Analytical investigation of vibration attenuation with a nonlinear tuned mass damper

by

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ABSTRACT

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Vibration attenuation devices are used to reduce the vibrations of various mechanical systems and structures. In this work, an analytical method is proposed to provide the means to investigate the influence of system parameters on the dynamic response of a system. The method of multiple scales is used to calculate an approximate broadband solution for a two degree-of-freedom system consisting of a linear primary structure and a nonlinear tuned mass damper. The model is decoupled, approximate analytical solutions are calculated, and then they are combined to produce the desired frequency-response information. The approach is initially applied to a linear two degree-of-freedom system in order to verify its performance. The approach is then applied to the nonlinear system in order to study how varying the values of parameters associated with the nonlinear absorber affect its ability to attenuate the response of the primary structure. Finally, the analytical solution is compared to a numerical solution in order to determine how well it approximates the nonlinear system frequency-response.
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Chapter 1

Introduction

Tuned mass dampers have become a very useful tool for protecting structures from unwanted vibrations. This thesis focuses on the derivation of an analytical approximate solution for the frequency-response of a primary structure with a nonlinear tuned mass damper that has softening Duffing characteristics. The results show that the analytical method is capable of producing frequency-response curves nearly 500 times faster than a basic numerical method. The accuracy of the analytical method is very high for systems with low nonlinearity and damping. As expected, the accuracy declines as the magnitudes of the nonlinearities of the system increase. However, the drastic speed improvement can outweigh the reduced accuracy if appropriate parameter ranges are tested. The results suggest that the analytical solution derived in this thesis for the Duffing vibration absorber system can save incredible amounts of time for engineers that design such systems.

The structure of this thesis is as follows. First, the motivation for this research is stated, along with a brief history of the previous work in this field. The second chapter explains how the system of interest is modeled, and how this system model is manipulated to achieve a form that is favorable for studying. In the next chapter, the method of multiple scales is applied to the linear version of the system in order to verify the accuracy of the approach. The method is then applied to the nonlinear system. In the fourth chapter, the derived solution is used to study how the values of the parameters of the system affect the frequency-response of the primary structure.
These studies are also compared to numerically derived solutions, and the similarities and differences are discussed. The final chapter contains a discussion of conclusions that can be drawn from this research, as well as future work that can be done to expand this study.

1.1 Motivation

Vibrations in a structure, whether they are caused by the system itself or by external forces, can result in significant damage. Vibration absorbers have become a popular means of mitigating such damage in the past century since Frahm designed the first of these devices in 1909 [1]. The theory behind the tuned mass damper (TMD) was developed further by Den Hartog [2]. In doing so, he provided analytical proof that linear vibration absorbers can greatly reduce the vibration amplitude of a periodically forced structure.

One of the major motivations for the development of vibration absorber theory and technology has been the protection of large structures from earthquakes. This technology is currently used for this purpose in many countries [3]. A famous example of this application is the Taipei 101 building in Taiwan, which has a vibration absorber in the form of a massive pendulum. This absorber has successfully protected the building, which was the tallest building in the world at the time of its construction, from wind and earthquake induced shaking [4].

Figure 1.1 provides a lumped-mass model of a traditional linear TMD mounted on a single degree-of-freedom (DoF) structure. In this system, there are two masses: the primary structure (PS) and the absorber mass. Most practical applications of the TMD require that the absorber mass be much smaller than the PS mass. In structural engineering, for example, it is commonly accepted that the absorber mass
should be no larger than 10% of the PS mass [5]. There are linear spring and damping elements attached between a base and the PS to represent the stiffness and damping properties of the structure, and between the PS and the absorber mass to represent the stiffness and damping properties of the absorber. When an excitation force is applied to the PS, the coupling elements allow for energy to be transferred from the PS to the absorber mass. Figure 1.2 demonstrates the effect that a TMD can have on a system.

In this figure, the vibration amplitude of the PS is given as a function of the forcing frequency \( \Omega \), where \( f(t) = F \cos(\Omega t) \). One curve (solid) corresponds with the response of the PS without the attached TMD. As is expected for single-DoF systems, there is a single peak and it occurs near the natural frequency of the PS. The second curve (dashed) shows how adding the TMD can potentially improve the frequency-
Figure 1.2: Primary structure frequency-response with and without a TMD.

response of the PS. The most important feature is that the natural frequency of the PS is now a local minimum, meaning that the PS is far less likely to sustain damage when it undergoes forced oscillations at this frequency and the frequencies near it. However, Fig. 1.2 also makes it clear that there are frequency ranges where the response amplitude is actually increased by the addition of the TMD. In general, the better the attenuation of the PS near its natural frequency, the worse the response will be on either side of the natural frequency. Research conducted on nonlinear TMDs (NTMDs) has shown that this new type of vibration absorber can improve the response of the PS past what a TMD is capable of doing. As the name suggests, a NTMD is similar to a TMD, except that it also includes nonlinear spring and/or damping elements.

Nonlinear TMDs might be able to provide better vibration attenuation than a
TMD, but they can also be more difficult to design. Linear systems can be analytically expressed using transfer functions, which provide the exact steady-state amplitude of an oscillator undergoing forced vibration. Transfer functions are very simple to derive and they can be used to plot frequency-response curves for the PS-TMD system nearly instantly. Unfortunately, nonlinear systems cannot be modeled by using transfer functions, so numerical integration methods have become a popular tool for studying NTMDs. These methods are very accurate but they can be very time consuming.

The research presented in this thesis is focused on the Duffing-type NTMD which is characterized by the presence of a cubic stiffness element. Two cases are considered: one where the PS is completely linear and another where the PS also has an additional Duffing-type nonlinearity. Approximate analytical solutions for these two forms of the system are derived using a perturbation method called the method of multiple scales (MMS), although it is first validated by applying it to the completely linear TMD system. The resulting approximate solutions are used to quickly study the system dynamics associated with various design parameter values for the NTMD in physical applications. These solutions are also compared with more accurate numerical methods that produce the same amplitude response information. The goal of this work is to show that the analytically derived approximate solutions for the Duffing NTMD system have an acceptable level of accuracy within a certain range of potential design parameters, and that the increased speed of this method compared to numerical approaches offers a favorable trade-off between speed and accuracy. The application of this research can save time and money, and allow system designers to study more variations of the NTMD.
1.2 Previous Work

A literature review of relevant publications is discussed in this section. First, the development of linear TMDs and linear TMD theory is considered. The exploration of NTMDs as a more effective vibration absorber is examined next. After that, the use of analytical methods to create approximate solutions for NTMD responses is discussed. Finally, the history of research related to the softening Duffing system as it pertains to vibration absorbers is considered.

1.2.1 Linear TMD Development

After Frahm designed the first linear TMD, and Den Hartog further developed TMD theory, Jacquot and Hoppe created a method for determining parameter values that would optimize the performance of the simple linear absorber [6]. This made the process of designing TMDs quicker and more effective. At first, TMD theory was only applied to single-DoF systems, much like the lumped-mass structure shown in Fig. 1.1. Then, Warburton determined that two-DoF systems could also benefit from the addition of linear vibration absorbers, which further expanded the field [7].

1.2.2 Nonlinear TMD Development

Housner et al. suggested that nonlinear vibration absorbers could offer advantages over the linear versions [8]. Specifically, they claimed that the linear TMD could only effectively limit vibrations within a narrow frequency band compared to what could be achieved with nonlinear versions of the absorber technology. They also heavily considered actively controlled TMDs, which operate by changing the absorber’s parameters as the forcing term changes. Rice and McCraith proposed a design for a nonlinear vibration absorber that used an arched-spring and tested this design by
using numerical and analytical methods [9]. The arched-spring method takes advantage of a geometric nonlinearity, which is easy and inexpensive to produce. More recently, Sun et al. conducted numerical simulations that proved that a semi-active TMD (STMD) and NTMD with hardening Duffing-type stiffness properties could be combined in parallel to reduce vibration amplitudes past the point of an optimally tuned linear TMD [10]. In their testing, they also used a system that took advantage of geometric principles to create a nonlinear spring element. However, the geometry of their system could be controlled to create a range of nonlinear characteristics. Eason et al. performed similar experiments by using a NTMD with hardening Duffing characteristics and a STMD in a series configuration [11]. In this work, the nonlinear stiffness of the NTMD represents the degradation of the linear stiffness element in a TMD. Although this nonlinearity negatively affected the performance of the TMD at first, since it was a perturbation from an optimized linear state, the combination with the STMD was able to more than correct this issue. The results once again showed that systems with NTMDs are capable of having improved vibration attenuation beyond what linear TMDs can provide, even though it required an additional absorber component to do so. Sun et al. and Eason et al. conducted another set of experiments on the same system where the STMD was replaced with an adaptive-length pendulum TMD (ALPTMD) [12,13]. Once again, Sun et al. considered the system with the NTMD and ALPTMD in parallel, while Eason et al. used a series configuration. Eason et al. also performed a numerical study on a linear oscillator with a strongly nonlinear Duffing NTMD to determine the relative strength of the high and low amplitude solutions that can coexist in Duffing systems [14]. They discovered that the high amplitude solution has a significant influence.
1.2.3 Analytical Description of NTMDs

The numerical methods used in the studies mentioned above can be very accurate and quick to develop. However, the time it takes to complete simulations of these systems can make it difficult to gather large amounts of data. Analytical methods, in contrast, offer approximate solutions that can be accurate over a range of parameter values, but take very little time to produce results for that range of parameter values once the model is created. It might very well be in a NTMD designer’s best interest to trade the small amount of extra accuracy that can be achieved by using the numerically derived solution for the speed of the analytically derived solution. As stated before, the motivation for accepting this trade-off is to save both money and time on the design process.

Other researchers have studied various vibration absorber configurations using a variety of analytical methods. Sun et al. developed a set of closed-form analytical expressions that defined the linear two-DoF system and were used to conduct a detailed parametric study of how each design variable affects the system response [15]. This information can be visually portrayed by using contour plots, which allow vibration absorber designers to immediately identify the parameter values that lead to the ideal response they are trying to achieve. Elías-Zúñiga and Martínez-Romero created analytical models for an asymmetrically forced, damped Heimholts-Düffing system by using Jacobi elliptic functions, the method of elliptic balance, and Fourier series analysis [16]. Ji created a first order approximation of a weakly nonlinear absorber with a cubic nonlinearity by using the method of multiple scales (MMS) to suppress primary resonance vibration. MMS is a perturbation method which provides the means to rewrite the ordinary differential equation for a weakly nonlinear system as a series of linear, inhomogeneous ordinary differential equations which can
be solved to obtain an approximate solution [17–19]. The model prepared by Ji was used to show the effectiveness of the nonlinear absorber in suppressing vibrations under primary resonance conditions [20]. Ji and Zhang implemented a first order MMS solution to determine the amplitude and phase of a nonlinear primary structure (PS) with a linear TMD [21]. Maccari used the asymptotic perturbation method to create two slow-flow equations for the amplitude and phase of a van der Pol-Düffing system at fundamental resonance [22]. Kojima and Saito used the method of harmonic balance to analytically solve a simply supported beam that had a NTMD with hardening Düffing characteristics attached to it [23]. Sayed and Hamed modeled a two-DoF pitch-roll ship system with quadratic nonlinear coupling by using a MMS solution [24].

1.2.4 Softening Düffing NTMDs

While there has been more research related to the hardening Düffing system than to the softening type, the softening Düffing system has been the focus of studies in the past. Luo et al. used the method of harmonic balance to describe a periodically forced softening Düffing system with asymmetric periodic motions [25]. Kovacic and Brennan conducted a study on which physical systems can be described by both the hardening and softening Düffing equation [26]. They found that beams are included in this set of physical systems, which suggested that understanding the Düffing system can have applications in Structural Engineering. Guo et al. modeled the softening Düffing system by using the Monte-Carlo method and found that external stochastic excitation of the system causes chaos to arise more easily [27]. Okabe et al. created an improved averaging method for softening single-DoF Düffing systems by using the Jacobian elliptic sine function as a generating solution [28]. Their method is applicable
to systems with high levels of nonlinearity, and it was verified by using numerical methods. Tsuda *et al.* analyzed the resonant frequency region of a harmonically forced softening Duffing oscillator by using the method of harmonic balance and numerical simulation [29] [30]. Natsiavas and Hagler used numerical methods to observe the steady-state response of a two-DoF Duffing system [31]. Their research focused on identifying the interaction between the two modes.

This work is focused on the derivation of an approximate first order MMS solution of a periodically forced PS with a NTMD. Two cases are considered: one where the PS is completely linear and another where the PS has an additional Duffing-type nonlinearity. All Duffing elements considered in this study have softening stiffness properties.
Chapter 2

Modeling

This chapter is devoted to the first three steps involved in creating the MMS frequency-response solution. It details the derivation of the non-dimensionalized and decoupled form of the equations of motion that define the NTMD system studied throughout this paper. A flowchart of all the steps included in creating the MMS solution can be seen in Fig. 2.1. First, the dimensional equations of motion of the nonlinear PS - nonlinear absorber system are provided. The steps involved in non-dimensionalizing the full system of equations are then discussed. Afterwards, the equations are decoupled. The resulting set of equations has completely decoupled mass and stiffness matrices, while the damping matrix and the vector of nonlinear terms remain coupled.

2.1 Equations of Motion

In order to study the dynamics of this system, a simple lumped-mass model is used. As illustrated in Fig. 2.2, the PS and NTMD are represented as coupled spring-mass-damper oscillators with additional nonlinear springs.

By using either Newton’s method or the Lagrange equation, the equations of motion for this system are obtained. The dimensional form of these equations is
Figure 2.1: Flowchart illustrating the method used to calculate an approximate analytical solution for the nonlinear PS - nonlinear absorber system.

presented as Eqn. (2.1) and Eqn. (2.2).

\[ \hat{m}_1 \dddot{x}_1 (\hat{t}) + \hat{c}_1 \ddot{x}_1 (\hat{t}) + \hat{k}_1 \dot{x}_1 (\hat{t}) + \hat{c}_2 [\ddot{x}_1 (\hat{t}) - \ddot{x}_2 (\hat{t})] + \hat{k}_2 [\dot{x}_1 (\hat{t}) - \dot{x}_2 (\hat{t})] + \hat{\alpha}_1 [x_1 (\hat{t})] = 0, \]

\[ \hat{m}_2 \dddot{x}_2 (\hat{t}) + \hat{c}_2 [\ddot{x}_2 (\hat{t}) - \ddot{x}_1 (\hat{t})] + \hat{k}_2 [\dot{x}_2 (\hat{t}) - \dot{x}_1 (\hat{t})] + \hat{\alpha}_2 [x_2 (\hat{t}) - x_1 (\hat{t})] = \hat{f} (\hat{t}), \]

where \( \hat{m}_n, \hat{c}_n, \hat{k}_n, \hat{\alpha}_n, \) and \( \hat{x}_n \) represent the mass, damping, stiffness, nonlinear stiffness, and displacement of the \( n \)th oscillator and \( n = 1 \) and \( n = 2 \) correspond the PS and the NTMD, respectively. Time is represented by \( \hat{t} \). The function \( \hat{f} \) is the excitation force applied to the system and has an assumed form of \( \hat{f} (\hat{t}) = \hat{f}_0 \cos (\hat{\omega} \hat{t}) \).

The hat symbol (\( \hat{\cdot} \)) is used to indicate that these are dimensional variables and the
prime symbol (′) is used to indicate a derivative with respect to time.

2.2 Non-dimensionalization

In order to generalize the results of this study and to facilitate the analysis, the model presented in Eqn. (2.1) and Eqn. (2.2) is non-dimensionalized with respect to both time and length. The characteristic time used in the non-dimensionalization is the period associated with the natural frequency of the PS, such that \( \hat{t} = t/\hat{\omega}_1 \). The characteristic length used in the non-dimensionalization is the static deformation of the PS, such that \( \hat{L} = f_0/\hat{k}_1 \). Additional parameters introduced through the non-
The natural frequency, damping ratios, and non-dimensional nonlinear stiffness coefficients are represented as \( \hat{\omega}_n \), \( \zeta_n \), and \( \alpha_n \), respectively. The variable \( \Omega_{21} \) is the ratio of the natural frequency of the NTMD to the natural frequency of the PS, while \( \Omega \) is the ratio of the excitation frequency to the natural frequency of the PS. The ratio of the mass of the NTMD to the mass of the PS is represented by \( \epsilon_{21} \). The non-dimensional system model produced by this process is presented as Eqn. (2.4).

\[
\begin{align*}
\ddot{X} + [C] \cdot \dot{X} + [S] \cdot X + \{A\} &= \{F\}, \\
\end{align*}
\]  

(2.4)

where \( \{X\} \) is a vector of the non-dimensional states, \([C]\) is the non-dimensional damping matrix, \([S]\) is the non-dimensional linear stiffness matrix, \(\{A\}\) is the non-dimensional nonlinear stiffness vector, and \(\{F\}\) is the non-dimensional excitation force vector. The form of these matrices and vectors are presented in Eqn. (2.5) through
Before MMS can be applied, it is necessary to apply a coordinate transformation in order to produce a pair of equations which are not coupled through the linear stiffness terms.

### 2.3 Decoupling the System of Equations

A standard approach is used to decouple the linear stiffness terms within the system model. The transform is performed by using the eigenvectors of the linear stiffness matrix \([S]\) in order to obtain a set of equations for the modal responses of the two-DoF system. This decoupling process can be found in standard vibrations textbooks such as [32, 33]. The two eigenvectors of the linear stiffness matrix are combined to produce matrix \([P]\), which is then used to define the responses of the PS and the NTMD in terms of the modal responses, \(\{X\} = [P] \cdot \{Y\}\). The new state vector consists of the modal response of the first and second mode shapes of the system, as

\[
\{X\} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},
\]

\[
[C] = \begin{bmatrix} 2(\zeta_1 + \zeta_2 \epsilon_{21} \Omega_{21}) & -2\zeta_2 \epsilon_{21} \Omega_{21} \\ -2\zeta_2 \Omega_{21} & 2\zeta_2 \Omega_{21} \end{bmatrix},
\]

\[
[S] = \begin{bmatrix} 1 + \epsilon_{21} \Omega_{21}^2 & -\epsilon_{21} \Omega_{21}^2 \\ -\Omega_{21}^2 & \Omega_{21}^2 \end{bmatrix},
\]

\[
\{A\} = \begin{bmatrix} \alpha_1 [x_1(t)]^3 + \alpha_2 [x_1(t) - x_2(t)]^3 \\ -\epsilon_{21}^{-1} \alpha_2 [x_1(t) - x_2(t)]^3 \end{bmatrix},
\]

\[
\{F\} = \begin{bmatrix} \cos(\Omega t) \\ 0 \end{bmatrix}.
\]
defined in Eqn. (2.10).

\[\{Y\} = \begin{bmatrix} y_1(t) & y_2(t) \end{bmatrix}^T \] (2.10)

The resulting set of equations describing the modal responses of the system, presented as Eqn. (2.11), have a similar form as Eqn. (2.4) with the important exception that the \(\tilde{S}\) matrix is a diagonal matrix.

\[\begin{bmatrix} \ddot{Y} \end{bmatrix} + \begin{bmatrix} \tilde{C} \end{bmatrix} \cdot \begin{bmatrix} \dot{Y} \end{bmatrix} + \begin{bmatrix} \tilde{S} \end{bmatrix} \cdot \{Y\} + \begin{bmatrix} \tilde{A} \end{bmatrix} = \begin{bmatrix} \tilde{F} \end{bmatrix} \] (2.11)

The parameter matrices and vectors for Eqn. (2.11) may be calculated by using Eqn. (2.12) through Eqn. (2.15). The matrix \([Q]\) is introduced as the inverse of matrix \([P]\).

\[\begin{bmatrix} \tilde{C} \end{bmatrix} = [Q] \cdot [C] \cdot [P], \] (2.12)

\[\begin{bmatrix} \tilde{S} \end{bmatrix} = [Q] \cdot [S] \cdot [P], \] (2.13)

\[\begin{bmatrix} \tilde{A} \end{bmatrix} = [Q] \cdot \{A\}, \] (2.14)

\[\begin{bmatrix} \tilde{F} \end{bmatrix} = [Q] \cdot \{F\}. \] (2.15)

The pair of nonlinear ordinary differential equations produced by applying the coordinate transformation are completely decoupled in the stiffness terms. The resulting damping matrix and vector of nonlinear terms, however, remain coupled. Therefore, it is expected that each mode will have an effect on the other.

After being non-dimensionalized and decoupled, the equations of motion are now in a form that is suitable for studying the dynamics of the system. This is performed by applying MMS in order to derive approximate analytical solutions of the PS and NTMD frequency-response functions. Throughout the following derivation, terms within \(\{\tilde{A}\}\) are referenced. An expanded version of this vector is provided in
Eqn. (2.16).

\[
\begin{align*}
\left\{ \tilde{A} \right\} &= \begin{cases}
S_3 [S_1 y_1 (t) + S_2 y_2 (t)]^3 + S_5 [P_{11} y_1 (t) + P_{12} y_2 (t)]^3 \\
S_4 [S_1 y_1 (t) + S_2 y_2 (t)]^3 + S_6 [P_{11} y_1 (t) + P_{12} y_2 (t)]^3
\end{cases}, \\
\end{align*}
\]  \tag{2.16}

where

\[
\begin{align*}
S_1 &= -P_{11} + P_{21}, \\
S_2 &= -P_{12} + P_{22}, \\
S_3 &= (-Q_{11} + \epsilon_1^{-1} Q_{12}) \alpha_2, \\
S_4 &= (-Q_{21} + \epsilon_2^{-1} Q_{22}) \alpha_2, \\
S_5 &= Q_{11} \alpha_1, \\
S_6 &= Q_{21} \alpha_1.
\end{align*}
\]  \tag{2.17}

In these equations, the variables \(Q_{nm}\) and \(P_{nm}\) refer to the element in the \(n^{th}\) row and the \(m^{th}\) column of matrices \([Q]\) and \([P]\), respectively. Likewise, anywhere \(\tilde{C}_{nm}\) is mentioned, this refers to the element in the \(n^{th}\) row and the \(m^{th}\) column of matrix \([\tilde{C}]\).
Chapter 3

Application of the Method of Multiple Scales

This chapter discusses the use of MMS to create approximate analytical solutions for the PS and NTMD frequency-response functions. First, the steps required to perform MMS analysis are laid out. Before applying the method to the full nonlinear system, it is verified by applying it to the linear version of the system. Finally, the same MMS procedure is applied to the nonlinear system to arrive at the desired approximate solution.

3.1 Summary of the Method

Application of MMS is based on the assumption that the solution can be written as an asymptotic series. For a two-DoF system, the responses are assumed to take the form of the series presented in Eqn. (3.1). The expansion is written by using the bookkeeping parameter $\epsilon$, where $\epsilon$ has a small positive value.

$$
X_1 = x_{1,0} + \epsilon x_{1,1} + \epsilon^2 x_{1,2} + \ldots,
$$
$$
X_2 = x_{1,0} + \epsilon x_{2,1} + \epsilon^2 x_{2,2} + \ldots
$$

(3.1)

If one response is expected to be much smaller than the other, it can be scaled with extra orders of $\epsilon$. When the method is applied, the order of the solution must be chosen. The order of an MMS solution corresponds to how many terms of the expansion shown in Eqn. (3.1) are included in the approximation. A $0^{th}$ order solution...
would only include the first term of $X_1$ and $X_2$, while a 1st order solution would include the first two terms, and so on.

Application of MMS requires the use of multiple time scales which are also defined by using the bookkeeping parameter $\epsilon$. These time scales are defined by $T_n = \epsilon^n t$ and $n = 0, 1$ are considered in this work. In order to incorporate the multiple time scales into the equation of motion, the time derivative is rewritten as a series expansion, as shown in Eqn. (3.2). In this equation, $D_n$ is used to represent a partial derivative with respect to the $T_n$ time scale.

$$\frac{d}{dt} \to D_0 + \epsilon D_1 + \epsilon^2 D_2 + \ldots \quad (3.2)$$

The expansions in Eqn. (3.1) and Eqn. (3.2) are substituted into the system’s equations of motion. System parameters may also be scaled with additional orders of $\epsilon$ to reflect any assumptions of small parameter values. The terms in the resulting equations are separated and regrouped according to the orders of $\epsilon$, which form the series of linear, inhomogeneous ordinary differential equations (ODEs) that can be solved to obtain an approximate solution.

The lowest order ODE provides the form of the solution, which includes an amplitude term and a phase term that are functions of the higher order $\epsilon$ time scales. This solution form is then substituted into the ODE for next highest order of $\epsilon$. The secular terms, which are terms that share the same frequency as the natural frequency of the left hand side of the ODE, must be set to zero because otherwise the amplitude of the solution would increase endlessly with time. Isolating all secular terms and forcing them equal to zero to ensure a periodic solution allows for the creation of modulation equations that are used to solve for the amplitude and phase of the solution in terms of some detuning parameter. In this study, the detuning parameter is always chosen to be $\sigma$. The detuning parameter represents the deviation of the non-dimensional
excitation frequency \( \Omega \) from the linear natural frequency. This provides the full amplitude and phase information for the first term of the expansion stated in Eqn. (3.1). The solution for that term can then be substituted into the next ODE to solve for the next term. This process of substitution and solving continues until all of the desired terms for the approximate solution are known.

In order to create a frequency-response curve from the MMS approximation, the terms in Eqn. (3.1) must be combined into a single term with one expression for the amplitude of the response. Any time two cosine terms at the same frequency but with different amplitudes and phases are added together in this study, the wave combination equations are used to combine the two terms into a single cosine term with one amplitude and one phase. This set of equations, which will be referenced throughout this paper, is given below

\[
A_{cmb} = \left( [A_1 \cos (\gamma_1) + A_2 \cos (\gamma_2)]^2 + [A_1 \sin (\gamma_1) + A_2 \sin (\gamma_2)]^2 \right)^{1/2}, \quad (3.3)
\]

\[
\gamma_{cmb} = \tan^{-1} \left[ \frac{A_1 \sin (\gamma_1) + A_2 \sin (\gamma_2)}{A_1 \cos (\gamma_1) + A_2 \cos (\gamma_2)} \right]. \quad (3.4)
\]

In these equations, the two waves being added have the form \( A_n \cos (\Omega t + \gamma_n) \) for \( n = 1, 2 \). The resulting combined cosine term is \( A_{cmb} \cos (\Omega t + \gamma_{cmb}) \).

A MMS solution is derived for two cases: one where the excitation frequency is originally \( \Omega = \Phi_1 \), and one where it is originally \( \Omega = \Phi_2 \), where \( \Phi_n \) is the square root of the element in the \( n \)th row and \( n \)th column of \( S \). These are the linear natural frequencies of the system. The two sets of solutions are then combined to obtain one broad-band solution. The principles outlined above are now applied to the linear system.
3.2 Linear System MMS

This section explains the derivation of the MMS approximate solution for the linear system where \( \alpha_1 = 0 \) and \( \alpha_2 = 0 \). This process is split into two subsections: performing MMS analysis based around the first natural frequency of the PS-TMD system and performing the same analysis based around the second natural frequency.

3.2.1 First Natural Frequency Excitation

For this two-DoF system, the modal responses of the PS-TMD system stated in Eqn. (2.10) are assumed to take the form of the series presented in Eqn. (3.5) and Eqn. (3.6). As mentioned before, MMS is applied twice, once to study the resonance response about each of the two natural frequencies of the system. In order to study each of these resonance conditions, the responses of the two modes are scaled differently, with the off-resonance response scaled by an additional \( \epsilon \). The expansion form presented in Eqn. (3.5) and Eqn. (3.6) corresponds to excitation at the first natural frequency of the system. Solving for all terms in this expansion will provide a 1\(^{st}\) order approximate solution.

\[
Y_1 \approx y_{1,0} + \epsilon y_{1,1}, \quad (3.5)
\]
\[
Y_2 \approx \epsilon y_{2,1} + \epsilon^2 y_{2,2}. \quad (3.6)
\]

In order to apply MMS, the bookkeeping parameter is also used to scale the coefficients in Eqn. (2.11). In this work, the damping terms and excitation terms are scaled with \( \epsilon \). Further, the off-diagonal terms of \( \tilde{C} \) are scaled with an additional \( \epsilon \). This is shown in Eqn. (3.7) and Eqn. (3.8).
\[
\begin{bmatrix}
\tilde{C}
\end{bmatrix} = \epsilon \begin{bmatrix}
\tilde{C}_{11} & \epsilon \tilde{C}_{12} \\
\epsilon \tilde{C}_{21} & \tilde{C}_{22}
\end{bmatrix}, \quad (3.7)
\]

\[
\begin{bmatrix}
\tilde{F}
\end{bmatrix} = \epsilon \begin{bmatrix}
Q_{11}\cos(\Omega t) \\
Q_{21}\cos(\Omega t)
\end{bmatrix}, \quad (3.8)
\]

For the first resonance case, the non-dimensional excitation frequency is defined as \( \Omega = \Phi_1 + \epsilon \sigma \). Under these conditions, the order \( \epsilon^0 \), order \( \epsilon^1 \), and order \( \epsilon^2 \) terms are identified, as shown in the following equations.

Order \( \epsilon^0 \) Terms:

\[
D_0^2 y_{1,0} + \Phi_1^2 y_{1,0} = 0. \quad (3.9)
\]

Order \( \epsilon^1 \) Terms:

\[
D_0^2 y_{1,1} + \Phi_1^2 y_{1,1} = \frac{1}{2} Q_{11} \left( e^{i(\Phi_1 T_0 + \sigma T_1)} + e^{-i(\Phi_1 T_0 + \sigma T_1)} \right) - \tilde{C}_{11} D_0 y_{1,0} - 2D_0 D_1 y_{1,0}, \quad (3.10)
\]

\[
D_0^2 y_{2,1} + \Phi_2^2 y_{2,1} = \frac{1}{2} Q_{21} \left( e^{i(\Phi_1 T_0 + \sigma T_1)} + e^{-i(\Phi_1 T_0 + \sigma T_1)} \right). \quad (3.11)
\]

Order \( \epsilon^2 \) Terms:

\[
D_0^2 y_{2,2} + \Phi_2^2 y_{2,2} = -\tilde{C}_{21} D_0 y_{1,0} - \tilde{C}_{22} D_0 y_{2,1} - 2D_0 D_1 y_{2,1}. \quad (3.12)
\]

From Eqn. (3.9) and Eqn. (3.11), the form of the solution to \( y_{1,0}(T_0, T_1) \) and \( y_{2,1}(T_0, T_1) \) will be

\[
y_{1,0}(T_0, T_1) = A_1(T_0, T_1)e^{i\Phi_1 T_0} + cc, \quad (3.13)
\]

\[
y_{2,1}(T_0, T_1) = A_2(T_0, T_1)e^{i\Phi_2 T_0} + \Lambda e^{i\Phi_1 T_0} + cc, \quad (3.14)
\]
where $cc$ refers to the complex conjugate of the terms that appear before it, and

\begin{align}
A_1(T_0, T_1) &= \frac{1}{2} a_1(T_1) e^{i\beta_1(T_1)}, \quad (3.15) \\
A_2(T_0, T_1) &= \frac{1}{2} a_2(T_1) e^{i\beta_2(T_1)}, \quad (3.16) \\
\Lambda &= \frac{Q_{21}}{2(\Phi_2^2 - \Phi_1^2)}. \quad (3.17)
\end{align}

The next step in the procedure is to substitute the terms from Eqn. (3.13) and Eqn. (3.14) into Eqn. (3.10) and Eqn. (3.12). Then, the secular terms are identified and equated to zero. The resulting system of equations is given below.

\begin{align}
\frac{1}{2} Q_{11} e^{i(\sigma T_1 - \beta_1(T_1))} - \frac{1}{2} i \tilde{C}_{11} \Phi_1 a_1(T_1) - i \Phi_1 a_1'(T_1) + \\
\Phi_1 a_1(T_1) \beta_1'(T_1) &= 0, \quad (3.18) \\
-\frac{1}{2} i \tilde{C}_{22} \Phi_2 a_2(T_1) - i \Phi_2 a_2'(T_1) + \Phi_2 a_2(T_1) \beta_2'(T_1) &= 0. \quad (3.19)
\end{align}

These two equations can be converted into a set of four by separating the real and imaginary terms and setting each equal to zero. This system of equations are referred to as the amplitude and phase modulation equations. They can be seen in Eqn. (3.20) through Eqn. (3.23).

\begin{align}
a_1'(T_1) &= \frac{Q_{11}}{2\Phi_1} \sin(\phi_1(T_1)) - \frac{\tilde{C}_{11}}{2} a_1(T_1), \quad (3.20) \\
a_1(T_1) \phi_1'(T_1) &= a_1(T_1) \sigma + \frac{Q_{11}}{2\Phi_1} \cos(\phi_1(T_1)), \quad (3.21) \\
a_2'(T_1) &= -\frac{\tilde{C}_{22}}{2} a_2(T_1), \quad (3.22) \\
a_2(T_1) \phi_2'(T_1) &= a_2(T_1) \sigma, \quad (3.23)
\end{align}
where

\[
\phi_1(T_1) = \sigma T_1 - \beta_1(T_1), \quad (3.24)
\]

\[
\phi_2(T_1) = \sigma T_1 - \beta_2(T_1). \quad (3.25)
\]

Since the solutions in a frequency-response curve are at steady-state, all derivative terms in Eqn. (3.20) through Eqn. (3.23) are equal to 0. Therefore, \( a_2 = 0 \). The remaining non-zero terms, shown in Eqn. (3.26) and Eqn. (3.27), can be used to solve for the steady-state values of \( a_1 \) and \( \phi_1 \).

\[
\sin(\phi_1) = \frac{\tilde{C}_{11} \Phi_1 a_1}{Q_{11}}, \quad (3.26)
\]

\[
\cos(\phi_1) = \frac{-2 \Phi_1 a_1 \sigma}{Q_{11}}. \quad (3.27)
\]

With the solutions to the first terms of Eqn. (3.5) and Eqn. (3.6) now known, the secular terms from Eqn. (3.10) and Eqn. (3.12) are completely removed, and the remaining differential equations are solved to obtain the solution for the second terms of Eqn. (3.5) and Eqn. (3.6). The differential equations are given in trigonometric form below.

\[
D_0^2 y_{1,1} + \Phi_1^2 y_{1,1} = 0, \quad (3.28)
\]

\[
D_0^2 y_{2,2} + \Phi_2^2 y_{2,2} = \tilde{C}_{21} \Phi_1 a_1 \sin(\Omega T_0 - \sigma). \quad (3.29)
\]

Solving these differential equations gives

\[
y_{1,1} = 0, \quad (3.30)
\]

\[
y_{2,2} = -\frac{a_1 \tilde{C}_{21} \Phi_1}{\Phi_1^2 - \Phi_2^2} \cos(\Omega T_0 - \sigma - \frac{\pi}{2}). \quad (3.31)
\]
Now, all terms of the first order MMS solution centered around an excitation frequency of $\Omega = \Phi_1$ have been solved. Note that in Eqn. (3.5) and Eqn. (3.6), one of the terms in each expansion is equal to zero.

### 3.2.2 Second Natural Frequency Excitation

Now, the case where $\Omega = \Phi_2 + \epsilon\sigma$ is considered. The expansion form presented in Eqn. (3.32) and Eqn. (3.33) corresponds to excitation at the second natural frequency of the system. Solving for all terms in this expansion will provide a 1st order approximate solution.

$$Y_1 \approx \epsilon y_{1,1} + \epsilon^2 y_{1,2}, \quad (3.32)$$
$$Y_2 \approx y_{2,0} + \epsilon y_{2,1}. \quad (3.33)$$

Once again, the bookkeeping parameter is also used to scale the coefficients in Eqn. (2.11). The damping and excitation terms are scaled as they were in the previous section. Under these conditions, the order $\epsilon^0$, order $\epsilon^1$, and order $\epsilon^2$ terms are identified, as shown in the following equations.

**Order $\epsilon^0$ Terms:**

$$D_0^2 y_{2,0} + \Phi_2^2 y_{2,0} = 0. \quad (3.34)$$

**Order $\epsilon^1$ Terms:**

$$D_0^2 y_{2,1} + \Phi_2^2 y_{2,1} = \frac{1}{2}Q_21 \left( e^{i(\Phi_2 T_0 + \sigma T_1)} + e^{-i(\Phi_2 T_0 + \sigma T_1)} \right) - \tilde{C}_{22} D_0 y_{2,2} - 2D_0D_1 y_{2,0} \quad (3.35)$$

$$D_0^2 y_{1,1} + \Phi_1^2 y_{1,1} = \frac{1}{2}Q_{11} \left( e^{i(\Phi_2 T_0 + \sigma T_1)} + e^{-i(\Phi_2 T_0 + \sigma T_1)} \right). \quad (3.36)$$

**Order $\epsilon^2$ Terms:**

$$D_0^2 y_{1,2} + \Phi_1^2 y_{1,2} = -\tilde{C}_{12} D_0 y_{2,0} - \tilde{C}_{11} D_0 y_{1,1} - 2D_0D_1 y_{1,1}. \quad (3.37)$$
From Eqn. (3.34) and Eqn. (3.36), the form of the solution to \( y_{1,1}(T_0, T_1) \) and \( y_{2,0}(T_0, T_1) \) will be

\[
y_{1,1}(T_0, T_1) = A_1(T_0, T_1)e^{i\Phi_1 T_0} + \Lambda e^{i\Phi_2 T_0} + cc, \tag{3.38}
\]

\[
y_{2,0}(T_0, T_1) = A_2(T_0, T_1)e^{i\Phi_2 T_0} + cc, \tag{3.39}
\]

where \( cc \) refers to the complex conjugate of the terms that appear before it, and

\[
A_1(T_0, T_1) = \frac{1}{2}a_1(T_1)e^{i\beta_1(T_1)}, \tag{3.40}
\]

\[
A_2(T_0, T_1) = \frac{1}{2}a_2(T_1)e^{i\beta_2(T_1)}, \tag{3.41}
\]

\[
\Lambda = \frac{Q_{11}}{2(\Phi_1^2 - \Phi_2^2)}. \tag{3.42}
\]

The next step in the procedure is to substitute the terms from Eqn. (3.38) and Eqn. (3.39) into Eqn. (3.35) and Eqn. (3.37). Then, the secular terms are identified and equated to zero. The resulting system of equations is given below.

\[
-\frac{1}{2}i\tilde{C}_{11}\Phi_1 a_1(T_1) - i\Phi_1 a_1' (T_1) + \Phi_1 a_1(T_1)\beta_1' (T_1) = 0, \tag{3.43}
\]

\[
\frac{1}{2}Q_{21}e^{i(\sigma T_1 - \beta(T_2))} - \frac{1}{2}i\tilde{C}_{22}\Phi_2 a_2(T_1) - i\Phi_2 a_2' (T_1) + \Phi_2 a_2(T_1)\beta_2' (T_1) = 0. \tag{3.44}
\]

The resulting amplitude and phase modulation equations can be seen in Eqn. (3.45) through Eqn. (3.48).
\[ a'_1(T_1) = -\frac{\tilde{C}_{11}}{2}a_1(T_1), \quad (3.45) \]
\[ a_1(T_1)\phi'_1(T_1) = a_1(T_1)\sigma, \quad (3.46) \]
\[ a'_2(T_1) = \frac{Q_{21}}{2\Phi_2}\sin(\phi_2(T_1)) - \frac{\tilde{C}_{22}}{2}a_2(T_1), \quad (3.47) \]
\[ a_2(T_1)\phi'_2(T_1) = a_2(T_1)\sigma + \frac{Q_{21}}{2\Phi_2}\cos(\phi_2(T_1)), \quad (3.48) \]

where
\[ \phi_1(T_1) = \sigma T_1 - \beta_1(T_1), \quad (3.49) \]
\[ \phi_2(T_1) = \sigma T_1 - \beta_2(T_1). \quad (3.50) \]

Since the solutions in a frequency-response curve are at steady-state, all derivative terms in Eqn. (3.45) through Eqn. (3.48) are equal to 0. Therefore, \(a_1 = 0\). The remaining non-zero terms, shown in Eqn. (3.51) and Eqn. (3.52), can be used to solve for the steady-state values of \(a_2\) and \(\phi_2\).

\[ \sin(\phi_2) = \frac{\tilde{C}_{22}\Phi_2 a_2}{Q_{21}}, \quad (3.51) \]
\[ \cos(\phi_2) = \frac{-2\Phi_2 a_2\sigma}{Q_{21}}. \quad (3.52) \]

With the solutions to the first terms of Eqn. (3.32) and Eqn. (3.33) now known, the secular terms from Eqn. (3.35) and Eqn. (3.37) are completely removed, and the remaining differential equations are solved to obtain the solution for the second terms of Eqn. (3.32) and Eqn. (3.33). The differential equations are given in trigonometric form below.

\[ D_0^2 y_{1,2} + \Phi_1^2 y_{1,2} = \tilde{C}_{12}\Phi_2 a_2 \sin(\Omega T_0 - \sigma), \quad (3.53) \]
\[ D_0^2 y_{2,1} + \Phi_2^2 y_{2,1} = 0. \quad (3.54) \]
Solving these differential equations gives

\[ y_{1,2} = -\frac{a_2 \tilde{C}_{12} \Phi_2}{\Phi_2^2 - \Phi_1^2} \cos(\Omega T_0 - \sigma - \frac{\pi}{2}), \quad (3.55) \]

\[ y_{2,1} = 0. \quad (3.56) \]

Now, all terms of the first order MMS solution centered around an excitation frequency of \( \Omega = \Phi_2 \) have been solved. The MMS solutions derived for the two natural frequencies of the system can be combined into one broad-band solution. First, \( Y_{m,n} \) is defined as the 1\(^{st}\) order MMS approximation of the \( m^{th} \) mode centered around the \( n^{th} \) natural frequency. The wave combination equations are then applied to \( Y_{1,1} \) and \( Y_{1,2} \) and to \( Y_{2,1} \) and \( Y_{2,2} \). The resulting broad-band solutions are referred to as \( Y_{ cmb,1} \) and \( Y_{ cmb,2} \). Next, the coordinate transformation stated in Sec. (2.3) is reversed to achieve the broad-band approximate solution to the frequency-response of the PS. The reversal equation is shown below.

\[ X_1 = P_{11}Y_{ cmb,1} + P_{12}Y_{ cmb,2}. \quad (3.57) \]

The wave combination equations are applied one final time to the terms in the right hand side of Eqn. (3.57), which results in a single amplitude term that is a function of the linear non-dimensional system parameters defined in Eqn. (2.3). Solving this frequency-response equation is done by using the Newton-Raphson method.

### 3.3 Nonlinear System MMS

This section explains the derivation of the MMS approximate solution for the nonlinear system given in Eqn. (2.1) and Eqn. (2.2). It is split into two further subsections: performing MMS analysis for excitation around the first natural frequency of the PS-NTMD system and performing the same analysis for excitation around the second
natural frequency. The final recoupled solution will be used to perform a parametric study that determines the effect that each system parameter has on the PS frequency-response.

3.3.1 First Natural Frequency Excitation

For this two-DoF system, the modal responses of the PS-NTMD system stated in Eqn. (2.10) are assumed to take the form of the series presented in Eqn. (3.58) and Eqn. (3.59). Solving for all terms in this expansion will provide a 1st order approximate solution.

\[
Y_1 \approx y_{1,0} + \epsilon y_{1,1}, \quad (3.58)
\]
\[
Y_2 \approx \epsilon y_{2,1} + \epsilon^2 y_{2,2}. \quad (3.59)
\]

In this work, the damping, excitation, and nonlinear terms are scaled with \( \epsilon \). Further, the off-diagonal terms of \( \tilde{C} \) and the off-resonant terms of \( \tilde{A} \) are scaled with an additional \( \epsilon \). This is shown in Eqn. (3.60) through Eqn. (3.62).

\[
\left[ \tilde{C} \right] = \epsilon \begin{bmatrix}
\tilde{C}_{11} & \epsilon \tilde{C}_{12} \\
\epsilon \tilde{C}_{21} & \tilde{C}_{22}
\end{bmatrix}, \quad (3.60)
\]
\[
\left\{ \tilde{F} \right\} = \epsilon \begin{bmatrix}
Q_{11} \cos(\Omega t) \\
Q_{21} \cos(\Omega t)
\end{bmatrix}, \quad (3.61)
\]
\[
\left\{ \tilde{A} \right\} = \epsilon \begin{bmatrix}
S_3 \left[ S_1 y_1(t) + S_2 y_2(t) \right]^3 + S_5 \left[ P_{11} y_1(t) + P_{12} y_2(t) \right]^3 \\
\epsilon S_4 \left[ S_1 y_1(t) + S_2 y_2(t) \right]^3 + \epsilon S_6 \left[ P_{11} y_1(t) + P_{12} y_2(t) \right]^3
\end{bmatrix}. \quad (3.62)
\]

With the non-dimensional excitation frequency defined as \( \Omega = \Phi_1 + \epsilon \sigma \), the order \( \epsilon^0 \), order \( \epsilon^1 \), and order \( \epsilon^2 \) terms are identified, as shown in the following equations.
Order $\epsilon^0$ Terms:

\[ D_0^2 y_{1,0} + \Phi_1^2 y_{1,0} = 0. \] (3.63)

Order $\epsilon^1$ Terms:

\[
D_0^2 y_{1,1} + \Phi_1^2 y_{1,1} = \frac{1}{2} Q_{11} \left( e^{i(\Phi_1 T_0 + \sigma T_1)} + e^{-i(\Phi_1 T_0 + \sigma T_1)} \right) \\
- (S_1^3 S_3 + P_{11}^3 S_5) y_{1,0}^3 - \tilde{C}_{11} D_0 y_{1,0} - 2D_0 D_1 y_{1,0},
\] (3.64)

\[
D_0^2 y_{2,1} + \Phi_2^2 y_{2,1} = \frac{1}{2} Q_{21} \left( e^{i(\Phi_1 T_0 + \sigma T_1)} + e^{-i(\Phi_1 T_0 + \sigma T_1)} \right) \). \] (3.65)

Order $\epsilon^2$ Terms:

\[
D_0^2 y_{2,2} + \Phi_2^2 y_{2,2} = -(S_1^3 S_4 + P_{11}^3 S_6) y_{1,0}^3 - \tilde{C}_{21} D_0 y_{1,0} - \tilde{C}_{22} D_0 y_{2,1} - 2D_0 D_1 y_{2,1}.
\] (3.66)

From Eqn. (3.63) and Eqn. (3.65), the form of the solution to $y_{1,0}(T_0, T_1)$ and $y_{2,1}(T_0, T_1)$ will be

\[
y_{1,0}(T_0, T_1) = A_1(T_0, T_1)e^{i\Phi_1 T_0} + cc,
\] (3.67)

\[
y_{2,1}(T_0, T_1) = A_2(T_0, T_1)e^{i\Phi_2 T_0} + \Lambda e^{i\Phi_1 T_0} + cc,
\] (3.68)

where $cc$ refers to the complex conjugate of the terms that appear before it, and

\[
A_1(T_0, T_1) = \frac{1}{2} a_1(T_1) e^{i\beta_1(T_1)},
\] (3.69)

\[
A_2(T_0, T_1) = \frac{1}{2} a_2(T_1) e^{i\beta_2(T_1)},
\] (3.70)

\[
\Lambda = \frac{Q_{21}}{2(\Phi_2^2 - \Phi_1^2)}.
\] (3.71)
Next, the terms from Eqn. (3.67) and Eqn. (3.68) are substituted into Eqn. (3.64) and Eqn. (3.66). Then, the secular terms are identified and equated to zero. The resulting system of equations is given below.

\[
\frac{1}{2}Q_{11}e^{i(\sigma T_1 - \beta(T_1))} - \frac{3}{8}(S_3^3 - P_{11}^3 S_5)(a_1(T_1))^3 - \\
\frac{1}{2}i\tilde{C}_{11}\Phi_1 a_1(T_1) - i\Phi_1 a_1'(T_1) + \Phi_1 a_1\beta_1'(T_1) = 0, \tag{3.72}
\]

\[
-\frac{1}{2}i\tilde{C}_{22}\Phi_2 a_2(T_1) - i\Phi_2 a_2'(T_1) + \Phi_2 a_2\beta_2'(T_1) = 0. \tag{3.73}
\]

Separating the real and imaginary terms and setting each equal to zero provides the amplitude and phase modulation equations seen in Eqn. (3.74) through Eqn. (3.77).

\[
a_1'(T_1) = \frac{Q_{11}}{2\Phi_1} \sin(\phi_1(T_1)) - \frac{\tilde{C}_{11}}{2} a_1(T_1), \tag{3.74}
\]

\[
a_1(T_1) \phi_1'(T_1) = a_1(T_1)\sigma + \frac{Q_{11}}{2\Phi_1} \cos(\phi_1(T_1)) - \\
\frac{3}{8\Phi_1} (S_3^3 + P_{11}^3 S_5)(a_1(T_1))^3, \tag{3.75}
\]

\[
a_2'(T_1) = -\frac{\tilde{C}_{22}}{2} a_2(T_1), \tag{3.76}
\]

\[
a_2(T_1) \phi_2'(T_1) = a_2(T_1)\sigma, \tag{3.77}
\]

where

\[
\phi_1(T_1) = \sigma T_1 - \beta_1(T_1), \tag{3.78}
\]

\[
\phi_2(T_1) = \sigma T_1 - \beta_2(T_1). \tag{3.79}
\]

Since the solutions in a frequency-response curve are at steady-state, all derivative terms in Eqn. (3.74) through Eqn. (3.77) are equal to 0. Therefore, \(a_2 = 0\). The
remaining non-zero terms, shown in Eqn. (3.80) and Eqn. (3.81), can be used to solve for the steady-state values of $a_1$ and $\phi_1$.

$$\sin(\phi_1) = \frac{\tilde{C}_{11} \Phi_1 a_1}{Q_{11}}, \quad (3.80)$$

$$\cos(\phi_1) = \frac{\frac{3}{4} (S_1^3 S_3 + P_{11}^3 S_5) a_1^3 - 2 \Phi_1 a_1 \sigma}{Q_{11}}. \quad (3.81)$$

With the solutions to the first terms of Eqn. (3.58) and Eqn. (3.59) now known, the secular terms from Eqn. (3.64) and Eqn. (3.66) are completely removed, and the remaining differential equations are solved to obtain the solution for the second terms of Eqn. (3.58) and Eqn. (3.59). The differential equations are given in trigonometric form below.

$$D_0^2 y_{1,1} + \Phi_1^2 y_{1,1} = -\frac{1}{4} (S_1^3 S_3 + P_{11}^3 S_5) a_1^3 \cos(3(\Omega T_0 - \sigma)), \quad (3.82)$$

$$D_0^2 y_{2,2} + \Phi_2^2 y_{2,2} = -\frac{3}{4} (S_1^3 S_4 + P_{11}^3 S_6) a_1^3 \cos(\Omega T_0 - \sigma) - \frac{1}{4} (S_1^3 S_4 + P_{11}^3 S_6) a_1^3 \cos(3(\Omega T_0 - \sigma)) + \tilde{C}_{21} \Phi_1 a_1 \sin(\Omega T_0 - \sigma). \quad (3.83)$$

Solving these differential equations gives

$$y_{1,1} = \frac{(S_1^3 S_3 + P_{11}^3 S_5) a_1^3 \cos(3(\Omega T_0 - \sigma))}{32 \Phi_1^2}, \quad (3.84)$$

$$y_{2,2} = \frac{1}{4(9 \Phi_1^4 - 10 \Phi_1^2 \Phi_2^2 + \Phi_2^4)} \times [3(S_1^3 S_4 + P_{11}^3 S_6)(9 \Phi_1^2 - \Phi_2^2) a_1^3 \cos(\Omega T_0 - \sigma) + (S_1^3 S_4 + P_{11}^3 S_6)(\Phi_1^2 - \Phi_2^2) a_1^3 \cos(3(\Omega T_0 - \sigma)) + 4 \tilde{C}_{21} \Phi_1 (-9 \Phi_1^2 + \Phi_2^2) a_1 \cos(\Omega T_0 - \sigma - \frac{\pi}{2})]. \quad (3.85)$$
Now, all terms of the first order MMS solution centered around an excitation frequency of \( \Omega = \Phi_1 \) have been solved. Note that in each expansion there are terms with a frequency of \( 3\Omega \). This means the frequency-response has super-harmonic resonance peaks at the 3\(^{rd} \) harmonic. Since, this study is only concerned with the frequency-response in the neighborhood of the natural frequencies of the PS-NTMD system, these higher harmonic terms will be ignored.

Leaving out the higher harmonic terms from the \( y_{2,2} \) solution still leaves two cosine terms with different phases. The wave combination equations are used to convert these into a single cosine term.

### 3.3.2 Second Natural Frequency Excitation

When there is excitation at the second natural frequency of the system, the modal response expansion takes the form of Eqn. (3.86) and Eqn. (3.87). Solving for all terms in this expansion will provide a 1\(^{st} \) order approximate solution.

\[
Y_1 \approx \epsilon y_{1,1} + \epsilon^2 y_{1,2}, \quad (3.86)
\]
\[
Y_2 \approx y_{2,0} + \epsilon y_{2,1}. \quad (3.87)
\]

The damping and excitation terms are scaled as they were in the previous section. However, \( \{\tilde{A}\} \) is scaled differently as shown in Eqn. (3.88).

\[
\{\tilde{A}\} = \epsilon \begin{cases} 
\epsilon S_3 [S_1 y_1 (t) + S_2 y_2 (t)]^3 + \epsilon S_5 [P_{11} y_1 (t) + P_{12} y_2 (t)]^3 \\
S_4 [S_1 y_1 (t) + S_2 y_2 (t)]^3 + S_6 [P_{11} y_1 (t) + P_{12} y_2 (t)]^3 
\end{cases}. \quad (3.88)
\]

With the non-dimensional excitation frequency defined as \( \Omega = \Phi_2 + \epsilon \sigma \), the order \( \epsilon^0 \), order \( \epsilon^1 \), and order \( \epsilon^2 \) terms are identified, as shown in the following equations.
Order $\epsilon^0$ Terms:

\[ D_0^2 y_{2,0} + \Phi_2^2 y_{2,0} = 0. \]  
(3.89)

Order $\epsilon^1$ Terms:

\[ D_0^2 y_{2,1} + \Phi_2^2 y_{2,1} = \frac{1}{2} Q_{21} \left( e^{i(\Phi_2 T_0 + \sigma T_1)} + e^{-i(\Phi_2 T_0 + \sigma T_1)} \right) - (S_2^3 S_4 + P_{12}^3 S_6) y_{2,0}^3 - 2D_0 D_{1y_{2,0}}. \]  
(3.90)

\[ D_0^2 y_{1,1} + \Phi_1^2 y_{1,1} = \frac{1}{2} Q_{11} \left( e^{i(\Phi_2 T_0 + \sigma T_1)} + e^{-i(\Phi_2 T_0 + \sigma T_1)} \right). \]  
(3.91)

Order $\epsilon^2$ Terms:

\[ D_0^2 y_{1,2} + \Phi_1^2 y_{1,2} = - (S_2^3 S_3 + P_{12}^3 S_5) y_{2,0}^3 - \tilde{C}_{12} D_0 y_{2,0} - \tilde{C}_{11} D_0 y_{1,1} - 2D_0 D_{1y_{1,1}}. \]  
(3.92)

From Eqn. (3.89) and Eqn. (3.91), the form of the solution to $y_{1,1}(T_0, T_1)$ and $y_{2,0}(T_0, T_1)$ will be

\[ y_{1,1}(T_0, T_1) = A_1(T_0, T_1) e^{i\Phi_1 T_0} + \Lambda e^{i\Phi_2 T_0} + cc, \]  
(3.93)

\[ y_{2,0}(T_0, T_1) = A_2(T_0, T_1) e^{i\Phi_2 T_0} + cc, \]  
(3.94)

where $cc$ refers to the complex conjugate of the terms that appear before it, and

\[ A_1(T_0, T_1) = \frac{1}{2} a_1(T_1) e^{i\beta_1(T_1)}, \]  
(3.95)

\[ A_2(T_0, T_1) = \frac{1}{2} a_2(T_1) e^{i\beta_2(T_1)}, \]  
(3.96)

\[ \Lambda = \frac{Q_{11}}{2(\Phi_1^2 - \Phi_2^2)}. \]  
(3.97)
Next, the terms from Eqn. (3.93) and Eqn. (3.94) are substituted into Eqn. (3.90) and Eqn. (3.92). Then, the secular terms are identified and equated to zero. The resulting system of equations is given below.

\[-\frac{1}{2}i\tilde{C}_{11}\Phi_1a_1(T_1) - i\Phi_1a_1' (T_1) + \Phi_1a_1\beta_1' (T_1) = 0, \quad (3.98)\]

\[\frac{1}{2}Q_{21}e^{i(\sigma T_1 - \beta_2(T_1))} - \frac{3}{8}(S_2^3S_4 - P_{12}^3S_6)(a_2(T_1))^3 - \]

\[\frac{1}{2}i\tilde{C}_{22}\Phi_2a_2(T_1) - i\Phi_2a_2' (T_1) + \Phi_2a_2\beta_2' (T_1) = 0. \quad (3.99)\]

Separating the real and imaginary terms and setting each equal to zero provides the amplitude and phase modulation equations seen in Eqn. (3.100) through Eqn. (3.103).

\[a_1' (T_1) = -\frac{\tilde{C}_{11}}{2}a_1(T_1), \quad (3.100)\]

\[a_1(T_1)\phi_1' (T_1) = a_1(T_1)\sigma, \quad (3.101)\]

\[a_2' (T_1) = \frac{Q_{21}}{2\Phi_2}\sin(\phi_2(T_1)) - \frac{\tilde{C}_{22}}{2}a_2(T_1), \quad (3.102)\]

\[a_2(T_1)\phi_2' (T_1) = a_2(T_1)\sigma + \frac{Q_{21}}{2\Phi_2}\cos(\phi_2(T_1)) - \]

\[\frac{3}{8\Phi_2}(S_2^3S_4 + P_{12}^3S_6)(a_2(T_1))^3, \quad (3.103)\]

where

\[\phi_1(T_1) = \sigma T_1 - \beta_1(T_1), \quad (3.104)\]

\[\phi_2(T_1) = \sigma T_1 - \beta_2(T_1). \quad (3.105)\]

Since the solutions in a frequency-response curve are at steady state, all derivative terms in Eqn. (3.100) through Eqn. (3.103) are equal to 0. Therefore, \(a_1 = 0\). The
remaining non-zero terms, shown in Eqn. (3.106) and Eqn. (3.107), can be used to solve for the steady-state values of $a_2$ and $\phi_2$.

\[
\sin(\phi_2) = \frac{\tilde{C}_{22} \Phi_2 a_2}{Q_{21}}, \quad (3.106)
\]

\[
\cos(\phi_2) = \frac{\tilde{\Phi}_2}{4} \left( \frac{S_3^2 S_4 + P_{12}^3 S_6}{a_2} - 2 \Phi_2 a_2 \sigma \right) \quad (3.107)
\]

With the solutions to the first terms of Eqn. (3.86) and Eqn. (3.87) now known, the secular terms from Eqn. (3.90) and Eqn. (3.92) are completely removed, and the remaining differential equations are solved to obtain the solution for the second terms of Eqn. (3.86) and Eqn. (3.87). The differential equations are given in trigonometric form below.

\[
D_0^2 y_{1,2} + \Phi_1^2 y_{1,2} = -\frac{3}{4} (S_2^3 S_3 + P_{12}^3 S_6) a_2^3 \cos(\Omega T_0 - \sigma) - \frac{1}{4} (S_2^3 S_3 + P_{12}^3 S_6) a_2^3 \cos(3(\Omega T_0 - \sigma)) + \tilde{C}_{12} \Phi_2 a_2 \sin(\Omega T_0 - \sigma), \quad (3.108)
\]

\[
D_0^2 y_{2,1} + \Phi_2^2 y_{2,1} = -\frac{1}{4} (S_2^3 S_4 + P_{12}^3 S_6) a_2^3 \cos(3(\Omega T_0 - \sigma)). \quad (3.109)
\]

Solving these differential equations gives

\[
y_{1,2} = \frac{1}{4(9\Phi_2^4 - 10\Phi_2^2 \Phi_1^2 + \Phi_1^4)} \times \left[ 3(S_3^2 S_3 + P_{12}^3 S_5)(9\Phi_2^2 - \Phi_1^2)a_2^3 \cos(\Omega T_0 - \sigma) + \right.
\]

\[
\left. (S_2^3 S_3 + P_{12}^3 S_6)(\Phi_2^2 - \Phi_1^2)a_2^3 \cos(3(\Omega T_0 - \sigma)) + \right]
\]

\[
4\tilde{C}_{12} \Phi_2 (-9\Phi_2^2 + \Phi_1^2)a_2 \cos(\Omega T_0 - \sigma - \frac{\pi}{2}) \right], \quad (3.109)
\]

\[
y_{2,1} = \frac{(S_2^3 S_4 + P_{12}^3 S_6)a_2^3 \cos(3(\Omega T_0 - \sigma))}{32\Phi_2^2}. \quad (3.110)
\]
Now, all terms of the first order MMS solution centered around an excitation frequency of $\Omega = \Phi_2$ have been solved. Just as in the last section, there are terms with a frequency of $3\Omega$, which will be ignored because they are super-harmonic terms. Further, the remaining two terms from $y_{1,2}$ are converted into a single cosine term using the wave combination equations.

Determining the broad-band solution for the PS frequency-response is achieved by following the same procedure that is stated at the end of Sec. (3.2.2). The wave combination equations are applied to $Y_{1,1}$ and $Y_{1,2}$ and to $Y_{2,1}$ and $Y_{2,2}$ to create the decoupled broad-band solutions $Y_{cmb,1}$ and $Y_{cmb,2}$. These solutions are recoupled using Eqn. (3.57), and the wave combination equation is applied one final time to the terms in the right hand side of Eqn. (3.57), which results in a single amplitude term that is a function of the non-dimensional system parameters defined in Eqn. (2.3). The approximate frequency-response solution for the linear PS - NTMD system is created by setting $\alpha_1 = 0$. The PS frequency-response function is now complete, and it can be used to conduct a parametric study of the linear PS-NTMD and nonlinear PS - NTMD systems.
Chapter 4

Analysis and Results

The completed analytical models for the linear PS - TMD, linear PS - NTMD, and nonlinear PS - NTMD systems are tested in this chapter. First, the models are used to create contour plots that show how each parameter affects the frequency-response of the system to which they are applied. These plots reveal the qualitative trends associated with varying each parameter and also make it possible to easily identify the values that lead to an optimized equal-peak frequency-response. Next, similar contour plots are created for the nonlinear systems by using a numerical method. From these more accurate figures, it is clear that the qualitative trends observed in the MMS contour plots are accurate for the range of parameter values studied. The optimal responses obtained through the numerical method are also quantitatively compared to those derived analytically. This further confirms that the approximate solution is increasingly accurate for lower levels of NTMD nonlinearity. After the qualitative and quantitative studies, the time improvement of the MMS solution is examined. It is confirmed to be several orders of magnitude faster than the numerical method.

4.1 Linear Structure - Linear Absorber System

As stated in Sec. (3.2), the model for the linear system is obtained by using Eqn. (2.1) and Eqn. (2.2) and setting the nonlinear coefficients $\alpha_1 = 0$ and $\alpha_2 = 0$. In the
non-dimensional form of the equations of motion, three non-dimensional parameters associated with the vibration absorber are selected and studied in order to identify the best attenuation of the primary structure. These parameters are the mass ratio \( \epsilon_{21} \), the frequency ratio \( \Omega_{21} \), and the damping ratio of the absorber \( \zeta_2 \). The influence of each of the parameters is studied by investigating its influence on the calculated frequency-response curves. These investigations are performed by preparing two-dimensional plots illustrating the maximum response amplitude as a function of the excitation frequency and the parameter being studied. The remaining system parameter, the damping ratio of the primary structure, is held constant at a value of \( \zeta_1 = 0.0079 \). This value is chosen in order to satisfy the low damping assumptions required by the scaling that is used in this MMS study. However, varying this parameter to some small degree would not invalidate the study. While multiple iterations are required in order to find the best value for each of the three parameters, the results presented correspond to the final values. The figure produced to illustrate the influence of the mass ratio \( \epsilon_{21} \) is presented in Fig. 4.1. This figure and all other similar figures are prepared by calculating the frequency-response of the PS by using the analytical results presented in the previous section for 200 values of the parameter being studied and 500 frequency values. By using the approximate analytical solution provided by MMS, this can be done much more quickly than a numerical integration-based approach.

The results presented in Fig. 4.1 show that varying the value of the mass ratio \( \epsilon_{21} \) influences both the peak amplitudes and locations. Increasing the mass ratio causes each frequency peak to move farther away from the natural frequency of the PS, \( \Omega = 1 \). For the same changes, the low frequency peak is seen to increase in magnitude, while the high frequency peak decreases in magnitude. While this information can
be used in the design of the absorber, a lower mass ratio value is generally preferable in order to minimize the size of the absorber. A value of $\epsilon_{21} = 0.03$ is selected and illustrated in the figure by the red dashed horizontal line. The influence of the next parameter, the frequency ratio $\Omega_{21}$, is presented in Fig. 4.2.

The information presented in Fig. 4.2 reveals that the frequency ratio $\Omega_{21}$ influences both the position and the magnitude of the peaks. As the frequency ratio is increased, the lower frequency peak increases in magnitude and shifts to approach $\Omega = 1$. For the same changes, the higher frequency peak is observed to decrease in magnitude and shift to the right away from $\Omega = 1$. An ideal value of $\Omega_{21} = 0.97$, illustrated in the figure by the red dashed horizontal line, is selected such that the maximum values of the two peaks are equal. The influence of the final parameter, the absorber’s damping ratio $\zeta_2$, is presented in Fig. 4.3.
Figure 4.2: Influence of the frequency ratio $\Omega_{21}$ on the frequency-response amplitude of the linear system.

Figure 4.3: Influence of the absorber damping ratio $\zeta_2$ on the frequency-response amplitude of the linear system.
Table 4.1: Non-dimensional parameters for the linear system.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>PS Damping Ratio</td>
<td>$\zeta_1$</td>
<td>0.79%</td>
</tr>
<tr>
<td>TMD Damping Ratio</td>
<td>$\zeta_2$</td>
<td>0.86%</td>
</tr>
<tr>
<td>Mass Ratio</td>
<td>$\epsilon_{21}$</td>
<td>0.03</td>
</tr>
<tr>
<td>Frequency Ratio</td>
<td>$\Omega_{21}$</td>
<td>0.97</td>
</tr>
</tbody>
</table>

The influence of the damping ratio of the absorber $\zeta_2$ is observed to affect the magnitudes of the peaks but not the separation between them. As illustrated in Fig. 4.3, increasing the damping ratio of the absorber results in a decrease in the magnitude of both peaks. The strength of this effect is greater for the lower frequency peak, although the difference is small enough to be difficult to be perceived visually in Fig. 4.3. A value of $\zeta_2 = 0.86\%$ is selected and illustrated in the figure by the red dashed horizontal line. At this value, the magnitude of both peaks is nearly equal.

The values identified through each of these studies are presented in Table 4.1. These parameters were chosen to match the $\epsilon$ scaling choices that were made in the MMS solution derivation. This requires a small mass ratio and small damping ratios where $\zeta_1$ is of the same order of magnitude as $\zeta_2$. Typically the mass ratio used in vibration absorbers is between $\epsilon_{21} = 0.01$ and $\epsilon_{21} = 0.1$, so $\epsilon_{21} = 0.03$ was chosen because this value is close to the lower end of that range, but also allows for values on either side of it to be used in parametric analysis. By using these values, the corresponding ideal frequency-response curve is calculated and presented in Fig. 4.4.

The frequency-response curve presented in Fig. 4.4 illustrates very strong attenuation of the response at the natural frequency of the primary structure, $\Omega = 1$, ...
and similar maximum values of the two peaks. This curve agrees with the responses published in the literature supporting the validity of the proposed method. Figure 4.5 shows a comparison between the ideal linear frequency-response generated by the MMS solution (solid) and the exact solution provided by the transfer function approach (dashed).

The difference between the two plots is nearly indistinguishable by sight, which suggests that the assumptions behind the MMS procedure outlined in Chapter 3 are valid. As expected, the MMS solution deviates from the exact solution as $\Omega$ moves farther away from the range between $\Phi_1$ and $\Phi_2$.

The results shown in this section prove that this style of broad-band MMS can be very accurate for a PS-TMD system if correct parameters are chosen. With that
established, the same parametric study is now applied to the nonlinear systems.

### 4.2 Linear Structure - Nonlinear Absorber System

The nonlinear system is first studied by using Eqn. (2.1) and Eqn. (2.2) and setting the nonlinear coefficient \( \alpha_1 = 0 \). The influence of the three non-dimensional parameters considered for the linear system as well as the nonlinear coefficient \( \alpha_2 \) are studied for the nonlinear system. The damping ratio of the primary structure is again held constant at a value of \( \zeta_1 = 0.0079 \). The results presented in the following study correspond to the final values obtained after multiple iterations. The figure produced to illustrate the influence of the mass ratio \( \epsilon_{21} \) is presented in Fig. 4.6.

The results presented in Fig. 4.6 show that varying the value of the mass ratio \( \epsilon_{21} \)
Figure 4.6: Influence of the mass ratio $\epsilon_{21}$ on the maximum frequency-response amplitude of the linear structure - nonlinear absorber system.

significantly influences the maximum values of the peaks. As the mass ratio increases, the amplitude of the lower frequency peak increases and the amplitude of the higher frequency peak decreases. The separation between the two peaks is also observed to increase as the value of the mass ratio is increased. Unlike the results in Fig. 4.1 for the linear system, high contrast changes exist in the results presented in Fig. 4.6 for each value of the mass ratio. This high contrast change corresponds to the drop-down from a high amplitude response to a low amplitude response resulting from the nonlinearity in the system. When multiple solutions coexist at the same excitation frequency, the response amplitude associated with the larger amplitude response is represented within the figure. Due to the softening nonlinear behavior represented within the model, the response peaks lean to the left and the figures correspond to
decreasing frequency sweeps. This softening behavior becomes more pronounced for lower values of the mass ratio. A value of $\epsilon_{21} = 0.03$ is selected and illustrated by the red dashed horizontal line in the figure. The influence of the next parameter, the frequency ratio $\Omega_{21}$, is presented in Fig. 4.7.

The information presented in Fig. 4.7 reveals that the frequency ratio $\Omega_{21}$ also influences both the separation between the peaks and the magnitude of the peaks for the nonlinear system. As the value of the frequency ratio deviates from $\Omega_{21} = 1.05$, the separation between the two peaks increases. The low frequency peak approaches $\Omega = 1$ while the high frequency peak moves away from $\Omega = 1$ toward larger frequency values. With regard to the maximum values of the peaks, increasing the frequency ratio causes the lower frequency peak to increase while the high frequency peak decreases. The same nonlinear softening characteristics observed in Fig. 4.6 also appear

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**Figure 4.7**: Influence of the frequency ratio $\Omega_{21}$ on the maximum frequency-response amplitude of the linear structure - nonlinear absorber system.
Figure 4.8: Influence of the absorber damping ratio $\zeta_2$ on the maximum frequency-response amplitude of the linear structure - nonlinear absorber system.

in Fig. 4.7. A value of $\Omega_{21} = 1.05$, illustrated by the red dashed horizontal line, is identified through this study. The influence of the next parameter, the damping ratio of the absorber $\zeta_2$, on the dynamic behavior of the nonlinear system is presented in Fig. 4.8.

The influence of the damping ratio of the absorber $\zeta_2$ is observed to affect the magnitudes of the peaks as well as the separation between the peaks due to the nonlinearity. As illustrated in Fig. 4.8, increasing the damping ratio of the absorber results in a decrease in the magnitude of both peaks, although the higher frequency peak is affected to a much greater degree. A value of $\zeta_2 = 0.86\%$ is selected and illustrated in the figure by the red dashed horizontal line. This value results in an equal-peak response. The influence of the final parameter being studied, the nonlinear coefficient $\alpha_2$, on the dynamic behavior of the nonlinear system is presented in Fig. 4.9.
As illustrated in Fig. 4.9, the nonlinear coefficient $\alpha_2$ is observed to influence the maximum values of the peaks and the degree to which the peaks of the frequency-response lean. When the magnitude of the nonlinear coefficient increases, the maximum value of the lower frequency peak is observed to decrease and the maximum value of the higher frequency peak increases. The red dashed horizontal line in Fig. 4.9 illustrates a value of $\alpha_2 = -6.0 \times 10^{-7}$ selected for the nonlinear system. The values identified through each of these studies are presented in Table 4.2. By using these values, the corresponding frequency-response curve is calculated and presented in Fig. 4.10.

The frequency-response curve presented in Fig. 4.10 illustrates the same strong attenuation of the response at the natural frequency of the primary structure $\Omega = 1$. 
Table 4.2: Non-dimensional parameters for the linear structure - nonlinear absorber system.

<table>
<thead>
<tr>
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</tr>
<tr>
<td>Mass Ratio</td>
<td>$\epsilon_{21}$</td>
<td>0.03</td>
</tr>
<tr>
<td>Frequency Ratio</td>
<td>$\Omega_{21}$</td>
<td>1.05</td>
</tr>
<tr>
<td>Nonlinear Coefficient</td>
<td>$\alpha_2$</td>
<td>$-6 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

Figure 4.10: Ideal frequency-response of the linear structure - nonlinear absorber system calculated with the identified parameters. High and low frequency peaks are labeled.
The parameter values selected for the nonlinear system also produce equal maximum values for the two peaks. Compared to the response seen in Fig. 4.4, Fig. 4.10 shows a maximum amplitude response approximately 10% lower, with an attenuated response near $\Omega = 1$ that is also approximately 10% lower.

4.3 Nonlinear Structure - Nonlinear Absorber System

The fully nonlinear system is studied by using the complete system defined by Eqn. (2.1) and Eqn. (2.2). The four non-dimensional parameters considered for the linear PS with a NTMD are studied for the fully nonlinear system as well. The damping ratio of the primary structure is again held constant at a value of $\zeta_1 = 0.79\%$. The nonlinear coefficient of the primary structure is also held constant at a value of $\alpha_1 = -2 \times 10^{-5}$. This value is very large compared to the range of values studied for $\alpha_2$, but it was chosen in order to make the qualitative effects of having a nonlinearity in the PS more visually distinguishable. Figure 4.11 through Fig. 4.14 illustrate the effects of varying the mass ratio $\epsilon_{21}$, frequency ratio $\Omega_{21}$, damping ratio $\zeta_2$, and nonlinear coefficient $\alpha_2$, respectively.

The results presented in Fig. 4.11 through Fig. 4.14 are very similar to those shown in Fig. 4.6 through Fig. 4.9. The key difference between the two sets is the more pronounced leaning effect caused by the added nonlinearity in the PS. This effect is seen for all parameter values over the entire range of driving frequency values. Once again, the best parameter values identified by the study are illustrated by red dashed horizontal lines on each figure.

Note that the horizontal lines in Fig. 4.11 through Fig. 4.13 appear at the same values where they appear in Fig. 4.6 through Fig. 4.8. This suggests that the best values for the mass ratio, frequency ratio, and second damping ratio identified by the
Figure 4.11: Influence of the mass ratio $\epsilon_{21}$ on the maximum frequency-response amplitude of the fully nonlinear system.

Figure 4.12: Influence of the frequency ratio $\Omega_{21}$ on the maximum frequency-response amplitude of the fully nonlinear system.
Figure 4.13 : Influence of the absorber damping ratio $\zeta_2$ on the maximum frequency-response amplitude of the fully nonlinear system.

Figure 4.14 : Influence of the nonlinear coefficient $\alpha_2$ on the maximum frequency-response amplitude of the fully nonlinear system.
Table 4.3: Non-dimensional parameters for the fully nonlinear system.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>PS Damping Ratio</td>
<td>$\zeta_1$</td>
<td>0.79%</td>
</tr>
<tr>
<td>TMD Damping Ratio</td>
<td>$\zeta_2$</td>
<td>0.86%</td>
</tr>
<tr>
<td>Mass Ratio</td>
<td>$\epsilon_{21}$</td>
<td>0.03</td>
</tr>
<tr>
<td>Frequency Ratio</td>
<td>$\Omega_{21}$</td>
<td>1.05</td>
</tr>
<tr>
<td>PS Nonlinear Coef.</td>
<td>$\alpha_1$</td>
<td>$-2 \times 10^{-5}$</td>
</tr>
<tr>
<td>Absorber Nonlinear Coef.</td>
<td>$\alpha_2$</td>
<td>$-6.4 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

study were unchanged by the addition of nonlinearity in the PS at the level considered. However, a value of $\alpha_2 = -6.4 \times 10^{-7}$ is identified in Figure 4.14, while a value of $\alpha_2 = -6 \times 10^{-7}$ is identified in Figure 4.9. Despite the fact that more softening stiffness was added to the system, the magnitude of $\alpha_2$ had to be increased in order for the best response to be realized. Therefore, the nonlinearity of the PS causes the resonant peaks to lean in the same direction as the nonlinearity of the NTMD, but the two nonlinearities exhibit opposite qualitative trends in terms of how they cause the peak magnitudes to change.

The values identified through each of these studies are presented in Table 4.3. By using these values, the corresponding frequency-response curve is calculated and presented in Fig. 4.15.

This frequency-response shows that the nonlinearity of the PS has caused the peak amplitudes to rise compared to what was shown in Fig. 4.10. Just as the previous contour plots suggest, it is also clear that these peaks lean farther to the left.
4.4 Numerical Method Comparison

Approximate analytical results by themselves are not very meaningful because it is possible that they may not represent the behavior of the actual system very well. This section is devoted to verifying how closely the MMS solution derived in the previous chapter correlates to the actual system frequency-response predicted by the system model. A numerical model developed by Verner Viisainen for a journal paper on this material is used as the basis of comparison [34]. Several types of plots are created with the numerical method to show where the MMS solution is successful and where it misrepresents the actual system that it approximates.
4.4.1 Contour Plots

Several plots are selected from the two nonlinear systems (linear and nonlinear PS) as representative examples of how the true response compares to the analytical model. The default parameter values used in the plots related to the linear PS - NTMD system are the same as those given in Table 4.2. Figure 4.16 shows how the nonlinear coefficient associated with the linear PS - NTMD system affects the frequency-response. The red dashed line identifies the value of $\alpha_2$ that the approximate model determined to give the best response. The green dashed line identifies the value that the numerical model determined. While the analytical model illustrates a value of $\alpha_2 = -6 \times 10^{-7}$, the numerical model illustrates a value of $\alpha_2 = -2.16 \times 10^{-7}$. This means the analytical model predicted a value for $\alpha_2$ nearly three times higher than the true ideal value.

There are clearly differences between this figure and the MMS counterpart, Fig. 4.9. However, the MMS model does succeed at predicting the qualitative trend observed from approximately $\alpha_2 = 0$ to $\alpha_2 = -4 \times 10^{-7}$. Specifically, the amount which the peaks lean changes at an approximately linear rate for that range of $\alpha_2$. Also, as the magnitude of $\alpha_2$ increases, the lower frequency peak decreases in magnitude, while the opposite trend is observed for the higher frequency peak. This trend is true for the entire range of $\alpha_2$ in both Fig. 4.9 and Fig. 4.16.

Another thing to note about Fig. 4.16 is that there appear to be data points missing. This is caused by the occurrence of quasi-periodic responses at those points. Since this is not considered a steady state, no value was entered when quasi-periodic behavior was observed. This reflects one of the drawbacks of the MMS model: the inability to determine when such states will occur. Another example of potential issues with the analytical model can be seen in Figure 4.17.


Figure 4.16: Influence of the nonlinear coefficient $\alpha_2$ on the maximum frequency-response amplitude of the linear structure - nonlinear absorber system (numerical method).

Figure 4.17: Influence of the frequency ratio $\Omega_{21}$ on the maximum frequency-response amplitude of the linear structure - nonlinear absorber system (numerical method).
This figure shows how the frequency ratio $\Omega_{21}$ associated with the linear PS - NTMD system affects the frequency-response. The results are quite different from what is seen in the MMS version of this figure, Fig. 4.7. This is most likely due to the fact that the value of $\alpha_2$ identified by the analytical study was used to create this figure. That value of $\alpha_2$ is too high for the actual system, which was demonstrated by Fig. 4.16. The range of Fig. 4.17 does not reach the full upper bound of the analytically derived version because the system becomes unstable past that point. Softening Duffing systems can have that quality because after a certain level of displacement, the restoring force becomes negative and the nonlinear spring starts to push the mass away from the stable equilibrium point. This is yet another quality of the studied system that the MMS model cannot predict. Once again, the red dashed line identifies the value of $\Omega_{21}$ that the approximate model determined to give the best response, and the green dashed line identifies the value that the numerical model determined. While the analytical model illustrates a value of $\Omega_{21} = 1.05$, the numerical model identifies the upper boundary of the figure, $\Omega_{21} = 1.1$. However, this value does not actually provide an equal-peak response. For the value of $\alpha_2$ considered in this figure, there is no best value for $\Omega_{21}$ because it would fall in the unstable region.

For the nonlinear PS - NTMD system, only the contour plot in which $\alpha_2$ is varied is presented. This choice is made because the other numerical studies of this system do not offer as much additional information related to the similarities and differences between the MMS and numerical models. Figure 4.18 shows how $\alpha_2$ affects the frequency-response of the numerical model. The default parameter values used in this figure are the same as those given in Table 4.3. The comparison between this and Fig. 4.14 is similar to the comparison between Fig. 4.16 and Fig. 4.9. There is still a range where the leaning of the peaks is approximately linear in relation to $\alpha_2$, and the
predicted ideal value of $\alpha_2$ from the analytical study is nearly three times larger than the numerically determined value. The main benefit of considering this comparison is that Fig. 4.18 makes it clear that the qualitative trends associated with the presence of the PS nonlinearity are the same as predicted by the approximate analytical solution. Specifically, the extra softening nonlinearity causes a higher degree of leaning, and the value of $\alpha_2$ required to create the ideal frequency-response to increase.

These contour plots offer a good qualitative comparison between the analytical and numerical results. A quantitative comparison is conducted by studying several sets of frequency-response plots.
4.4.2 Individual Frequency-Response Plots

First, the linear PS - NTMD system is considered. The exact parameter values for plots related to this system can be seen in Table 4.4. Figure 4.19 shows the analytically derived equal-peak frequency-response (solid line), as well as the numerical result by using the same parameter values (x markers). Since the analytically chosen value of $\alpha_2$ is much higher than the true value which leads to an equal-peak response, the numerically created frequency-response curve has been over-corrected. This results in a response that is not equal-peak. The two plots are quantitatively compared using a normalized root mean square (RMS) equation, which is stated in Eqn. 4.1

$$RMS = \sqrt{\frac{\sum_{i=1}^{N}(X_{a,i} - X_{n,i})^2}{\sum_{i=1}^{N}(X_{n,i})^2}}.$$  \ \ (4.1)

In the equation above, $N$ is the number of data points used in each plot, $X_{a,i}$ are frequency-response magnitudes from the analytical model, and $X_{n,i}$ are magnitudes from the numerical model. For each value of $i$, $X_{a,i}$ and $X_{n,i}$ correspond to the same value of $\Omega$.

The RMS value for the plots in Fig. 4.19 is 0.486. The best possible value is 0, whereas a value of 1 would mean all $X_{a,i}$ values are zero. Therefore, the RMS result shows a poor correlation between the two plots. In this case, large error exists because the peak amplitudes and peak alignment are very dissimilar. The numerical equal-peak response (x markers) is compared to the analytical response with equivalent parameter values (solid line) in Fig. 4.20. For this case, the two frequency peaks are aligned very well, but the peak magnitudes are still fairly far removed. The RMS value for this pair is 0.379, which is a large improvement compared to the previous
Figure 4.19: Numerical and analytical frequency-responses using the value of $\alpha_2$ determined by the MMS solution for the linear PS-NTMD system.

Table 4.4: Non-dimensional parameters for the numerical vs. analytical comparison of the linear PS-NTMD system.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>PS Damping Ratio</td>
<td>$\zeta_1$</td>
<td>0.79%</td>
</tr>
<tr>
<td>TMD Damping Ratio</td>
<td>$\zeta_2$</td>
<td>0.86%</td>
</tr>
<tr>
<td>Mass Ratio</td>
<td>$\epsilon_{21}$</td>
<td>0.03</td>
</tr>
<tr>
<td>Frequency Ratio</td>
<td>$\Omega_{21}$</td>
<td>1.05</td>
</tr>
<tr>
<td>Numerically Determined Absorber Nonlinear Coef.</td>
<td>$\alpha_2$</td>
<td>$-2.16 \times 10^{-7}$</td>
</tr>
<tr>
<td>Analytically Determined Absorber Nonlinear Coef.</td>
<td>$\alpha_2$</td>
<td>$-6 \times 10^{-7}$</td>
</tr>
</tbody>
</table>
Figure 4.20: Numerical and analytical frequency-responses using the value of $\alpha_2$ determined by the numerical solution for the linear PS - NTMD system.

case. This is due to the fact that $\alpha_2$ is smaller for the plots in Fig. 4.20, so the MMS approximate solution is more accurate.

The same two sets of plots are compared for the nonlinear PS - NTMD system. The exact parameter values for plots related to this system can be seen in Table 4.5. Figure 4.21 shows the analytically derived equal-peak frequency-response (solid line), as well as the numerical result using the same parameter values (x markers). The numerical response shows over-correcting, as it did for the system with the linear PS. The RMS value for this pair of plots is 0.416. The presence of additional nonlinearity is expected to make the RMS value higher. However, it appears as though $\alpha_1$ has a greater influence on the peak leaning of the numerical model than it does for the
Table 4.5: Non-dimensional parameters for the numerical vs. analytical comparison of the fully nonlinear system.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>PS Damping Ratio</td>
<td>ζ₁</td>
<td>0.79%</td>
</tr>
<tr>
<td>TMD Damping Ratio</td>
<td>ζ₂</td>
<td>0.86%</td>
</tr>
<tr>
<td>Mass Ratio</td>
<td>ǫ₂₁</td>
<td>0.03</td>
</tr>
<tr>
<td>Frequency Ratio</td>
<td>Ω₂₁</td>
<td>1.05</td>
</tr>
<tr>
<td>PS Nonlinear Coef.</td>
<td>α₁</td>
<td>−2 × 10⁻⁵</td>
</tr>
<tr>
<td>Numerically Determined Absorber Nonlinear Coef.</td>
<td>α₂</td>
<td>−2.28 × 10⁻⁷</td>
</tr>
<tr>
<td>Analytically Determined Absorber Nonlinear Coef.</td>
<td>α₂</td>
<td>−6.4 × 10⁻⁷</td>
</tr>
</tbody>
</table>

MMS solution. Therefore, this extra nonlinearity is actually causing the two peaks to align more closely with each other compared to the plots shown in Fig. 4.19.

The numerical equal-peak response (x markers) is compared to the analytical response with equivalent parameter values (solid line) in Fig. 4.22. The RMS value for this pair of plots is 0.397. Of the four sets of plots studied in this section, the ones with the best RMS value are those that use the smaller value of α₂ identified by the numerical model. Even for these comparisons, though, the difference in peak magnitudes is fairly large. One way to improve the agreement would be to use even smaller nonlinear coefficient values, but this limits design options. Another possibility is to improve the model itself, possibly by incorporating a higher order solution.

Despite the apparent inaccuracy of the analytical method for higher levels of nonlinearity, Fig (4.23) shows that the analytical method can still be used to predict the peak magnitudes of the equal-peak response. In this figure, the equal-peak response
Figure 4.21: Numerical and analytical frequency-responses using the value of $\alpha_2$ determined by the MMS solution for the fully nonlinear system.

Figure 4.22: Numerical and analytical frequency-responses using the value of $\alpha_2$ determined by the numerical solution for the fully nonlinear system.
Figure 4.23: Numerical and analytical equal-peak frequency-responses for the fully nonlinear system.

of the fully nonlinear system given by the numerical method (x markers) is plotted with the equal-peak response predicted by the analytical solution (solid line). As discussed previously, the value of $\alpha_2$ used to create each of these plots is very different. However, the peak magnitudes are nearly identical. This knowledge is valuable for NTMD designers because it allows them to accurately predict how much they can improve a TMD by adding a softening Duffing nonlinearity.

4.4.3 Time Study

So far, this paper has shown that the analytical solution of the softening Duffing NTMD system has many qualitative similarities with the numerically predicted response. The quantitative study showed that some adjustments should be explored
Table 4.6: Time required to create the $\alpha_2$ contour plot for the fully nonlinear system.

<table>
<thead>
<tr>
<th>Method</th>
<th>Resolution</th>
<th>Time to completion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numerical</td>
<td>500 $\times$ 200</td>
<td>58.120s</td>
</tr>
<tr>
<td>MMS</td>
<td>500 $\times$ 200</td>
<td>124.64s</td>
</tr>
</tbody>
</table>

to potentially create a more accurate model. The time study determining how much quicker the analytical model can produce results is the final measure of the model’s success.

The comparison in this section is based on creating a contour plot with the same parameter ranges shown in Fig. 4.14 and Fig. 4.18. Both the analytical model and the numerical model are used to create this plot with a high resolution; 500 values of $\Omega$ are studied for each of 200 values of $\alpha_2$. Each model is timed, and the results show that the analytical model is able to produce contour plots 466 times faster than the numerical model. The time study is shown in more detail in Table 4.6. To phrase these results another way, the analytical model is capable of doing several days worth of calculations in several minutes. This level of improvement could allow vibration absorber design engineers to do much more thorough investigations and still be able to save a significant amount of time. Even if the MMS solution is not as accurate as the numerical method, it can still be used to study a broad range of parameter values to determine a narrower range that can be tested by using the numerical method.
Chapter 5

Conclusions

5.1 Concluding Remarks

In this work, a method has been proposed to calculate the broadband response of a two degree-of-freedom system consisting of a primary structure and a nonlinear absorber. Two cases were considered for the system: one where the primary structure is linear, and another where the primary structure has an additional softening Duffing nonlinearity. The proposed approach uses the method of multiple scales to calculate an approximate analytical solution for the system after it has been decoupled by using the linear mode shapes of the system. The amplitude and phase values were numerically calculated from the derived modulation equations by using the Newton-Raphson method. The amplitude and phase information was then recombined to produce the system response.

The performance of the method was verified by applying it to the linear structure-absorber system. Comparing the linear approximate solution to the exact transfer function solution showed very good agreement between the method of multiple scales model and the exact solution. This result suggested that the proposed method of multiple scales approximation could effectively define the nonlinear system as well.

The method was applied to the fully nonlinear system, and the approximate solution was then used to study the effects that each non-dimensionalized absorber parameter has on the frequency-response of the studied systems. By using contour
plots, the qualitative trends associated with each absorber parameter was studied. Also, parameter values were identified for the vibration absorber which provide the best attenuation of the primary structure. The approximate solution suggested that the nonlinear vibration absorber is capable of producing an equal peak response with lower peak and trough magnitudes compared to what a linear absorber can achieve. However, the primary structure nonlinearity reduced this advantage.

In order to test the accuracy of the analytical solution, a numerical model of the same system was developed. The same contour plots that were produced by using the method of multiple scales approximate solution were recreated by using the numerical model. Several of these plots were provided as representative example of the numerical results. When compared to the numerical model, it is clear that the analytical solution has matching qualitative results for absorber nonlinear coefficient values between $\alpha_2 = 0$ and $\alpha_2 = -4 \times 10^{-7}$. However, the numerical model also demonstrated how the approximate solution cannot predict the presence of quasi-periodic oscillations or unstable responses. Further, the absorber nonlinear coefficient value that led to the optimal equal-peak frequency-response with the numerical model was nearly three times smaller than the analytically determined value.

A quantitative analysis was conducted by comparing several sets of numerically and analytically derived frequency-responses. In each set, a normalized root mean square calculation was used to provide a measure of how similar the analytical solution was to the numerical method. These results once again suggested that the approximate solution is increasingly more accurate for lower levels of nonlinearity. However, the normalized root mean square value is fairly high even for relatively small levels of nonlinearity. The qualitative results are promising, but the quantitative results suggest that the method of multiple scales approximate solution presented in this
thesis should not be solely relied upon for designing a nonlinear vibration absorber.

Finally, a time study was conducted to determine how much faster the analytical solution can produce results compared to the numerical method. It was discovered that the analytical solution works 466 times faster, which is a huge improvement. Therefore, the method of multiple scales approximate solution can still have practical uses, even though the quantitative results showed that the approximation could be improved. In its current form, the approximate solution could be used to study a broad range of absorber parameter values. Then, a narrower range of parameters can be identified that should be tested with the more accurate numerical method. This would not sacrifice any accuracy, and the time would still be improved.

5.2 Future Work

One possible way to improve the accuracy of the approximate solution would be to use a higher order method of multiple scales approximate solution instead of a first order solution. In most cases, the second order solution does not add much additional information after the first order. However, Nayfeh discovered that using a second order solution has the potential to significantly improve the accuracy and robustness of approximations that involve softening nonlinearities [35]. It could also be possible to further improve the speed of the analytical method by expressing the modulation equations in a recoupled form, as opposed to recoupling the results of the modulation equations after they are solved with the Newton-Raphson method. Experimentation with a physical model could also add to the integrity of the study, as many of the numerical studies cited in the Introduction were verified by such experiments.
Bibliography


