VI

THEOREMS OF LIOUVILLE AND PICARD TYPE
IN A CURVILINEAR STRIP

The considerations of the preceding chapter allow an extensive study of the behavior of a function represented by a Dirichlet series, or even, more generally, of a function represented asymptotically with respect to an asymptotic sequence in a strip, or in a channel of which the width depends on the upper density of the exponents. Theorems XV and XVI will play in our study the same rôle as is played in the study of general analytic functions by the well known estimate of Cauchy:

\[ |a_n| < \frac{M}{R^n}, \]

where \( f(z) = \sum a_n z^n \), \( |f(z)| \leq M \), (where \( R \) is smaller than the radius of convergence of the Taylor series). The results obtained by these theorems are, in nature, much more general than the classical theorems, even if only Taylor series are involved, especially when "gap series" are studied. One of the simplest results is that which generalizes the famous theorem of Liouville.

Let us first give the following definition.

Suppose that \( s(u) = \sigma(u) + it(u) \) is a continuous complex function of the real variable \( u ( - \infty < u < \infty ) \). Let us suppose that for \( u > u_0 : t(u) = t_0 \) (constant), and that \( \sigma(u) \to - \infty \) as \( u \to - \infty \); let also \( R \) be a positive quantity. The region

\[ \sum = \sum (s(u), R) = \bigcup_{- \infty < u < \infty} C(s(u), R) \]

(the union of the open circles of radius \( R \) and center \( s(u) \))
as \( u \) takes all real values) shall be called a *curvilinear strip of width* 2\( R \), horizontal at the right and extending to \( -\infty \) at the left. The curve composed of the points of which the affixes are given by \( s(u) \) shall be called the *central line of the strip*. By definition each strip contains a horizontal half-strip.

**Theorem XIX.** (a) If \( F(s) \) is a holomorphic and bounded function in a curvilinear strip \( \Sigma \) of width 2\( \pi a \) horizontal at the right and extending to \(-\infty\) at the left, (b) if in the horizontal half-strip, contained in \( \Sigma \), \( F(s) \) is represented by \( \sum a_ne^{-\lambda_n s} \) asymptotically with respect to an asymptotic sequence \( \{\varphi_n(x)\} \), (c) if, on denoting by \( D \) the upper density of \( \{\lambda_n\} \), \( D < a \), (d) and if, on denoting by \( \varphi(x) \) the envelope of \( \{\varphi_n(x)\} \), by \( \Lambda^*(u) \) the growth function of the excess distribution of \( \{\lambda_n\} \), and by \( \omega \) the quantity \( \frac{1}{2(a-D)} \), one of the following conditions holds:

(I) there exists a positive constant \( \rho \) such that

\[
\Lambda^*(\rho u^\omega) \varphi(u) = O(1)
\]

\[
\liminf_{t \to 0^+} \int_t^1 \log(\Lambda^*(\rho u^\omega) \varphi(u)) u^{\omega-1} du = -\infty,
\]

(II) there exists a constant \( \omega' > \omega \) such that

\[
\int_0^1 \log \varphi(u) u^{\omega'-1} du = -\infty,
\]

then \( a_n = 0 \) \((n \geq 1)\).

It follows indeed from Theorem XV that the inequality (54) is satisfied for every \( n \geq 1 \) with \( \sigma_1 \) negative numerically arbitrarily large. Thus, on fixing \( n \) in (54) and on making \( \sigma_1 \) tend to \(-\infty\) we see that \( a_n = 0 \) \((n \geq 1)\).

From this theorem, by means of Lemma VI, or directly from Theorem XVI (in the same manner as Theorem XIX follows from Theorem XV) follows the next theorem which concerns converging Dirichlet series.
Theorems of Picard Type

Theorem XX. If $F(s)$ is holomorphic and bounded in a curvilinear strip of width $2\pi a$ horizontal at the right and extending to $-\infty$ at the left, and if this function is represented, for $\sigma$ sufficiently large, by the series $\sum a_n e^{-\lambda_n s}$, the upper density of $\{\lambda_n\}$ satisfying the inequality $D < a$, then $F(s)$ is identically zero.

Indeed $a_n = 0 (n \geq 1)$, and, since $F(s) = \sum a_n e^{-\lambda_n s}$, we see that $F(s) \equiv 0$.

It should be remarked that from the hypotheses of Theorem XIX it also follows that $F(s) \equiv 0 [11]$.

Each theorem which follows, and which is stated for converging Dirichlet series, is also true for series represented asymptotically with respect to an asymptotic sequence. We shall, however, not give those longer statements, since the reader will be able to form such statements, and to prove them by starting from Theorems XV or XIX in the same way as we do in starting from theorems on Dirichlet series and on basing ourselves on Theorems XVI or XX. We did, however, state Theorem XIX because it plays a very important rôle in the theory of quasi-analyticity (see [11]), and it could not be replaced, for that purpose, by the weaker theorems bearing on Dirichlet series. Theorem XIX can be considered as a far-going generalization of a theorem which gives a sufficient condition for the solution of a classical problem of Watson (see [11, 13]).

We shall suppose all through the remaining part of this course of lectures that the envisaged Dirichlet series have an abscissa of convergence $\sigma_0$ not equal to $\infty$; $\sigma_0 < \infty$. We shall also denote systematically by $D$ the upper density of $\{\lambda_n\}$.

An expression of the form $\sum_{n=1}^{m} a_n e^{-\lambda_n s} (0 < \lambda_1 < \lambda_2 \cdots \lambda_m)$ shall be called a Dirichlet polynomial of degree $\lambda_m$.

Theorem XX can be generalized in the following manner:
THEOREM XXI. If $F(s)$ is holomorphic in a curvilinear strip $\Sigma$ of width $2\pi a$, horizontal at the right and extending to $-\infty$ at the left, if, for $\sigma$ sufficiently large, $F(s) = \sum a_n e^{-\lambda_n s}$ with $D < a$, and if there exists a constant $c$ such that $\lambda_m \leq c < \lambda_m + 1$ and such that in $\Sigma$, $e^{\sigma |F(s)|} \leq M < \infty$, then $F(s)$ is a Dirichlet polynomial of degree $\lambda_m$.

It follows from the definition of $c$ that $(\sum_{n=1}^{m} a_n e^{-\lambda_n s}) e^{cs}$ is bounded in each half-plane $\sigma \leq \sigma'$ ($\sigma'$ arbitrary), and from the hypotheses of the theorem it follows then that the function

$$F_1(s) = (F(s) - \sum_{n=1}^{m} a_n e^{-\lambda_n s}) e^{cs}$$

is holomorphic and bounded in $\Sigma$, since, as is readily seen, for $\sigma > \sigma_0$, $(\sum_{m+1}^{\infty} a_n e^{-\lambda_n s}) e^{cs}$ is also bounded. On the other hand, for $\sigma > \sigma_0$:

$$F_1(s) = \sum_{m+1}^{\infty} a_n e^{-(\lambda_n - c)s} = \sum_{1}^{\infty} d_n e^{-\mu_n s}$$

with $\mu_n = \lambda_{m+n} - c > 0$ ($n \geq 1$). The function $F_1(s)$ satisfies the conditions of Theorem XX (in which $F(s)$ is replaced by $F_1(s)$) and therefore $F_1(s) \equiv 0$, which proves that

$$F(s) = \sum_{n=1}^{m} a_n e^{-\lambda_n s}.$$

THEOREM XXII. If $F(s)$ is not a Dirichlet polynomial but is holomorphic in the part of the strip $\Sigma = \Sigma(s(u), \pi a)$ horizontal at the right and extending to $-\infty$ at the left, for which $\sigma \geq \sigma_0$, and if, in this part of the strip, $F(s) = \sum a_n e^{-\lambda_n s}$, with $D < a$, then the analytic continuation of $F(s)$ through $\Sigma$ satisfies one of the three conditions:

a) $F(s)$ admits in $\Sigma$ at least one singularity.

b) For each positive constant $c$, $e^{\sigma |F(s)|}$ tends to $\infty$ as $\sigma$ tends to $-\infty$, uniformly with respect to $t$, $s = \sigma + it$ belonging to the strip $\Sigma(s(u), \pi(a - \epsilon))$, where $\epsilon > 0$ is arbitrary.
c) \( F(s) \) takes in \( \Sigma \) each value, except at most one, infinitely many times.

The proof of this theorem depends on certain facts of the theory of normal families of functions.

**Lemma IX.** Let \( f(z, u) \) represent a holomorphic function of \( z \) in the circle \( |z| < 1 \) for each value of \( u \) varying in an interval \([a, \infty)\). If the family composed of all the functions \( f(z, u) \) is normal in \( |z| < 1 \), and if, on denoting by \( M(u, \alpha) \) the maximum of \( |f(z, u)| \) in \( |z| \leq 1 - \alpha \), where \( \alpha \) is a fixed quantity \((0 < \alpha < 1)\), we have

\[
\lim_{u \to \infty} M(u, \alpha) = \infty,
\]

then

\[
\lim_{u \to \infty} f(z, u) = \infty,
\]

uniformly in every circle \( |z| \leq 1 - \delta \) \((0 < \delta < 1)\).

Suppose that the statement is not true. There exists then a quantity \( 0 < \delta_1 < 1 \) such that for a suitable sequence \( \{u_i\} \) with \( u_i \to \infty \), a sequence \( \{z_i\} \) with \( |z_i| \leq 1 - \delta_1 \), and a constant \( M < \infty \) the inequality \(|f(z_i, u_i)| < M < \infty \) holds. But, since the family composed of the functions \( f(z, u) \) is normal in \( |z| < 1 \), it is possible to extract from the sequence \( \{f(z_i, u_i)\} \) a subsequence \( \{f(z_i, u_{i_n})\} \) which tends, in each circle \( |z| < 1 - \eta \) \((0 < \eta < 1)\), uniformly either to a holomorphic function or to infinity. If we choose \( \eta \leq \min (\alpha, \delta_1) \) we see a contradiction, since by (88) this subsequence cannot tend to a holomorphic function in \( |z| < 1 - \eta \), and from

\[
|f(z_{i_n}, u_i)| < M < \infty
\]

it follows that the limit cannot be infinity.

**Lemma X.** [12] Let \( f(z, u) \) be a holomorphic function of \( z \) in \( |z| < 1 \), for each \( u \in [0, \infty) \). If in each circle

\[
|z| \leq 1 - \delta \ (0 < \delta < 1), \ \lim_{u \to \infty} f(z, u) = \infty
\]
uniformly, then there exists a positive function of \( \delta : A(\delta) < \infty \) such that

\[
\frac{1}{A(\delta)} \leq \frac{\log |f(z_1, u)|}{\log |f(z_2, u)|} < A(\delta),
\]

for each couple of values \( z_1, z_2 \) with \( |z_1| \leq 1 - \delta, \ |z_2| \leq 1 - \delta \) and for \( u \) sufficiently large \( (u > u(\delta)) \).

For \( u \) sufficiently large, \( u > u_0, \ |f(z, u)| > 1 \) for \( |z| \leq 1 - \frac{\delta}{2} \).

The family of harmonic functions of the variables \((x, y), \ (z = x + iy)\),

\[
P_{x, u}(x, y) = \frac{\log |f(z, u)|}{\log |f(\xi, u)|},
\]

where \( \xi \leq 1 - \delta, \ u > u_0 \) (\( \xi \) and \( u \) are considered as parameters, each of which, when fixed, defines one of the functions of the family) is normal, since \( P_{x, u}(x, y) > 0 \) \([15]\). But on setting \( \xi = x + iy \), we have

\[
P_{\xi, u}(x, y) = \frac{\log |f(\xi, u)|}{\log |f(\xi, u)|} = 1,
\]

hence no subsequence extracted from a sequence of the family can tend, in \( |z| \leq 1 - \delta \), uniformly to infinity. In other words, each such subsequence is bounded in such a circle.

This proves that the whole family is bounded in \( |z| \leq 1 - \frac{3}{\frac{1}{4}} \delta \).

(This last statement is seen immediately, since, if the contrary were true, we should have a contradiction with the statement that each subsequence is bounded.) We have thus proved that

\[
\log \left| \frac{f(z, u)}{f(\xi, u)} \right| < A(\delta) \ (|z| \leq 1 - \delta, \ |\xi| \leq 1 - \delta).
\]

On interchanging, in this inequality, \( z \) and \( \xi \), and on putting \( z = z_1, \ \xi = z_2 \), these inequalities give (90).

Let us now proceed to the proof of Theorem XXII. Sup-
Theorems of Picard Type

pose that \(a)\) does not hold. We shall then have, by Theorem XVI, on setting \(D < a' < a:\)

\[
|a_n| \leq 2\pi a' \Lambda^*(a'-D)M(s(u))\Lambda_n e^{\lambda_n\sigma(u)},
\]

for each \(u \in (-\infty, \infty)\), where

\[
M(s(u)) = \max_{s \in C(s(u))} |F(s)|.
\]

Since \(F(s)\) is not a Dirichlet polynomial, there exists an infinite sequence \(\{n_i\}\) such that \(a_{n_i} \neq 0\). On fixing \(n_i\), we see from (91) that, if \(0 < c < \lambda_{n_i}:\)

\[
\frac{|a_{n_i}| e^{(c-\lambda_{n_i})\sigma(u)}}{2\pi a' \Lambda^*(a'-D)\Lambda_{n_i}} \leq M(s(u))e^{\lambda_{n_i}\sigma(u)}.
\]

The left-hand expression tends (\(n_i\) fixed) to infinity, as \(u\) (and \(\sigma(u)\)) tends to \(-\infty\). Thus \(M(s(u))e^{\lambda_{n_i}\sigma(u)} \to \infty\) as \(u \to -\infty\), for each \(c > 0\). Let us set

\[
f(z, u) = F(s(-u) + \pi a z).
\]

The functions \(f(z, u)\) are holomorphic in \(|z| < 1\), and, on putting \(\alpha = 1 - \frac{a'}{a}\), we see by the remarks just made that, if \(M(u, \alpha) = \max |f(z, u)|, (|z| \leq 1 - \alpha)\), then, for each \(c > 0\):

\[
\lim_{u \to \infty} e^{\lambda_n\sigma(-u)}M(u, \alpha) = \infty.
\]

Let us now suppose that \(c)\) does not hold either. Then the family of functions \(f(z, u)\) is normal in \(|z| < 1\), and we shall have, by Lemma IX, \(\lim_{u \to \infty} f(z, u) = \infty\), in every circle \(|z| \leq 1 - \delta\) \((0 < \delta < 1)\). By Lemma X, (90) holds for each \(\sigma\) with \(0 < \delta < 1\), \(z_1, z_2\) being chosen arbitrarily in the circle \(|z| \leq 1 - \delta\). Let us set \(\delta < \alpha\). From (92) it follows that to each \(u\) there corresponds a quantity \(z_u\), \(|z_u| \leq 1 - \alpha < 1 - \delta\), such that for \(c\) arbitrary:

\[
\lim_{u \to \infty} (\lambda_n\sigma(-u) + \log |f(z_u, u)|) = \infty,
\]

\((z_u\) is such that \(\max_{|z| \leq 1 - \alpha} |f(z, u)| = M(u, \alpha), z_u\) is thus independent
of $c)$. Since $c$ is arbitrary we have also
\[
\lim_{n \to \infty} (\lambda_n c k(u) \sigma(-u) + \log |f(z_u, u)|) = \infty,
\]
where $k(u)$ is any function such that $\frac{1}{A(\delta)} \leq k(u) \leq A(\delta)$. It
follows then from (90), in which we set $z_1 = z_u$, $z_2 = z$, that, for
$|z| \leq 1 - \delta$:
\[
\lim_{n \to \infty} (\lambda_n c \sigma(-u) + \log |f(z, u)|) = \infty,
\]
that is to say, that, for $|s'| < \pi a (1 - \delta)$:
\[
\lim_{n \to \infty} e^{\lambda_n c \sigma(-u)} F(s(-u) + s') = \infty.
\]
On choosing $\delta = \frac{e}{a}$, this condition is equivalent to $b)$. We
have thus proved that if neither $a)$ nor $c)$ holds, then $b)$
must hold, and the theorem is proved.

Theorem XXII was proved by the author [10]. In a
weaker form it was proved in an earlier paper written in
common with G. Gergen [14]. In this paper the authors
considered strips of width $2\pi a$ with $a > \frac{1}{h}$, where
\[
h = \lim_{n \to \infty} \inf (\lambda_{n+1} - \lambda_n).
\]
The Theorem XXII is more general since $hD \leq 1$. On the
other hand, it was supposed in this paper that $\sigma_c = -\infty$, and naturally the condition $a)$ is eliminated a priori.

The question still remains open, whether or not the condition $b)$ can really occur in the general case. We shall see, however, in the next chapter that, in a horizontal strip of which the width is determined by certain other considera-
tions the condition $b)$ certainly does not occur.

REMARK. In the proof of Theorem XXII we have rea-
soned in the following fashion: on assuming the conditions of the theorem satisfied we have proved that if $a)$ and $c)$
do not hold, then the family of functions
Theorems of Picard Type

\[ f(z, u) = F(s(-u) + \pi az) \]
considered as functions of \( z \) in \( |z| < 1 \) is normal and from that follows that b) must hold (by Lemmas IX and X). In other words, with the hypotheses of Theorem XXII one of the three following conditions is satisfied: a), b) or c'), which amounts to the property that the family of functions \( F(s(-u) + \pi az) \) (with the parameter \( u \in (-\infty, \infty) \)) is not normal in \( |z| < 1 \). But then this family admits in the unit circle an irregular point \([15]\), that is to say, a point \( z_0 (|z_0| < 1) \) such that in each neighborhood of this point the family is not normal and therefore in each such neighborhood of \( z_0 \) this family takes infinitely many times each value except at most one.

This remark will be useful in the next chapter.