DIRICHLET SERIES

I

GENERAL PROPERTIES OF CONVERGENCE

Let \( \{\lambda_n\} (n \geq 1) \) be a sequence of numbers all positive, except perhaps \( \lambda_1 \), which can be positive or zero, strictly increasing to infinity, and let \( \{a_n\}, (n = 1, 2 \ldots) \), be a sequence of complex numbers.

The series

\[
\sum_{n=1}^{\infty} a_n e^{-\lambda_ns},
\]

where \( s = \sigma + it \) is a complex number (with \( \sigma \) and \( t \) real), is called a Dirichlet series.

If in a Taylor series

\[
\sum_{n=1}^{\infty} a_n z^n
\]

we put \( z = e^{-s} \), the series becomes a Dirichlet series

\[
\sum_{n=1}^{\infty} a_n e^{-ns}
\]

which we shall call a \textit{Taylor-D} series. Here \( \lambda_n = n (n \geq 1) \). Another example of Dirichlet series is furnished by the series

\[
\sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} e^{-(\log n)s},
\]

\(^1\)A series of lectures delivered at the Rice Institute during the academic year 1942-43 by S. Mandelbrojt, Docteur ès Sciences (Paris), Professor at the Collège de France, Visiting Professor of Mathematics at the Rice Institute.

In the present course the author does not pretend to give a general theory of Dirichlet series. This was masterfully done some ten years ago by Vladimir Bernstein (see bibliography [3]). Apart from certain elementary results concerning convergence of Dirichlet series, we give here general results, some very recent, based essentially upon methods introduced by the author in his previous papers.
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which represents the famous $\xi(s)$ function of Riemann. In this case $\lambda_n = \log n$. The function $\xi(s)$, as well as the functions represented by the series of the more general form

$$\sum_{n=1}^{\infty} a_n \frac{1}{n^s} = \sum_{n=1}^{\infty} a_n e^{-(\log n)s}$$

plays a very important rôle in the theory of numbers.

Let us recall that the Taylor series (2) admits a radius of convergence, $R$, which may be equal to 0, to infinity, or to a positive finite number. In the first case, the series converges only for $z = 0$, in the second case it converges for every value of $z$, in the case when $0 < R < \infty$ the series converges for $|z| < R$ and converges for no value $z$ such that $|z| > R$. The circle $|z| < R$ (if $0 < R$) is the circle of convergence of (2). In each case $R$ is given by the formula:

$$\frac{1}{R} = \limsup_{n \to \infty} |a_n|^{1/n}.$$ 

It is, moreover, well known that the series (2) converges absolutely inside the circle of convergence, and converges uniformly in each closed bounded region situated inside the circle of convergence.

Since the transformation $z = e^{-t}$ gives $\sigma = -\log |z|$, $t = -\text{Arg}z$, we see that, if $z$ varies on a curve $L$ inside a circle $|z| < a$ the variable $s$ varies on a curve $L'$ which is situated in the half-plane $\sigma > \log a$. (If to a point $z_0$ of $L$ we make correspond the point $s_0 = \sigma_0 + it_0$ with $0 \leq t_0 < 2\pi$ the curve $L'$ is well defined, since we agree to consider $s$ as varying in a continuous way as $z$ varies on $L$).

The series (3) converges therefore absolutely for

$$\sigma > \limsup_{n \to \infty} \frac{\log |a_n|}{n} = \sigma_c,$$

if $\sigma_c < \infty$, and does not converge if $\sigma < \sigma_c$ (if $-\infty < \sigma_c$). The series (3) converges also uniformly in every closed bounded
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region situated in the half-plane (5). But the series (3) does not converge at any point if \( \sigma_0 = \infty \).

The facts just mentioned cannot be translated directly into general Dirichlet series, in replacing simply the quantities \( n \) by \( \lambda_n \). The most striking difference between general Dirichlet series and Taylor-D series, from the point of view of convergence, consists in the non-coincidence of the half-planes of different kinds of convergence.

The following theorem furnishes the greater part of results concerning convergence properties of a Dirichlet series (see for instance \([3]\)):¹

**Theorem I.** If \( \sum a_n e^{-\lambda_n s_0} \), \( (s_0 = \sigma_0 + it_0) \) converges, then \( \sum a_n e^{-\lambda_n s} \) converges uniformly in every closed angle given by

\[
|\text{Arg} \ (s - s_0)| \leq \gamma < \frac{\pi}{2}.
\]

Let us set \( s = s_0 + s', \ s' = \sigma' + it', \ \sigma' > 0, \ |\text{Arg} \ s'| < \gamma \), and let us write

\[
\sum_{1}^{m} a_n e^{-\lambda_n s_0} = A_m(s_0) = A_m
\]

\[
\sum_{1}^{\infty} a_n e^{-\lambda_n s_0} = S = \lim_{m \to \infty} A_m.
\]

We have: \( \sigma' > 0, \ \frac{|t'|}{\sigma'} \leq \tan \gamma = M < \infty \).

We have for \( s = s_0 + s' \) and \( q > p \geq 2 \):

\[
\sum_{p}^{q} a_n e^{-\lambda_n s} = \sum_{p}^{q} a_n e^{-\lambda_n s_0} e^{-\lambda_n s'}
\]

\[
= \sum_{p}^{q} (A_n - A_{n-1}) e^{-\lambda_n s'} = \sum_{p}^{q} [(A_n - S) - (A_{n-1} - S)] e^{-\lambda_n s'}
\]

\[
= \sum_{p}^{q-1} (A_n - S)(e^{-\lambda_n s'} - e^{-\lambda_{n+1} s'}) + (A_q - S) e^{-\lambda_{q} s'}
\]

\[
- (A_{p-1} - S) e^{-\lambda_{p} s'}.
\]

¹Numbers in brackets refer to the bibliography at the end of the Pamphlet.
If for \( \varepsilon > 0 \) given, \( p \) is chosen such as to have \( |A_n - S| < \varepsilon \) for \( n \geq p - 1 \), then

\[
|\sum_{\rho}^{\infty} a_n e^{-\lambda_n s}| \leq \sum_{\rho}^{p-1} |A_n - S| \cdot |e^{-\lambda_n s'} - e^{-\lambda_{n+1}s'}| + |A_{p-1} - S| e^{-\lambda_{p-1}s'}
\]

(7)

\[
\leq \varepsilon \sum_{\rho}^{p-1} |e^{-\lambda_n s'} - e^{-\lambda_{n+1}s'}| + 2\varepsilon.
\]

But

\[
|e^{-\lambda_n s'} - e^{-\lambda_{n+1}s'}| = |s'| \int_{\lambda_n}^{\lambda_{n+1}} e^{-us'} du \leq |s'| \int_{\lambda_n}^{\lambda_{n+1}} e^{-us'} du
\]

\[
= |s'| \left( e^{-\lambda_n s'} - e^{-\lambda_{n+1}s'} \right) \leq (M + 1) \left( e^{-\lambda_n s'} - e^{-\lambda_{n+1}s'} \right),
\]

and it follows from (7) that

\[
|\sum_{\rho}^{\infty} a_n e^{-\lambda_n s}| \leq 2\varepsilon + \varepsilon (M + 1) \sum_{\rho}^{p-1} \left( e^{-\lambda_n s'} - e^{-\lambda_{n+1}s'} \right)
\]

\[
= 2\varepsilon + \varepsilon (M + 1) \left( e^{-\lambda_n s'} - e^{-\lambda_{n+1}s'} \right) \leq \varepsilon (M + 3).
\]

This proves the theorem.

The following theorem is an immediate corollary of Theorem I.

**Theorem II.** If \( \sum a_n e^{-\lambda_n s}(s_0 = \sigma_0 + it_0) \) converges, the series \( \sum a_n e^{-\lambda_n s} \) converges at each point \( s = \sigma + it \) with \( \sigma > \sigma_0 \), and converges uniformly in each closed bounded region which is situated in the half-plane \( \sigma > \sigma_0 \).

If we now replace the series (1) by the series

(8)

\[
\sum_{1}^{\infty} |a_n| e^{-\lambda_n s},
\]

we see from Theorem II that if (8) converges for \( s_0 = \sigma_0 \) (real), it converges also for \( s = \sigma > \sigma_0 \). Since \( |e^{-\lambda_n s}| = e^{-\lambda_n \sigma} \), we see that the following theorem holds.

**Theorem III.** If the series (1) converges absolutely for \( s = s_0 = \sigma_0 + it_0 \), it converges absolutely for \( s = \sigma + it \) with \( \sigma > \sigma_0 \).

By a classical reasoning, in which the notion of Dedekind's
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cuts is involved, Theorems II and III lead immediately to

**Theorem IV.** If there exists a point at which the series (1) converges, then either (1) converges for each value of \( s \) or there exists a real number \( \sigma_C (\infty < \sigma_C < \infty) \) such that (1) converges for \( \sigma > \sigma_C \) and converges for no \( s \) with \( \sigma < \sigma_C \). If \( \sum |a_n|e^{-\lambda_n \sigma} \) converges for one value of \( \sigma \), then either this series converges for all the values of \( \sigma \), or there exists a quantity \( \sigma_A (\infty < \sigma_A < \infty) \) such that this series converges for \( \sigma > \sigma_A \) and diverges for \( \sigma < \sigma_A \).

If the series (1) converges at no point we shall write \( \sigma_C = \infty \). If (1) converges at every point we shall write \( \sigma_C = -\infty \). If \( \sum |a_n|e^{-\lambda_n \sigma} \) converges for no value of \( \sigma \) we shall write \( \sigma_A = \infty \), and if it converges for each value of \( \sigma \) we shall write \( \sigma_A = -\infty \).

The quantities \( \sigma_C \) and \( \sigma_A \) are called, respectively, *abscissa of convergence*, and *abscissa of absolute convergence* of the series (1). The straight-lines given by \( \sigma = \sigma_C \), \( \sigma = \sigma_A \) are called *axis of convergence* and *axis of absolute convergence* of the series. Obviously \( \sigma_A \geq \sigma_C \). For a Taylor-D series \( \sigma_A = \sigma_C \), but this is not true in general. It is, for instance, known that the series \( \sum \frac{(-1)^n}{n^\sigma} \) converges for \( \sigma > 0 \) and does not converge for \( \sigma \leq 0 \). On the other hand, \( \sum \frac{1}{n^\sigma} \) converges for \( \sigma > 1 \) and diverges for \( \sigma \leq 1 \). For the series \( \sum \frac{(-1)^n}{n^\sigma} : \sigma_C = 0, \sigma_A = 1 \). The values of \( \sigma_C \) and \( \sigma_A \) are furnished by the following theorem.

**Theorem V.** Let us set

\[
(9) \quad a = \lim \sup_{n=\infty} \log \left| \sum_{1}^{n} a_m \right| \frac{1}{\lambda_n}.
\]

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The series \( (1) \) converges for \( \sigma > \max (0, a) \), and if \( a > 0 \), then \( \sigma_C = a \). Let us set

\[
\log \sum_{n=1}^{\infty} |a_n| = \limsup_{n \to \infty} \frac{\log \sum_{1}^{n} |a_n|}{\lambda_n}.
\]

The series \( (1) \) converges absolutely for \( \sigma > \max (0, b) \), and if \( b > 0 \), then \( \sigma_A = b \).

The part concerning the absolute convergence follows immediately from the first part if we replace the series \( (1) \) by \( \sum |a_n| e^{-\lambda_n s} \).

Let us set \( A_n = A_n(0) = \sum_{1}^{n} a_n \). Let us suppose that

\[
\limsup_{n \to \infty} \frac{\log |A_n|}{\lambda_n} = a < \infty.
\]

If \( \epsilon > 0 \), then for \( n > n_{\epsilon} \),

\[
\frac{\log |A_n|}{\lambda_n} < a + \frac{\epsilon}{2}, \quad \text{if} \quad a > -\infty, \quad \frac{\log |A_n|}{\lambda_n} < \frac{\epsilon}{2}, \quad \text{if} \quad a = -\infty.
\]

Hence:

\[
|A_n| < e^{\lambda_n(a' + \frac{\epsilon}{2})} \quad \text{or} \quad |A_n| < e^{\lambda_n \frac{\epsilon}{2}} \quad \text{(if} \quad a = -\infty).\]

Let us set \( a' = \max (0, a) \).

We may write for \( q > p \geq 2 \):

\[
\sum_{p}^{q} a_n e^{-\lambda_n(a' + \epsilon)} = \sum_{p}^{q} (A_n - A_{n-1}) e^{-\lambda_n(a' + \epsilon)}
\]

\[
= \sum_{p}^{q-1} A_n (e^{-\lambda_n(a' + \epsilon)} - e^{-\lambda_{n+1}(a' + \epsilon)}) + A_{q} e^{-\lambda_{q}(a' + \epsilon)} - A_{p-1} e^{-\lambda_{p}(a' + \epsilon)}.
\]

It follows then from (11) that if \( p > 1 + n_{\epsilon} \) then:

\[
|\sum_{p}^{q} a_n e^{-\lambda_n(a' + \epsilon)}| \leq \sum_{p}^{q-1} e^{\lambda_n(a' + \frac{\epsilon}{2})} (e^{-\lambda_n(a' + \epsilon)} - e^{-\lambda_{n+1}(a' + \epsilon)})
\]

\[
+ e^{\lambda_{q}(a' + \frac{\epsilon}{2})} e^{-\lambda_{q}(a' + \epsilon)} + e^{\lambda_{p-1}(a' + \frac{\epsilon}{2})} e^{-\lambda_{p}(a' + \epsilon)}
\]
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(12) \[ \leq \sum_{p}^{q-1} e^{\lambda_n(a'+\varepsilon)} (a'+\varepsilon) \int_{\lambda_n}^{\lambda_{n+1}} e^{-u(a'+\varepsilon)} \, du + e^{-\lambda_n^2} + e^{-\lambda_{n+1}^2} \]

\[ \leq (a'+\varepsilon) \sum_{p}^{q-1} \int_{\lambda_n}^{\lambda_{n+1}} e^{-\frac{u^2}{2}} \, du + e^{-\lambda_n^2} + e^{-\lambda_{n+1}^2} \]

\[ = (a'+\varepsilon) \int_{\lambda_n}^{\lambda_{n+1}} e^{-\frac{u^2}{2}} \, du + e^{-\lambda_n^2} + e^{-\lambda_{n+1}^2}, \]

and since

\[ \int_{0}^{\infty} e^{-\frac{u^2}{2}} \, du < \infty, \]

one sees immediately that the expression

\[ (a'+\varepsilon) \int_{\lambda_n}^{\lambda_{n+1}} e^{-\frac{u^2}{2}} \, du + e^{-\lambda_n^2} + e^{-\lambda_{n+1}^2} \]

tends to zero as \( p \) tends to infinity. This proves that the series (1) converges for \( s = a'+\varepsilon \), thus that \( \sigma_0 \leq a' \).

We shall now prove that if (1) converges for \( s = \sigma_0 > 0 \), then \( a \leq \sigma_0 \). This will prove that if \( a = \infty \), then \( \sigma_C = \infty \), and that if \( 0 < a < \infty \), then \( a \geq \sigma_C \). This together with \( \sigma_C \leq a' \) proves the theorem.

We can write

\[ A_n = \sum_{1}^{n} a_m = \sum_{2}^{n-1} A_m(\sigma_0)(e^{\lambda_m \sigma_0} - e^{\lambda_{m+1} \sigma_0}) + A_n(\sigma_0)e^{\lambda_n \sigma_0} + a_1 e^{-\lambda_n \sigma_0}(e^{\lambda_1 \sigma_0} - e^{\lambda_n \sigma_0}). \]

If \( \sum a_n e^{-\lambda_n \sigma_0} \) converges, there exists a constant \( M \) such that \( |A_n(\sigma_0)| < M, (n \geq 1) \), and

\[ |A_n| \leq M \left[ \sum_{1}^{n-1} (e^{\lambda_m+1} \sigma_0 - e^{\lambda_m} \sigma_0) + e^{\lambda_n} \sigma_0 \right] \]

\[ = M(2e^{-\lambda_n \sigma_0} - e^{\lambda_1 \sigma_0}) < 2Me^{\lambda \sigma_0}, \]

which proves that

\[ a = \limsup_{n \to \infty} \frac{\log |A_n|}{\lambda_n} \leq \sigma_0. \]

The next theorem is an immediate corollary of Theorem V.
THEOREM VI. If \( \sum a_n e^{-\lambda_n \sigma_0} \) does not converge, then
\[
\log | \sum_{1}^{n} a_m e^{-\lambda_m \sigma_0} | = \sigma_0 + \limsup_{n \to \infty} \frac{\log n}{\lambda_n}.
\]

If \( \sum |a_n| e^{-\lambda_n \sigma_0} = \infty \), then:
\[
\log (\sum_{1}^{n} |a_m| e^{-\lambda_m \sigma_0}) = \sigma_0 + \limsup_{n \to \infty} \frac{\log n}{\lambda_n}.
\]

It should be remarked that in Theorem V the hypothesis \( a > 0 \) (as well as \( b > 0 \)) is essential. Suppose indeed that the series (2) represents a function \( f(z) \) holomorphic in \( |z| < R_1 \) with \( R_1 > 1 \), and such that \( f(1) \neq 0 \). The series
\[
\varphi(z) = \frac{1}{1-z} f(z) = \sum A_n z^n, \quad (A_n = \sum_{1}^{n} a_m),
\]
has then the radius of convergence equal to unity, since the point \( z = 1 \) is singular for \( \varphi(z) \), and
\[
\limsup_{n \to \infty} \frac{\log |A_n|}{n} = 0,
\]
but, on the other hand, for the series (3), \( \sigma_0 \leq -\log R_1 < 0 \).

We shall have also to use the following theorem which furnishes cases in which the abscissas of convergence are obtained by formulas analogous to those for Taylor-D series.

THEOREM VII. If
\[
\limsup_{n \to \infty} \frac{n}{\lambda_n} < \infty,
\]
then
\[
\sigma_A = \sigma_C = \limsup_{n \to \infty} \frac{\log |a_n|}{\lambda_n}.
\]

Let us suppose that
\[
\frac{n}{\lambda_n} < \frac{1}{L} < \infty \quad (n \geq 2),
\]
and let us first suppose that \( a = \limsup \frac{\log |a_n|}{\lambda_n} < \infty \). If \( b_1 > a \), we have, for \( n > n_{b_1} \): 
\[ |a_n| e^{-\lambda_n \sigma} < e^{\lambda_n b_1} \]
and for \( \sigma > b_1 \):
\[ |a_n| e^{-\lambda_n \sigma} \leq e^{\lambda_n (b_1 - \sigma)} < e^{\lambda (b_1 - \sigma) n}. \]
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Thus \( \sum |a_n| e^{-\lambda n} < \infty \) for \( \sigma > a \). On the other hand, if \( -\infty < a \leq \infty \), and if \( -\infty < \sigma < b_1 < a \), we have for an infinity of \( n \): \( |a_n| > e^{\lambda_n b_1} \), and \( |a_n| e^{-\lambda n} \geq e^{\lambda_n (b_1 - \sigma)} > 1(n > 1) \), the series (1) does not converge therefore for \( \sigma < a \). This proves the theorem.

Theorem VII holds also, as G. Valiron has proved, if the condition \( \lim \sup \frac{n}{\lambda_n} < \infty \) is replaced by the less restrictive condition \( \lim \frac{\log n}{\lambda_n} = 0 \) (see [3]).

A Dirichlet series may converge at every point and converge absolutely at no point, that is to say, the case may occur in which \( \sigma_C = -\infty, \sigma_A = \infty \).

Let us set: \( a_n = \frac{(-1)^n}{n}, \lambda_n = \sqrt{\log \log n}, (n \geq 3) \).

If \( k > 0 \), the following relationships hold:

(\( \alpha \)) \( \lim_{n \to \infty} a_n e^{\lambda_n k} = 0 \)

(\( \beta \)) \( |a_n| e^{\lambda_n k} < |a_{n-1}| e^{\lambda_{n-1} k} (n > n_k) \).

The first relationship is obvious. The second can be written in the form:

\[ -\log \left(1 - \frac{1}{n}\right) > k \left(\sqrt{\log \log n} - \sqrt{\log \log (n-1)}\right), \quad (n > n_k) \]

which follows from \( -\log \left(\frac{n-1}{n}\right) > \frac{1}{n} \) and from the relationship

\[ \sqrt{\log \log n} - \sqrt{\log \log (n-1)} \approx \frac{1}{2n \log n \sqrt{\log \log n}}. \]

The series \( \sum a_n e^{\lambda_n k} \) (alternating, for \( n \) sufficiently large) converges therefore for \( k > 0 \) arbitrary. This proves that \( \sigma_C = -\infty \). On the other hand, \( \sum a_m |\infty \log n \), hence

\[ \sigma_A = \lim \sup \frac{\log \sum_{m=1}^n |a_m|}{\lambda_n} = \lim \frac{\log \log n}{\sqrt{\log \log n}} = \infty. \]
In this case $\limsup \frac{\log n}{\lambda_n} = \infty$. Let us remark that in this example the quantity $a$ of Theorem V is equal to zero.

The following theorem gives a general relationship between the two abscissas of convergence.

**Theorem VIII.** For every Dirichlet series

$$\sigma_A - \sigma_C \leq \limsup_{n \to \infty} \frac{\log n}{\lambda_n}. \quad (13)$$

The examples above show that the sign of equality cannot be omitted in this theorem.

Let us set

$$\limsup_{n \to \infty} \frac{\log n}{\lambda_n} = b.$$

Any positive $\epsilon$ being given, we have to prove that

$$\sum |a_n| e^{-\lambda_n(\sigma + b + \epsilon)} < \infty.$$

From the convergence of (1) for $s = \sigma + \frac{\epsilon}{2}$ it follows that

$$|a_n| e^{-\lambda_n(\sigma + \frac{\epsilon}{2})} < A < \infty,$$

and

$$|a_n| e^{-\lambda_n(\sigma + b + \epsilon)} < Ae^{-\lambda_n(b + \frac{\epsilon}{2})};$$

but for $n > n_e$:

$$\frac{\log n}{\lambda_n} < b + \frac{\epsilon}{4}.$$ 

Hence

$$|a_n| e^{-\lambda_n(\sigma + b + \epsilon)} < Ae^{-\log n \left( \frac{b + \frac{\epsilon}{2}}{b + \frac{\epsilon}{4}} \right)} = An^{b + \frac{\epsilon}{4}},$$

which proves the theorem.

If $\sigma_C < \infty$, the series (1) converges uniformly in every closed bounded region situated in the half-plane $\sigma > \sigma_C$. This series represents then an analytic function $f(s)$ holomorphic for $\sigma > \sigma_C$:

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}. \quad (14)$$
By $f(s)$ we shall understand not only the sum of the series, but generally the function given by its analytic continuation, and we shall call it the function represented by the series $\sum a_n e^{-\lambda_n s}$. The actual sum of this series in $\sigma > \sigma_c$ will be called the principal branch of $f(s)$ in $\sigma > \sigma_c$. To be more specific, the corresponding abscissas $\sigma_c, \sigma_A$ will be denoted, when needed, by $\sigma_c', \sigma_A'$. 

It is obvious that the principal branch of $f(s)$ is bounded in $\sigma \geq \sigma_A + \varepsilon$, if $\sigma_A > -\infty$, $(\sigma_A < \infty)$, the quantity $\varepsilon > 0$ being chosen arbitrarily. We have as a matter of fact, in this half-plane:

$$|f(s)| \leq \sum |a_n| e^{-\lambda_n (\sigma_A + \varepsilon)}.$$

Before we close this chapter we should remark that in contrast to a Taylor-D series, a general Dirichlet series does not converge uniformly in every half-plane $\sigma \geq \sigma_0 > \sigma_c (\sigma_c < \infty)$, although it does converge uniformly in every bounded closed region of such a half-plane. This is, for instance, the case of the series $\sum \frac{(-1)^n}{n^s}$, which does not converge uniformly in the half-plane $\sigma \geq \sigma_0 > 0$ with $\sigma_0 < 1$, and yet here $\sigma_c = 0$. A Dirichlet series does converge uniformly, as is readily seen, in each half-plane $\sigma > \sigma_1 > \sigma_A (\sigma_A < \infty)$. H. Bohr introduced the notion of the abscissa $\sigma_u$ of uniform convergence which is the g. l. b. of quantities $\sigma_0$, such that (1) converges uniformly for $\sigma \geq \sigma_0$. Obviously $\sigma_c \leq \sigma_u \leq \sigma_A$. Bohr has shown (see [3]) that

$$\sigma_u = \lim \sup_{n \to \infty} \log \left( \text{l. u. b. } \left| \sum_{-\infty < \lambda < \infty} a_m e^{i \lambda n} \right| \right),$$

if the right-hand expression is positive.