are put together to be divided. Let $\mathcal{E}$ denote the set of all economies.

Given (N, ω), the set of all feasible allocations is usually defined as $A(N, \omega) = \{x \in \mathbb{R}^{|N| \times l}_+ | \sum\limits_{i \in N} x_i \leq \omega\}$, where x_i is the l dimensional bundle for agent i. We further require that the bundle of each agent is in his critical set. The reason is that the Leontief preferences are not strictly monotone, so society would like to keep the redundant goods in this economy for alternative use, in the spirit of non-wastefulness. Note that our main result does not hold when the allocations are allowed to be wasteful. A counter-example will be given at the end of Section 3.

Formally, for any economy $E = (N, u_N, \omega)$, we consider the restriction of $A(N, \omega)$ on the critical sets, $A^*(E) = A(N, \omega) \cap \prod\limits_{i \in N} \gamma_i$ where γ_i is the critical set of u_i. Let $\mathcal{A}^* = \{A^*(E) | E \in \mathcal{E}\}$.

[§]We normalize the utility functions so that our rules only care about the ordinal properties. However, it is not necessary for our result. It can be easily shown that any rule satisfying efficiency, strategy-proofness and consistency only takes into account the ordinal properties.

Definition 2 *An allocation rule (or rule for simplicity) is a mapping* $\mu : \mathcal{E} \to \mathcal{A}^*$ *with* $\mu(E) \in A^*(E)$*, assigning to each economy a non-wasteful feasible allocation. For any* $i \in N$*,* $\mu_i(E)$ *denotes the bundle allocated to agent* i*.*

For notational simplicity, we write $\mu(u_N)$ (or $\mu(\omega)$) to denote $\mu(N, u_N, \omega)$, when (N, ω) (or (N, u_N)) is fixed.

Our normative requirements on rules are all very familiar in the literature (see the Introduction).

(I) Efficiency

Efficiency naturally requires that a rule always assigns Pareto optimal allocations.

Given $E = (N, u_N, \omega)$, an allocation $x \in A(N, \omega)$ is *efficient* if there exists no $y \in A(N, \omega)$ such that $u_i(y_i) \geq u_i(x_i)$ for all $i \in N$, and $u_j(y_j) > u_j(x_j)$ for some $j \in N$. A rule μ is *efficient* (EFFN) if $\mu(E)$ is efficient for every $E \in \mathcal{E}$.

Lemma 1 *Given* $E = (N, u_N, \omega)$*, an allocation* $x \in A^*(E)$ *is efficient if and only if* $\sum_{i \in N} x_i^k = \omega^k$ *for some* $k \in L$*, where* x_i^k *denotes the amount of good* k *given to agent* i*.*

Proof 1 *For sufficiency, suppose the contrary that there exists* $y \in A(N, \omega)$ *such that* $u_i(y_i) \geq u_i(x_i)$ *for all* $i \in N$*, and* $u_j(y_j) > u_j(x_j)$ *for some* $j \in N$*. Then* $y_i \geq x_i$ *for all* $i \in N$ *and* $y_j > x_j$ *for some* $j \in N$*, since* $x_i \in \gamma_i$*,* $\forall i \in N$*. Hence,* $\sum_{i \in N} y_i > \sum_{i \in N} x_i$*, and thus* $\sum_{i \in N} y_i^k > \sum_{i \in N} x_i^k = \omega^k$*, which contradicts feasibility. For necessity, suppose the contrary that* $\sum_{i \in N} x_i < \omega$*. Then consider the allocation* $y \in A(N, \omega)$ *such that* $y_i = x_i$*,* $\forall i \in N \setminus \{j\}$*, and* $y_j = x_j + \omega - \sum_{i \in N} x_i > x_j$*. Clearly, it implies that* x *is not efficient, which is a contradiction.*

(II) Incentive compatibility

We require the familiar strategy-proofness and its strengthening as group strategy-proofness.

Let $\mathcal{U}_S = \mathcal{U}^{|S|}$, $\forall S \subseteq N$, and $\mathcal{U}_N$ is the set of all preference profiles. For any $S \subseteq N$, we denote by (u'_S, u_{-S}) the vector $u_N \in U_N$ with u_i replaced by u'_i, $\forall i \in S$. If $S = \{i\}$, we simply write (u'_i, u_{-i}).

A rule μ is *strategy-proof* (SP) if $\forall (N, u_N, \omega)$, $\forall i \in N$, $\forall u'_i \in \mathcal{U}$, $u_i(\mu_i(u_N)) \geq u_i(\mu_i(u'_i, u_{-i}))$.

A rule μ is *group strategy-proof* (GSP) if $\forall (N, u_N, \omega)$, there does not exist $S \subseteq N$ and $u'_S \in \mathcal{U}_S$ such that $u_i(\mu_i(u_N)) \leq u_i(\mu_i(u'_S, u_{-S}))$, $\forall i \in S$, and at least one inequality is strict.

(III) Fairness

There are four classic fairness axioms: anonymity, envy-freeness, resource monotonicity and population monotonicity. Envy-freeness and resource monotonicity are known to be very demanding and usually incompatible.

Let π be a bijection on $\mathbb{N}$. A rule μ is *anonymous* (ANON) if $\forall \pi$, $\forall (N, u_N, \omega)$, $\forall i \in N$, $\mu_i(N, u_N, \omega) = \mu_{\pi(i)}(\pi(N), (u_{\pi(j)})_{\pi(N)}, \omega)$ where $u_{\pi(j)} = u_j$, $\forall j \in N$.

Remark 1 *If μ is ANON, then for any (N, u_N, ω) such that $u_i = u_j$, $i, j \in N$, $\mu_i(N, u_N, \omega) = \mu_j(N, u_N, \omega)$.*

A rule μ is *envy-free* (EF) if $\forall (N, u_N, \omega)$, $\forall i, j \in N$, $u_i(\mu_i(N, u_N, \omega)) \geq u_i(\mu_j(N, u_N, \omega))$.

A rule μ is *resource monotonic* (RM) if $\forall (N, u_N)$, $\forall \omega, \omega' \in \mathbb{R}^l_+$, $\omega > \omega'$ implies that $u_i(\mu_i(\omega)) > u_i(\mu_i(\omega'))$, $\forall i \in N$.

There is another version of resource monotonicity. It states that $\forall (N, u_N)$, $\forall \omega, \omega' \in \mathbb{R}^l_+$, $\omega \geq \omega'$ implies that $u_i(\mu_i(\omega)) \geq u_i(\mu_i(\omega'))$, $\forall i \in N$. In general, these two versions do not imply each other. However, our rules below satisfy both of them, and the first one combined with the other axioms implies the second by our characterization result.

A rule μ is *population monotonic* (PM) if $\forall (N, u_N, \omega)$, $\forall N' \subseteq N$ and $N' \neq \emptyset$, $\forall i \in N'$, $u_i(\mu_i(N', u_{N'}, \omega)) \geq u_i(\mu_i(N, u_N, \omega))$.

(IV) Consistency

Consistency has played an important role in the rationing literature and also in the fair division problems of discrete goods.

A rule μ is *consistent* (CST) if $\forall(N, u_N, \omega)$, $\forall N' \subseteq N$ and $N' \neq \emptyset$, $\forall i \in N'$, $\mu_i(N, u_N, \omega) = \mu_i(N', u_{N'}, \omega - \sum_{j \in N \setminus N'} \mu_i(N, u_N, \omega))$.

Note that to check consistency, it is equivalent to check whether the corresponding condition holds when $|N'| = |N| - 1$.

Remark 2 *It is easy to see that if a rule is consistent and resource monotonic (no matter which version of resource monotonicity is adopted), then it must be population monotonic. In the following, if a rule is CST and RM, we will just keep in mind that it is also PM without even mentioning in the theorems.*

1.3 Generalized Egalitarian Rules

Let $f : \mathbb{D} \to \mathbb{R}^n$ where $\mathbb{D} \subseteq \mathbb{R}^m$ and $m, n \in \mathbb{N}$ be an arbitrary function. We say f is strictly increasing if $\forall x, y \in \mathbb{R}^m,\ x > y$ implies that $f(x) > f(y)$.

Suppose that $W : \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\} \to \mathbb{R}_+$ is a strictly increasing and continuous function. Given an economy $E = (N, u_N, \omega)$, let $A^W(E) = \{x \in A^*(E) | W(x_i) = W(x_j), \forall i, j \in N\}$.

Lemma 2 *$A^W(E)$ is a totally ordered and closed set in $\mathbb{R}^{|N| \times l}$. In particular, $\max A^W(E)$ exists.*

Proof 2 *To show that $A^W(E)$ is totally ordered, let $x, y \in A^W(E)$ such that $x \neq y$ be given. Suppose without loss of generality (WLOG) that $x_j < y_j$ for some $j \in N$. By the definition of $A^W(E)$ and the properties of W, we know that $\forall i \in N,\ x_i, y_i \in \gamma_i$, and $W(x_i) = W(x_j) < W(y_j) = W(y_i)$. Since the γ_i's are totally ordered sets and W is strictly increasing, then $x_i < y_i,\ \forall i \in N$, and thus $x < y$.*

To see that $\max A^W(E)$ exists, note that $A^*(E)$ is closed and W is continuous. Moreover, $A^W(E)$ is nonempty and bounded. Thus, $\max A^W(E)$ exists.

Lemma 2 guarantees that the following rule is well-defined.

Definition 3 *A rule μ is called a generalized egalitarian rule, if there is a strictly increasing continuous function $W : \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\} \to \mathbb{R}_+$ such that for all $E \in \mathcal{E}$, $\mu(E) = \max A^W(E)$.*

Let $\mathcal{M}$ denote the class of generalized egalitarian rules. We write μ^W when we want to indicate that μ is generated by W.

We give two interpretations of our rules. One is in terms of a benchmark preference on the commodity space. The other is related to "equal opportunity allocations".

First, suppose that society has a benchmark preference over the commodity space which is represented by W.[¶] Then μ^W assigns to each agent the same welfare level according to this benchmark preference of society. We use two examples to explain.

Example 1 : *Equalizing total wealth.*

Fix a price vector $p \in \mathring{\mathbb{R}}^l_+$. Let $W(x) = p \cdot x$, $\forall x \in \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\}$. In this case, society wants the agents to get the same total wealth. The indifference classes of the benchmark preference are just the budget lines. See Figure 1 for an illustration in a two-good economy.

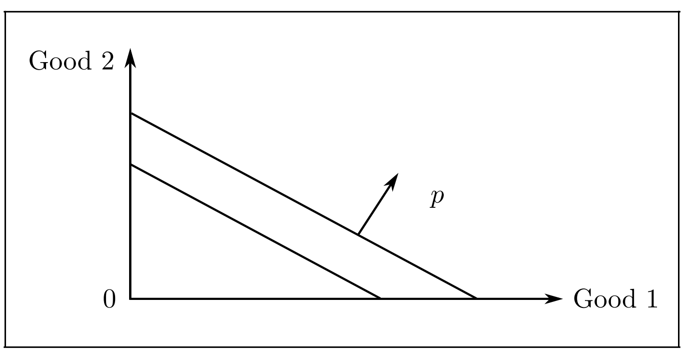

Figure 1.1 : Equalizing total wealth

[¶]The value of W on $\partial\mathbb{R}^l_+ \backslash \{\mathbf{0}\}$ is irrelevant, since $A^*(E) \cap \partial\mathbb{R}^l_+ = \{\mathbf{0}\}$. More rigorously, W represents a benchmark preference on the interior and the origin of the commpodity space.

Example 2 : *Egalitarian − equivalent (EE) rules.*

The spirit of the classic EE rule is that every agent should get "equal" share of the social endowment. The difficulty is to find a way of measuring these shares in a world of ordinal preferences (Moulin, 1995). Pazner and Schmeidler (1978) were the first to propose a solution. It assigns an allocation at which the agents are indifferent between their bundles and the same fraction of the social endowment. In our context, that is, $\mu(E) = \max\{x \in A^*(E) | u_i(x_i) = u_i(t\omega), \forall i \in N, t \in \mathbb{R}_+\}$. However, the classic EE rule is not resource monotonic. Then the e-EE rule is proposed to overcome this drawback. The e-EE rule fixes an arbitrary reference bundle $e \in \mathring{\mathbb{R}}^l_+$, and gives the agents the shares between which and te they feel indifferent, where t is taken as high as possible. Even more generally, fix a strictly increasing continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}^l_+$ such that $\lim_{t\to\infty} \varphi_k(t) = \infty, \forall k \in L$, and $\varphi(0) = \mathbf{0}$. We can define the φ-EE rule by $\mu(E) = \max\{x \in A^*(E)|u_i(x_i) = u_i(\varphi(t)), \forall i \in N, t \in \mathbb{R}_+\}$. The φ-EE rule makes all agents indifferent between their shares and the same commodity bundle on the reference curve, i.e., $\varphi(t^*)$ for some $t^* \in \mathbb{R}_+$. Hence, these shares are "equal" as viewed by society. Note that the e-EE rule is the φ-EE rule with $\varphi(t) = te$.

We check that on the domain of Leontief preferences, the φ-EE rule is a special case of the generalized egalitarian rules. Note that when ω is fixed, the classic EE rule with $\varphi(t) = t\omega$ is also a special case.

Lemma 3 *Let μ be a φ-EE rule. Define for all $x \in \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\}$, $W(x) = t$ if and only if $x \in \{\varphi(t)\} - \partial\mathbb{R}^l_+$. Then $\mu = \mu^W$.*

Proof 3 *First, since φ is continuous and strictly increasing, W is well-defined, and moreover, continuous and strictly increasing.*

Next, fix $E = (N, u_N, \omega)$. Observe that $\forall x \in A^*(E)$ and $\forall i \in N$, $W(x_i) = t$ if and only if $x \in \{\varphi(t)\} - \partial\mathbb{R}^l_+$, which is equivalent to $u_i(x_i) = u_i(\varphi(t))$ since $x_i \in \gamma_i$. Hence, $\mu = \mu^W$ by the definitions.

Figure 2 shows in a two-commodity space the indifference classes of the benchmark preference W which is defined from φ.

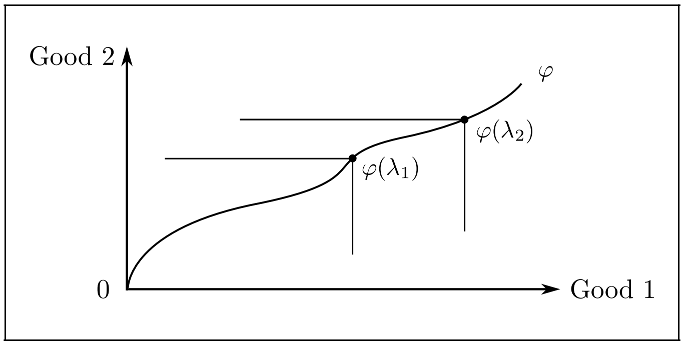

Figure 1.2 : φ-EE rules

The second interpretation relates to "equal opportunity allocations" proposed by Thomson (1994). Such an allocation is obtained by having each agent choose by himself in *a common choice set*. In this way, it gives the agents equal opportunities. We reformulate the definition in our context.

Let $\mathcal{C}$ be a family of choice sets, where each $C \in \mathcal{C}$ is a nonempty subset of $\mathbb{R}^l_+$.

Definition 4 (Thomson, 1994) *Given an economy $E = (N, u_N, \omega)$, a feasible allocation x is an equal opportunity allocation relative to the family $\mathcal{C}$ if there exists $C \in \mathcal{C}$ such that $\forall i \in N$, $x_i \in \arg\max_{y \in C} u_i(y)$.*

Lemma 4 *Let μ^W be given. Suppose $C(t) = \{y \in \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\} | W(y) \leq t\}$ where $t \in \mathbb{R}_+$. Let $\mathcal{C} = \{C(t) | t \in \mathbb{R}_+\}$. Then $\mu^W(E) = \max\{x \in A^*(E) | x \text{ is an equal opportunity allocation relative to } \mathcal{C}\}$ for all $E \in \mathcal{E}$.*

Proof 4 *Let E be given. We only need to show that if $x \in A^*(E)$, then $W(x_i) = W(x_j), \forall i, j \in N$ is equivalent to that x is an equal opportunity allocation relative to the family $\mathcal{C}$. If $W(x_i) = W(x_j), \forall i, j \in N$, then let $t = W(x_i)$, and thus x_i is the optimal bundle in $C(t)$ for* *all i since both u_i and W are strictly increasing. Conversely, suppose that x_i is the optimal bundle in $C(t)$ for all i. If WLOG there exist x_1 and x_2, such that $W(x_1) > W(x_2)$, then we must have $t \geq W(x_1) > W(x_2)$. Thus there must exist $x'_2 \in \gamma_2$ such that $x'_2 > x_2$ and $W(x'_2) < t$. It contradicts that x_2 is the optimal bundle in $C(t)$.*

Hence, a generalized egalitarian rule always picks the Pareto optimal equal opportunity allocation relative to the family of nested choice sets generated by W. In example 1, $\mathcal{C}$ is the class of all budget sets with a fixed price. In example 2, $\mathcal{C}$ is the class of box-shaped sets C with $C = \{y | y \leq \varphi(\lambda)\}$.

Our first main result is a characterization of generalized egalitarian rules.

Theorem 1 *(i) If a rule μ is in $\mathcal{M}$, then it is efficient, resource monotonic, consistent, group strategy-proof, anonymous and envy-free.*

(ii) Let a rule μ be efficient, resource monotonic and consistent. If μ is either strategy-proof and anonymous, or envy-free, then $\mu \in \mathcal{M}$.

In fact, Theorem 1 also holds for a much larger preference domain which is the object of the next section.

The requirement of non-wasteful allocation is very important for Theorem 1. Consider a natural extension of our rules to those which divide up every good. That is, first apply a generalized egalitarian rule μ^W and then allocate the remaining goods equally among the agents. More precisely, this extended rule $\bar{\mu}$ assigns for all $E = (N, \mu_N, \omega)$ and for all $i \in N$, $\bar{\mu}_i(E) = \mu_i^W(E) + \frac{1}{|N|}(\omega - \sum_{i \in N} \mu_i^W(E))$. We show that $\bar{\mu}$ is not SP by a counter-example. For simplicity suppose that $W = p \cdot x$ where $p > \mathbf{0}$. Let $E = (\{1,2\}, (u_1, u_2), \omega)$ where (i) $\omega \in \mathbb{R}^2_+$ and ω_2 is large enough so that good 2 is always available in the following discussion; (ii) the slope of the critical set of u_1 is greater than that of u_2. Let u'_1 be such that the slope of its critical set is in between those of u_1 and u_2. See Figure 3 for an illustration. Let E' be E with u_1 replaced by u'_1. Suppose that $\mu^W(E) = (x_1, x_2)$, $\mu^W(E') = (y_1, y_2)$, $\bar{\mu}(E) = (\bar{x}_1, \bar{x}_2)$ and $\bar{\mu}(E') = (\bar{y}_1, \bar{y}_2)$. Since ω_2 is large enough, then it is always good 1 that is divided up.

We check that $y_1^1 > x_1^1$. If $y_1^1 \leq x_1^1$, then $W(y_2) = W(y_1) < W(x_1) = W(x_2)$. Hence, $y_2 < x_2$, and thus $y_1^1 + y_2^1 < x_1^1 + x_2^1 = \omega_1$, which violates the efficiency of μ^W. Once again let ω_2 be large enough such that $\bar{y}_1^2 = y_1^2 + \frac{1}{2}(\omega_2 - y_1^2 - y_2^2) > x_1^2$. Then after dividing the remaining good 2, $\bar{y}_1 > x_1$, and thus $u_1(\bar{y}_1) > u_1(x_1) = u_1(\bar{x}_1)$. This example can be easily extended to economies with more goods.

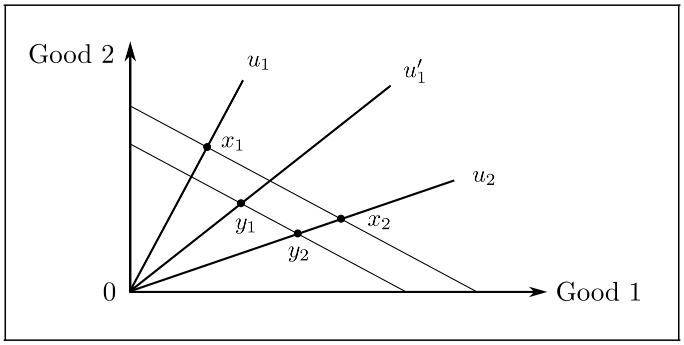

Figure 1.3 : A counter-example for wasteful allocation

Hence, if one wants a rule to allocate all the goods and be EFFN and SP, then one must carefully design the way that the useless goods are divided. Nicolò (2004) provides such a rule in a two-agent two-good economy. However, there is no result yet in a general economy.

Remark 3 *In the characterization of Nicolò (2004), he introduces an incentive compatibility axiom stronger than strategy-proofness — fully implementability in truthful strategies. It requires that a rule is strategy-proof and moreover when a misreport of an agent does not change his own utility, the whole allocation is unaffected. Our rules satisfy this axiom if and only if* $\forall x, y \in \mathring{\mathbb{R}}_+^l \cup \{\mathbf{0}\}$, $x \geq y$ *and* $x \neq y$ *imply that* $W(x) > W(y)$.

1.4 Generalized Leontief Preferences

All the proofs of the results in this section are in the Appendix.

Let $\succcurlyeq$ be a complete and transitive binary relation on $\mathbb{R}^l_+$, $\succ$ and $\sim$ be the corresponding strict and indifferent relations. For all $x \in \mathbb{R}^l_+$, denote by $U_{\succcurlyeq}(x) = \{y \in \mathbb{R}^l_+ | y \succcurlyeq x\}$ the upper contour set of x, and $I_{\succcurlyeq}(x) = \{y \in \mathbb{R}^l_+ | y \sim x\}$ its indifference class. For any subsets S_1 and S_2 of $\mathbb{R}^l_+$, $S_1 + S_2 = \{s_1 + s_2 | s_1 \in S_1,\ s_2 \in S_2\}$.

Definition 5 *The set of generalized Leontief preferences is defined by* $\mathcal{D} = \{\succcurlyeq \text{ on } \mathbb{R}^l_+ |\ \succcurlyeq$ *is continuous and locally non-satiated, and* $\forall x \in \mathbb{R}^l_+, U_{\succcurlyeq}(x) = \{a\} + \mathbb{R}^l_+$ *for some* $a \in \mathbb{R}^l_+\}$.

Lemma 5 *If* $\succcurlyeq \in \mathcal{D}$, *then*

(i) $\succcurlyeq$ *is monotone, i.e.,* $\forall x, y \in \mathbb{R}^l_+, x > y$ *implies that* $x \succ y$;

(ii) for any $x \in \mathbb{R}^l_+$, $U_{\succcurlyeq}(x) = \{a\} + \mathbb{R}^l_+$ *implies that* $I_{\succcurlyeq}(x) = \{a\} + \partial\mathbb{R}^l_+$.

Definition 6 *For any* $\succcurlyeq \in \mathcal{D}$, *define* $\gamma_{\succcurlyeq} = \{a \in \mathbb{R}^l_+ : U_{\succcurlyeq}(x) = \{a\} + \mathbb{R}^l_+$ *for some* $x \in \mathbb{R}^l_+\}$ *to be the critical set of the preference* $\succcurlyeq$.

Clearly, Definition 6 generalizes Definition 1 on the domain of generalized Leontief preferences.

Lemma 6 *For any* $\succcurlyeq \in \mathcal{D}$,

(i) $\boldsymbol{0} \in \gamma_{\succcurlyeq}$, *and* $\gamma_{\succcurlyeq}$ *is unbounded;*

(ii) if $a, b \in \gamma_{\succcurlyeq}$ *and* $a \neq b$, *then either* $a < b$ *or* $a > b$, *i.e.,* $\gamma_{\succcurlyeq}$ *is totally ordered;*

(iii) $\gamma_{\succcurlyeq}$ *is connected;*

(iv) $\gamma_{\succcurlyeq}$ *is closed.*

Figure 4 shows the typical upper contour set, the indifference class and the critical set of a generalized Leontief preference in a two-good economy.

Proposition 1 *For any* $\succcurlyeq \in \mathcal{D}$, $\succcurlyeq$ *is represented by* $u(x) = \max\{t \in \mathbb{R}_+ | x \geq \zeta(t)\}$, $\forall x \in \mathbb{R}^l_+$, *where* $\zeta : \mathbb{R}_+ \to \gamma_{\succcurlyeq}$ *is a strictly increasing homeomorphism such that* $\sum\limits_{k \in L} \zeta^k(t) = t$, $\forall t \in \mathbb{R}_+$.

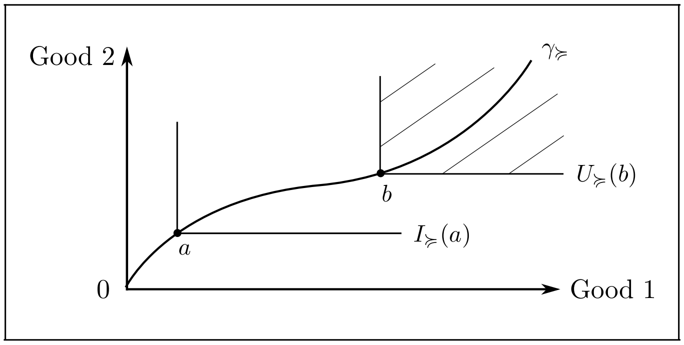

Figure 1.4 : A generalized Leontief preference in a two-good economy

For any $x \in \gamma_{\succsim}$, $x = \zeta(t)$ for some t, and thus $u(x) = t = \sum_{k \in L} x^k$. Hence, u restricted on $\gamma_{\succsim}$ is a strictly increasing continuous function.

Let $\tilde{\mathcal{U}}$ be the set of all utility functions representing generalized Leontief preferences in the way specified in Proposition 1. Note that $\tilde{\mathcal{U}}$ is a generalization of $\mathcal{U}$, since for any standard Leontief preference represented by $u \in \mathcal{U}$ with $u(x) = \min_{k \in L}\{\frac{x^k}{\lambda_k}\}$, $\zeta(t) = (\lambda_1 t, ..., \lambda_l t)$, $\forall t \in \mathbb{R}_+$, and thus $u(x) = \max\{t \in \mathbb{R}_+ | x \geq \zeta(t)\}$, $\forall x \in \mathbb{R}^l_+$, as well.

It is easy to see that under the larger preference domain $\tilde{\mathcal{U}}$, all the previous notions such as economy, rule and generalized egalitarian rule are still well-defined. Moreover, as we mentioned before, Theorem 1 still holds when $\mathcal{U}$ is replaced by $\tilde{\mathcal{U}}$.

Let $\tilde{\mathcal{M}}$ denote the class of generalized egalitarian rules under the domain $\tilde{\mathcal{U}}$. For simplicity, we will still use notations such as E, $A^*(E)$ and μ to denote the corresponding notions under the generalized preference domain.

Theorem 2 *(i) If a rule μ is in $\tilde{\mathcal{M}}$, then it is efficient, resource monotonic, consistent, group strategy-proof, anonymous and envy-free.*

(ii) Let a rule μ be efficient, resource monotonic and consistent. If μ is either strategy-proof and anonymous, or envy-free, then $\mu \in \tilde{\mathcal{M}}$.

1.5 The Proofs

Generally speaking, the structure of our problem has some resemblance to the "fixed path" methods in the rationing literature, such as the parametric method in Young (1987), and the fixed path rationing method in Moulin (1999). The essential idea of the proof is to investigate how the given axioms impact the range of the rules. We find that the range can be identified with some features which enable us to construct a benchmark preference.

Here we prove Theorem 2. In fact, the result of every step in the following is true under both preference domains. The proofs under $\mathcal{U}$ just involve less cases to check. For the simplicity of presentation, we assume that a rule assigns to every agent an unbounded bundle when the endowment increases, i.e., $\forall(N, u_N)$, $\forall i \in N$, $\mu_i(N, u_N, \mathbb{R}^l_+)= \{x_i \in \mathbb{R}^l_+ | \mu_i(N, u_N, \omega) = x_i$ for some $\omega \in \mathbb{R}^l_+\}$ is an unbounded subset in $\mathbb{R}^l_+$. This assumption is not necessary. The relaxation of it will be discussed in the Appendix.

Step 1. If μ is EFFN and RM, then

(i) $\forall(N, u_N)$, $\forall x, x' \in \mu(N, u_N, \mathbb{R}^l_+)$ such that $x \neq x'$, either $x_i < x'_i$, $\forall i \in N$, or $x_i > x'_i$, $\forall i \in N$;

(ii) $\forall(N, u_N),\ \forall i \in N,\ \mu_i(N, u_N, \mathbb{R}^l_+) = \gamma_i$.

Proof 5 *Let (N, u_N) be given. Suppose WLOG that $N = \{1, ..., n\}$.*

(i) Assume that $\mu(\omega) = x$, $\mu(\omega') = x'$, $\omega, \omega' \in \mathbb{R}^l_+$, and $x \neq x'$.

First observe that if $x_j < x'_j$ for some $j \in N$, then $x_i \leq x'_i$ for all $i \in N$. Suppose the contrary WLOG that $x_1 < x'_1$ and $x_2 > x'_2$. Then $\sum_{i\in N} \min\{x_i, x'_i\} < \sum_{i\in N} x_i \leq \omega$, and $\sum_{i\in N} \min\{x_i, x'_i\} < \sum_{i\in N} x'_i \leq \omega'$. Since μ is RM, then $\mu_i(\sum_{i\in N} \min\{x_i, x'_i\}) < \min\{x_i, x'_i\}$, $\forall i \in N$, which violates the efficiency of μ.

Next note that if $y \in \mu(\mathbb{R}^l_+)$, then $\mu(\sum_{i\in N} y_i) = y$. Suppose the contrary WLOG that $\mu(\sum_{i\in N} y_i) = y'$ and $y_1 < y'_1$. By our previous result, $y_i \leq y'_i$, $\forall i \in N$. Thus $\sum_{i\in N} y'_i > \sum_{i\in N} y_i$, which violates feasibility.

Hence, we can take $\omega = \sum_{i \in N} x_i$ *and* $\omega' = \sum_{i \in N} x'_i$*. Suppose WLOG that* $x_1 \neq x'_1$*. If* $x_1 < x'_1$*, then we know that* $x_i \leq x'_i$*,* $\forall i \in N$*. Thus,* $\omega < \omega'$*. Since* μ *is RM, then* $x_i < x'_i$*,* $\forall i \in N$*. Similarly, if* $x_1 > x'_1$*, then* $x_i > x'_i$*,* $\forall i \in N$*.*

(ii) Suppose the contrary WLOG that $a \in \gamma_1 \setminus \mu_1(\mathbb{R}^l_+)$*. Since* $\boldsymbol{0} \in \mu_1(\mathbb{R}^l_+)$ *and* $\mu_1(\mathbb{R}^l_+)$ *is unbounded, then* $\underline{\nu} = \{x \in \mu(\mathbb{R}^l_+) | x_1 < a\}$ *and* $\bar{\nu} = \{x \in \mu(\mathbb{R}^l_+) | x_1 > a\}$ *are nonempty. Let* $\underline{\omega} = \sup\{\sum_{i \in N} x_i | x \in \underline{\nu}\}$[‖] *and* $\bar{\omega} = \inf\{\sum_{i \in N} x_i | x \in \bar{\nu}\}$*. By (i),* $\underline{\nu} \cup \bar{\nu} = \mu(\mathbb{R}^l_+)$ *is totally ordered, so* $\underline{\omega}$ *and* $\bar{\omega}$ *are well-defined, and* $\underline{\omega} \leq \bar{\omega}$*. If* $\underline{\omega} < \bar{\omega}$*, then pick* ω *such that* $\underline{\omega} < \omega < \bar{\omega}$*. By the choice of* ω*,* $\mu(\omega) \notin \underline{\nu} \cup \bar{\nu}$*, which is a contradiction. If* $\underline{\omega} = \bar{\omega}$*, let* $y = \sup \underline{\nu} = \inf \bar{\nu}$*, and then* $y_1 = a$*. Let* $(y'_i)_{i \in N} = \mu(\sum_{i \in N} y_i)$*. By assumption* $y'_1 \neq y_1$*. If* $y_1 < y'_1$*, then* $y' \in \bar{\nu}$ *and thus* $y'_i \geq y_i$*,* $\forall i \in N$*. Hence,* $\sum_{i \in N} y'_i > \sum_{i \in N} y_i$*, which violates the feasibility. If* $y_1 > y'_1$*, then by a similar argument the efficiency is violated.*

Step 2. If $\mu \in \tilde{\mathcal{M}}$ is EFFN, RM and CST, then

(i) $\forall (N, u_N)$, $\forall N' \subseteq N$, $(x_i)_{i \in N} \in \mu(N, u_N, \mathbb{R}^l_+)$ implies that $(x_i)_{i \in N'} \in \mu(N', u_{N'}, \mathbb{R}^l_+)$;

(ii) $\forall (N_1, u_{N_1})$ and (N_2, u_{N_2}) such that $N_1 \cap N_2 = \emptyset$, $\forall (x_i)_{i \in N_1} \in \mu(N_1, u_{N_1}, \mathbb{R}^l_+)$ and $(x_i)_{i \in N_2} \in \mu(N_2, u_{N_2}, \mathbb{R}^l_+)$, if for some $i_1 \in N_1$ and $i_2 \in N_2$, $(x_{i_1}, x_{i_2}) \in \mu(\{i_1, i_2\}, (u_{i_1}, u_{i_2}), \mathbb{R}^l_+)$, then $(x_i)_{i \in N} \in \mu(N, u_N, \mathbb{R}^l_+)$ where $N = N_1 \cup N_2$ and $u_N = (u_{N_1}, u_{N_2})$.

Proof 6 *Obviously, (i) follows from Step 1 (i) and the definition of consistency.*

For (ii), suppose the contrary that under the required condition, $(x_i)_{i \in N} \notin \mu(N, u_N, \mathbb{R}^l_+)$*. Then assume that* $\mu(N, u_N, \sum_{i \in N} x_i) = (x'_i)_{i \in N} \neq (x_i)_{i \in N}$*. Thus, there must exist some* $j \in N$ *such that* $x'_j < x_j$*. Suppose WLOG that* $j \in N_1$*. By (i), we know that* $(x'_i)_{i \in N_1} \in \mu(N_1, u_{N_1}, \mathbb{R}^l_+)$*,* $(x'_{i_1}, x'_{i_2}) \in \mu(\{i_1, i_2\}, (u_{i_1}, u_{i_2}), \mathbb{R}^l_+)$*, and* $(x'_i)_{i \in N_2} \in \mu(N_2, u_{N_2}, \mathbb{R}^l_+)$*. From our assumption and Step 1, we have that* $x'_i < x_i$*,* $\forall i \in N_1$*, and thus* $x'_{i_2} < x_{i_2}$*, and finally* $x'_i < x_i$*,* $\forall i \in N_2$*. Hence,* $\sum_{i \in N} x'_i < \sum_{i \in N} x_i$*, which violates that* μ *is EFFN.*

[‖]For all $A \subseteq \mathbb{R}^m$, $m \in \mathbb{N}$, $(\sup A)_k = \sup\{a_k : a \in A\}$, $k = 1, ..., m$; $\inf A$ is similarly defined.

Remark 4 *It can also be shown that if μ is EFFN and RM, then both (i) and (ii) of Step 2 are sufficient conditions for μ to be CST.*

Step 3. Suppose that μ is EFFN, RM and CST. Then μ is SP if and only if $\forall(N, u_N)$ such that $|N| = 2$, $\forall i \in N$, $\forall u'_i \in \tilde{\mathcal{U}}$, if $(x_i, x_{-i}) \in \mu(N, u_N, \mathbb{R}^l_+)$ and $(x'_i, x_{-i}) \in \mu(N, u'_N, \mathbb{R}^l_+)$ where $u'_N = (u'_i, u_{-i})$, then $x_i \not< x'_i$.

Proof 7 *For necessity, suppose the contrary WLOG that $N = \{1, 2\}$, and under the required condition, $x_1 < x'_1$. Since γ'_1 is connected, we can find $y'_1 \in \gamma'_1$ such that $y'_1 \in \{x_1\} + \partial\mathbb{R}^l_+$. See Figure 5 for an illustration in a two-good economy.*

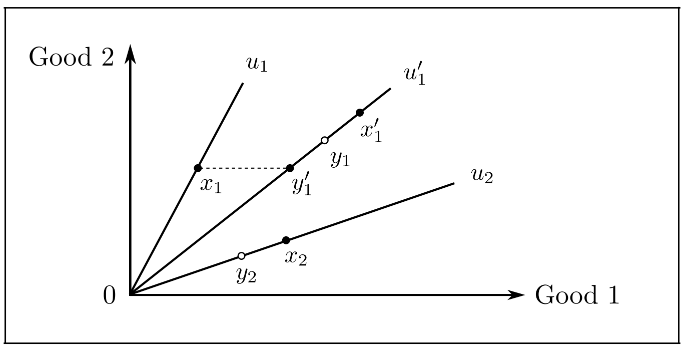

Figure 1.5 : Necessity for Step 3 (strategy-proofness)

Let $\omega = y'_1 + x_2$. By Step 1 (i), $\mu(N, u_N, \omega) = (x_1, x_2)$. We assume that $\mu(N, u'_N, \omega) = (y_1, y_2)$. Since $(x'_1, x_2) \in \mu(N, u'_N, \mathbb{R}^l_+)$ and $x'_1 + x_2 > x_1 + x_2$, then $y_1 < x'_1$ and $y_2 < x_2$. Thus by efficiency, $y_1 > y'_1 \geq x_1$. This means that in the economy (N, u_N, ω), agent 1 has incentive to misreport his preference, which violates that μ is SP.

For sufficiency, given the required assumption, we want to show that μ is SP. WLOG let (N, u_N, ω) where $N = \{1, ..., n\}$, and $u'_1 \in \tilde{\mathcal{U}}$ be given. Let $\mu(N, u_N, \omega) = (x_i)_{i\in N}$, and $\mu(N, u'_N, \omega) = (x'_i)_{i\in N}$ where $u'_N = (u'_1, u_{-1})$. See Figure 6.

We can find $y_1 \in \gamma'_1$ such that $(y_1, x_2) \in \mu(\{1, 2\}, (u'_1, u_2), \mathbb{R}^l_+)$. By consistency, $(x_1, x_2) \in \mu(\{1, 2\}, (u_1, u_2), \mathbb{R}^l_+)$. By the required assumption, $x_1 \not< y_1$. Hence, if $x'_1 \leq y_1$, then

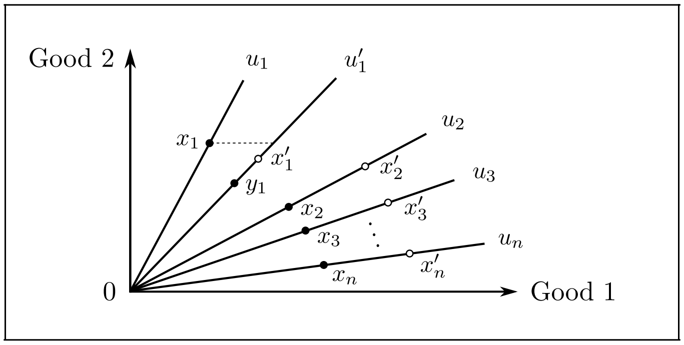

Figure 1.6 : Sufficiency for Step 3 (strategy-proofness)

$x_1 \nless x'_1$. *Consider the other case that* $x'_1 > y_1$. *By consistency,* $(x_2, ..., x_n) \in \mu(N \setminus \{1\}, u_{N\setminus\{1\}}, \mathbb{R}^l_+)$. *From Step 2, consider* $(N_1, u_{N_1}) = (\{1\}, u'_1)$, $(N_2, u_{N_2}) = (N \setminus \{1\}, u_{N\setminus\{1\}})$, *and thus* $(y_1, x_2, ..., x_n) \in \mu(N, u'_N, \mathbb{R}^l_+)$. *Hence,* $x'_i > x_i$, $\forall i = 2, ..., n$. *If* $x'_1 > x_1$, *then* $\omega \geq \sum_{i\in N} x'_i > \sum_{i\in N} x_i$, *which violates the efficiency. Hence,* $x_1 \nless x'_1$ *and agent 1 has no incentive to misreport his preference.*

Step 4. A rule $\mu \in \tilde{\mathcal{M}}$ if and only if μ is EFFN, RM, CST, SP and ANON.

Proof 8 *For necessity, let* $\mu \in \tilde{\mathcal{M}}$ *and* (N, u_N, ω) *be given. To check efficiency, by Lemma 1, we only need to check that some commodity is divided up. Suppose the contrary that* $\mu(N, u_N, \omega) = x$ *and* $\sum_{i\in N} x_i < \omega$. *We can find for each* $i \in N$ $x'_i \in \gamma_i$ *such that* $x'_i > x_i$ *and* $\sum_{i\in N} x'_i \leq \omega$, *since* γ_i*'s are connected. Pick* $t \in \mathbb{R}_+$ *such that* $W(x_i) < t < W(x'_i)$, $\forall i \in N$. *Since* W *is continuous and* γ_i*'s are connected, then* $W(\gamma_i)$*'s are connected. Thus for each* $i \in N$ *there exists* $y_i \in \gamma_i$ *such that* $W(y_i) = t$. *Clearly,* $\sum_{i\in N} y_i < \sum_{i\in N} x'_i \leq \omega$, *which contradicts that* $\mu(N, u_N, \omega) = x$ *by the definition of* μ.

To verify that μ *is RM, fix* ω' *such that* $\omega' > \omega$. *Then use the similar argument as above, we can show that the bundle allocated to every agent is strictly increased.*

Consistency follows from the definition of μ, *the efficiency of* μ, *and the assumption that* W *is strictly increasing.*

Strategy-proofness follows from Step 3 and strict increasingness of W.

Lastly, anonymity is simply because μ does not depend on agents' names, but their preferences.

For sufficiency, suppose that μ is EFFN, RM, CST, SP and ANON. Fix $\bar{u} \in \tilde{\mathcal{U}}$. Define $W : \mathring{R}^l_+ \cup \{\boldsymbol{0}\} \to \mathbb{R}_+$ as follows. For any $x \in \mathring{R}^l_+ \cup \{\boldsymbol{0}\}$, choose $u_x \in \tilde{\mathcal{U}}$ such that its critical set γ_x contains x. Choose $N = \{1,2\}$, $u_1 = \bar{u}$, and $u_2 = u_x$. From Step 1, we know that there uniquely exists $\bar{x} \in \bar{\gamma}$ such that $(\bar{x}, x) \in \mu(N, u_N, \mathbb{R}^l_+)$. Define $W(x) = \bar{u}(\bar{x})$. The choice of u_x does not matter, since for any other $u'_x \in \tilde{\mathcal{U}}$ such that $x \in \gamma'_x$ and the corresponding $\bar{x}' \neq \bar{x}$, WLOG say $\bar{x}' < \bar{x}$, then there must be an $x' \in \gamma_x$ such that $x' < x$ and $(\bar{x}', x') \in \mu(N, (\bar{u}, u_x), \mathbb{R}^l_+)$, which contradicts that μ is SP by Step 3. See Figure 7 for an illustration in a two-good economy. Hence, W is well-defined. Note that for any $x \in \bar{\gamma}$, we can pick $u_x = \bar{u}$. Since μ is ANON, then $\mu_i(N, u_N, 2x) = x$, $i = 1, 2$, and thus $W(x) = \bar{u}(x)$ for all $x \in \bar{\gamma}$.

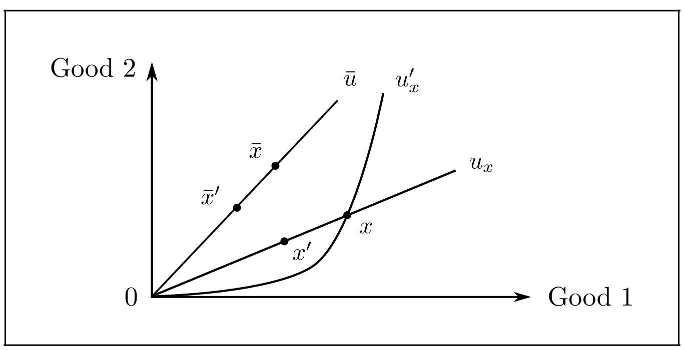

Figure 1.7 : Independence of the choice of u_x

To check that W is strictly increasing, let $x, y \in \mathbb{R}^l_+$ such that $x < y$. We can find $u \in \tilde{\mathcal{U}}$ whose critical set contains both x and y. Find $\bar{x}, \bar{y} \in \bar{\gamma}$ such that $(x, \bar{x}), (y, \bar{y}) \in \mu(\{1,2\}, (\bar{u}, u), \mathbb{R}^l_+)$. Clearly, $\bar{x} < \bar{y}$, and thus $W(x) = \bar{u}(\bar{x}) < \bar{u}(\bar{y}) = W(y)$.

To verify that W is continuous, we only need to check that $W^{-1}((t, \infty))$ and $W^{-1}([0, s))$ are open sets in $\mathring{\mathbb{R}}^l_+ \cup \{\boldsymbol{0}\}$ when $t \geq 0$ and $s > 0$. Let $t \geq 0$ and $x \in W^{-1}((t, \infty))$ be *given. Let* u_x *and* $\bar{x}$ *be correspondingly given. By Proposition 1, we can find* $\bar{x}_t \in \bar{\gamma}$ *such that* $\bar{u}(\bar{x}_t) = t$*. By Step 1 (ii), there exists* $x_t \in \gamma_x$ *such that* $W(x_t) = t$*. See Figure 8. Since* $x \in \gamma_x$ *and* $W(x) > t$*, then* $x > x_t$*. Thus there exists* $\epsilon > 0$ *such that* $B_\epsilon(x) = \{y \in \mathbb{R}^l_+| \, ||y - x|| < \epsilon\} \subseteq \{x_t\} + \mathring{\mathbb{R}}^l_+$*. For all* $y \in B_\epsilon(x)$*,* $y > x_t$*, and thus* $W(y) > W(x_t) = t$*. Hence,* $B_\epsilon(x) \subseteq W^{-1}((t, \infty))$*, which implies that* $W^{-1}((t, \infty))$ *is open. Similarly, we have that* $W^{-1}([0, s))$ *is open for all* $s > 0$*.*

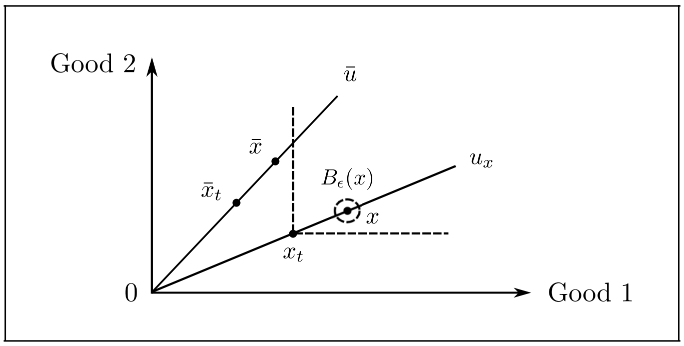

Figure 1.8 : The continuity of W

Finally, we check that $\forall E = (N, u_N, \omega)$*,* $\mu(E) = \max\{x \in A^*(E) | W(x_i) = W(x_j),\ \forall i, j \in N\}$*. Suppose that* $\mu(E) = (x^*_i)_{i \in N}$*. Fix* $i, j \in N$*, and* $i \neq j$*. Assume WLOG that* $1 \notin N$*. By the construction of* W *and the anonymity of* μ*, there exists* $\bar{x}$ *such that* $(\bar{x}, x^*_i) \in \mu(\{1, i\}, (\bar{u}, u_i), \mathbb{R}^l_+)$*. Since* μ *is CST,* $(x^*_i, x^*_j) \in \mu(\{i, j\}, (u_i, u_j), \mathbb{R}^l_+)$*. Using Step 2 (ii), consider* $N_1 = \{1\}$*,* $N_2 = \{i, j\}$*, we get that* $(\bar{x}, x^*_i, x^*_j) \in \mu(\{1, i, j\}, (\bar{u}, u_i, u_j), \mathbb{R}^l_+)$*. By the consistency of* μ*,* $(\bar{x}, x^*_j) \in \mu(\{1, j\}, (\bar{u}, u_j), \mathbb{R}^l_+)$*. Since* μ *is ANON,* $W(x^*_i) = W(x^*_j)$*. Since* μ *is EFFN,* $(x^*_i)_{i \in N} = \max\{x \in A^*(E) | W(x_i) = W(x_j),\ \forall i, j \in N\}$*.*

Step 5. If μ is in $\tilde{\mathcal{M}}$, then μ is GSP.

Proof 9 *Let* (N, u_N, ω)*,* $S \subseteq N$*, and* $u'_N = (u'_S, u_{-S})$ *where* $u'_S \in \tilde{\mathcal{U}}_S$ *be given. Assume that* $\mu(N, u_N, \omega) = x$ *and* $\mu(N, u'_N, \omega) = x'$*. Suppose the contrary that* $\forall i \in S$*,* $u_i(x'_i) \geq u_i(x_i)$*, and* $\exists j \in S$ *such that* $u_j(x'_j) > u_j(x_j)$*. Hence,* $\forall i \in S$*,* $x'_i \geq x_i$ *and* $x'_j > x_j$*. Thus*

$W(x'_j) > W(x_j)$, which by the definition of μ implies that $\forall i \in N \setminus S$, $x'_i > x_i$. Therefore, $\sum_{i \in N} x_i < \sum_{i \in N} x'_i \leq \omega$, which contradicts the efficiency of μ.

Step 6. A rule μ is in $\tilde{\mathcal{M}}$ if and only if μ is EFFN, RM, CST and EF.

Proof 10 *For necessity, let $\mu \in \tilde{\mathcal{M}}$ be given. We only need to check that μ is EF. This simply follows from the definition of μ and the assumption that W is strictly increasing.*

For sufficiency, suppose that μ is EFFN, RM, CST and EF. First we show that μ is ANON. Let a bijection π on $\mathbb{N}$, and an economy $E = (N, u_N, \omega)$ be given. Let $E' = (\pi(N), (u_{\pi(i)})_{\pi(N)}, \omega)$ where $u_i = u_{\pi(i)}$, $\forall i \in N$. Assume that $\mu(E) = x$ and $\mu(E') = x'$. Suppose the contrary WLOG that $1, 2 \in N$ and $x_1 < x'_{\pi(1)}$ and $x_2 > x'_{\pi(2)}$. We can find $x'_1 \in \gamma_1$ such that $(x'_1, x'_{\pi(2)}) \in \mu(\{1,2\}, (u_1, u_2), \mathbb{R}^l_+)$. Note that $x'_1 < x_1 < x'_{\pi(1)}$ since $x'_{\pi(2)} < x_2$. See Figure 9.

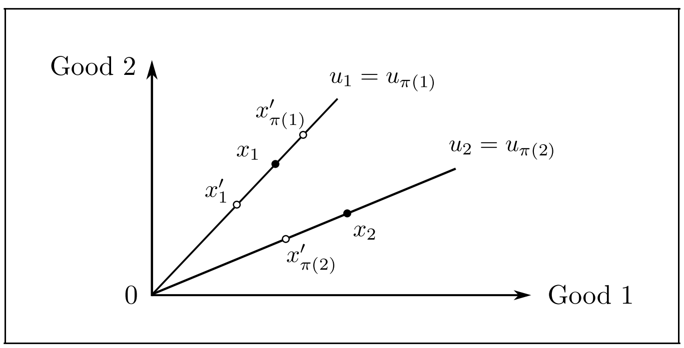

Figure 1.9 : The anonymity of μ

Suppose that $\{1,2\} \cap \{\pi(1), \pi(2)\} = \emptyset$. Since μ is EFFN and EF, then $\mu(\{2, \pi(2)\}, (u_2, u_2), 2x'_{\pi(2)}) = (x'_{\pi(2)}, x'_{\pi(2)})$. Thus by Step 2 (ii), $(x'_1, x'_{\pi(2)}, x'_{\pi(2)}, x'_{\pi(1)}) \in \mu(\{1, 2, \pi(2), \pi(1)\}, (u_1, u_2, u_2, u_1), \mathbb{R}^l_+)$, and agent 1 will envy agent $\pi(1)$ which is a contradiction. If $\{1,2\} \cap \{\pi(1), \pi(2)\} \neq \emptyset$, then pick $i_1, i_2 \in \mathbb{N}$ such that $\{1, 2, \pi(1), \pi(2)\} \cap \{i_1, i_2\} = \emptyset$. From the above result, we know that $(x'_{\pi(1)}, x'_{\pi(2)}) \in \mu(\{i_1, i_2\}, (u_1, u_2), \mathbb{R}^l_+)$. Applying the same argument to the agents $1, 2, i_1, i_2$ with the preferences u_1, u_2, u_1, u_2 respectively, we again will get a contradiction.

Now we only need to show that μ is SP. By Step 3, suppose WLOG that $\mu(\{1,2\},(u_1,u_2),\omega) = (x_1,x_2)$ *and* $\mu(\{1,2\},(u'_1,u_2),\omega') = (x'_1,x_2)$*, and we want to check whether* $x_1 \not< x'_1$*. Let* $u_3 = u'_1$*. Since μ is ANON, then* $\mu(\{3,2\},(u_3,u_2),\omega) = (x'_1,x_2)$*. Since μ is CST, then by Step 2* $\mu(\{1,2,3\},(u_1,u_2,u_3),\omega'') = (x_1,x_2,x'_1)$ *for some* $\omega'' \in \mathbb{R}^l_+$*. Since μ is EF, then* $u_1(x_1) \geq u_1(x'_1)$*, and thus* $x_1 \not< x'_1$*.*

1.6 Tightness of the Characterization

By Theorem 2, a rule is in $\tilde{\mathcal{M}}$ if and only if one of the following equivalent conditions holds:

(i) it is EFFN, RM, CST, ANON and SP;

(ii) it is EFFN, RM, CST, ANON and GSP;

(iii) it is EFFN, RM, CST and EF.

Our characterization is tight with respect to all these axioms when there are at least two goods in the economy.** The tightness result for Theorem 1 is the same.

Drop the efficiency, and consider the rule $\bar{\mu}$ such that for all $E = (N, u_N, \omega)$, $\bar{\mu}(E) = \max\{x \in A^*(E) | W(x_i) = t, \forall i \in N; \sum_{i \in N} x_i \leq \omega - te\}$ where W is as in Example 1, and e is the unit vector in the commodity space. It can be checked that $\bar{\mu}$ is well-defined, and is RM, CST, ANON, GSP and EF. The key fact used to verify these properties is that if $W(\bar{\mu}_i(E)) = t$, $\forall i \in N$, then $\sum_{i \in N} \bar{\mu}^k_i(E) = \omega^k - t$ for some $k \in L$. However, the allocation given by this rule is never efficient when $\omega > \mathbf{0}$.

Drop the resource monotonicity, and the following rule $\bar{\mu}$ is EFFN, CST, ANON, GSP and EF. Here we define $\bar{\mu}$ in a two-good economy for simplicity, and it can be easily extended to the economies with more than two goods. Consider for each $t \in \mathbb{R}_+$, a parameterized indifference curve $q(t)$ such that: $q(t) = \{x \in \mathbb{R}^2_+ | x^1 + x^2 = t\}$ when $t \in [0,2]$; $q(t) = \{x \in$

**It is easy to see that if there is only one good in the economy, then efficiency and either anonymity or envy-freeness will suffice to characterize the rules tightly.

$\mathbb{R}^2_+|x^1 + (t-1)x^2 = t$, where $x^1 \geq 1$ or $(t-1)x^1 + x^2 = t$, where $x^1 \leq 1\}$ when $t \in [2,4]$, and $q(t) = \{x \in \mathbb{R}^2_+|x^1 + 3x^2 = t$, where $x^1 \geq \frac{t}{4}$ or $3x^1 + x^2 = t$, where $x^1 \leq \frac{t}{4}\}$ when $t \in [4, +\infty)$. Let $W : \mathbb{R}^2_+ \to \mathbb{R}_+$ be a set-valued correspondence such that $W(x) = \{t|x \in q(t)\}$. Notice that $W(x)$ is always single-valued except for $W(1,1) = [2,4]$. For each $E = (N, u_N, \omega)$, let $\bar{\mu}(E) = \max\{x \in A^*(E)| \prod_{i \in N} I_{W(x_i)}(t) \neq 0$ for some $t \in \mathbb{R}\}$ where $I_{W(x_i)}(t) = 1$ when $t \in W(x_i)$, and $I_{W(x_i)}(t) = 0$ when $t \in \mathbb{R} \setminus W(x_i)$ for all i. It is a well-defined rule, satisfies all the axioms except for resource monotonicity. For a counter-example, consider a preference profile (u_1, u_2) such that γ_2 contains $x_2 = (1,1)$, as shown in Figure 10. Let $x_1 \in \gamma_1 \cap q(2)$ and $x_1' \in \gamma_1 \cap q(t)$ for some $t \in (2,4)$. Then $\bar{\mu}(x_1 + x_2) = (x_1, x_2)$, $\bar{\mu}(x_1' + x_2) = (x_1', x_2)$. In this case, $x_1 + x_2 < x_1' + x_2$ but agent 2 is not better off. Note that this rule still satisfies the second version of resource monotonicity.

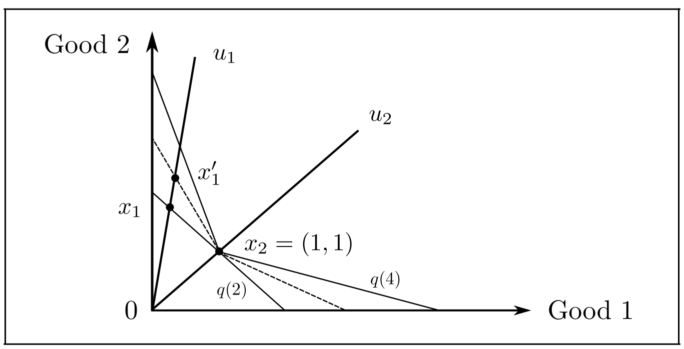

Figure 1.10 : Tightness of resource monotonicity

Drop the consistency, and consider the rule $\bar{\mu}$ such that for all $E = (N, u_N, \omega)$, $\bar{\mu}(E) = \mu^{W_1}(E)$ if $|N|$ is even and $\bar{\mu}(E) = \mu^{W_2}(E)$ if $|N|$ is odd where W_1 and W_2 are as in Example 1 with different p's. Obviously, $\bar{\mu}$ is EFFN, RM, ANON, GSP and EF, but not CST.

Drop the anonymity, and consider the rule $\bar{\mu}$ such that for all $E = (N, u_N, \omega)$ with $1 \notin N$, $\bar{\mu}(E) = \mu^W(E)$ where W is as in Example 1, and for all $E = (N, u_N, \omega)$ with $1 \in N$, $\bar{\mu}(E) = \max\{x \in A^*(E)|2W(x_1) = W(x_i),\ \forall i \in N \setminus \{1\}\}$. It is a well-defined rule, and is EFFN, RM, CST, GSP, but not ANON. We will prove this result for a general class of such rules in the next section.

Drop the strategy-proofness (and thus the group strategy-proofness), and consider the following rule $\bar{\mu}$ which is EFFN, ANON, RM and CST. Let $\bar{u} \in \tilde{\mathcal{U}}$ be fixed. For all $E = (N, u_N, \omega)$, $\bar{\mu}(E) = \mu^W(E)$ if $\forall i \in N$, $u_i \neq \bar{u}$, and if $S = \{j \in N | u_j = \bar{u}\} \neq \emptyset$, $\bar{\mu}(E) = \max\{x \in A^*(E) | 2W(x_j) = W(x_i),\ j \in S,\ i \in N \setminus S\}$ where W is as in Example 1. It is easy to check that $\bar{\mu}$ is well-defined and satisfies the above axioms. Figure 11 illustrates that $\bar{\mu}$ is not SP (and thus not GSP) in a two-commodity space. Consider a two-agent economy where their utility profile is as given in Figure 11. Suppose that $\bar{\mu}(\{1,2\}, (u_1, u_2), \omega) = (x_1, x_2)$ for some $\omega \in \mathbb{R}^2_+$. Then agent 1 prefers to report u'_1 which is very "close" to u_1. The point on γ'_1 "moves" faster than on γ_1, so after agent 1's misreport, it must be that his allocated bundle $x'_1 > x_1$.

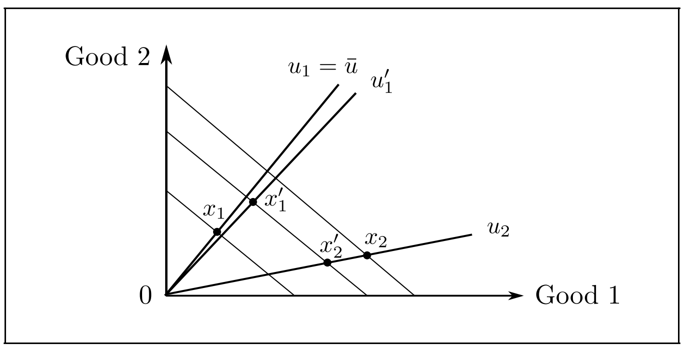

Figure 1.11 : Tightness of strategy-proofness

Drop the envy-freeness, the above two rules also work as counter-examples. This is because envy-freeness implies anonymity and strategy-proofness when a rule is EFFN, RM and CST.

1.7 Agent-specific Egalitarian Rules

Now we consider a natural extension of $\tilde{\mathcal{M}}$ to a class of non-anonymous rules. While generalized egalitarian rules equalize the agents' final welfare levels according to a benchmark preference over the commodity space, society may measure the welfare of each agent differently. It may attach to each agent i a utility function W_i and equalize the agents' final welfare according to these agent-specific utility functions.

Formally, for all $i \in \mathbb{N}$, let $W_i : \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\} \to \mathbb{R}_+$ be a strictly increasing continuous function such that $W_i(\mathbf{0}) = 0$. Let $\mathcal{W}^a = \{W_i | i \in \mathbb{N}\}$ be a set of all agents' welfare indices.

Definition 7 *A rule μ is called an agent-specific egalitarian rule if there exists $\mathcal{W}^a$ such that for all $E \in \mathcal{E}$,*

$$\mu(E) = \max\{x \in A^*(E) | W_i(x_i) = W_j(x_j),\ \forall i, j \in N\}$$

where $W_i \in \mathcal{W}^a$, $\forall i \in N$. Let $\mathcal{M}^a$ denote the class of agent-specific egalitarian rules.

Using the similar argument as in the proof of Lemma 2, it is easy to see that the analogous result holds, and $\mathcal{M}^a$ is well-defined.

Theorem 3 *If μ is in $\mathcal{M}^a$, then μ is efficient, resource monotonic, consistent and group strategy-proof.*

Proof 11 *The proof is almost the same as what we did for generalized egalitarian rules. Just by replacing $W(x_i)$ with $W_i(x_i)$ in Step 4 and 5 of Section 5, we can get the desired results.*

1.8 Endowment-specific Egalitarian Rules and Private Property

Another extension of $\tilde{\mathcal{M}}$ is natural when we drop the common property assumption. We first introduce the model where every agent has a private endowment. For notational simplicity, we will abuse the previous symbols again to denote the corresponding notions in the model with private property.

An economy E is a triple (N, u_N, ω_N) where $N \subseteq \mathbb{N}$ is a nonempty finite set of agents, $u_N = (u_i)_{i \in N}$ with $u_i \in \tilde{\mathcal{U}}$, $\forall i \in N$, is a preference profile, and $\omega_N = (\omega_i)_{i \in N}$ with $\omega_i \in \mathbb{R}^l_+$, $\forall i \in N$, denotes a vector of private endowments of the agents. Let $\mathcal{E}$ be the set of all economies.

Given (N, ω_N), the set of all feasible allocations is $A(N, \omega_N) = \{x \in \mathbb{R}^{|N| \times l}_+ | \sum_{i \in N} x_i \leq \sum_{i \in N} \omega_i\}$. For any economy $E = (N, u_N, \omega_N)$, the set of non-wasteful feasible allocations is $A^*(E) = A(N, \omega_N) \cap \prod_{i \in N} \gamma_i$ where γ_i is the critical set of u_i. Let $\mathcal{A}^* = \{A^*(E) | E \in \mathcal{E}\}$. A rule is a mapping $\mu : \mathcal{E} \to \mathcal{A}^*$ such that $\mu(E) \in A^*(E)$ for all $E \in \mathcal{E}$.

When the private property is introduced, an important problem is whether the agents are willing to put their own endowments together and participate in the social reallocation. Hence, here we need the individual rationality axiom to guarantee the voluntary participation.

A rule μ is *individually rational* (IR) if $\forall (N, u_N, \omega_N)$, $\forall i \in N$, $u_i(\mu_i(N, u_N, \sum_{i \in N} \omega_i)) \geq u_i(\omega_i)$.

The efficiency, incentive compatibility and fairness axioms are defined in the same way as the previous ones, except a little modification on anonymity and resource monotonicity.

Let π be a bijection on $\mathbb{N}$. A rule μ is *anonymous* if $\forall \pi$, $\forall (N, u_N, \omega_N)$, $\forall i \in N$, $\mu_i(N, u_N, \omega_N) = \mu_{\pi(i)}(\pi(N), (u_{\pi(j)})_{\pi(N)}, (\omega_{\pi(j)})_{\pi(N)})$ where $u_j = u_{\pi(j)}$ and $\omega_j = \omega_{\pi(j)}$, $\forall j \in N$.

A rule μ is *resource monotonic* if $\forall (N, u_N)$, $\forall \omega_N, \omega'_N \in \mathbb{R}^{|N| \times l}_+$, $\omega_i > \omega'_i$ for all $i \in N$ implies that $u_i(\mu_i(\omega_N)) > u_i(\mu_i(\omega'_N))$.

Resource monotonicity is shown to be incompatible with efficiency and individual rationality in Moulin and Thomson (1988). Although they assume a larger preference domain and use another version of resource monotonicity, it is easy to check that with a slight modification their counter-example still works in our context.

Our last result shows that when we allow the welfare index of an agent to depend on his private endowment, we obtain a class of rules which is EFFN, GSP, ANON and IR.

For all $x \in \mathbb{R}^l_+$, let $W_x : \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\} \to \mathbb{R}_+$ be a strictly increasing and continuous function such that for all $y \in \mathring{\mathbb{R}}^l_+ \cup \{\mathbf{0}\}$ with $y \leq x$ and $y^k = x^k$ for some $k \in L$, $W_x(y) = 1$.[††] Let $\mathcal{W}^e = \{W_x | x \in \mathbb{R}^l_+\}$.

Definition 8 *A rule μ is called an endowment-specific egalitarian rule, if there exists $\mathcal{W}^e$ such that for all $E \in \mathcal{E}$,*

$$\mu(E) = \max\{x \in A^*(E) | W_{\omega_i}(x_i) = W_{\omega_j}(x_j),\ \forall i, j \in N\}$$

where $W_{\omega_i} \in \mathcal{W}^e$, $\forall i \in N$. Let $\mathcal{M}^e$ denote the class of endowment-specific egalitarian rules.

By the analogous result of Lemma 2, $\mathcal{M}^e$ is well-defined.

Theorem 4 *If a rule μ is in $\mathcal{M}^e$, then it is efficient, group strategy-proof, anonymous and individually rational.*

Proof 12 *The proof of efficiency, group strategy-proofness and anonymity is basically the same as in Step 4 and 5 of Section 5.*

To see that μ is IR, note that for all $E = (N, u_N, \omega_N)$, there exists the allocation $x \in A^*(E)$ such that $\forall i \in N$, $u_i(x_i) = u_i(\omega_i)$ and $W_{\omega_i}(x_i) = 1$. Hence, $\forall i \in N$, $\mu_i(E) \geq x_i$ and then $u_i(\mu_i(E)) \geq u_i(\omega_i)$.

1.9 Concluding Remarks

In this paper, we investigate fair allocation rules on the generalized Leontief preference domain and achieve very positive results. Nevertheless, there are still some immediate open questions. The characterization of the agent-specific and endowment-specific egalitarian rules

[††]Essentially, what we need is that $W_x(y)$ is some constant which is independent of x.

remains open. A more intriguing question is how we could drop the non-wastefulness assumption of the rules and still get some positive results.

1.10 Appendix

1. The proofs of the results in Section 3

Lemma 5. *If* $\succcurlyeq \in \mathcal{D}$*, then*

(i) $\succcurlyeq$ *is monotone, i.e.,* $\forall x, y \in \mathbb{R}^l_+, x > y$ *implies that* $x \succ y$*;*

(ii) for any $x \in \mathbb{R}^l_+$*,* $U_{\succcurlyeq}(x) = \{a\} + \mathbb{R}^l_+$ *implies that* $I_{\succcurlyeq}(x) = \{a\} + \partial\mathbb{R}^l_+$*.*

Proof 13 *Let* $\succcurlyeq \in \mathcal{D}$ *be given.*

(i) Suppose that $x, y \in \mathbb{R}^l_+$ *and* $x > y$*. Since* $\succcurlyeq$ *is locally non-satiated, we can find* $y' < x$ *such that* $y' \succ y$*. Let* $U_{\succcurlyeq}(y') = \{a\} + \mathbb{R}^l_+$*,* $a \in \mathbb{R}^l_+$*. Since* $y' \in U_{\succcurlyeq}(y')$ *and* $x > y'$*, then* $x \geq a$*, and thus* $x \in U_{\succcurlyeq}(y')$*. Hence,* $x \succcurlyeq y' \succ y$*.*

(ii) Suppose that $x \in \mathbb{R}^l_+$ *and* $U_{\succcurlyeq}(x) = \{a\} + \mathbb{R}^l_+$*. By (i),* $\forall y \in \{a\} + \mathring{\mathbb{R}}^l_+$*,* $y \succ x$*. Now let* $y \in \{a\} + \partial\mathbb{R}^l_+$*. Since* $\succcurlyeq$ *is continuous, if* $y \succ x$*, then there exists* $y' < y$ *such that* $y' \succ x$*, which contradicts that* $U_{\succcurlyeq}(x) = \{a\} + \mathbb{R}^l_+$*. Hence,* $I_{\succcurlyeq}(x) = \{a\} + \partial\mathbb{R}^l_+$*.*

Lemma 6. *For any* $\succcurlyeq \in \mathcal{D}$*,*

(i) $\boldsymbol{0} \in \gamma_{\succcurlyeq}$*, and* $\gamma_{\succcurlyeq}$ *is unbounded;*

(ii) if $a, b \in \gamma_{\succcurlyeq}$ *and* $a \neq b$*, then either* $a < b$ *or* $a > b$*, i.e.,* $\gamma_{\succcurlyeq}$ *is totally ordered;*

(iii) $\gamma_{\succcurlyeq}$ *is connected;*

(iv) $\gamma_{\succcurlyeq}$ *is closed.*

Proof 14 *Let* $\succcurlyeq \in \mathcal{D}$ *be given.*

(i) To see $\boldsymbol{0} \in \gamma_{\succcurlyeq}$*, it suffices to show that* $U_{\succcurlyeq}(\boldsymbol{0}) = \{\boldsymbol{0}\} + \mathbb{R}^l_+$*. Suppose the contrary that* $U_{\succcurlyeq}(\boldsymbol{0}) = \{a\} + \mathbb{R}^l_+$ *where* $a \neq \boldsymbol{0}$*. Then it implies that* $\boldsymbol{0} \notin U_{\succcurlyeq}(\boldsymbol{0})$*, a contradiction.*

For unboundedness, suppose the contrary that there exists $y \in \mathbb{R}^l_+$ *such that* $\forall a \in \gamma_{\succcurlyeq}$*,* $a < y$*. Suppose* $U_{\succcurlyeq}(y) = \{b\} + \mathbb{R}^l_+$*. Then* $b \in \gamma_{\succcurlyeq}$ *and* $I_{\succcurlyeq}(y) = \{b\} + \partial\mathbb{R}^l_+$*. Thus* $b \leq y$ *and* $y^k = b^k$

for some $k \in \{1, ..., l\}$, which is a contradiction.

(ii) Let $a, b \in \gamma_{\succcurlyeq}$ and $a \neq b$. Suppose that $U_{\succcurlyeq}(x) = \{a\} + \mathbb{R}^l_+$ and $U_{\succcurlyeq}(y) = \{b\} + \mathbb{R}^l_+$, $x, y \in \mathbb{R}^l_+$. It is not true that $x \sim y$, otherwise $a = b$. By Lemma 5 (ii), $a \sim x$ and $b \sim y$. If $x \succ y$, then $a \succ y$ and thus $a \in \{b\} + \mathring{\mathbb{R}}^l_+$, which means $a > b$. Similarly, if $y \succ x$, then $a < b$.

(iii) Define $\rho : \gamma_{\succcurlyeq} \to \mathbb{R}_+$ such that $\rho(x) = \sum_{k \in L} x^k$, $\forall x \in \gamma_{\succcurlyeq}$. It suffices to show that ρ is a homeomorphism.

The injectivity of ρ follows from (ii). We first prove that ρ is surjective. Suppose the contrary that there exists $t \in \mathbb{R}_+ \setminus \rho(\gamma_{\succcurlyeq})$. Then $\gamma_{\succcurlyeq} = \alpha \cup \beta$ where $\alpha = \{a \in \gamma_{\succcurlyeq} | \rho(a) < t\}$ and $\beta = \{b \in \gamma_{\succcurlyeq} | \rho(b) > t\}$. By (i) we know that $\rho(\mathbf{0}) = 0$, and $\sup \rho(\gamma_{\succcurlyeq}) = \infty$. Hence, $\alpha, \beta \neq \emptyset$. Let $\bar{a} = \sup \alpha$ and $\underline{b} = \inf \beta$. Clearly, $\bar{a}, \underline{b} \in \mathbb{R}^l_+$ and $\bar{a} \leq \underline{b}$. If there exists $h \in L$ such that $\bar{a}^h < \underline{b}^h$, then pick $x \in \mathbb{R}^l_+$ such that $\bar{a} < x$ and $x^h < \underline{b}^h$. Suppose $I_{\succcurlyeq}(x) = \{c\} + \partial\mathbb{R}^l_+$. Thus $c \in \beta$ and $x \geq c$, which contradicts that $x^h < \underline{b}^h$. Hence, $\bar{a} = \underline{b}$. Then by (ii), $I_{\succcurlyeq}(\bar{a}) = \{\bar{a}\} + \partial\mathbb{R}^l_+$. Thus, either $\bar{a} \in \alpha$ or $\bar{a} \in \beta$. If $\bar{a} \in \alpha$, then $\rho(\bar{a}) < t$. We can choose $b \in \beta$ such that $\rho(b)$ is arbitrarily close to $\rho(\bar{a})$, and this contradicts that $\rho(b) > t$. Similarly, if $\bar{a} \in \beta$, we can also get a contradiction.

Next observe that for any $x, y \in \gamma_{\succcurlyeq}$, $||x - y|| \leq |\rho(x) - \rho(y)| \leq l||x - y||$ ‡‡, since either $x < y$ or $x > y$. Hence, ρ is a continuous open mapping.

(iv) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of elements in $\gamma_{\succcurlyeq}$ such that $\lim_{n \to \infty} a_n = a$. If $a \notin \gamma_{\succcurlyeq}$, then $\gamma_{\succcurlyeq} = [\gamma_{\succcurlyeq} \cap (\{a\} + \mathbb{R}^l_+)] \cup [\gamma_{\succcurlyeq} \cap (\{a\} - \mathbb{R}^l_+)]$, since $\gamma_{\succcurlyeq}$ is totally ordered and a is the limit of a sequence of elements in $\gamma_{\succcurlyeq}$. This contradicts that $\gamma_{\succcurlyeq}$ is connected.

Proposition 1. *For any $\succcurlyeq \in \mathcal{D}$, $\succcurlyeq$ is represented by $u(x) = \max\{t \in \mathbb{R}_+ | x \geq \zeta(t)\}$, $\forall x \in \mathbb{R}^l_+$, where $\zeta : \mathbb{R}_+ \to \gamma_{\succcurlyeq}$ is a strictly increasing homeomorphism such that $\sum_{k \in L} \zeta^k(t) = t$, $\forall t \in \mathbb{R}_+$.*

Proof 15 *Let $\succcurlyeq \in \mathcal{D}$ be given. Suppose that ρ is defined as in the proof of Lemma 6 (iii).*

‡‡ $||\cdot||$ is the standard Euclidean norm.

Clearly, ρ is strictly increasing since $\gamma_{\succcurlyeq}$ is totally ordered. Let $\zeta = \rho^{-1}$. Hence, all the properties of ζ follows from those of ρ. Since $\zeta(\mathbb{R}_+)$ is unbounded and ζ is continuous, then $\{t \in \mathbb{R}_+ | x \geq \zeta(t)\}$ is bounded and closed for any $x \in \mathbb{R}^l_+$, and thus $u : \mathbb{R}^l_+ \to \mathbb{R}_+$ is well defined.

Now we show that u represents $\succcurlyeq$. If $x \sim y$ and $I_{\succcurlyeq}(x) = \{a\} + \partial\mathbb{R}^l_+$, then $u(x) = u(y) = \sum_{k\in L} a^k$, since ζ is strictly increasing. If $x \succ y$, $I_{\succcurlyeq}(x) = \{a\} + \partial\mathbb{R}^l_+$ and $I_{\succcurlyeq}(y) = \{b\} + \partial\mathbb{R}^l_+$, then by Lemma 6 (ii) and Lemma 5 (i), $a > b$. Thus $u(x) = \sum_{k\in L} a^k > \sum_{k\in L} b^k = u(y)$.

2. The relaxation of the unbounded allocation assumption

There are several places in the steps of the proofs to be modified when we drop the assumption that $\forall (N, u_N),\ \forall i \in N,\ \mu_i(N, u_N, \mathbb{R}^l_+)$ is unbounded.

Step 1. (ii) Suppose that μ is EFFN and RM. If for (N, u_N) and $i \in N$, $\mu_i(N, u_N, \mathbb{R}^l_+)$ is bounded, then $\mu_i(N, u_N, \mathbb{R}^l_+) = \{x_i \in \gamma_i | x_i < x_i^*\}$ for some $x_i^* \in \gamma_i$, and moreover, there exists $j \in N$ such that $\mu_j(N, u_N, \mathbb{R}^l_+) = \gamma_j$.

Proof 16 *Let (N, u_N) and $i \in N$ be given. Suppose that $\mu_i(\mathbb{R}^l_+)$ is bounded. Let $x_i^* = \sup \mu(\mathbb{R}^l_+)$. Since γ_i is closed, then $x_i^* \in \gamma_i$. Note that if $x_i \in \mu_i(\mathbb{R}^l_+)$, then $x_i + \epsilon \in \mu_i(\mathbb{R}^l_+)$ for some $\epsilon > \mathbf{0}$, since μ is RM. Hence, $x_i^* \notin \mu_i(\mathbb{R}^l_+)$. Then using the similar argument as in the proof of Step 1, we get that $\forall x_i \in \gamma_i$ such that $x_i < x_i^*$, $x_i \in \mu_i(\mathbb{R}^l_+)$. If $\forall i \in N,\ \mu_i(\mathbb{R}^l_+)$ is bounded, then pick $\omega \geq \sum_{i\in N} x_i^*$, and thus $\sum_{i\in N} \mu_i(\omega) < \sum_{i\in N} x_i^* \leq \omega$, which contradicts that u is EFFN.*

Step 3. The sufficiency part.

Proof 17 *Let all the assumptions as in the sufficiency proof of Step 3 be given. we only need to check the case when there does not exist $y_1 \in \gamma'_1$ such that $(y_1, x_2) \in \mu(\{1,2\}, (u'_1, u_2), \mathbb{R}^l_+)$. Pick $y'_1 \in \gamma'_1$ such that $y'_1 > x_1$. By the modified Step 1 (ii), we can find $y_2 \in \gamma_2$ such that $(y'_1, y_2) \in \mu(\{1,2\}, (u'_1, u_2), \mathbb{R}^l_+)$, and $y_2 < x_2$. Since μ is CST, $(x_1, x_2) \in \mu(\{1,2\}, (u_1, u_2), \mathbb{R}^l_+)$.*

Again by the modified Step 1, there exist $y_1 \in \gamma_1$ *such that* $(y_1, y_2) \in \mu(\{1,2\},(u_1,u_2),\mathbb{R}^l_+)$. *Since* $y_2 < x_2$, *then* $y_1 < x_1$, *and thus* $y_1 < y_1'$, *which contradicts our assumption.*

Step 4. The sufficiency part.

Proof 18 *We first show the following two statements:*

(i) If μ *is EFFN, RM and ANON, then* $\forall (N, u_N)$ *such that* $\forall i \in N,\ u_i = u,\ \mu_i(N, u_N, \mathbb{R}^l_+) = \gamma_i,\ \forall i \in N$;

(ii) If μ *is EFFN, RM, ANON and SP, then* $\forall (N, u_N)$ *such that* $|N| = 2$ *and* γ_i *is unbounded in every commodity for some* $i \in N$, $\mu_j(N, u_N, \mathbb{R}^l_+) = \gamma_j$ *where* $j \in N$ *and* $j \neq i$.

The result (i) follows from Remark 1 and the modified Step 1.

For (ii), let (N, u_N) *which satisfies the required conditions be given. By the modified Step 1, suppose the contrary that* $\mu_j(\mathbb{R}^l_+)$ *is bounded where* $j \in N$ *and* $j \neq i$. *Thus when* ω *is big enough, agent* j *would pretend to have agent* i*'s preference, since his allocation would be unbounded in every dimension by statement (i) and the assumption on* γ_i. *This contradicts that* μ *is SP.*

Then the construction of W *is basically the same except that* $\bar{u}$ *should be chosen such that its critical set is unbounded in every dimension. By statement (ii),* W *is well-defined. The rest of the proof is the same.*

Chapter 2

Monopoly pricing of experience goods under endogenous social learning

2.1 Introduction

For a new durable experience good, from electronic products to first-run movie tickets, very often the quality of the good would remain unknown not only to the consumers, but also the producers. It is not until some purchases happen that the true information of the product is revealed. And such information spreads through reviews, comments, or simply word-of-mouth. This phenomenon is even more significant and effective due to various Internet forums and platforms for communication. To be more precise, some consumers who engaged in purchase at earlier time would "voluntarily" provide product information for the consumers who wait or hesitate in purchase.

In this paper, we consider a stylized two-period monopolistic model with two consumers who may have different or similar tastes. The monopolist sets a fixed price when the true quality is unknown to everyone. Upon price announcement, the consumers decide whether to accept it immediately in the first period or wait for the second period. A consumer who opts to purchase in early stage conveys product information attained through his own personal experience with certainty to the other consumer in later stage through social communication. Based on the prediction of such consumer behavior, the monopolist decide the price to this the profit. We show that there always exists a unique symmetric equilibrium for the consumers facing different prices. The choice of optimal price of the monopolist depends on the parameters. When the patience is moderate, it is more likely for the monopolist to choose a price to induce social learning process.

This paper is inspired mainly by three strands of literature. The first is pricing problem with forward-thinking consumers with rational expectation (Coase 1972, Bulow 1982, Stokey 1979, 1981). The second is monopolistic pricing of experience goods with information transmission (Shapiro 1983, Vettas 1997, Bergemann and Valimaki 2006, Bhalla 2012, Gunay 2014). The third is social learning experimentation (Kelley et al 2005, Che and Horner 2014). The literature usually adopts signaling/dynamic pricing approaches, which imply the information advantage of the monopolist. However, we argued that the monopolist may not have such power in this situation. For example, the adjustment of the price of new products may be costly or constrained by regulations and laws. The consumer communication also happens beyond the control of the monopolist. The monopolist may have difficulty in gathering related information or making timely responses. And as discussed above, the monopolist may not know the true quality of the product. Here we present a simple model in which the monopolist deliberately decides the price to affect the informational choices of the consumers, hence the social learning process is emphasized here.

The rest of the paper is organized as follows. The model is introduced in section 2. The results of the equilibrium of the consumers (with low θ) is presented in section 3. The seller's pricing problem is analyzed in section 4. In section 5, we complete the analysis by including high θ case. Section 6 concludes the paper.

2.2 The Model

We consider a two-period model. The discount factor is $\delta \in (0, 1)$. There is one monopolist producer who produces and sells a new experience good to two customers. Both the producer and the consumers are risk neutral. We assume that the producer sells this product at constant marginal cost, without loss of generality, to be zero. The consumers have unit demand.

There are two possible states of the quality of the product: high type H and low type L, denoted by $q \in \{1, 0\}$ respectively. Nature draws the quality type at the beginning of period 1. The prior i.e. the probability of the good being high type is $\lambda \in (0, 1)$, which is common knowledge. But the result of drawing is unknown for both parties at this time. The consumers are also characterized by two taste types: good type G and bad type B, denoted by 1 and $\theta \in (0, 1)$ respectively. The prior i.e. the probability of one consumer being good type is $\frac{1}{2}$, which is common knowledge. The result of independent type-drawing is private information of each consumer. For a consumer with good taste, his utility is 0 for low type product, 1 for high type product. For a consumer with bad taste, his utility is still 0 for low type product, but θ for high type product.

Now the game proceeds as following. The producer first announces the price p at the beginning of period 1, which is fixed during both periods. The consumers decide whether to "stay" i.e. complete their purchase in period 1, or "delay" i.e. wait until period 2, upon observation of the price announcement and their own type realizations. If both consumers choose stay, the game ends. If both consumers choose delay, the game goes into period 2 with nothing changed other than timing. If one consumer chooses stay and another chooses delay, then the later consumer will enter period 2 alone and know the true quality from the former consumer for sure. Now the remaining consumer(s) choose to accept or reject the price again, and then the game ends. The consumers are allowed to use mixed strategies. If a consumer is indifferent between Stay and Delay, we assume he always chooses Stay.

2.3 Consumer Choices

Now we want to find all the symmetric equilibria of the consumers facing different prices. For simplicity we denote a strategy profile in form of (S_G, S_B), where S_G represents the choice of a good consumer and S_B represents the choice of a bad consumer. $S_G, S_B \in \{Stay, Delay\} \cup \{\alpha \in (0, 1)\}$ where α means a mixed strategy with probability of α choosing stay, and probability $1 - \alpha$ choosing delay.

(i) If $p > \lambda$, the consumers will not stay in period 1 no matter they are of good type or bad type, since the price is higher than the expected utility in period 1. Both of them will delay into period 2. And since no one stays in period 1, no information disclosure happens in period 2 and both of them will eventually quit. In conclusion, $(Delay, Delay)$ is the only equilibrium when $p > \lambda$.

(ii) If $\lambda \geq p > \lambda\theta$, a bad consumer will not stay in period 1 for the same reason. Now we check the choice of a good consumer.

For a good consumer, he should not choose delay. Otherwise, when he delay into period 2 he will never receive new information, and will be strictly worse off from discounting.

If a good consumer chooses stay, He will always get payoff $\lambda - p$ in period 1. If he deviates to delay, his payoff is,

$$\frac{\delta(\lambda - p)}{2} + \frac{\delta\lambda(1 - p)}{2},$$

where the first term happens when the other consumer is bad and chooses delay. In this case no information is revealed, the good consumer has the same prior λ in period 2 and will purchase. The second term happens when the other consumer is good and chooses stay. The true quality type is revealed in period 2, and the deviating good consumer would buy only if he knows the quality is high. To make the good consumer not to deviate, We should have,

$$\lambda - p \geq \frac{\delta(\lambda - p)}{2} + \frac{\delta\lambda(1-p)}{2}.$$

The solution of p is $p \leq \dfrac{(1-\delta)\lambda}{1 - \frac{1+\lambda}{2}\delta} < \lambda$. Hence if we assume,

$$\frac{1-\delta}{1 - \frac{1+\lambda}{2}\delta} > \theta,$$

we have that when $p \in (\lambda\theta, \dfrac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}]$, an equilibrium is $(Stay, Delay)$. Note that we shall focus on this case from now on.

Now we suppose a good consumer chooses a mixed strategy α, which means stay with probability α and delay with probability $1-\alpha$. Give the other consumer will choose equilibrium strategy α when he is good, the expected payoff of a good consumer choosing an arbitrary mixed strategy a is,

$$a(\lambda - p) + \frac{(1-a)\delta(\lambda - p)}{2} + \frac{(1-a)(1-\alpha)\delta(\lambda - p) + (1-a)\alpha\delta\lambda(1-p)}{2}.$$

In an optimization problem, we notice that a is a strictly mixed strategy only if the coefficient of a is zero,

$$(\lambda - p) - \tfrac{1}{2}\delta((\lambda - p) + (1-\alpha)(\lambda - p) + \alpha(\lambda - \lambda p)) = 0.$$

Solve it and we have $\alpha = \dfrac{2(1-\delta)(\lambda-p)}{\delta(1-\lambda)p}$. α needs to be in $[0,1]$. Hence we need to solve,

$$0 \leq \frac{2(1-\delta)(\lambda-p)}{\delta(1-\lambda)p} \leq 1.$$

The left hand inequality is trivially satisfied. For the right hand inequality, we have,

$$p \geq \frac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}.$$

When $p = \dfrac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}$, $\alpha = 1$ which means that the good consumers will stay. This is the case just covered when the good consumer chooses stay. Hence we conclude that when $p \in (\dfrac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}, \lambda]$, an equilibrium is $(\dfrac{2(1-\delta)(\lambda-p)}{\delta(1-\lambda)p}, Delay)$. This result exactly covers the range between the first two results.

(iii) Now we only need to check when $p \leq \lambda\theta$. In this case, it is convenient to consider the general case. Suppose a good consumer's equilibrium strategy is $\alpha \in [0,1]$, and a bad consumer's equilibrium strategy is $\beta \in [0,1]$. Given the other consumer will always follow equilibrium strategy, the expected payoff of a good consumer choosing an arbitrary mixed strategy a is,

$$a(\lambda-p) + \delta(1-a)(\tfrac{\alpha+\beta}{2}(\lambda-\lambda p) + (1-\tfrac{\alpha+\beta}{2})(\lambda-p)).$$

for a bad consumer choosing an arbitrary mixed strategy b, it is,

$$b(\lambda\theta-p) + \delta(1-b)(\tfrac{\alpha+\beta}{2}(\lambda\theta-\lambda p) + (1-\tfrac{\alpha+\beta}{2})(\lambda\theta-p)).$$

Therefore, the consumers' optimal problems are respectively,

$\max_a a((\lambda - p) - \delta(\frac{\alpha+\beta}{2}(\lambda - \lambda p) + (1 - \frac{\alpha+\beta}{2})(\lambda - p)))$ for good consumers and,
$\max_b b((\lambda\theta - p) - \delta(\frac{\alpha+\beta}{2}(\lambda\theta - \lambda p) + (1 - \frac{\alpha+\beta}{2})(\lambda\theta - p)))$ for bad consumers.

For optimization of good consumers, we see that $a = 1$ if $(\lambda - p) - \delta(\frac{\alpha+\beta}{2}(\lambda - \lambda p) + (1 - \frac{\alpha+\beta}{2})(\lambda - p)) > 0$, and $a = 0$ if the inequality reverses. And $a \in [0, 1]$ if the equality holds. Similar observation holds for bad consumers. Furthermore, we notice that $(\lambda - p) - \delta(\frac{\alpha+\beta}{2}(\lambda - \lambda p) + (1 - \frac{\alpha+\beta}{2})(\lambda - p)) > (\lambda\theta - p) - \delta(\frac{\alpha+\beta}{2}(\lambda\theta - \lambda p) + (1 - \frac{\alpha+\beta}{2})(\lambda\theta - p))$, which implies $a \geq b$ and $a > b$ if either inequality holds as equality. Now it is not difficult to conclude that we have the following cases to check,

(a) $(\lambda - p) - \delta(\frac{\alpha+\beta}{2}(\lambda - \lambda p) + (1 - \frac{\alpha+\beta}{2})(\lambda - p)) > (\lambda\theta - p) - \delta(\frac{\alpha+\beta}{2}(\lambda\theta - \lambda p) + (1 - \frac{\alpha+\beta}{2})(\lambda\theta - p)) \geq 0$, denoted by "(+,+)". In this case $\alpha = \beta = 1$ *, which means both types of consumers choose stay.

We only need to check the bad consumers where,

$(\lambda\theta - p) - \delta(\frac{\alpha+\beta}{2}(\lambda\theta - \lambda p) + (1 - \frac{\alpha+\beta}{2})(\lambda\theta - p)) \geq 0.$
$\Rightarrow (\lambda\theta - p) - \delta(1 \cdot (\lambda\theta - \lambda p) + 0 \cdot (\lambda\theta - p)) \geq 0.$

The solution is $p \in (0, \dfrac{(1-\delta)\lambda\theta}{1-\delta\lambda}]$.

(b) $(\lambda - p) - \delta(\frac{\alpha+\beta}{2}(\lambda - \lambda p) + (1 - \frac{\alpha+\beta}{2})(\lambda - p)) \geq 0$, $(\lambda\theta - p) - \delta(\frac{\alpha+\beta}{2}(\lambda\theta - \lambda p) + (1 -$

*It is also possible for $\beta < 1$, which will be covered in following cases where the second inequality strictly holds as equality. Similar cases will be treated in the same way later.

$\frac{\alpha+\beta}{2})(\lambda\theta - p)) \leq 0$, and both inequalities cannot hold as equalities simultaneously. This cases is denoted by "$(+,-)$", and we have $\alpha = 1$ and $\beta = 0$, which means good consumers choose stay and bad consumers choose delay.

For good consumers,

$$(\lambda - p) - \delta(\tfrac{\alpha+\beta}{2}(\lambda - \lambda p) + (1 - \tfrac{\alpha+\beta}{2})(\lambda - p)) \geq 0,$$
$$\Rightarrow (\lambda - p) - \delta(\tfrac{1}{2}(\lambda - \lambda p) + \tfrac{1}{2}(\lambda - p)) \geq 0.$$

The solution is $p \leq \frac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}$.

For bad consumers,

$$(\lambda\theta - p) - \delta(\tfrac{\alpha+\beta}{2}(\lambda\theta - \lambda p) + (1 - \tfrac{\alpha+\beta}{2})(\lambda\theta - p)) \leq 0,$$
$$\Rightarrow (\lambda\theta - p) - \delta(\tfrac{1}{2}(\lambda\theta - \lambda p) + \tfrac{1}{2}(\lambda\theta - p)) \leq 0.$$

The solution is $p \geq \frac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}$. Since $\frac{(1-\delta)}{1-\frac{1+\lambda}{2}\delta} > \theta$ as we have already assumed, the final solution of p is actually $p \in [\frac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}, \lambda\theta]$.

(c) $(\lambda - p) - \delta(\frac{\alpha+\beta}{2}(\lambda - \lambda p) + (1 - \frac{\alpha+\beta}{2})(\lambda - p)) > 0$, $(\lambda\theta - p) - \delta(\frac{\alpha+\beta}{2}(\lambda\theta - \lambda p) + (1 - \frac{\alpha+\beta}{2})(\lambda\theta - p)) = 0$, denoted by "$(+,0)$". In this case $\alpha = 1$ and $\beta \in [0,1]$, which means good consumers choose stay and bad consumers choose a mixed strategy.

We only need to check the bad consumers where,

$$(\lambda\theta - p) - \delta(\tfrac{\alpha+\beta}{2}(\lambda\theta - \lambda p) + (1 - \tfrac{\alpha+\beta}{2})(\lambda\theta - p)) = 0.$$

$\Rightarrow (\lambda\theta - p) - \delta(\frac{1+\beta}{2}(\lambda\theta - \lambda p) + \frac{1-\beta}{2}(\lambda\theta - p)) = 0.$

The solution of β is $\beta = \dfrac{2(1-\delta)\lambda\theta - (2-\delta(1+\lambda))p}{\delta(1-\lambda)p}$. We need to have that $\beta \in [0,1]$. So,

$$\beta \geq 0 \Rightarrow p \leq \frac{(1-\delta)\lambda\theta}{1-\delta\lambda}$$

,

$$\beta \leq 1 \Rightarrow p \geq \frac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}$$

.

The solution is $p \in [\dfrac{(1-\delta)\lambda\theta}{1-\delta\lambda}, \dfrac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}]$.

(d) $(\lambda - p) - \delta(\frac{\alpha+\beta}{2}(\lambda - \lambda p) + (1 - \frac{\alpha+\beta}{2})(\lambda - p)) = 0$, $(\lambda\theta - p) - \delta(\frac{\alpha+\beta}{2}(\lambda\theta - \lambda p) + (1 - \frac{\alpha+\beta}{2})(\lambda\theta - p)) < 0$, denoted by "$(0, -)$". In this case $\alpha \in [0,1]$ and $\beta = 0$, which means good consumers choose a mixed strategy and bad consumers choose delay.

We only need to check the good consumers where,

$$(\lambda - p) - \delta(\frac{\alpha+\beta}{2}(\lambda - \lambda p) + (1 - \frac{\alpha+\beta}{2})(\lambda - p)) = 0,$$

$$\Rightarrow (\lambda - p) - \delta(\frac{\alpha}{2}(\lambda - \lambda p) + (1 - \frac{\alpha}{2})(\lambda - p)) = 0.$$

The solution of α is $\alpha = \dfrac{2(1-\delta)(\lambda - p)}{\delta(1-\lambda)p}$, but we will have that $p > \dfrac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}$ which is a contradiction. Hence, this case fails to exist as an equilibrium here.

(e) $0 \geq (\lambda - p) - \delta(\frac{\alpha+\beta}{2}(\lambda - \lambda p) + (1 - \frac{\alpha+\beta}{2})(\lambda - p)) > (\lambda\theta - p) - \delta(\frac{\alpha+\beta}{2}(\lambda\theta - \lambda p) + (1 - \frac{\alpha+\beta}{2})(\lambda\theta - p))$, denoted by "$(-, -)$". In this case $\alpha = \beta = 0$, which means both good consumers and bad consumers choose delay.

We only need to check the good consumers where,

$$(\lambda - p) - \delta(\tfrac{\alpha+\beta}{2}(\lambda - \lambda p) + (1 - \tfrac{\alpha+\beta}{2})(\lambda - p)) \leq 0,$$
$$\Rightarrow (\lambda - p)(1 - \delta) \leq 0,$$

which is impossible.

Now we have exhausted all possibilities and reach the following conclusion by sorting out the above cases,

When $p \in (\lambda, 1)$, $(Delay, Delay)$;

When $p \in (\dfrac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}, \lambda]$, $(\dfrac{2(1-\delta)(\lambda-p)}{\delta(1-\lambda)p}, Delay)$;

When $p \in (\lambda\theta, \dfrac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}]$, $(Stay, Delay)$;

When $p \in [\dfrac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}, \lambda\theta]$, $(Stay, Delay)$;

When $p \in [\dfrac{(1-\delta)\lambda\theta}{1-\delta\lambda}, \dfrac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}]$, $(Stay, \dfrac{2(1-\delta)\lambda\theta - (2-\delta(1+\lambda))p}{\delta(1-\lambda)p})$;

When $p \in (0, \dfrac{(1-\delta)\lambda\theta}{1-\delta\lambda}]$, $(Stay, Stay)$.

By combining these case, we have the following result,

Proposition 2 *Suppose* $\theta < \dfrac{1-\delta}{1-\frac{1+\lambda}{2}\delta}$. *Then the following strategy profile of consumers make a unique symmetric equilibrium for each price* $p \in (0, 1)$ *chosen by the seller.*

(i) For the good consumers, they choose delay when $p \in (\lambda, 1)$*, choose mixed strategy* $\alpha = \dfrac{2(1-\delta)(\lambda-p)}{\delta(1-\lambda)p}$ *when* $p \in (\dfrac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}, \lambda]$ *and choose stay when* $p \in (0, \dfrac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}]$*;*

(ii) For the bad consumers, they choose delay when $p \in (\dfrac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}, 1)$*, choose mixed strategy* $\beta = \dfrac{2(1-\delta)\lambda\theta-(2-\delta(1+\lambda))p}{\delta(1-\lambda)p}$ *when* $p \in (\dfrac{(1-\delta)\lambda\theta}{1-\delta\lambda}, \dfrac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}]$ *and choose stay when* $p \in (0, \dfrac{(1-\delta)\lambda\theta}{1-\delta\lambda}]$.

The strategy profile is "monotone" in a sense since it is more likely for a consumer to choose delay as the price climbs up. The good consumers play the lead here since they are always more likely to choose stay than the bad consumers, who play the role of followers.

Now we check the symmetric equilibrium of the limiting case, in which both consumers are only of good type. First we only need to consider $p \le \lambda$. Otherwise it is the trivial case that no consumer buys at all.

Now suppose one consumer will choose a mixed strategy $\alpha \in [0,1]$ (prob α stay in period 1, $1-\alpha$ delay into period 2) in a equilibrium. Notice that the consumer will purchase in either period.

Then the other consumer chooses $a \in [0,1]$ to maximize,

$$\pi = a(\lambda-p) + (1-a)\alpha\lambda(1-p) + (1-a)(1-\alpha)(\lambda-p)\delta$$

$$\Rightarrow \pi = a((\lambda-p) - \alpha\lambda(1-p) - (1-\alpha)(\lambda-p)\delta) + \alpha\lambda(1-p) + (1-\alpha)(\lambda-p)\delta.$$

Obviously the optimal choice of a depends on the sign of $(\lambda-p)-\alpha\lambda(1-p)-(1-\alpha)(\lambda-p)\delta$.

If $(\lambda - p) - \alpha\lambda(1-p) - (1-\alpha)(\lambda-p)\delta > 0$, then we should have $a = 1$. Hence, $\alpha = 1$. But we will have $\lambda p > p$, which is impossible.

If $(\lambda - p) - \alpha\lambda(1-p) - (1-\alpha)(\lambda-p)\delta < 0$, then we should have $a = 0$. Hence, $\alpha = 0$. We will have $(\lambda - p)(1-\delta) < 0$ which is also impossible.

If $(\lambda - p) - \alpha\lambda(1-p) - (1-\alpha)(\lambda-p)\delta = 0$, then $\alpha = \dfrac{(\lambda-p)-(\lambda-p)\delta}{\lambda(1-p)-(\lambda-p)\delta}$.

The inconsistency indeed arises from the fact that good consumers and bad consumers are not differently labelled. Another possible limiting case is when the probability of being good consumer goes to 1.

2.4 Seller's Problem

Now we consider the corresponding seller's profit maximization problem. Basically we will check and compare the price choices of the seller in the six different price ranges discussed in the last section. Note that we only need to check the cases when $p \in (0, \lambda]$. The profit is trivially zero when $p > \lambda$ since no consumer will purchase at all at equilibrium.

(A) $p \in [\frac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}, \lambda]$. In this case the bad consumers will choose delay, and the good consumers will choose a mixed strategy $q = \dfrac{2(1-\delta)(\lambda-p)}{\delta(1-\lambda)p}$. We can show that the candidate solutions of the seller are on the boundaries indeed. In another word, he would induce the consumers to pure equilibria.

Lemma 7 *When* $p \in [\frac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}, \lambda]$*, the seller's profit function of* p *is strictly convex on this interval. Hence the local maximum is at either endpoint of the price range.*

Proof 19 *Denote the strategic choice of good consumers as the function of* p*,* $q(p) = \frac{2(1-\delta)(\lambda-p)}{\delta(1-\lambda)p}$*.*

There are four cases of possible consumer profiles with equal probabilities at $\frac{1}{4}$*.*

(i) Both of them are bad consumers. Then they will choose delay and quit. No purchase happens and the profit is zero.

(ii) Both of them are good consumers. Then there are further four subcases,

Both of them choose stay with probability q^2*. The profit is simply* $2p$ *since they will stay and buy in period 1.*

Both of them choose delay with probability $(1-q)^2$*. The profit is* $2p\delta$ *since they will delay and buy in period 2.*

The rest two are symmetric. One of them chooses stay, and another chooses delay. The total probability of these two cases is $2q(1-q)$*. The profit is* $(1+\delta\lambda)p$*. One of them buys in period 1, another one knows the true quality in period and only buys when the quality is high.*

(iii) and (iv) are symmetric. One of them is a good consumer, another one is a bad consumer. Similar to case (ii), when the good consumer chooses stay, the profit is $(1+\delta\lambda)p$ *with probability* q*. When the good consumer chooses delay, the profit is* λp *with probability*

$1 - q$.

Sum up the above cases, the seller's profit function is,

$$\begin{aligned}
\pi(p) &= \frac{1}{4}(0 + q^2 \cdot 2p + (1-q)^2 \cdot 2p\delta + 2q(1-q) \cdot (1+\delta\lambda)p + 2(q(1+\delta\lambda)p + (1-q)\delta p)) \\
&= \frac{1}{4}(4qp + (1-q)(2-q)2\delta p + q(2-q)2\delta\lambda p) \\
&= \frac{1}{4}(4qp + 4\delta p - 6\delta qp + 2\delta q^2 p + 4\delta\lambda qp - 2\delta\lambda q^2 p) \\
&= \frac{1}{4}((4 - 6\delta + 4\delta\lambda)qp + 4\delta p + 2\delta(1-\lambda)q^2 p)
\end{aligned} \tag{2.1}$$

Take second-order derivative of $\pi(p)$ *is,*

$$\pi''(p) = \tfrac{1}{4}((4 - 6\delta + 4\delta\lambda)(2q' + q''p) + 4\delta(1-\lambda)q(2q' + q''p) + 2\delta(1-\lambda)2(q')^2 p).$$

Notice that $q = \dfrac{2(1-\delta)(\lambda - p)}{\delta(1-\lambda)p}$, *which is in form of* $f(p) = \dfrac{a - bp}{cp}$. *From the following fact,*

$$2f'(p) + f''(p)p = 2\frac{a}{cp^2} + (-\frac{2a}{cp^3})p = 0,$$

We have $2q' + q''p = 0$.

Hence $\pi''(p) = \delta(1-\lambda)(q')^2 p > 0$, *and* $\pi(p)$ *is strictly convex. Proved.*

When $p = \lambda$, the profit $\pi_{A_1} = \delta\lambda$. The bad consumers always choose delay. For the good consumers, this is indeed an example of Prisoners' Dilemma. Suppose they choose stay in a symmetric equilibrium and get zero net utility from period 1. Then any good consumer knows that the other consumer is possibly of good type and chooses stay. Then he will deviate to choose delay, in order to take this potential information advantage from the staying consumer.

When $p = \frac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}$, $q = 1$. The profit $\pi_{A_2} = \frac{1}{4}(4+2\delta\lambda)p = \frac{1}{4}\frac{(4+2\delta\lambda)(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}$.

(B) $p \in (\lambda\theta, \frac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}]$. In this case, the good consumers always choose stay, and the bad consumers always choose delay. A bad consumer buys in period 2 only if that the other consumer is good and reveal the information of high quality to him. Since the demand is fixed, the seller will set a price as high as possible. The resulting choice is exactly the same as the second case in (A), and the profit $\pi_B = \pi_{A_2}$. Hence we could also redefine $\pi_A = \pi_{A_1}$ for convenience.

(C) $p \in (\frac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}, \lambda\theta]$. Period 1 strategy choices are the same as in (B). The difference here is that the bad consumers will always buy in period 2 unless information of low quality is revealed, i.e. the bad consumers purchase in period 2 even if no information is available. The seller again will set the price as high as possible at $\lambda\theta$, and the profit $\pi_C = \frac{1}{4}(4+2\delta+2\delta\lambda)\lambda\theta$.

(D) $p \in (\frac{(1-\delta)\lambda\theta}{1-\delta\lambda}, \frac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}]$. In this case the good consumers will choose stay, and the bad consumers will play a mixed strategy $\alpha_B = \frac{(1-\delta)\lambda\theta - (1-\frac{\delta}{2}(1+\lambda))p}{\frac{\delta}{2}(1-\lambda)p}$. We can show that the candidate solutions of the seller are also on the boundaries, similar to (A).

Lemma 8 *When $p \in (\frac{(1-\delta)\lambda\theta}{1-\delta\lambda}, \frac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}]$, the seller's profit function of p is strictly convex on this interval. Hence the local maximum is at either endpoint of the price range.*

Proof 20 *Denote the strategic choice of bad consumers as the function of p,*
$q(p) = \frac{(1-\delta)\lambda\theta - (1-\frac{\delta}{2}(1+\lambda))p}{\frac{\delta}{2}(1-\lambda)p}$.

There are four cases of possible consumer profiles with equal probability at $\frac{1}{4}$*.*

(i) Both of them are good consumers. Then they will choose stay and purchase in period 1. The profit is $2p$*.*

(ii) Both of them are bad consumers. Then there are further four subcases.

Both of them choose stay, with probability q^2*. The profit is simply* $2p$ *since they will stay and buy in period 1.*

Both of them choose delay, with probability $(1-q)^2$*. The profit is* $2p\delta$ *since they will delay and buy in period 2.*

The rest two are symmetric. One of them chooses stay, and another chooses delay. The total probability of these two cases is $2q(1-q)$*. The profit is* $(1+\delta\lambda)p$*. One of them buys in period 1, another one knows the true quality in period and only buys when the quality is high.*

(iii) and (iv) are symmetric. One of them is a good consumer, another one is a bad consumer. Similarly, when the bad consumer chooses stay, the profit is $2p$ *with probability* q*. When the bad consumer chooses delay, the profit is* $(1+\delta\lambda)p$ *with probability* $1-q$*.*

Sum up the above cases, the seller's profit function is,

$$\begin{aligned}\pi(p) &= \frac{1}{4}(2p + q^2 \cdot 2p + (1-q)^2 \cdot 2p\delta + 2q(1-q)\cdot(1+\delta\lambda)p + 2((1-q)(1+\delta\lambda)p + qp)) \\ &= \frac{1}{4}(4(1+q)p + 2(1-q)^2\delta p + 2(1-q^2)\delta\lambda p) \\ &= \frac{1}{4}(4qp + 4\delta p - 6\delta qp + 2\delta q^2 p + 4\delta\lambda qp - 2\delta\lambda q^2 p) \\ &= \frac{1}{4}(4 + (4-4\delta)qp + 2\delta(1+\lambda)p + 2\delta(1-\lambda)q^2 p)\end{aligned} \tag{2.2}$$

Take second-order derivative of $\pi(p)$ *is,*

$\pi''(p) = \frac{1}{4}((4-4\delta)(2q' + q''p) + 4\delta(1-\lambda)q(2q' + q''p) + 4\delta(1-\lambda)(q')^2 p).$

Notice that for $q = \dfrac{(1-\delta)\lambda\theta - (1 - \frac{\delta}{2}(1+\lambda))p}{\frac{\delta}{2}(1-\lambda)p}$ *which is in form of* $f(p) = \dfrac{a - bp}{cp}$*, we also have* $2q' + q''p = 0$ *as shown in Lemma 7.*

Hence $\pi''(p) = \delta(1-\lambda)(q')^2 p > 0$*, and* $\pi(p)$ *is strictly convex. Proved.*

When $p = \dfrac{(1-\delta)\lambda\theta}{1 - \frac{1+\lambda}{2}\delta}$, the profit is just π_C. When $p = \dfrac{(1-\delta)\lambda\theta}{1-\delta\lambda}$, the profit $\pi_D = \dfrac{2(1-\delta)\lambda\theta}{1-\delta\lambda}$ as both consumers will always stay.

(E) $p \in (0, \dfrac{(1-\delta)\lambda\theta}{1-\delta\lambda}]$. Since both consumers will always stay, the seller would set the price as high as possible. The profit is just π_D.

Now we have obtained the potential candidates of the seller's optimal price choice.

(A) $p_A = \lambda$, $\pi_A = \delta\lambda$;

(B) $p_B = \dfrac{(1-\delta)\lambda}{1 - \frac{1+\lambda}{2}\delta}$, $\pi_B = \dfrac{1}{4}\dfrac{(4 + 2\delta\lambda)(1-\delta)\lambda}{1 - \frac{1+\lambda}{2}\delta}$;

(C) $p_C = \lambda\theta$, $\pi_C = \dfrac{1}{4}(4+2\delta+2\delta\lambda)\lambda\theta$;

(D) $p_D = \dfrac{(1-\delta)\lambda\theta}{1-\delta\lambda}$, $\pi_D = \dfrac{2(1-\delta)\lambda\theta}{1-\delta\lambda}$;

To reach a global maximization, an applicable approach is simply first comparing π_A and π_B, then comparing π_C and π_D, finally comparing the winners from both comparisons.

Suppose $\pi_A \geq \pi_B$, we have,

$$\delta\lambda \geq \frac{1}{4}\frac{(4+2\delta\lambda)(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}.$$

After simplification, we get $\delta^2 + (\lambda-4)\delta + 2 \leq 0$. Solve it in δ,

$$\Rightarrow \delta \in [\frac{4-\lambda-\sqrt{\lambda^2-8\lambda+8}}{2}, \frac{4-\lambda+\sqrt{\lambda^2-8\lambda+8}}{2}].$$

Obviously $\dfrac{4-\lambda+\sqrt{\lambda^2-8\lambda+8}}{2} > 1$, so we have,

$\pi_A \geq \pi_B$ if $\delta \in [\dfrac{4-\lambda-\sqrt{\lambda^2-8\lambda+8}}{2}, 1)$; $\pi_A \leq \pi_B$ if $\delta \in (0, \dfrac{4-\lambda-\sqrt{\lambda^2-8\lambda+8}}{2}]$.

Now suppose $\pi_C \geq \pi_D$, we have,

$$\frac{1}{4}(4+2\delta+2\delta\lambda)\lambda\theta \geq \frac{2(1-\delta)\lambda\theta}{1-\delta\lambda}.$$

After simplification, we get $(\lambda+\lambda^2)\delta^2 + (\lambda-5)\delta + 2 \leq 0$. Solve it in δ by taking λ as parameter,

$$\Rightarrow \delta \in [\frac{5-\lambda-\sqrt{25-18\lambda-7\lambda^2}}{2(\lambda^2+\lambda)}, \frac{5-\lambda+\sqrt{25-18\lambda-7\lambda^2}}{2(\lambda^2+\lambda)}].$$

Obviously $\frac{5-\lambda+\sqrt{25-18\lambda-7\lambda^2}}{2(\lambda^2+\lambda)} > 1$, so we have,

$$\pi_C \geq \pi_D \text{ if } \delta \in [\frac{5-\lambda-\sqrt{25-18\lambda-7\lambda^2}}{2(\lambda^2+\lambda)}, 1); \pi_C \leq \pi_D \text{ if } \delta \in (0, \frac{5-\lambda-\sqrt{25-18\lambda-7\lambda^2}}{2(\lambda^2+\lambda)}].$$

These conditions divide the parameter space into three regions. See Figure 2.1 we can see the three regions at this stage.

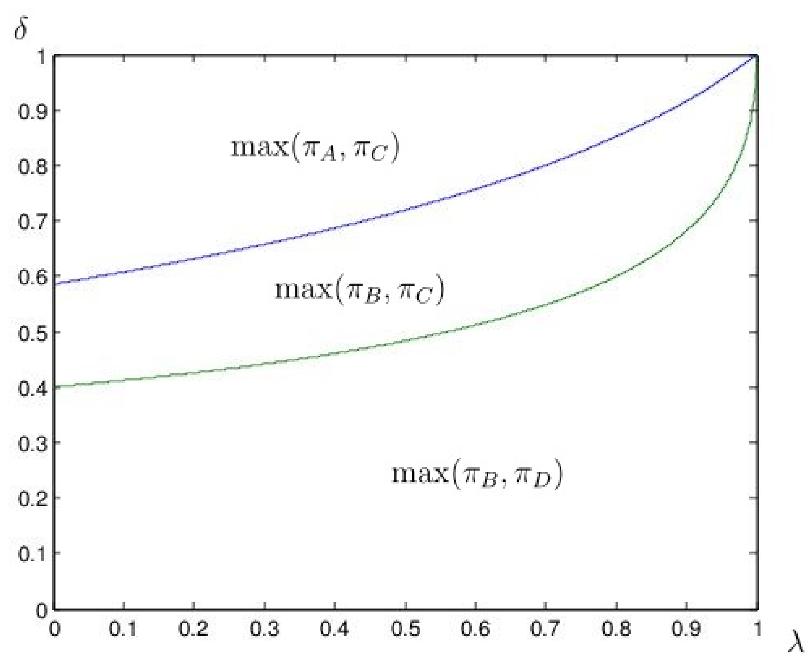

Figure 2.1 : The three regions in first-round comparison

In region $\max(\pi_A, \pi_C)$, If $\pi_A \geq \pi_C$,

$$\delta\lambda \geq \frac{1}{4}(4+2\delta+2\delta\lambda)\lambda\theta,$$

$\Rightarrow \theta \leq \dfrac{2\delta}{2+\delta+\delta\lambda}$.

Recall that we assume $\theta < \dfrac{1-\delta}{1-\frac{1+\lambda}{2}\delta}$. Hence if $\dfrac{1-\delta}{1-\frac{1+\lambda}{2}\delta} \leq \dfrac{2\delta}{2+\delta+\delta\lambda} \Rightarrow (3-\lambda)\delta \geq 2$, we must have $\pi_A \geq \pi_C$.

In conclusion we have that $\pi_A \geq \pi_C$ when $(3-\lambda)\delta \geq 2$, or when $(3-\lambda)\delta \leq 2$ and $\theta \leq \dfrac{2\delta}{2+\delta+\delta\lambda}$; $\pi_C \geq \pi_A$ when $(3-\lambda)\delta \leq 2$ and $\theta \geq \dfrac{2\delta}{2+\delta+\delta\lambda}$.

In region $\max(\pi_B, \pi_C)$, If $\pi_B \geq \pi_C$,

$$\frac{1}{4}\frac{(4+2\delta\lambda)(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta} \geq \frac{1}{4}(4+2\delta+2\delta\lambda)\lambda\theta,$$

$$\Rightarrow \theta \leq \frac{(4+2\delta\lambda)(1-\delta)}{(2+\delta+\delta\lambda)(2-\delta-\delta\lambda)}.$$

Notice that $\dfrac{1-\delta}{1-\frac{1+\lambda}{2}\delta} > \dfrac{(4+2\delta\lambda)(1-\delta)}{(2+\delta+\delta\lambda)(2-\delta-\delta\lambda)}$. Hence we have that $\pi_B \geq \pi_C$ when $\theta \leq \dfrac{(4+2\delta\lambda)(1-\delta)}{(2+\delta+\delta\lambda)(2-\delta-\delta\lambda)}$; $\pi_C \geq \pi_B$ when $\theta \geq \dfrac{(4+2\delta\lambda)(1-\delta)}{(2+\delta+\delta\lambda)(2-\delta-\delta\lambda)}$.

In region $\max(\pi_B, \pi_D)$, If $\pi_B \geq \pi_D$,

$$\frac{1}{4}\frac{(4+2\delta\lambda)(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta} \geq \frac{2(1-\delta)\lambda\theta}{1-\delta\lambda},$$

$$\Rightarrow \theta \leq \frac{(2+\delta\lambda)(1-\delta\lambda)}{2(2-\delta-\delta\lambda)}.$$

When $\dfrac{1-\delta}{1-\frac{1+\lambda}{2}\delta} \leq \dfrac{(2+\delta\lambda)(1-\delta\lambda)}{2(2-\delta-\delta\lambda)}$, We must have $\pi_B \geq \pi_D$. The solution is $\delta \in [\dfrac{4-\lambda-\sqrt{(4-\lambda)^2-8\lambda^2}}{2\lambda^2}, 1)$.

However, it is verified that $\frac{4-\lambda-\sqrt{(4-\lambda)^2-8\lambda^2}}{2\lambda^2} > \frac{5-\lambda-\sqrt{25-18\lambda-7\lambda^2}}{2(\lambda^2+\lambda)}$, which means such scene never happens in this region. Hence we only have that $\pi_B \geq \pi_D$ when $\theta \leq \frac{(2+\delta\lambda)(1-\delta\lambda)}{2(2-\delta-\delta\lambda)}$; $\pi_B \leq \pi_D$ when $\theta \geq \frac{(2+\delta\lambda)(1-\delta\lambda)}{2(2-\delta-\delta\lambda)}$.

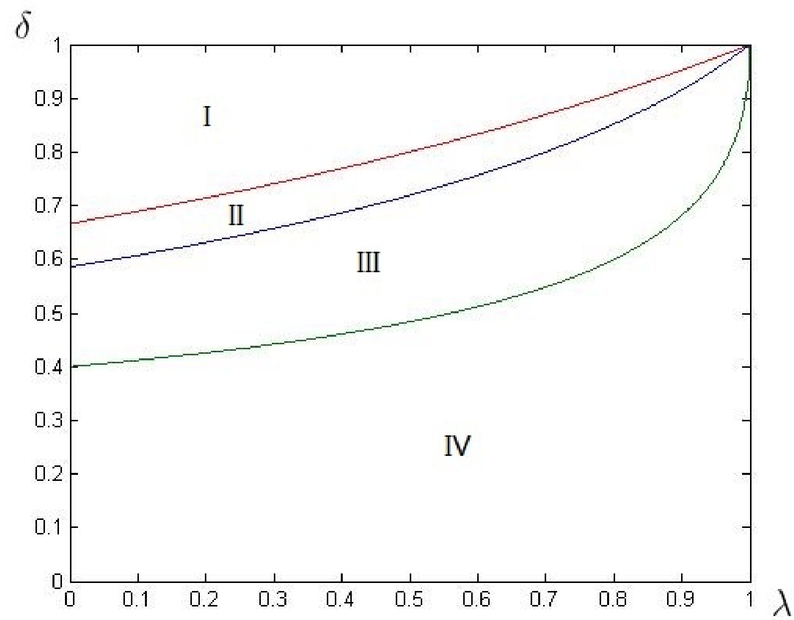

Figure 2.2 : The four regions in low θ case

See Figure 2.2 for the resulting 4 specific regions:

Region I: (A) is always chosen. It means that the seller would sell at price equal to expected quality, also expected utility for the good consumers. Only the good consumers are going to buy it in period 2.

Region II: (A) is chosen if θ is relatively small, (C) is chosen if θ is relatively large. The latter case means that the seller would sell at price equal to expected utility for the bad consumers, hence they will buy as long as they do not hear bad news.

Region III: (B) is chosen if θ is relatively small, (C) is chosen if θ is relatively large. The former case means The seller would sell at price which is just enough for the good consumers to choose to stay, but the bad consumers do not want to buy unless they hear good news.

Region IV: (B) is chosen if θ is relatively small, (D) is chosen if θ is relatively large. The latter case means the the seller would sell at price low enough to make consumers of both types stay in period 1.

2.5 Further Analysis

Now we carry on to investigate the $\theta > \dfrac{1-\delta}{1-\frac{1+\lambda}{2}\delta}$ case. Again we want to find all the symmetric equilibria of the consumers facing different prices. We briefly go over since the analysis is rather similar to the $\theta < \dfrac{1-\delta}{1-\frac{1+\lambda}{2}\delta}$ case.

(i) If $p > \lambda$, similarly the consumers will not stay in period 1 no matter they are of good type or bad type, since the price is higher than the expected utility in period 1, and $(Delay, Delay)$ is the only equilibrium.

(ii) If $\lambda \geq p > \lambda\theta$, the bad consumers will not stay in period 1 for the same reason. Similarly, we suppose the equilibrium strategy of the good consumers is α, then the expected payoff of a good consumer choosing an arbitrary mixed strategy a is still,

$$a(\lambda - p) + \frac{(1-a)\delta(\lambda - p)}{2} + \frac{(1-a)(1-\alpha)\delta(\lambda-p) + (1-a)\alpha\delta\lambda(1-p)}{2}.$$

The solution is $\alpha = \dfrac{2(1-\delta)(\lambda-p)}{\delta(1-\lambda)p}$ and we shall also have $p \geq \dfrac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}$. By assumption it is automatically satisfied since $p > \lambda\theta > \frac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}$. Hence we conclude that when $p \in (\lambda\theta, \lambda]$, an equilibrium is $(\frac{2(1-\delta)(\lambda-p)}{\delta(1-\lambda)p}, Delay)$.

(iii) If $p \leq \lambda\theta$, we still check the cases as in last section.

(a)"$(+,+)''$. In this case, We still have that $p \in (0, \frac{(1-\delta)\lambda\theta}{1-\delta\lambda}]$.

(b) "$(+,-)''$. In this case the solution is $p \leq \frac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}$ for the good consumers, and $p \geq \frac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}$ for the bad consumers. By assumption, we conclude that when $p \in [\frac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}, \frac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}]$, an equilibrium is $(Stay, Delay)$.

(c) "$(+,0)''$. In this case we have that $p \leq \frac{(1-\delta)\lambda\theta}{1-\delta\lambda}$ and $geq\frac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}$, We conclude that when $p \in [\frac{(1-\delta)\lambda\theta}{1-\delta\lambda}, \frac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}]$, an equilibrium is $(Stay, \frac{2(1-\delta)\lambda\theta-(2-\delta(1+\lambda))p}{\delta(1-\lambda)p})$.

(d) "$(0,-)''$. In this case we have that $p > \frac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}$. By assumption, we conclude that when $p \in (\frac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}, \lambda\theta]$, an equilibrium is $(\frac{2(1-\delta)(\lambda-p)}{\delta(1-\lambda)p}, Delay)$.

(e) "$(-,-)''$. Similarly we find this case impossible.

When $p \in (\lambda, 1)$, $(Delay, Delay)$;

When $p \in (\lambda\theta, \lambda]$, $(\frac{2(1-\delta)(\lambda-p)}{\delta(1-\lambda)p}, Delay)$;

When $p \in (\frac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}, \lambda\theta]$, $(\frac{2(1-\delta)(\lambda-p)}{\delta(1-\lambda)p}, Delay)$;

When $p \in [\frac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}, \frac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}]$, $(Stay, Delay)$;

When $p \in [\frac{(1-\delta)\lambda\theta}{1-\delta\lambda}, \frac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}]$, $(Stay, \frac{2(1-\delta)\lambda\theta-(2-\delta(1+\lambda))p}{\delta(1-\lambda)p})$;

When $p \in (0, \frac{(1-\delta)\lambda\theta}{1-\delta\lambda}]$, $(Stay, Stay)$.

We see that the strategy profiles is indeed the same to the previous case, and have the following result,

Proposition 3 *The following strategy profile of consumers make a unique symmetric equilibrium for each price $p \in (0,1)$ chosen by the seller.*

(i) For the good consumers, they choose delay when $p \in (\lambda, 1)$, choose mixed strategy $\alpha = \frac{2(1-\delta)(\lambda-p)}{\delta(1-\lambda)p}$ when $p \in (\frac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}, \lambda]$ and choose stay when $p \in (0, \frac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}]$;

(ii) For the bad consumers, they choose delay when $p \in (\frac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}, 1)$, choose mixed strategy $\beta = \frac{2(1-\delta)\lambda\theta-(2-\delta(1+\lambda))p}{\delta(1-\lambda)p}$ when $p \in (\frac{(1-\delta)\lambda\theta}{1-\delta\lambda}, \frac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}]$ and choose stay when $p \in (0, \frac{(1-\delta)\lambda\theta}{1-\delta\lambda}]$.

Notice that the difference here is that the bad consumers would buy more even choosing delay in the high θ case.

Now we consider the corresponding seller's profit maximization problem. Note that we only need to check the cases when $p \in (0, \lambda]$. The profit is trivially zero when $p > \lambda$ since no consumer will purchase at all at equilibrium.

(A) $p \in (\lambda\theta, \lambda]$. In this case the bad consumers will choose delay, and the good consumers will choose a mixed strategy $q = \dfrac{2(1-\delta)(\lambda - p)}{\delta(1-\lambda)p}$. By Lemma 7, we know that the candidate solutions of the seller are on the boundaries indeed.

When $p = \lambda$, the profit $\pi_A = \delta\lambda$. When $p = \lambda\theta$, we refer to (B).

(B) $p \in (\dfrac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}, \lambda\theta]$. In this case the bad consumers will still choose delay, and the good consumers will still choose a mixed strategy $q = \dfrac{2(1-\delta)(\lambda - p)}{\delta(1-\lambda)p}$. The difference from (A) is that the bad consumers will always buy as long as not hearing bad news. In another word, the seller is making extra profit.

Similar to Lemma 7 and 8, we can show that the candidate solutions of the seller are still on the boundaries.

When $p = \lambda\theta$, the profit $\pi_B = \dfrac{1}{4}(4qp + 2(2-q)^2\delta p + 2q(2-q)\delta\lambda p)$, where $q = \dfrac{2(1-\delta)(1-\theta)}{\delta(1-\lambda)\theta}$.

When $p = \dfrac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}$, we refer to (C).

(C) $p \in [\dfrac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}, \dfrac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}]$. In this case, the good consumers always choose stay, and the bad consumers always choose delay. The seller will set a price as high as possible at $p = \dfrac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}$ with $\pi_C = \dfrac{1}{4}(4 + 2\delta + 2\delta\lambda)\dfrac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}$.

(D) $p \in [\dfrac{(1-\delta)\lambda\theta}{1-\delta\lambda}, \dfrac{(1-\delta)\lambda\theta}{1-\frac{1+\lambda}{2}\delta}]$. Lemma 8 applies and we have $p_D = \dfrac{(1-\delta)\lambda\theta}{1-\delta\lambda}$, $\pi_D = \dfrac{2(1-\delta)\lambda\theta}{1-\delta\lambda}$, which is also the optimal choice for $p \in (0, \dfrac{(1-\delta)\lambda\theta}{1-\delta\lambda}]$.

Now we have obtained the potential candidates of the sellers optimal price choice.

(A) $p_A = \lambda$, $\pi_A = \delta\lambda$;

(B) $p_B = \lambda\theta$, $\pi_B = \frac{1}{4}(4qp + 2(2-q)^2\delta p + 2q(2-q)\delta\lambda p)$;

(C) $p_C = \frac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}$, $\pi_C = \frac{1}{4}(4 + 2\delta + 2\delta\lambda)\frac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}$;

(D) $p_D = \frac{(1-\delta)\lambda\theta}{1-\delta\lambda}$, $\pi_D = \frac{2(1-\delta)\lambda\theta}{1-\delta\lambda}$;

To solve the global optimization, we first compare π_A and π_C.

Suppose $\pi_A \geq \pi_C$, we have,

$$\delta\lambda \geq \frac{1}{4}(4 + 2\delta + 2\delta\lambda)\frac{(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta}.$$

After simplification, we get $\delta \geq \frac{2}{3-\lambda}$.

Next we compare π_B and π_D, which is a bit complicated. First we calculate the profit π_B,

$$\pi_B = 2(\delta + \frac{(1-\delta)(1-\theta)}{\delta(1-\lambda)\theta}(1 - 2\delta + \delta\lambda) + \frac{(1-\delta)^2(1-\theta)^2}{\delta(1-\lambda)\theta^2})\lambda\theta.$$

Suppose $\pi_B \geq \pi_D$, we have,

$$2(\delta + \frac{(1-\delta)(1-\theta)}{\delta(1-\lambda)\theta}(1 - 2\delta + \delta\lambda) + \frac{(1-\delta)^2(1-\theta)^2}{\delta(1-\lambda)\theta^2})\lambda\theta \geq \frac{2(1-\delta)\lambda\theta}{1-\delta\lambda}.$$

After simplification, we get,

$$\delta^2(1-\lambda)^2\theta^2 - (1-\delta)(1-\delta\lambda)^2\theta + (1-\delta)^2(1-\delta\lambda) \geq 0.$$

Solve it we have,

$$\theta \geq \frac{(1-\delta)((1-\delta\lambda)^2 + \sqrt{(1-\delta\lambda)^4 - 4(1-\delta\lambda)(1-\lambda)^2\delta^2})}{2\delta^2(1-\lambda)^2},$$

$$\text{or } \theta \leq \frac{(1-\delta)((1-\delta\lambda)^2 - \sqrt{(1-\delta\lambda)^4 - 4(1-\delta\lambda)(1-\lambda)^2\delta^2})}{2\delta^2(1-\lambda)^2}.$$

Notice that (i) $\theta > \frac{1-\delta}{1-\frac{1+\lambda}{2}\delta}$ should be satisfied; (ii) If the condition $(1-\delta\lambda)^3 < 4(1-\lambda)^2\delta^2$ is satisfied, then $\pi_B > \pi_D$ always holds. So we only need to check the case when $(1-\delta\lambda)^3 \geq 4(1-\lambda)^2\delta^2$.

For the first inequality we show that $\frac{(1-\delta)(1-\delta\lambda)^2}{2\delta^2(1-\lambda)^2} > 1$. By (ii) above we have that $\frac{(1-\delta\lambda)^3}{4(1-\lambda)^2\delta^2} \geq 1$. Then $\frac{\frac{(1-\delta)(1-\delta\lambda)^2}{2\delta^2(1-\lambda)^2}}{\frac{(1-\delta\lambda)^3}{4(1-\lambda)^2\delta^2}} = \frac{2-2\delta}{1-\delta\lambda}$. From (ii) above we can also derive that $\delta \leq \frac{1}{2} \Rightarrow \frac{2-2\delta}{1-\delta\lambda} > 1$. Hence we have that $\frac{(1-\delta)(1-\delta\lambda)^2}{2\delta^2(1-\lambda)^2} > 1$, which implies the first inequality will never be satisfied, and we only need to consider the second inequality. From the second inequality we can see: if $\frac{(1-\delta)((1-\delta\lambda)^2 - \sqrt{(1-\delta\lambda)^4 - 4(1-\delta\lambda)(1-\lambda)^2\delta^2})}{2\delta^2(1-\lambda)^2} < \frac{1-\delta}{1-\frac{1+\lambda}{2}\delta}$, then we must have $\pi_B < \pi_D$; otherwise it depends on the value of θ. We solve,

$$\frac{(1-\delta)((1-\delta\lambda)^2 - \sqrt{(1-\delta\lambda)^4 - 4(1-\delta\lambda)(1-\lambda)^2\delta^2})}{2\delta^2(1-\lambda)^2} \geq \frac{1-\delta}{1-\frac{1+\lambda}{2}\delta},$$

which is simplified to,

$$4(1-\lambda)^3(\delta^2\lambda^2 + \delta^2\lambda + \delta\lambda - 5\delta + 2) \leq 0.$$

The solution is $\delta \in [\frac{5-\lambda-\sqrt{25-18\lambda-7\lambda^2}}{2(\lambda^2+\lambda)}, \gamma(\lambda))$, where $\gamma(\lambda)$ is the solution of $(1-\delta\lambda)^3 = 4(1-\lambda)^2\delta^2$, which could be verified to be always greater than $\frac{5-\lambda-\sqrt{25-18\lambda-7\lambda^2}}{2(\lambda^2+\lambda)}$.

In conclusion we have that $\pi_B > \pi_D$ when $(1-\delta\lambda)^3 < 4(1-\lambda)^2\delta^2$; $\pi_B < \pi_D$ when $\delta \in (0, \frac{5-\lambda-\sqrt{25-18\lambda-7\lambda^2}}{2(\lambda^2+\lambda)})$. When $\delta \in [\frac{5-\lambda-\sqrt{25-18\lambda-7\lambda^2}}{2(\lambda^2+\lambda)}, \gamma(\lambda))$, $\pi_B \geq \pi_D$ if
$\theta \leq \frac{(1-\delta)((1-\delta\lambda)^2-\sqrt{(1-\delta\lambda)^4-4(1-\delta\lambda)(1-\lambda)^2\delta^2})}{2\delta^2(1-\lambda)^2}$;
$\pi_B \leq \pi_D$ if $\theta \geq \frac{(1-\delta)((1-\delta\lambda)^2-\sqrt{(1-\delta\lambda)^4-4(1-\delta\lambda)(1-\lambda)^2\delta^2})}{2\delta^2(1-\lambda)^2}$.

Easy to see that when $\pi_A \geq \pi_C$, the other candidate solution is always π_B.

Compare them we find that,
suppose $\pi_A \geq \pi_C \Rightarrow \delta \in [\frac{(1-\theta)(2+\theta-\theta\lambda)-\sqrt{(1-\theta)\theta(1-\lambda)(\theta^2\lambda-\theta^2-\theta\lambda-3\theta+2)}}{2-\theta-\theta\lambda}$,
$\frac{(1-\theta)(2+\theta-\theta\lambda)+\sqrt{(1-\theta)\theta(1-\lambda)(\theta^2\lambda-\theta^2-\theta\lambda-3\theta+2)}}{2-\theta-\theta\lambda}]$.
It could be show that this range decrease when either θ or λ increases.

Now we compare π_B and π_C, suppose $\pi_B \geq \pi_C$,

$$\theta\delta^2(2\theta+\delta-1)\lambda^2+(-4\theta^2\delta+2\theta\delta^3-10\theta\delta^2+8\theta\delta-2\delta^3+4\delta^2-2\delta)\lambda+(-2\theta^2\delta^2+4\theta^2\delta+\theta\delta^3-\theta\delta^2+4\theta\delta-4\theta-2\delta^3+8\delta^2-10\delta+4) \geq 0.$$

Notice that we assume $\delta \leq \frac{2}{3-\lambda}$ here since we want that $\pi_C \geq \pi_A$. Combined with the assumption $\theta > \frac{1-\delta}{1-\frac{1+\lambda}{2}\delta}$, we get $\theta > \frac{1}{2}$. Hence $2\theta+\delta-1 > 0$ and we can directly solve this inequality,

$$\lambda \geq \frac{2\theta+2\delta-\delta\theta-2}{\delta\theta}, \text{ or } \lambda \leq \frac{2\delta\theta+3\delta-\delta^2-2}{\delta(2\theta+\delta-1)}.$$

Notice the first inequality is indeed $\theta \leq \dfrac{1-\delta}{1-\frac{1+\lambda}{2}\delta}$, which is impossible. So we only need to have the second inequality satisfied.

Now we compare π_C and π_D, suppose $\pi_D \geq \pi_C$,

$$\theta \geq \frac{(1-\delta\lambda)(2+\delta+\delta\lambda)}{2(2-\delta-\delta\lambda)}.$$

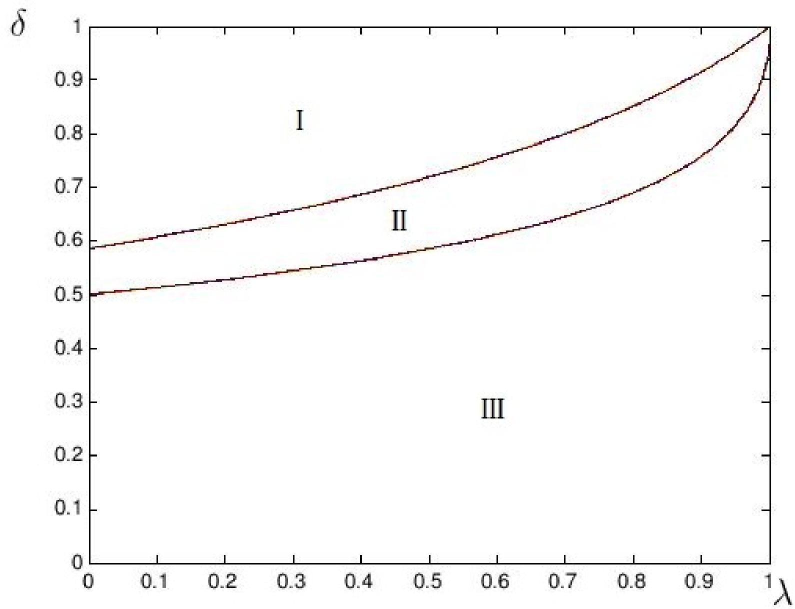

Figure 2.3 : The three regions for the whole parameter range

Now we have fully characterize the final equilibrium outcome for the whole parameter range. See Figure 2.3. Roughly speaking When θ is small, the seller would set the price to induce consumers of both types to choose delay in Region I, since he is patient enough and the value of the bad consumers is ignorable. For similar reason, he would lower the price to induce the good consumers to buy in period 1 in Region II and III, but not low enough to induce the bad consumers to buy even if there is no news at all. As θ increases, the market value of the bad consumers weighs more for the seller, and he will lower the price to induce the bad consumers to buy even there is no news in Region I and II. But when he is not patient enough in Region III, he will make both consumers induce consumers of both types to choose stay in period 1. Only in the belt-shaped Region II, the seller would always induce the consumers into social learning no matter how much θ is.

It is also interesting to examine whether the social learning process benefits the seller or not. Suppose the consumer in period 1 could not convey information to period 2, then obviously a consumer either buys in period 1 or never buy. The profit of the seller under this information nondisclosure situation is then $\pi = \max(\lambda, 2\lambda\theta)$. If $\theta < \dfrac{1-\delta}{1-\frac{1+\lambda}{2}\delta}$,

$$\pi_A = \delta\lambda < \lambda \leq \pi,$$

$$\pi_C = \frac{1}{4}(4 + 2\delta + 2\delta\lambda)\lambda\theta < 2\lambda\theta \leq \pi,$$

$$\pi_D = \frac{2(1-\delta)\lambda\theta}{1-\delta\lambda} < 2\lambda\theta \leq \pi.$$

So we only need to check π_B. We check,

$$\frac{1}{4}\frac{(4+2\delta\lambda)(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta} \geq \lambda,$$

$$\Rightarrow \delta \leq 2 - \frac{1}{\lambda}.$$

$$\frac{1}{4}\frac{(4+2\delta\lambda)(1-\delta)\lambda}{1-\frac{1+\lambda}{2}\delta} \geq 2\lambda\theta.$$

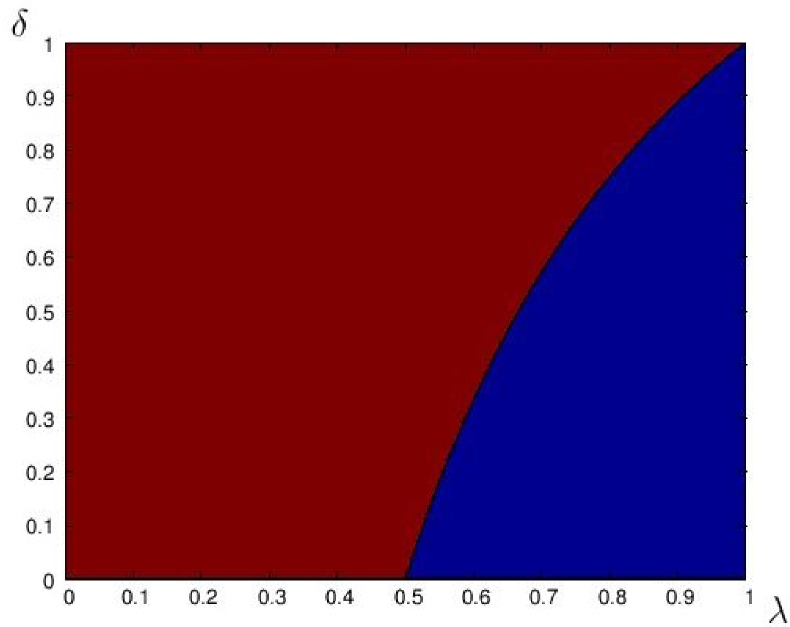

Figure 2.4 : When $\theta < \frac{1}{2}$, the seller prefers information disclosure in blue region, otherwise in red region.

Similar results also hold for high θ case. Information disclosure only benefits the seller when he chooses the price just to let the good consumers stay in period 1. Combined with results in section 4 and 5, we can find that when $\theta < \frac{1}{2}$, the seller prefers information disclosure to information nondisclosure if and only if $\delta \leq 2 - \frac{1}{\lambda}$(See Figure 2.4). When $\theta \geq \frac{1}{2}$, the seller becomes less likely to prefer information disclosure as θ. When θ is high enough, the seller would always prefer information nondisclosure(See Figure 2.5).

From these results we can find some intuitive conclusions. When information disclosure exists, the seller has to post a low price to make consumers of both types buy in period 1 because of the informational free-ride effect. If he could make more profit by making the bad consumers choosing delay, it shall be the case that the bad consumers who previously will not buy engage in purchase when they hear good news, but not that the bad consumers

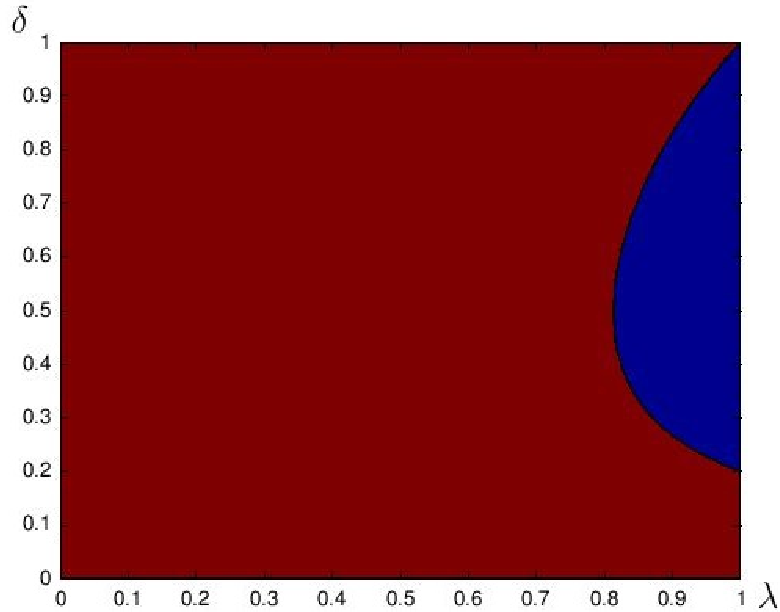

Figure 2.5 : When $\theta \geq \frac{1}{2}$, the area of blue region decreases as θ increases.

who previously will buy avoid purchase when they hear bad news. When λ is high, the seller will benefit more from possible truth revelation. However, the sales to the bad consumers in period 2 is still some kind of "side-product" to the seller. When the θ increases, the seller will become more and more interested to post a relative low price to induce the bad consumers. Information disclosure then becomes negative to the seller since it provides opportunities for the bad consumers to escape the seller's plan.

2.6 Conclusion

In this paper we study a monopolistic model with social learning. We show that there always exists a unique symmetric equilibrium of consumers for each price. The whole picture of monopolist's optimal price choice depending on the parameters is fully characterized. Based on this we could further study the seller's policy and profit function depending on different

λ and δ.

We also discuss the impact of information disclosure on the monopolistic profit. In future work, we plan to consider more general distributions of quality and consumer types. We are also going to investigate cases in which the consumer communication is imperfect.

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