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Bounding The Forcing Number of a Graph

by

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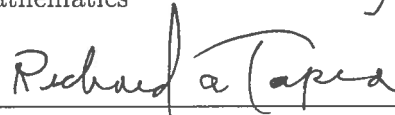
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ABSTRACT

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The *forcing number*, denoted $F(G)$, is an upper bound for the maximum nullity of all symmetric matrices with a sparsity pattern described by the simple graph G . Simple lower and upper bounds are $\delta \leq F(G)$ where δ is the minimum degree and $F(G) \leq n - 1$ where n is the order of the graph. This thesis provides improvements on the minimum degree lower bound in the case that G has girth of at least 5. In particular, it is shown that $2\delta - 2 \leq F(G)$ for graphs with girth of at least 5; this can be further improved when G has a small cut set. Further, this thesis also conjectures a lower bound on $F(G)$ as a function of the girth, g , and δ .

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Chapter 1

Preliminaries

1.1 Introduction

Originally called the *geometry of position* [7], graph theory, as it came to be known, is the theory of relationships. Because many systems can be modeled with simple relationships, graph theory has found a verity of applications in both pure and applied mathematics. For example, one may use a graph to model the sparsity pattern of a matrix, or one may use a random graph to study the structure of large social networks.

One of the most widely studied branches of graph theory is extremal graph theory; which attempts to determine the relationship between graph invariants and graph properties. For instance, given a graph property P , a class of graphs \hat{G} , and graph invariant i , what is the least value x such that every graph G in \hat{G} satisfies $i(G) \geq x$?

Some of the earliest invariants that extremal graph theorists studied were those involving graph coloring. That is, assigning colors either to the vertex set, or to the edge set of a graph, and then determining the minimal and maximal sets of such a coloring. Many variations of these colorings exist, some examples are: vertex and edge chromatic numbers, face coloring number, and rainbow coloring number.

For the duration of the 20th century, all graph colorings were static. That is, once a coloring of a graph was given, it did not change. However in recent years, new variations of graph coloring have emerged that allow a coloring of a graph to change in discrete time intervals. More generally, this means a graph may undergo

many different colorings given a single initial coloring. This dynamic approach to graph coloring has led to a wide variety of new and exciting applications of graph theory such as applications in logic circuits, physics, coding theory, and power network monitoring.

The *zero forcing number*, originally introduced in 2006 by a group of linear algebraists [4], is currently one of the most highly studied graph dynamic coloring invariants and is the main subject of this thesis. Roughly speaking, the zero forcing number is the smallest number of initially colored vertices so that any colored vertex with exactly one non-colored neighbor changes its non-colored neighbor to be colored and through repeating this process all of the vertices become colored.

This thesis consists of finding, and conjecturing, previously unknown lower and upper bounds and is outlined as follows: The rest of Chapter 1 consists of necessary definitions and terminology needed for the study of zero forcing, Chapter 2 gives a literature review of zero forcing, Chapter 3 introduces a generalization of zero forcing along with known results, Chapter 4 gives new theorems and conjectures, and Chapter 5 provides suggestions for future work.

1.2 Graph Preliminaries

A graph $G = (V, E)$ consists of a finite vertex set V (sometimes called node set) and finite edge set E . The elements of E are non-ordered pairs of vertices from V ; for the purposes of this thesis we will not consider graphs with loops. The cardinality of the vertex set is the *order* of G , and is denoted $n := |V|$. Whereas the cardinality of the edge set is the *size* of G , and is denoted $m := |E|$. Two vertices $v, w \in V$ are said to be *adjacent*, or *neighbors*, if $vw \in E$. The *open neighborhood* of $v \in V$ is $N(v) := \{w : w \text{ is adjacent to } v\}$. The *closed neighborhood* of $v \in V$ is $N[v] :=$

$N(v) \cup \{v\}$. The *degree* of $v \in V$ is the cardinality of its open neighborhood, and is denoted $d(v) := |N(v)|$. The *minimum degree* and *maximum degree* of G , are denoted δ and Δ , respectively. For $S \subseteq V$, the *induced subgraph* obtained by S , is the graph obtained by deleting $V \setminus S$ from G , and is denoted $G[S]$.

A *path* is a finite set of vertices v_1, \dots, v_k such that $v_i \neq v_j$ for all $i \neq j$ and there exists edges $v_i v_{i+1}$ for all $i = 1, \dots, k-1$. The *path cover number* of G , denoted $P(G)$, is the minimum number of vertex disjoint paths, occurring as induced subgraphs of G , that cover all the vertices. A *cycle* is a path v_1, \dots, v_k ($k \geq 3$) with the additional edge $v_1 v_k$. The *girth* of a graph is the size of a smallest induced cycle, and is denoted $g := g(G)$. By convention, if G does not contain an induced cycle subgraph we say $g = \infty$. A graph which does not contain cycle is called a *tree*. A degree one vertex of a tree is called a *leaf*, and a degree one vertex of a graph which is not a tree is called a *pendent*. *Complete graphs* are graphs for which all pairs of vertices are adjacent, whereas *empty graphs* are graphs for which no pair of vertices are adjacent.

Let $k \geq 1$ be an integer. We say that $S \subseteq V$ is a *k-dominating set* if $v \in S$, or v has at least k neighbors in S for all $v \in V$. The *k-domination number* of G is the cardinality of a smallest k -dominating set of G , and is denoted $\gamma_k(G)$. If a k -dominating set induces a connected subgraph, then we call the set in question a *connected k-dominating set*. The *connected k-domination number* of G is the cardinality of a minimum connected k -dominating set, and is denoted $\gamma_{c,k}(G)$. When $k = 1$, we use the terminology *domination number* in place of 1-domination number, and *connected domination number* in place of connected 1-domination number. Furthermore, we use $\gamma(G)$ in place of $\gamma_1(G)$, and $\gamma_c(G)$ in place of $\gamma_{c,1}(G)$.

Let $k \geq 1$ be an integer. For an initial set $S \subseteq V$, the sets $(\mathcal{P}_G^i(S))_{i \geq 0}$ of vertices *monitored* by S at the i -th step are defined recursively by,

1. $\mathcal{P}_G^0 = N[S]$.
2. $\mathcal{P}_G^{i+1} = \bigcup \{N[v] : v \in \mathcal{P}_G^i(S) \text{ such that } |N[v] \setminus \mathcal{P}_G^i(S)| \leq k\}$.

If $\mathcal{P}_G^{i_0} = \mathcal{P}_G^{i_0+1}$, for some i_0 , then $\mathcal{P}_G^j = \mathcal{P}_G^{i_0}$, for all $j \geq i_0$. We define $\mathcal{P}_G^\infty = \mathcal{P}_G^{i_0}$. If $\mathcal{P}_G^\infty(S) = V$, we say that S is a *k-power dominating set* of G . The cardinality of a smallest *k-power dominating set* is known as the *k-power domination number* of G , and is denoted $\gamma_{P,k}(G)$.

Let each vertex of G be either colored or non-colored. Let $S \subseteq V$ be the set of initially colored vertices. The *color-change rule* changes a non-colored vertex, w (say), to be colored if w is the only non-colored neighbor of some colored vertex, v (say): in this case we say v *forces* w . If after applying the the color-change rule iteratively all of the vertex becomes colored, we say that S was a *zero forcing set*. The minimum cardinality of a zero forcing set of G is called the *zero forcing number* of G , and is denoted $Z(G)$.

1.3 Linear Algebra Preliminaries

This thesis will cover many more graph theoretic results than linear algebraic. But the literature review makes extensive use of some linear algebra terminology. We state the needed definitions here.

Let \mathcal{F} be any field. Denote the set of all symmetric matrices with entries from \mathcal{F} by $S_n(\mathcal{F})$. For $\mathbf{A} \in S_n(\mathcal{F})$, the graph of \mathbf{A} , denoted $\mathcal{G}(\mathbf{A})$, is the graph with vertex set $\{1, \dots, n\}$ and edge set $\{\{i, j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$, where the diagonal is ignored. The set of symmetric matrices of a graph G over the field \mathbb{R} is defined as

$$\mathcal{L}(G) = \{\mathbf{A} \in S_n(\mathbb{R}) : \mathcal{G}(\mathbf{A}) = G\}.$$

Whereas the set of symmetric matrices of a graph G over an arbitrary field \mathcal{F} is defined as

$$\mathcal{L}(\mathcal{F}, G) = \{\mathbf{A} \in S_n(\mathcal{F}) : \mathcal{G}(\mathbf{A}) = G\}.$$

The *minimum rank* of G over \mathbb{R} is defined to be

$$mr(G) = \min\{\text{rank}(\mathbf{A}) : \mathbf{A} \in \mathcal{L}(G)\}.$$

And more generally, the minimum rank over an arbitrary field \mathcal{F} is defined to be

$$mr^{\mathcal{F}}(G) = \min\{\text{rank}(\mathbf{A}) : \mathbf{A} \in \mathcal{L}(\mathcal{F}, G)\}.$$

For $\mathbf{A} \in \mathbb{R}^{n \times n}$ the *maximum nullity* of \mathbf{A} is defined to be

$$M(G) = \max\{\text{corank}(\mathbf{A}) : \mathbf{A} \in \mathcal{L}(G)\}.$$

And more generally the maximum nullity over an arbitrary field \mathcal{F} is defined to be

$$M^{\mathcal{F}}(G) = \max\{\text{corank}(\mathbf{A}) : \mathbf{A} \in \mathcal{L}(\mathcal{F}, G)\}.$$

One should observe for a field \mathcal{F} and a graph with order n , we have

$$mr^{\mathcal{F}}(G) + M^{\mathcal{F}}(G) = n.$$

Now that we have the necessary terminology, we may move to the literature review. We will then give the story of how zero forcing has progressed from linear algebraic research to pure graph theory research.

Chapter 2

Literature Review

Zero forcing has interested a diverse group of researchers in mathematics and the collection of papers on the subject could easily fill a large text book. This literature review will not cover every known result, but will tell the story of how zero forcing has made a journey from linear algebraists to graph theorists.

2.1 Initial Motivation

In October of 2006 the American Institute of Mathematics held the workshop entitled: “Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns”. During this workshop, the AIM Minimum Rank - Special Graphs Group (F. Barioli, W. Barret, S. Butler, S.M. Cioabă, D. Cvetković, S.M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelsen, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, and A. Wehe) formally introduced the definition of the zero forcing process and the associated zero forcing number [4]. The motivating idea behind defining the zero forcing process and the associated zero forcing number was to bound the maximum nullity from above, equivalently bounding the minimum rank from below, as illustrated by the following proposition:

Proposition 2.1 (AIM Minimum Rank - Special Graphs Group [4]). *Let $G = (V, E)$ be a graph and let $S \subseteq V$ be a zero forcing set. Then $M(G) \leq |S|$, and thus*

$$n - mr(G) = M(G) \leq Z(G).$$

Computing $M(G)$ requires the evaluation of an infinite number of matrices, whereas computing the zero forcing number requires only the evaluation of a finite set of vertices. Because of this, researchers interested in knowing the maximum nullity of a graph will find Proposition 2.1 very useful. In particular, the AIM Minimum Rank - Special Graphs Group sought to establish cases of equality for Proposition 2.1 and thus find cases of graphs for which the maximum nullity could be computed. Some of their general characterizations which appeared in [4] are listed in the following theorem:

Theorem 2.1 (AIM Minimum Rank - Special Graphs Group [4]). *For each of the following families of graphs, $Z(G) = M(G)$.*

1. *Any graph G with order at most 6.*
2. *K_n, P_n, C_n .*
3. *Any tree T_n .*
4. *Any G which is Hamiltonian.*
5. *The hypercube on n vertices.*

In general, characterizing equality for Proposition 2.1 is difficult. Addressing this, the AIM Minimum Rank - Special Graphs Group posed the following question which remains open:

Question 2.1. *What is the class of graphs G for which $M^{\mathcal{F}}(G) = Z(G)$ for some field \mathcal{F} ?*

In August 2007, zero forcing appeared again in the paper: “An upper bound for the minimum rank of a graph” by A. Berman, S. Friedland, L. Hogben, U.G. Rothblum, and B. Shader. This paper was mainly concerned with bounding the minimum rank of a graph, but the following proposition was also proven and is central to this thesis:

Proposition 2.2 (Berman-Friedland-Hogben-Rothblum-Shader [8]). *Let G be a graph with minimum degree δ . Then,*

$$\delta \leq Z(G).$$

2.2 Zero Forcing Properties

In the years following the publication of [4] and [8], there was an explosion of interest in zero forcing. This was in part inspired by the maximum nullity problem, but also because zero forcing may describe other phenomena. For example, it was observed that the zero forcing process was equivalent to that of graph infection already used by physicists in order to study control of quantum systems [10] and also exhibited applications to logic circuits [9]. Further, it would soon become clear that the zero forcing number was an interesting graph parameter on its own.

One of the first papers describing graph theoretic properties of zero forcing appeared in 2009 by F. Barioli, W. Barret S.M. Fallat, H.T. Hall, L. Hogben, B. Shader P. van den Driessche, and H. van der Holst [5]. In their paper, they defined a variant of the zero forcing number, the *positive semidefinite zero forcing number*. For the purposes of this thesis we will not address the positive semidefinite zero forcing number; we will however highlight the following questions which they asked:

1. Is there a graph that has a unique minimum zero forcing set?
2. Is there a graph G and a vertex $v \in V$ such that v is in every minimum zero forcing set?

In resolving these questions, Barioli et al., 2009 proved what could possibly be called the first purely graph theoretic theorems on zero forcing.

Theorem 2.2 (Barioli-Barret-Fallat-Hall-Hogben-Shader-Driessche-Holst [5]). *If S is a zero forcing set of G then so is any reversal of S .*

Theorem 2.3 (Barioli-Barret-Fallat-Hall-Hogben-Shader-Driessche-Holst [5]). *If G is a connected graph of order greater than one, then,*

$$\bigcap_{S \in ZFS(G)} S = \emptyset$$

where $ZFS(G)$ is the set of all minimum zero forcing sets of G .

As a corollary to Theorem 2.2, no connected graph of order greater than one has a unique minimum zero forcing set. As a corollary to Theorem 2.3, no connected graph of order greater than one can have a vertex which is contained in every minimum zero forcing set.

Besides the above mentioned theorems, the concept of *forcing chains* was also introduced in [5]. These chains allow one to keep track (in a theoretical and practical sense) of how a zero forcing set colors, or *propagates*, in a graph. One immediate example of the usefulness of forcing chains is the following theorem which relates zero forcing to the path cover number:

Theorem 2.4 (Barioli-Barret-Fallat-Hall-Hogben-Shader-Driessche-Holst [5]). *For any graph G ,*

$$P(G) \leq Z(G).$$

In graph theory, one is often concerned with changes in properties when an edge or vertex is deleted. This concept was addressed in 2010 when C.J. Edholm, L. Hogben, M. Huynh, J. LaGrange, and D. Row introduced the notion of *vertex spread* and *edge spread* of zero forcing [16]. Their paper proved various theorems on the zero forcing of graphs after edges and vertices are deleted. For the purposes of this thesis, we highlight the following theorem:

Theorem 2.5 (Edholm-Hogben-Huynh-LaGrange-Row [16]). *Let G be a graph on $n \geq 2$ vertices. Then*

1. *For $v \in V(G)$, $Z(G) - 1 \leq Z(G - v) \leq Z(G) + 1$.*
2. *For $e \in E(G)$, $Z(G) - 1 \leq Z(G - e) \leq Z(G) + 1$.*

In the years between 2009 and 2011 the research community took greater and greater interest in zero forcing as a graph parameter to study on its own. For example, in 2010, Row [23] offered purely graph theoretic techniques for computing the zero forcing number of graphs with a cut vertex. Notably, Row gave the following two characterizations:

Proposition 2.3 (Row [23]). *Let G be a connected graph with order $n \geq 2$. Then $Z(G) = n - 1$ if and only if $G = K_n$.*

Theorem 2.6 (Row [23]). *Let G be a graph. Then $Z(G) = 2$ if and only if G is a graph of two parallel paths.*

In October 2010, F. Barioli, W. Barrett, S.M. Fallout, H.T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst [6] gave several variations of the zero forcing number including the *enhanced zero forcing number* and the *loop zero forcing number*. Their paper establishes myriad of relations between these new definitions and their usefulness when considering variations of the maximum nullity problem. Two theorems related to this thesis are the following:

Theorem 2.7 (Barioli-Barrett-Fallout-Hall-Hogben-Shader-Driessche-Holst[6]). *For any graph G ,*

$$M(G) \leq \hat{Z}(G) \leq Z(G),$$

where $\hat{Z}(G)$ is the enhanced zero forcing number.

and,

Theorem 2.8 (Barioli-Barrett-Fallat-Hall-Hogben-Shader-van der Holst [6]).

$$tw(G) \leq Z(G)$$

In November of 2010, Meyer gave results for zero forcing sets in bipartite circulants [21]. In his paper, he posed the following question:

Question 2.2 (Meyer [21]). *Is there a simple function that depends only on n and d such that $Z(G) \leq f(n, d)$ for all connected d -regular bipartite circulants on $2n$ vertices?*

Meyer's question is closely related to this thesis. Namely, posing new questions that relate the zero forcing number to simple graph parameters such as degree and order.

In April 2013 L. Eroh, C.X. Kang, and E. Yi [17] introduced an edge-equivalent notion of zero forcing, *edge zero forcing*. In their paper, Eroh et al., made the observation that each edge zero forcing set of G corresponds to a zero forcing set of $L(G)$, where $L(G)$ is the line graph of G . Using this observation, they went on to prove the following theorem:

Theorem 2.9 (Eroh-Kang-Yi [17]). *For any connected graph G ,*

$$Z(G) \leq 2Z(L(G)).$$

And they conjectured the stronger statement:

Conjecture 2.1 (Eroh-Kang-Yi [17]). *For any graph G ,*

$$Z(G) \leq Z(L(G)).$$

Later in 2013, L. Eroh, C.X. Kang, and E. Yi expanded on their previous work by considering the zero forcing number of graphs and their complements [18]. In particular, they showed the following:

Theorem 2.10 (Eroh-Kang-Yi [18]). *Let G and \bar{G} be connected graphs of order $n \geq 4$. Then,*

$$Z(G) \leq n - 3.$$

From 2006 to 2013, the zero forcing number began to appear in papers completely separated from the maximum nullity problem. So by the beginning of 2014, the zero forcing number was set to be studied solely as a graph parameter.

Chapter 3

Generalization

The k -forcing number was introduced by the author, and is a generalization of the zero forcing number. In particular, the definition of the zero forcing number is equivalent to the definition of the k -forcing number when $k = 1$. This chapter mentions the known results on k -forcing.

3.1 Initial Bounds

Let $k \geq 1$ be an integer and let G be a graph with vertices that are either colored or non-colored; let $S \subseteq V$ be the set of all initially colored vertices. The k -color change rule is defined as follows: if v is a colored vertex with k or less non-colored neighbors; then change the non-colored neighbors of v to be colored. We say that S is a k -forcing set if by iteratively applying the k -color change rule all of the vertices become colored. The cardinality of a minimum k -forcing set of G is its k -forcing number, denoted $F_k(G)$. When $k = 1$, we use the terminology *forcing number* in place of 1-forcing number, and the notation $F(G)$ in place of $F_1(G)$. It should be clear that $F(G) = Z(G)$, since the 1-color change rule is equivalent to the color-change rule.

In January of 2015, D. Amos, Y. Caro, R. Davila, and R. Pepper [3] introduced a paper which studied the k -forcing number. One simple observation given in our paper is the following:

Proposition 3.1 (Amos-Caro-Davila-Pepper [3]). *Let G be a graph with minimum*

degree δ and let k be a positive integer. Then,

$$\delta - k + 1 \leq F_k(G).$$

Notice that when $k = 1$, Proposition 3.1 is precisely Proposition 4.1.

Many properties of the k -forcing number were studied in [3], yet our main goal was to bound the k -forcing number from above. By finding new upper bounds on the k -forcing number we resolved the question posed by Meyer [21] in the affirmative, and in a much more general fashion. To see this, consider the following two theorems:

Theorem 3.1 (Amos-Caro-Davila-Pepper [3]). *Let k be a positive integer and G be a graph on $n \geq 2$ vertices, maximum degree $\Delta \geq k$, and minimum degree $\delta \geq 1$. Then,*

$$F_k(G) \leq \frac{(\Delta - k + 1)n}{\Delta - k + 1 + \min\{\delta, k\}}.$$

And for connected graphs:

Theorem 3.2 (Amos-Caro-Davila-Pepper [3]). *Let k be a positive integer and let G be a k -connected graph with order $n > k$ and maximum degree $\Delta \geq 2$. Then,*

$$F_k(G) \leq \frac{(\Delta - 2)n + 2}{\Delta + k - 2}.$$

Taking $k = 1$ in Theorem 3.1 gives the following upper bound on the zero forcing number:

Theorem 3.3 (Amos-Caro-Davila-Pepper [3]). *Let G be a graph with minimum degree $\delta \geq 1$. Then,*

$$Z(G) = F(G) \leq \frac{\Delta}{\Delta + 1}n.$$

Taking $k = 1$ in Theorem 3.2 gives another upper bound on the zero forcing number, this time for connected graphs:

Theorem 3.4 (Amos-Caro-Davila-Pepper [3]). *Let G be a connected graph with maximum degree $\Delta \geq 2$. Then,*

$$Z(G) = F(G) \leq \frac{(\Delta - 2)n + 2}{\Delta - 1}.$$

Thus, there exists functions (at least two) which bound the zero forcing number from above in terms of degree and order. Therefore, Meyer's question has been answered.

The definition of the k -forcing number is more than just a generalization of zero forcing. In particular, it is related to other well studied graph parameters as the following two theorems show:

Theorem 3.5 (Amos-Caro-Davila-Pepper [3]). *Let k be a positive integer and let G be a k -connected graph with $n > k$ vertices. Then,*

$$F_k(G) \leq n - \gamma_{k,c}(G).$$

Taking $k = 1$ in Theorem 3.5 give the following simple relationship between zero forcing and connected domination.

Theorem 3.6 (Amos-Caro-Davila-Pepper [3]). *Let G be a connected graph with $n \geq 2$ vertices. Then,*

$$Z(G) = F(G) \leq n - \gamma_c(G),$$

and this inequality is sharp.

One should observe that $n - \gamma_c(G)$ is equal to the maximum number of leaves over all spanning trees of G . So, those interested in this spanning tree parameter may find the forcing process useful.

3.2 A Greedy Algorithm

Following [3], Caro and Pepper [11] gave a greedy algorithm which improved on Theorems 3.1 and Theorem 3.2. The basic idea of their algorithm is that you choose a minimum degree vertex v , and color all but k of its neighbors. By definition, v will then color its non-colored neighbors. Let the k -forcing process continue until no further color changes are possible. Now either all of the graph is colored, or there exist some colored vertex, w (say), with more than k non-colored neighbors. So greedily color all but k of w 's neighbors now. By definition w will color its neighbors in the same manor that v did. Continuing this algorithm, all of the graph will become colored. Keeping track of how many vertices are colored by the k -forcing or the greedy scheme gives the following theorem:

Theorem 3.7 (Caro-Pepper [11]). *Let k be a positive integer and let G be a connected graph with minimum degree δ and maximum degree $\Delta \geq k + 1$.*

- i. If $\delta < \Delta = k + 1$, then $F_k(G) = 1$.*
- ii. If $\delta = \Delta = k + 1$, then $F_k(G) = 2$.*
- iii. Otherwise, if $\Delta \geq k + 2$, then the following inequalities holds,*

$$F_k(G) \leq \frac{(\Delta - k - 1)n + \max\{\delta(k + 1\Delta) + k, k(\delta - \Delta + 2)\}}{\Delta - 1}.$$

Taking $k = 1$ in the Theorem 3.7 give the following upper bound on zero forcing:

Theorem 3.8 (Caro-Pepper [11]). *Let G be a connected graph with maximum degree Δ and minimum degree δ . Then,*

$$Z(G) = F(G) \leq \frac{(\Delta - 2)n - (\Delta - \delta) + 2}{\Delta - 1}.$$

3.3 Relationship to Power Domination

Electrical grids (systems) need be continually monitored by power companies. One method of monitoring the system is to place phase measurement units (PMUs) at selected locations. Because the cost of PMUs is typically high, finding the minimum number of PMUs needed to monitor a system is desirable. In 1998, T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi, and M.A. Henning [19] introduced a graph theoretic coloring problem to model the placement of PMUs in a system.

In 2012, G.J. Chang, P. Dorbec, M. Montassier, and A. Raspaud [12] introduced k -power domination as a natural generalization of power domination. We restate the definition of k -power domination here:

Let $k \geq 1$ be an integer. For an initial set $S \subseteq V$, the sets $(\mathcal{P}_G^i(S))_{i \geq 0}$ of vertices *monitored* by S at the i -th step are defined recursively by,

1. $\mathcal{P}_G^0 = N[S]$.
2. $\mathcal{P}_G^{i+1} = \bigcup \{N[v] : v \in \mathcal{P}_G^i(S) \text{ such that } |N[v] \setminus \mathcal{P}_G^i(S)| \leq k\}$.

If $\mathcal{P}_G^{i_0} = \mathcal{P}_G^{i_0+1}$, for some i_0 , then $\mathcal{P}_G^j = \mathcal{P}_G^{i_0}$, for all $j \geq i_0$. We define $\mathcal{P}_G^\infty = \mathcal{P}_G^{i_0}$. If $\mathcal{P}_G^\infty(S) = V$, we say that S is a *k -power dominating set* of G . The cardinality of a smallest k -power dominating set is known as the *k -power domination number* of G , and is denoted $\gamma_{P,k}(G)$.

The process for which k -power dominating sets monitor a system is intimately linked to how a k -forcing set colored a graph. In particular, we now show that a set is k -power dominating if and only if it is a dominating set of a k -forcing set.

Proposition 3.2 (Davila). *Let $G = (V, E)$ be a graph. A subset of vertices $S \subseteq V$ is a k -power dominating set of G if and only if $\mathcal{P}_G^1(S)$ is a k -forcing set of G .*

Proof. (\Rightarrow). Let $k \geq 1$ and let $S \subseteq V$ be a k -power dominating set of G . Then color $\mathcal{P}_G^1(S)$, i.e., color $N[S]$. Then, either all of V is colored, or there is some colored vertex v such that $|N[v] \setminus N[S]| \leq k$, i.e., v has at most k uncolored neighbors, and will k -force. This process will continue until we have reached a set equivalent to $\mathcal{P}_G^\infty = V$, since S was k -power dominating. Hence, $N[S]$ is a k -forcing set.

(\Leftarrow). Now suppose $\mathcal{P}_G^1(S)$ is a k -forcing set. Then either all of V is colored, and S is a dominating set, and hence also k -power dominating, or there is a vertex $v \in \mathcal{P}_G^1(S)$, such that v has at most k uncolored neighbors, i.e., $|N[v] \setminus N[S]| \leq k$. This is assured at each forcing step until all of V is colored. Hence, S must be k -power dominating. \square

Since any set of vertices is clearly a dominating set of itself, we have that every k -forcing set is a k -power dominating set, i.e., $\gamma_{P,k}(G) \leq F_k(G)$, where equality can occur (consider paths for example). One way to strengthen this relationship is to consider k -forcing sets that induce graphs without isolated vertices. We call these sets *total k -forcing sets*. We define the *total k -forcing number* of G to be the minimum cardinality of a total k -forcing set in G , denoted by $F_{t,k}(G)$.

Proposition 3.3 (Davila). *Let $G = (V, E)$ be a connected graph with $F_{t,k}(G) \geq 2$. Then,*

$$\gamma_{P,k}(G) \leq \frac{F_{t,k}(G)}{2}.$$

Proof. Let $G = (V, E)$ be a connected graph and let $k \geq 1$ be an integer. Let $D \subseteq V$ be a minimum total k -forcing set. Since D induces a graph with minimum degree at least 1, we can apply the well known theorem attributed to Ore (see [22]), and get the existence of a subset $S \subseteq D$ of cardinality at most $\frac{F_{t,k}}{2}$ which dominates D . Since S dominates D , $D \subseteq \mathcal{P}_G^1(S)$. The result follows from Proposition 3.2. \square

Proposition 3.3 is sharp for cycles, and in general whenever $F_{t,k} = F_k$, will yield large improvements over the trivial relation $\gamma_{P,k}(G) \leq F_k(G)$. Determining necessary and sufficient conditions for which $F_{c,k} = F_k$ is left as an open problem and is mentioned here:

Question 3.1. *What are necessary and sufficient conditions for $F_k(G)$ to equal $F_{t,k}(G)$?*

Total k -forcing is a new graph parameter, and because of its relationship to k -power domination should be of interest to those who study domination. Moreover, studying total k -forcing sets may shed light on the structure of general k -forcing sets, and help build theory for which new bounds can be constructed.

Chapter 4

Forcing of Graphs with Large Girth

In this chapter, we describe results that appear in the paper [14]. The main idea of this chapter is that the following proposition can be improved under mild requirements.

Proposition 4.1 (Berman-Friedland-Hogben-Rothblum-Shader [8]). *Let G be a graph with minimum degree δ . Then,*

$$\delta \leq Z(G).$$

4.1 Main Results

In order to keep track of what vertices are colored at a given time, we introduce the following notation

$$S_t = \{v : v \text{ is a colored vertex after } t \text{ iterations of the color-change rule}\}.$$

We take the initial set of colored vertices to be S_0 (Since zero iterations of the color-change rule have been applied). Further, we say that a colored vertex is *active* at time t , if it has exactly one non-colored neighbor at time t .

Lemma 4.1 (Davila-Kenter [14]). *Let G be a triangle-free graph with $\delta \geq 3$. Let S be a minimum zero forcing set of G . Let $v \in S$ force w at time $t = 1$, then w has at least one neighbor not in S .*

Proof. Let G be a graph with $\delta \geq 3$, and let S_0 be a minimum zero forcing set of G

such that v forces w at time $t = 1$. By way of contradiction, assume that $N(w) \subset S_0$. Since $\delta \geq 3$, we know there exists $z \in N(w) \setminus \{v\}$. Since G is triangle-free, we know $z \notin N(v)$. Starting with a uncolored copy of G , define an initial set of colored vertices $S'_0 = S_0 \setminus \{z\}$. Since v is not adjacent to z , v forces w at time $t = 1$ under the new coloring S'_0 . Since we had assumed that $N(w)$ is initially colored in S_0 , we know that $N(w) \setminus \{z\}$ is colored in S'_0 . It follows that at time $t = 2$, w will have $d(w) - 1$ neighbors colored, and hence will force z . So we have shown that $S_0 \subset S'_2$. Since S_0 was a zero forcing set of G , S'_0 must also be a zero forcing set of G , contradicting the minimality of S_0 . \square

With Lemma 4.1, we are able to improve on the trivial minimum degree lower bound whenever the minimum degree is at least 3. This is shown by the following theorem:

Theorem 4.1 (Davila-Kenter [14]). *Let G be a triangle-free graph with minimum degree $\delta \geq 3$. Then,*

$$\delta + 1 \leq Z(G).$$

Proof. Suppose G is triangle-free, and let S be a zero forcing set realizing $Z(G)$. Since G is not a complete graph, nor an empty graph, we know at least two forcing vertices exist, call them v and w (This is true since $Z(G) \leq n - 2$, i.e., there are at least two non-colored vertices for a given minimum zero forcing set). Note that v and w may force at separate times. Without loss of generality, suppose v forces v' , at time $t = 1$. Hence, $N[v] \setminus \{v'\} \subseteq S$. It suffices to show that there is an initially colored vertex not in $N[v] \setminus \{v'\}$ at time $t = 1$.

Since w is assumed to force eventually, we have the following cases:

1. w forces at time $t = 1$.

2. w forces at time $t \geq 2$.

Case 1. Suppose w forces w' , at time $t = 1$. If w is not in the neighborhood of v , then since w was initially colored, we are done. Hence, suppose w is in the neighborhood of v . Then, since G is triangle-free, w cannot be adjacent to any neighbors of v , otherwise G would have a triangle. Furthermore, by assumption $\delta \geq 3$, so w must have at least one neighbor other than v and w' , z (say). Since w forces w' at time $t = 1$, z must be colored at time $t = 0$. Since w is adjacent to v , z cannot be a neighbor of v , otherwise G would contain a triangle. Therefore, z is initially colored outside $N[v] \setminus \{w\}$, and we are done. Altogether if there are two vertices v and w that force at time $t = 1$, the theorem holds. For the remaining case we may assume that v is the unique forcing vertex at time $t = 1$.

Case 2. Suppose that w forces at some time $t \geq 2$. Since forcing steps occur at each time step, there must be a vertex that forces at time $t = 2$; without loss of generality, assume that it is w . Since w became active at time $t = 1$, and since v is assumed to be the unique forcing vertex at time $t = 1$, it must be the case that either v forced w or a neighbor of w . If v forces w , then we are done by Lemma ???. Otherwise v forced a neighbor of w , and therefore w cannot be in the neighborhood of v as G would contain a triangle. \square

The following lemma is a useful tool for finding large zero forcing sets.

Lemma 4.2 (Davila-Kenter [14]). *Let S be a set of colored vertices of G , and let $B \subseteq S$ be a set of vertices such that $N[B] \subseteq S$. Then S is a zero forcing set of G if and only if $S \setminus B$ is a zero forcing set of $G[V \setminus B]$.*

Proof. (\Rightarrow) Suppose that $S \setminus B$ is a zero forcing set of $G[V \setminus B]$. Then adding any set of already colored vertices back to $G[V \setminus B]$ will clearly still be a zero forcing set

of G .

(\Leftarrow) Suppose that $S \subseteq V$ is a zero forcing set of G . Let $B \subseteq S$ be a set of colored vertices with only colored neighbors. Note that every vertex of B is inactive. Furthermore, any currently active vertex will be active in the graph $G[V \setminus B]$. In addition, no vertex in B has any non-colored neighbors, and any active vertex in S will have exactly one white neighbor in $V \setminus B$. Hence, forcing chains will proceed in $G[V \setminus B]$ as if they were in G , and must force the rest of the graph. \square

Lemma 4.2 is especially useful when the graph in question has an induced subgraph for which the zero forcing number is known. For example, consider the following theorem:

Theorem 4.2 (Davila-Kenter [14]). *Let G be a graph with girth $g \geq 3$. Then,*

$$Z(G) \leq n - g + 2.$$

Proof. Let G be a graph with girth g . Let $C \subseteq V$, be set of vertices realizing the girth of G that induce a cycle. Color the set $V \setminus C$. Next color any two neighbors of C , and call this collection of colored vertices S . If B is the set of all colored vertices with only colored neighbors, observe that $S \setminus B$ is a zero forcing set of $G[V \setminus B]$. By Lemma 4.2, it follows that S is a zero forcing set of G . \square

Next we show that if a triangle-free graph has the property that a minimum zero forcing set requires two neighbors to initially force, then $2\delta - 2$ is a lower bound on $Z(G)$.

Lemma 4.3 (Davila-Kenter [14]). *Let G be a triangle-free graph with minimum degree $\delta \geq 2$. If there exists a minimum zero forcing set that requires at least two neighbors*

to force at time $t = 1$, then

$$2\delta - 2 \leq Z(G).$$

Proof. Since $\delta = 2$ satisfies the Lemma 4.3 trivially, assume G is a triangle-free graph with $\delta \geq 3$ such that every zero forcing set requires at least two neighbors to force at time $t = 1$. Let v_0 and w_0 be vertices of a minimum zero forcing set S which are neighbors and both force at time $t = 1$. Since G is triangle-free, v_0 and w_0 cannot share any neighbors. In particular, both v_0 and w_0 have $\delta - 2$ initially colored neighbors. Hence the colored neighbors of v_0 and w_0 together give $2\delta - 4 \leq |S| = Z(G)$. But v_0 and w_0 are also initially colored, and so $2\delta - 2 \leq |S| = Z(G)$. \square

For graphs with girth greater than 4, we are able to show that the minimum degree lower bound can be improved by a factor of almost 2.

Theorem 4.3 (Davila-Kenter [14]). *Let G be a graph with girth $g \geq 5$, and minimum degree $\delta \geq 2$. Then,*

$$2\delta - 2 \leq Z(G).$$

Proof. Notice that when $\delta = 2$, the theorem holds. So we consider graphs with $\delta \geq 3$.

Let G be a graph with girth $g \geq 5$, and minimum degree $\delta \geq 3$. Let S be a zero forcing set realizing $Z(G)$. Since G is not a complete graph, nor an empty graph, we know at least two forcing vertices exist, v and w (say). Note that v and w may force at separate times. Without loss of generality, suppose v forces v' (say), at time $t = 1$. Hence, $N[v] \setminus \{v'\} \subseteq S$.

Since w is assumed to force eventually, we have the following cases:

1. w forces at time $t = 1$.
2. w forces at time $t \geq 2$.

Case 1. Suppose w forces w' (say), at time $t = 1$. Note w is either in the neighborhood of v , or it isn't. If w is not in the neighborhood of v , then because w and all neighbors of w other than w' were colored at time $t = 0$, we have

$$|S| \geq \delta + \delta - k,$$

Where k is the (possibly zero) number of vertices in

$$(N[v] \setminus \{v'\}) \cap (N[w] \setminus \{w'\}).$$

Because $g \geq 5$, w cannot be adjacent to more than one neighbor of v , since otherwise v , w , and the two shared neighbors would induce a 4-cycle. Hence,

$$\begin{aligned} |S| &\geq \delta + \delta - 1 \\ &= 2\delta - 1 \\ &> 2\delta - 2, \end{aligned}$$

and the theorem holds.

Next suppose w is in the neighborhood of v . Then, because G is triangle-free with $\delta \geq 3$, we recall lemma 4.3, and get,

$$|S| \geq 2\delta - 2.$$

If w forces at time $t = 1$, the theorem holds. Since w was arbitrary, if any vertex other than v forces at time $t = 1$, then the theorem holds. Hence, for Case 2 we may assume that v is the unique forcing vertex at time $t = 1$.

Case 2. Suppose that w forces at some time $t \geq 2$. Since forcing steps occur at each time step, there must be a vertex that forces at time $t = 2$, without loss of generality we take w to be this vertex. Since w became active at time $t = 1$, and since v is assumed to be the unique forcing vertex at time $t = 1$, it must be the case that v forced a neighbor of w , i.e., v' is a neighbor of w . Because v forced a neighbor of w , w cannot be in the neighborhood of v , since otherwise v, v' , and w would induce a triangle. It follows that w was initially colored. Furthermore, since w is a forcing vertex at time $t = 2$, it must have at least $\delta - 1$ neighbors that are colored at time $t = 1$. Since v was the unique forcing vertex at time $t = 1$, the neighbors of w colored at time $t = 1$, must have been initially colored at time $t = 0$. Since $g \geq 5$, v and w cannot share neighbors, since otherwise the neighbors v and w share, along with v, v' , and w , would induce a 4-cycle. Hence,

$$\begin{aligned}
|S| &\geq |N[v] \setminus \{v'\} \cup N[w] \setminus \{w'\}| \\
&= |N[v] \setminus \{v'\}| + |N[w] \setminus \{w'\}| \\
&\geq \delta + \delta \\
&= 2\delta \\
&> 2\delta - 2,
\end{aligned}$$

and the theorem holds. □

Before moving to the proofs of Theorems 4.5 and 4.6, we make note of a theorem attributed to Edholm Hogben Hyunh LaGrange and Row:

Theorem 4.4 (Edholm-Hogben-Hyunh-LaGrange-Row [?]). *Let G be a graph on $n \geq 2$ vertices, then:*

1. For $v \in V(G)$, $Z(G) - 1 \leq Z(G - v) \leq Z(G) + 1$.

2. For $e \in E(G)$, $Z(G) - 1 \leq Z(G - e) \leq Z(G) + 1$.

First we consider graphs with a cut vertex, such that after removal of such a vertex, the resulting graph has at least one component with girth at least 5. We improve on Proposition 4.1 by close to a factor of 3 for such graphs.

Theorem 4.5 (Davila-Kenter [14]). *Let G be a graph with $\delta \geq 3$ and a cut vertex v such that $G - v$ has a component with girth at least 5. Then,*

$$3\delta - 6 \leq Z(G).$$

Proof. Let G be a graph with minimum degree $\delta \geq 3$ and a cut vertex v such that $G - v$ has a component with girth at least 5. Let $G - v$ have components H_1 and H_2 . Without loss of generality suppose H_1 has girth at least 5. Recall $Z(G - v) \leq Z(G) + 1$ by Theorem 4.4. Since zero forcing is additive with respect to disjoint components we have

$$Z(H_1) + Z(H_2) \leq Z(G) + 1.$$

Further, note that $\delta(H_1, H_2) \geq \delta - 1$, since deleting v from G could have reduced the minimum degree of the components by 1. By Theorem 4.3, we know that

$$2(\delta - 1) - 2 \leq Z(H_1).$$

By the minimum degree lower bound we know $\delta - 1 \leq Z(H_2)$, and hence

$$\begin{aligned} (2(\delta - 1) - 2) + (\delta - 1) &= 3\delta - 5 \\ &\leq Z(H_1) + Z(H_2) \\ &\leq Z(G) + 1. \end{aligned}$$

Rearranging terms, we get our desired result $3\delta - 6 \leq Z(G)$. \square

For graphs with a cut edge and girth at least 5, we can drastically improve the minimum degree lower bound, as illustrated by the following theorem.

Theorem 4.6 (Davila-Kenter [14]). *Let G be a graph with minimum degree $\delta \geq 3$, girth $g \geq 5$, and cut edge e . Then,*

$$4\delta - 9 \leq Z(G).$$

Proof. Let G be a graph with minimum degree $\delta \geq 3$, girth $g \geq 5$, and cut edge e such that $G - \{e\} = H_1 \cup H_2$. Since no cut edge of G may lie on a cycle, we know that each component H_1 and H_2 has girth at least 5. Recall $Z(G - e) \leq Z(G) + 1$, from Theorem 4.4. Since zero forcing is additive with respect to disjoint components we have

$$Z(H_1) + Z(H_2) \leq Z(G) + 1.$$

Note that $\delta(H_1, H_2) \geq \delta - 1$ since deletion of an edge at most reduced the degree by 1. By Theorem 4.3, we know that,

$$2(\delta - 1) - 2 \leq Z(H_i), \quad i = 1, 2.$$

Hence,

$$\begin{aligned}
(2(\delta - 1) - 2) + (2(\delta - 1) - 2) &= 4\delta - 8 \\
&\leq Z(H_1) + Z(H_2) \\
&\leq Z(G) + 1.
\end{aligned}$$

Rearranging, we get our desired result, $4\delta - 9 \leq Z(G)$. \square

If we consider k -connected graphs, we may iteratively apply these ideas and get the following general theorem.

Theorem 4.7 (Davila-Kenter [14]). *Let G be a graph and suppose there exists a set of vertices $K = \{v_1, \dots, v_k\}$ such that $\delta(G \setminus K) \geq 2$ and $G \setminus K$ has induced girth at least 5. Then,*

$$2\delta - 3k - 2 \leq Z(G).$$

Proof. Let K be a set of k vertices whose removal from G forms a subgraph with girth at least 5 and minimum degree at least 2. Next observe that $Z(G \setminus K) \leq Z(G) + k$. Since removing a vertex will most decrease the minimum degree of G by one, removing k vertices will at most reduce the minimum degree of G by k . By Theorem 4.3, we have $2\delta(G \setminus K) - 2 \leq Z(G \setminus K) \leq Z(G) + k$. Rearranging terms we get $2\delta - 3k - 2 \leq Z(G)$. \square

The results presented in this section are all related to a general idea. The idea is: restricting the size of a smallest cycle effects how currently colored vertices may “communicate” with previously colored vertices. We discuss this in the next section.

4.2 Two Conjectures

In this section we make two conjectures. First we conjecture a lower bound for $Z(G)$ that is a function of girth and minimum degree:

Conjecture 4.1 (Davila-Kenter [14]). *Let G be a graph with girth $g \geq 3$ and minimum degree $\delta \geq 2$. Then,*

$$(g - 3)(\delta - 2) + \delta \leq Z(G).$$

Conjecture 4.1 would imply Theorem 4.3, and is also true for large graphs with large minimum degree.

Proposition 4.2 (Davila-Kenter [14]). *Given girth $g > 6$, Conjecture 4.1 is true for sufficiently large minimum degree δ , regardless of n .*

In order to prove Proposition 4.2, we recall Theorem 2.8 and a result of Chandrana and Subrmanian which gives an exponential lower bound on tree-width in terms of δ and g , and a lower bound on $Z(G)$ in terms of tree-width:

Theorem 4.8 (Chandrana and Subramanian [13]). *Let G be a graph with average degree \bar{d} , girth g , and treewidth $tw(G)$. Then,*

$$tw(G) \geq \frac{(\bar{d} - 1)^{\lfloor (g-1)/2 \rfloor - 1}}{12(g + 1)}$$

Proof of Proposition 4.2. Given g , choose δ_{min} large enough such that

$$\frac{(\delta_{min} - 1)^{\lfloor (g-1)/2 \rfloor - 1}}{12(g + 1)} \geq \delta_{min} + (\delta_{min} - 2)(g - 3).$$

This is guaranteed for $g > 6$ as the left side has polynomial degree at least 2 in δ . In which case, by the previous two theorems, we have

$$\begin{aligned}
\delta_{min} + (\delta_{min} - 2)(g - 3) &\leq \frac{(\delta_{min} - 1)^{\lfloor (g-1)/2 \rfloor - 1}}{12(g + 1)} \\
&\leq \frac{(\bar{d} - 1)^{\lfloor (g-1)/2 \rfloor - 1}}{12(g + 1)} \\
&\leq tw(G) \\
&\leq Z(G).
\end{aligned}$$

Hence, given g , the conjecture holds true for all graphs with $\delta > \delta_{min}$. □

Furthermore, taking $g = 4$ in Conjecture 4.1 yields the following simple conjecture:

Conjecture 4.2 (Davila-Kenter [14]). *Let G be a triangle-free graph with minimum degree $\delta \geq 2$. Then,*

$$2\delta - 2 \leq Z(G).$$

Conjectures 4.1 and 4.2 summarize the ideas presented in this chapter. That is, the zero forcing number is intimately related to the girth of a graph. Moreover, the conjectures presented in this chapter are simple to state and seem to be true for many graphs. Because of this, these conjectures will be considered in the future research chapter of which we now consider.

Chapter 5

Conclusions

Previous known upper and lower bounds on the zero forcing number were either computationally impractical, or trivial and far from optimal. Results from this thesis give new computationally efficient and theoretically sharp upper and lower bounds on the zero forcing number. Moreover, this thesis provides several conjectures that may inspire future research, as well as possibly lead to new observations.

All bounds given in this thesis have been shown to hold with equality for very specific families of graphs. However, it is reasonable to believe that more complicated families of graphs may exist for which the bounds also hold with equality. Because of this, new research may focus on characterizing such families of graphs.

One particularly interesting discovery in this thesis is the relationship between the girth and minimum zero forcing sets. This discovery may help those researchers who wish to find algorithms which approximate the zero forcing number.

The total k -forcing number presented in this thesis is new, and there are no known results on this parameter. In particular, what are some sharp upper and lower bounds on $F_{t,k}(G)$? Answering this question will help understand the structure of k -forcing sets, and should also be included in any future work.

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