EVALUATION OF THE COEFFICIENTS

We shall extend to Dirichlet series the well known Cauchy formula for the coefficients of a Taylor series. We shall suppose that $\sigma_d < \infty$, although this condition is not essential. But we shall not have to use the most general cases in which the formulas, established here with this restrictive hypothesis, are still true.

**Theorem IX.** If

$$f(s) = \sum_{n=1}^{\infty} a_ne^{-\lambda ns}$$

with $\sigma_d' < \infty$, then

$$a_ne^{-\lambda_n t_0} = \lim_{T\to \infty} \frac{1}{T} \int_{t_0}^{T} f(\sigma_1 + it)e^{\lambda_n t} dt \quad (n \geq 1),$$

where $t_0$ is arbitrary, where $\sigma_1 > \sigma_d'$, otherwise arbitrary, and where $f(\sigma_1 + it)$ is the value of the principal branch of the function.

We have with $s = \sigma_1 + it$:

$$\frac{1}{T} \int_{t_0}^{T} f(s)e^{\lambda_n t} dt = \frac{1}{T} \int_{t_0}^{T} \left( \sum_{m=1}^{\infty} a_me^{(\lambda_n - \lambda_m)s} \right) dt$$

$$= \frac{1}{T} \int_{t_0}^{T} \left( \sum_{m=1}^{n-1} a_me^{(\lambda_n - \lambda_m)s} \right) dt + \frac{1}{T} \int_{t_0}^{T} a_n dt + \frac{1}{T} \int_{t_0}^{T} \left( \sum_{m=n+1}^{\infty} a_me^{(\lambda_n - \lambda_m)s} \right) dt.$$ 

Since, for $k$ real, $k \neq 0$:

$$\lim_{T\to \infty} \frac{1}{T} \int_{t_0}^{T} e^{k(\sigma_1 - it)} dt = 0,$$

and since the series under the integral signs converge uniformly with respect to $t(-\infty < t < \infty)$, we see that:

$$\lim_{T\to \infty} \frac{1}{T} \int_{t_0}^{T} \left( \sum_{m=1}^{n-1} a_me^{(\lambda_n - \lambda_m)s} \right) dt = 0.$$
Evaluation of Coefficients

\[ \lim_{T \to \infty} \int_{0}^{T} a_m e^{(\lambda_n - \lambda_m)T} dt = 0. \]

Hence

\[ \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f(s) e^{\lambda_n s} dt = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} a_n dt = a_n, \]

which is equivalent to the statement of the theorem.

As a matter of fact, it is seen that in Theorem IX, \( \sigma_n \) can be replaced by \( \sigma_n' \), since only the uniform convergence was used in the proof.

We shall have to use the following lemma.

**Lemma I.** If \( c > 0 \), if \( k \) is a positive integer, and if \( \omega \) is real, then:

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\omega(c+it)}}{(c+it)^k} dt = \begin{cases} \frac{\omega^{k-1}}{(k-1)!}, & (\omega > 0) \\
0, & (\omega \leq 0) \end{cases} \]

Let us first suppose \( \omega > 0 \). If \( \sigma_1 \) is real, we shall denote by \( I_1(\pm T, \sigma_1) \) the segments \( (\sigma_1 \leq \sigma \leq c, \ t = \pm T) \) and by \( I_2(T, \sigma_1), \ I_3(T) \) the segments \( (\sigma = \sigma_1, \ |t| \leq T) \) and \( (\sigma = c, \ |t| \leq T) \). By \( C(T, \sigma_1) \) we shall denote the rectangle composed of the four segments. We have by the theorem on residues:

\[ \frac{1}{2\pi i} \oint_{C(T, \sigma_1)} \frac{e^{\omega(s+it)}}{s^k} ds = \frac{\omega^{k-1}}{(k-1)!}. \]

Since for \( \sigma_1 \) fixed the integrals extended over \( I_1(T, \sigma_1) \) and \( I_1(-T, \sigma_1) \) tend to zero, we see that

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\omega(c+it)}}{(c+it)^k} dt - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\omega(\sigma_1+it)}}{(\sigma_1+it)^k} dt = \frac{\omega^{k-1}}{(k-1)!}, \]

and since the second integral in this equality tends to zero as \( \sigma_1 \) tends to \( -\infty \), we see that our equality holds for \( \omega > 0 \). The proof is similar if \( \omega \leq 0 \). Here \( C(T, \sigma_1) \) should be replaced by a contour \( C'(T, \sigma_2) \) with \( \sigma_2 > c \) and composed of segments \( I_1'(\pm T, \sigma_2) \equiv (c \leq \sigma \leq \sigma_2, \ t = \pm T), \ I_2(T, \sigma_2) \equiv (\sigma = \sigma_2, \)
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$|t| \leq T$, $I_3(T)$. The integral (17), in which $C(T, \sigma_1)$ is replaced by $C'(T, \sigma_2)$, is then zero, since the function under the integral sign is holomorphic inside $C'(T, \sigma_2)$, and on making first $T$ tend to $+\infty$ and then $\sigma_2$ to $+\infty$ we obtain the desired formula.

**Theorem X.** If

$$f(s) = \sum a_ne^{-\lambda_n s}$$

with $\sigma(s) < \infty$, if $v \geq 0$ and if $k$ is a positive integer, then

for $\nu > \lambda_1$: $\sum_{\lambda_n \leq \nu} (\nu - \lambda_n)^{k-1}a_n$ = $(k-1)! \int_{-\infty}^{\infty} \frac{f(c+it)e^{\nu(c+it)}}{(c+it)^k} dt$

for $0 \leq \nu \leq \lambda_1$: $0$

where $c > \max(\sigma(s), 0)$.

We have, with $s = c + it$ for $\nu > \lambda_1$:

$$\int_{-\infty}^{\infty} \frac{f(s)e^{\nu s}}{s^k} dt = \int_{-\infty}^{\infty} \left( \sum a_n e^{-(\lambda_n s)} \right) e^{\nu s} dt$$

$$= \int_{-\infty}^{\infty} \sum_{\lambda_n \leq \nu} a_n e^{(\nu - \lambda_n)s} \frac{dt}{s^k} + \int_{-\infty}^{\infty} \sum_{\lambda_n \geq \nu} a_n e^{(\nu - \lambda_n)s} \frac{dt}{s^k}.$$

From Lemma I and from the uniform convergence of the series under the last integral sign we see that the last integral is zero. Therefore by Lemma I:

$$\frac{(k-1)!}{2\pi} \int_{-\infty}^{\infty} \frac{f(s)e^{\nu s}}{s^k} dt = \frac{(k-1)!}{2\pi} \int_{-\infty}^{\infty} \left( \sum a_n e^{(\nu - \lambda_n)s} \right) \frac{dt}{s^k}$$

$$= \frac{(k-1)!}{2\pi} \sum a_n \int_{-\infty}^{\infty} \frac{e^{(\nu - \lambda_n)s}}{s^k} dt = \sum_{\lambda_n < \nu} a_n (\nu - \lambda_n)^{k-1}.$$

If $\nu \leq \lambda_1$ then

$$\int_{-\infty}^{\infty} \frac{f(s)e^{\nu s}}{s^k} dt = \int_{-\infty}^{\infty} \left( \sum a_n e^{(\nu - \lambda_n)s} \right) \frac{dt}{s^k} = 0.$$

In this theorem, too, $\sigma(s)$ can be replaced by $\sigma(s)$. 