MATHEMATICAL LECTURES[1]

I

CONSISTENCY IN MATHEMATICS

THERE are two circumstances which formally evoke the danger of contradiction within the system of mathematical propositions, because they prevent these propositions from being significant statements, in the sense that we know what we mean by asking whether they are true or false. The one circumstance, as Brouwer first made clear, is the unlimited application of the terms "all" and "any" to a field of mere possibilities which is open to infinity; the other is the leveling process which mathematics blindly performed on the Russell types. Especially with regard to the second point, mathematics manifests its full participation in the servile revolt of the positive sciences against philosophy, the revolt of the anti-spiritual mind, with its democratic leveling process, against the spiritual mind and its hierarchic structure, which changed the question: "What is your intrinsic nature and what does this nature bring forth?" into the other: "What can you be used for? What profit do you yield when you are made to play your part in the process of production standardized by such and such axioms?" Brouwer's intuitionistic mathematics represents the restoration of mind to its old and sacred rights. Hilbert's formalized mathematics, however, undertakes to show that

[1] Lectures delivered at the Rice Institute, May 20, 22, and 23, 1929, by Hermann Weyl, Professor of Mathematics at the Technische Hochschule in Zürich.

the opposite party, which indeed sinks far below the mind when it demands that its overflowing wealth of "results" be accepted as literally true, is ultimately right in spite of all—ultimately, however, meaning: before a transcendental forum which we realize symbolically. In mathematics the inquiry into the genuineness or non-genuineness of the inner working of our entire western culture urges towards a more rigorous decision than can be attained in the other hazier fields of knowledge.

With regard to the first point, the usage of the terms "all" and "any," I think one does not hit quite the right spot by referring to the validity or invalidity of the principle of the excluded middle. As you know, the point in question is the confrontation of the two assertions: "Being given a set M of objects, there exists an element in M with the property $\mathfrak{A}$" and "All elements of the set M have the property non-$\mathfrak{A}$." But the stress is not on the fact that two assertions are confronted one of which occurs as the negation of the other, but on the fact that these assertions involve the terms "there exists" and "all." Moreover, it is incorrect to describe the intuitionistic point of view by saying that the *tertium non datur* applies or does not apply in the case referred to, according as M is a finite or an infinite set. The issue does not lie in the distinction between finite and infinite, but it depends on whether M is given as an aggregate of objects which are individually exhibited, one by one (and is therefore indeed finite), or not. If several pieces of chalk lie in front of me, the assertion: "All these pieces are white" is merely an abbreviation for the assertion: "This piece is white, and this piece is white, and · · · " (while I exhibit them one by one); likewise "There exists among them one red one" is an abbreviated expression for: "This one is red, or this one is red, or · · · ." But such an interpretation is possible only for sets the elements of which are exhibited. If, in opposition to the given example, we take the sequence of natural numbers 1, 2, 3, · · · and consider an assertion such as "All numbers are even," the analogous interpretation leads to an infinite logical product (I put the logical "and" and "or" into analogy with the arithmetical × and +): 1 is even, and 2 is even, and 3 is even, · · · . But this obviously has no meaning. Wherever a general proposition of this kind occurs, it has a hypothetical meaning, it assures that if you are given any definite number, for example 18, you are certain of the correctness of the judgment that 18 is an even number. It is evidently impossible and without meaning to negate such a hypothetical proposition. This fact that the negation cannot be carried out, and not the invalidity of the *tertium non datur*, is the point on which the matter hinges. The formal negation of our general judgment: "there is an odd number" would be equivalent to an infinite logical sum: 1 is odd, or 2 is odd, or · · · ; it gains significance only with a view to the explicit construction of an individual definite number, for example 17, which is established to be odd. I have therefore called the existential proposition an abstract of judgment. If knowledge is a valuable treasure, I compare this abstract to a paper which informs us of the existence of a treasure but without disclosing where it lies. With regard to the transition from finite logical sums and products to infinite ones, matters stand much the same as in the domain of arithmetical operations, where the definition of a finite sum does not *a priori* determine the meaning of an infinite sum.

I do not want to go into too much discussion to convert you to this opinion of Brouwer's. It is entirely a matter of reflection (*Besinnung*),[1] which has nothing to do with any epistemological, or perhaps even metaphysical theories, nor indeed with any arbitrarily declared mathematical axioms and their technical manipulation. Everybody will admit its truth provided he understands it.

In the development of arithmetic, we can perhaps distinguish four stages with regard to the part played by the infinite. To the first stage belongs a concrete individual judgment like $2+3=5$. To the second, for example, the following judgment of hypothetical generality: "If m and n are any two concretely given number signs, then $m+n=n+m$." I can comprehend the meaning of this proposition and convince myself intuitively of its correctness, without "generating" any other number beside the two concretely given ones m and n; it is not even necessary to form the number $m+n$. In the third stage, the actually occurring number signs are imbedded into the sequence of all *possible* numbers, which originates by means of a generating process according to the principle that from a given number a new one, the following one, can always be formed by addition of the number 1. Here the existent is projected into the background of the *possible*, the background of a manifold of possibilities which is produced and ordered according to a fixed process but is open into infinity. To this point of view corresponds the method of definition and conclusion by means of complete induction. I cannot conceive of a grosser misunderstanding than that of making the legitimacy of this procedure which refers to the *possible* depend, as Russell does, on the actual existence of infinitely many objects in the real world. I believe that here we strike the root of the

[1] L. E. J. Brouwer, "*Intuitionistische Betrachtungen über den Formalismus*," Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1928, pp. 48-52. A bibliography is included.

mathematical method in general: the *a priori* construction of the *possible* in opposition to the *a posteriori* description of what is actually given. In the fourth place, however, and this is where according to Brouwer the fault of mathematics begins, the theory of sets declared the sequence of natural numbers, a sequence open into infinity, to be a closed complete aggregate of elements existing in themselves. Take the definition: "n is even or odd according as there does or does not exist a number x such that $n = 2x$." Whoever accepts this definition which appeals to the infinite totality of numbers as having a meaning, has passed into another sphere; for him the number system has become a realm of absolute existences which is "not of this world" and of which our consciousness catches a glimpse only here and there.

The second weak spot in the body of mathematics is due to the objectivation of properties, as Russell emphatically pointed out. As long as we deal with the natural numbers as objects of investigation, single definite properties such as being even, being a prime, etc., occur. The new step is made, when a judgment like "18 is even" is no longer subsumed under the propositional form, "x is even" by the substitution $x = 18$, but under the form "x has the property y" with two empty places, x and y. Properties have hereby become objects of a different order, to which the primary objects, the numbers, can entertain the relation ϵ of "participation," μέθεξις, as Plato says. This step becomes dangerous only when the terms "all" and "any" are applied without restriction to this new realm whose objects are "the possible properties." In this sense the second step is already included in the first. It is known that the constructive generation of properties establishes a hierarchy of types, the Russell types, the neglect of which leads to striking contradictions. An objectivated property is usually called a *set* in mathematics; and the real number of analysis is essentially equivalent to the notion "set of natural numbers." Therefore this second step is of decisive importance for the foundation of the theory of the continuum as the field of all possible real numbers. Usually the word "set" when used in opposition to "property" includes this further convention: the sets corresponding to the properties $\mathfrak{A}$ and $\mathfrak{A}'$ shall be considered equal if every object with the property $\mathfrak{A}$ also has the property $\mathfrak{A}'$, and vice versa. I shall here, however, leave aside the difficulties connected with carrying this identity principle into effect in a rigorous manner.

The intention of the Hilbert proof theory is to atone by an act performed once for all for the continual titanic offences which mathematics and all mathematicians have committed and will still commit against mind, against the principle of evidence; and this act consists of gaining the insight that mathematics, if it is not true, is at least consistent. Mathematics, as we saw, abounds in propositions that are not really significant judgments. But we must abstract from the content of *all* its propositions and consider only their formal structure when we intend to show that they involve no contradictions. Thus mathematics becomes, in Hilbert's theory, a game with signs and formulas; the formulas, which consist of signs, have no meaning which they wish to convey, but they are the material of the game of demonstration: according to the rules of the game new formulas are constructed from those already at hand. The formulas that one starts with are the axioms. Among the signs the negation $\backsim$ occurs. We would have a contradiction if of two proof games which both start from the axioms and are played according to the rules of the game, one ended up with the formula $\mathfrak{b}$ and the other with the contrary one $\backsim\mathfrak{b}$. The point is to gain the insight that this can never occur, and this is an act of cognition and not play. Exactly as we can convince ourselves that in a correctly played game of chess a position with nine white pawns can never occur. The insight into the consistency of the game of mathematics has to be attained in the same direct way as that referring to the chess game. Here the consideration proceeds as follows: in the initial position there are eight white pawns. According to the rules a move may decrease, but cannot increase, the number of pawns. Ergo ··· This ergo stands for the conclusion by complete induction which follows the concrete given chess game, move by move. It is self-evident for Hilbert that the considerations by means of which the consistency proof is given in "metamathematics" are throughout endowed with the finite character postulated by Brouwer. This intuitive thinking in terms of content matter is based on evidence and not on axioms; it is conveyed by means of language which is necessarily always an uncertain tool of communication. On the other hand, mathematics itself has no need of any language, since its formulas mean nothing and convey nothing.

But why go beyond the bounds of significant judgments, since what lies beyond is totally empty and cognition can gain nothing from it? A possible answer to this question appears to be that which assigns to the ideal judgments a part similar to that played by paper money in economy: it does not add new values to the real ones but it makes their handling easier. Whenever, in Hilbert's formalized mathematics a proof yields a final formula which admits interpretation as a significant judgment, this judgment is true. But I hardly think that this purely technical employment of the formulas for the deduction of significant propositions would sufficiently justify the method. Still this is not a controversy of principle but only a question of economy. Hilbert himself gives the somewhat obscure answer that the infinite plays the part of an idea in the Kantian sense, namely that it supplements the concrete in the sense of totality. I hope I am in agreement with Hilbert when I interpret this as analogous to the construction by which I imbed the objects which are actually given to me in my consciousness into the totality of an objective world which comprises many things that are not immediately present to my mind. From the point of view of pure consciousness, it is also here not at all easy to understand what this supplementation really means. Certain epistemological schools would like to interpret it as being only a technical artifice which enables one to find one's way about more easily in one's own consciousness. But there are enough people, and I belong to them, who have the firm belief that somehow the reality of the "you" and of the exterior world embodies a higher truth than this solipsistic point of view. Theoretical physics justifies and completes the construction of that intersubjective world we build up in our natural life. The conditions which prevail here by no means correspond to Brouwer's ideal of a science. An individual assertion, an individual physical law has no meaning that can be realized by intuition and verified by experience. Only the theoretical system as a whole is capable of confrontation with experience. And it holds good if *concordance* prevails, that is, if on the basis of our theories, all indirect determinations of the same physical quantity lead to the same result.

An example will make clear what I mean. Let us consider a definite oscillation of a pendulum; and let us assume that its period can be observed directly with an error less than 0.1 second, so that periods of oscillation which are described by the theoretical physicist as differing by less than 0.1 second are actually equal, i. e., equal for our direct perception. Still there is a simple way of increasing the exactness of observation a hundredfold: one waits until 100 oscillations have taken place and then divides the observed period by 100. But this indirect determination is dependent on a certain hypothesis, namely on the hypothesis that all oscillations last the same length of time. This can of course be tested by the direct observation with an exactness of 0.1 second. However, if the theory is used in the indicated manner, not this is meant, but instead that the periods are absolutely equal, or equal with a hundredfold precision. This assumption, just as well as the assertion with regard to the period of an individual oscillation to which it leads, is without meaning for the intuitionist who respects the limits of intuitive exactness. Still it is possible to test the hypothesis in a certain sense: one finds that the period of duration of m successive oscillations is to that of n oscillations as m is to n, where m and n are large numbers (for the test we arbitrarily choose several series of oscillations). If I interpret Hilbert correctly, an analogous situation is already prevailing in pure mathematics.

The formalist who abides by his principles must leave the question unanswered why he chooses just *these* axioms for the starting-point of his proof game. Also his interest in the fact that no contradiction occurs can hardly be justified or can at most be justified by the following remark. If two games lead to the formulas $\mathfrak{b}$ and $\backsim\mathfrak{b}$, then, if $\mathfrak{a}$ is an arbitrary given formula, it is possible to obtain the formula $\mathfrak{a}$ by two additional moves, as final result. It is consequently *a priori* certain that one can prove any arbitrary formula $\mathfrak{a}$ and one has a simple fixed rule according to which to do it. In this case the game would be tedious; still it would only be tedious if I knew the contradiction. If, however, we consider this game of formulas as a symbolic expression of a theory about the world, consistency is involved in the above described concordance. Thus we get a more satisfactory answer: only a consistent theory can lead to concordant results when it is applied to experience; the consistency is that part of the concordance which refers only to the theory itself, the part in which the sphere of what is sensually given is not yet touched. It is the task of the mathematician to see that the theories of the concrete sciences satisfy this condition *sine qua non* of being formally definite and consistent.

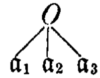
The development of science has shown clearly that different theoretical constructions of the world satisfy the postulate of concordance. The decision between the theories which compete in this manner is practically cogent for every open-minded scientist, yet it is hard to say precisely what brings it about. On the other hand it is scarcely up to us, the mathematicians and physicists, to account for this question; for in this respect we are at the mercy of the decisions cast in the history of mind, destitute of that ultimate insight that Brouwer postulates. What truth means in physical theories is a philosophical or epistemological problem rather than a physical one.

After these general remarks, I should like to go into a more detailed discussion of the structure of Hilbert's formalized mathematics.[1] As long as the transfinite is excluded, only two kinds of signs occur in the formulas; the *constants* like 1, 2; and the *operations.* Logical operations are $\backsim$ (negation), & (and), $\vee$ (or), $\rightarrow$ (implies); the first one is one-membered, the others are two-membered. The two-membered operations $=$ (is equal to) and ϵ (is element of) may be considered as logico-arithmetical ones. The operations σ (generates out of $\mathfrak{a}$ the natural number following $\mathfrak{a}$)

[1] This description follows, however, more closely a paper of J. v. Neumann, (*Math. Zeitschrift*, vol. 26, 1927, page 1) than Hilbert's own formal system.

and Z ($Z\mathfrak{a}$ read: $\mathfrak{a}$ is a natural number) are purely arithmetical one-membered operations. But I should explain the sense in which I speak throughout of operations and not of relations and properties besides. $\rightarrow$, for example, stands for the operation which generates the judgment: "$\mathfrak{a}$ implies $\mathfrak{b}$" from the two judgments $\mathfrak{a}$ and $\mathfrak{b}$; Z is the operation which generates the assertion $Z\mathfrak{a}$: "$\mathfrak{a}$ is a natural number" from $\mathfrak{a}$. These remarks are of course merely explanatory and are intended to recall the correspondence between the formulas of our formalized mathematics and certain propositions of ordinary mathematics which are meant as actual assertions of something.

If we include the infinite, two new kinds of signs become necessary: *variables* like x, y, and *integrations*. By means of the integration Σ_x the assertion $\Sigma_x\mathfrak{A}(x)$: "There is an x for which $\mathfrak{A}(x)$ holds," is obtained from the proposition $\mathfrak{A}(x)$ with one variable x. It is distinguished from the operations by the fact that it contains an arbitrary variable x as an index and "ties up" this variable in the formula standing behind it, i.e., deprives it of its capability of being substituted, exactly as though a definite thing, a constant, had been substituted for this variable. We are now in a position to describe in general what a *formula* is. Let it be written in the form of a genealogical tree such that an operation appears as the father of the terms on which it works. Thus a 1, 2, · · · membered operation is always followed by 1, 2, · · · signs respectively; the immediate progeny of an integration consists of a single sign, while a constant or a variable is always a last member without descendants. At the head of the genealogical tree we find a sign of integration or operation, and all its branches end with constants or variables. We can describe the same thing inductively as follows: An individual constant or variable in itself constitutes a basic formula. Out of these basic formulas, we construct derived formulas according to the two following principles: (1) If, for example, three formulas $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$, are given, and O is a definite three-membered operation, a new formula is obtained



by appending $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$, separately under the sign O. (2) If a formula $\mathfrak{a}$ is given, and if J_x is an integration which carries a certain variable x as an index, a new formula is obtained by hanging $\mathfrak{a}$ under the sign J_x. To make a homogeneous description possible, we thus, for example, write $\mathfrak{a} \overset{=}{\diagup\;\diagdown} \mathfrak{b}$ instead of $\mathfrak{a}=\mathfrak{b}$. But afterwards we may return to the habitual way of writing so as to prevent the formulas from having too strange an aspect. Let us do this, and also replace the operational symbol σ by the symbol $+1$ which shall be put after the formula to which it refers ($\mathfrak{a}+1$ instead of $\overset{\sigma}{\underset{\mathfrak{a}}{|}}$). Furthermore I have to describe the process of substitution. $\mathfrak{A}$ or $\mathfrak{A}(x)$ may be, as we always assume in the following, an arbitrary formula in which at most *one* variable x occurs free (not tied up), and $\mathfrak{b}$ (or $\mathfrak{c}$) a formula without a free variable, a so-called normal formula, then x shall be replaced by the entire formula $\mathfrak{b}$ everywhere in $\mathfrak{a}$ where it occurs *free*. The process of substitution which is thus intuitively described, again produces a formula; this is the formula we have in mind when we use the abbreviation $\mathfrak{A}(\mathfrak{b})$. The substitution rule would turn out to be more complicated if $\mathfrak{A}$ contained several variables free and if free variables also occurred in $\mathfrak{b}$. For example, let $\int_x$ stand for integration with respect to x from 0 to 1; then it is permissible to replace y in the correct formula

$$\int_x xy = \tfrac{1}{2}y$$

by a constant or a function which does not contain the variable x. On the other hand, nonsense results when y is replaced by a quantity containing x, for example by x itself: $\int_x xx = \frac{1}{2}x$ instead of the correct result $\int_x xx = \frac{1}{3}$. However, we avoid these complications by the restriction mentioned above.

German letters are used in general descriptions for means of communication; they belong to the language and are not signs in the same sense as 1 or x: men at our game of mathematics. In the course of the development of mathematics, new signs can continually be introduced; but of course we must always make the accompanying remark that the sign is a constant, a variable, a 1, or 2, or 3, · · · membered operation, or an integration.

Now for the axioms! First the finite logical axioms such as

$$\binom{*}{*} \qquad \mathfrak{b}\to(\mathfrak{c}\to\mathfrak{b}).$$

It states: if you have any two definite formulas $\mathfrak{b}$ and $\mathfrak{c}$ without free variables, put them together to the formula $\mathfrak{b}\to(\mathfrak{c}\to\mathfrak{b})$; you may then use this formula as an axiom. Thus $\binom{*}{*}$ is not an axiom itself, but a general rule for formation of axioms, an inexhaustible source of axioms. It is not necessary here to enumerate the few finite logical axiom rules. Logic appears in the mathematics game as playing still another entirely different part: it furnishes the *rules of the game*. The only rule of moves is the following: If you have produced a formula $\mathfrak{b}$ and a formula $\mathfrak{b}\to\mathfrak{c}$ in which the same formula $\mathfrak{b}$ stands to the left of the sign $\to$, you can put down the formula $\mathfrak{c}$. A mathematical proof consists in forming axioms according to the axiom rules and proceeding from them to new formulas by the repeated application of the syllogism rule just described. What is obtained in this manner are *provable* or rather *proved* formulas. One can judge from the looks of a complete genealogical tree of signs whether it is a formula or not. But one cannot judge from the looks of a complete formula whether it is provable. This is mainly caused by the fact that during its inductive construction a formula increases at every step, while in the syllogism two formulas $\mathfrak{b}$ and $\mathfrak{b} \rightarrow \mathfrak{c}$ combine to a new one $\mathfrak{c}$ which is shorter than the second premise, so that extension and contraction continually alternate in the proof game.

The axiom rules of equality, in which variables and the process of substitution already play a part, are the following:

$$\mathfrak{b} = \mathfrak{b}$$

$$(\mathfrak{b} = \mathfrak{c}) \rightarrow (\mathfrak{A}(\mathfrak{b}) = \mathfrak{A}(\mathfrak{c})).$$

They have an intermediate position between pure logic and mathematics. In the third place we have the purely arithmetical axioms with which we are well familiar and which relate to the notion of natural numbers, as for instance

$Z1$ (This is an individual definite axiom, not a rule for the formation of axioms.)

$Z\mathfrak{b} \rightarrow Z(\mathfrak{b}+1)$.

Now comes the transfinite part of logic. With regard to "there exists," Σ_x, and "all," Π_x, we can for the time being only establish the following rules:

(*) $\mathfrak{A}(\mathfrak{b}) \rightarrow \Sigma_x \mathfrak{A}(x)$ and $\Pi_x \mathfrak{A}(x) \rightarrow \mathfrak{A}(\mathfrak{b})$.

It is possible to infer existence from other assumptions, and it is possible to derive a particular application from a general assertion. But one cannot foresee at the moment how we can conclude in the opposite direction, how anything else can be inferred from the existence, or how a general assertion can follow from any other premise. I must describe how Hilbert extricates himself from this difficulty. (By the way, in our second axiom one can see very well how the German letters as tools of communication, so to speak, represent the hypothetically general, whereas the formal sign Π_x denotes the infinite logical product. This may help to clarify the distinction between the two ideas.) Let us take the property:

"x is honest" to be $\mathfrak{A}(x)$. If, in opposition to Brouwer, one appeals to the alternative that there must either be an honest man or that all men are dishonest, then an Aristides, who is the representative of honesty, can be found who is established to be honest if any man is honest. In the first case we choose for Aristides one of the honest men that exist, and in the second case an arbitrary one. But in order to be able to construct this Aristides, this representative, for every property, not only honesty, that is for every formula containing one single free variable x, we imagine that we have a divine automaton which accomplishes this task; when we insert the property $\mathfrak{A}$ into it, it produces the desired representative $\rho_x\mathfrak{A}$ which is sure to have the property $\mathfrak{A}$ if there is any individual of this kind. ρ_x is an integration sign. If we had an automaton like this at our disposal we would be free from the troubles caused by "all" and "any"; but of course the belief in its existence is pure nonsense. Mathematics, however, behaves as though it did exist. That can be expressed in an axiom rule, and the establishment of this rule is legitimate in formalized mathematics, provided its application does not lead to contradictions.

Thus we now add the following rules to (*):

$$\Sigma_x\mathfrak{A}(x)\rightarrow\mathfrak{A}(\rho_x\mathfrak{A}),\ \mathfrak{A}(\rho_x(\backsim\mathfrak{A}))\rightarrow\Pi_x\mathfrak{A}(x).$$ [1]

Naturally they do not accomplish the same as the fictitious automaton, for it does not divulge what $\rho_x\mathfrak{A}$ is for a given formula $\mathfrak{A}$. Only in special cases a formula like $\rho_x\mathfrak{A}=1$ can result as final formula of a proof which starts from the axioms.

Now a few words concerning the second difficulty of transfinite character, the objectivation of properties. $\mathfrak{A}(x)$ being an arbitrarily given property, a formula involving the free variable x, the creation and existence of a new thing, the

[1] If $\mathfrak{A}(x)$ is "x is honest," then, $\mathfrak{A}(\rho_x(\backsim\mathfrak{A}))\rightarrow\Pi_x\,\mathfrak{A}(x)$ means "If the representative, $\rho_x(\backsim\mathfrak{A})$, of dishonesty is honest, then all men are honest."

"set" y, such that the proposition $\mathfrak{A}(x)$ is equivalent to $x \epsilon y$, must be expressed by a formal axiomatic rule:

$$\Sigma_y \, \Pi_x \, \{(x \epsilon y) = \mathfrak{A}(x)\}.$$

But it soon becomes evident that its application irrevocably leads to a contradiction. For classical analysis the limitation of the argument x to the domain of natural numbers is, however, sufficient; so that we establish the *transfinite rule for sets in the restricted form:*

$$\Sigma_y \, \Pi_x \, \{Zx \rightarrow ((x \epsilon y) = \mathfrak{A}(x))\}.$$

This rule is qualified to overcome the Russell types, just as the transfinite logical axioms guarantee the free manipulation of "all" and "any" prohibited by Brouwer for actual thinking.

I should now like to go into a short discussion of the consistency proof. The attempts to secure it have revealed the vicious circle character of mathematics to its full extent. Only after we have brought it to light completely can we succeed in finding the path that cuts through the circles and enables us to gain the insight that, despite the circles, no actual contradiction arises. This can be accomplished without trouble as long as the transfinite axioms are left aside. Under this restriction we are able to decide whether a normal formula is true or false by following its inductive construction. This on the one hand makes the indirect process of proof superfluous and on the other hand shows that it cannot lead to a contradiction. I indicate the rule by means of which we determine the truth-value T or F, true or false, of normal formulas. (A formula which does not involve the transfinite symbols Σ_x, Π_x, ρ_x, may be called a finite one). α) We always assign the value F to a basic formula, that is, to an individual constant or a variable. (It is of course immaterial whether we here decide in favor of T or F). β) A derived formula begins with a sign of integration or operation. In the first case—let us be magnanimous—we shall always assign to it the value T, likewise in the second case if the operation sign is not one of the following six: $\backsim$ $\vee$ & $\rightarrow$ $=$ Z. The evaluation of derived formulas beginning with one of these signs shall take place according to the following instructions:

(1)

$\mathfrak{b}$	$\backsim\mathfrak{b}$
T	F
F	T

(2)

$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{b} \vee \mathfrak{c}$	$\mathfrak{b}$ & $\mathfrak{c}$	$\mathfrak{b} \rightarrow \mathfrak{c}$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	T
F	F	F	F	T

(3) $\mathfrak{b} = \mathfrak{c}$ shall have the value T only when the two formulas $\mathfrak{b}$ and $\mathfrak{c}$ agree completely, and that again must be verified on their structure step by step. (4) A formula $Z\mathfrak{b}$ shall have the value T if no other signs except σ and 1 occur in the formula $\mathfrak{b}$. This rule constitutes our *rational evaluation.* It satisfies the following conditions:

(1) Every *finite* axiom has the value T;

(2) If $\mathfrak{b}$ and $\mathfrak{b} \rightarrow \mathfrak{c}$ have the value T, $\mathfrak{c}$ also has the value T.

(3) If $\mathfrak{b}$ has the value T, $\backsim\mathfrak{b}$ has the value F.

It follows that a proof which avails itself of finite axioms only consists entirely of formulas having the value T. If therefore a certain proof of such a kind ends with $\mathfrak{b}$, $\mathfrak{b}$ has the value T and $\backsim\mathfrak{b}$ the value F; it is thus impossible for any other proof of the same kind to end with $\backsim\mathfrak{b}$.

This argument may give you the impression that you are being mocked by a farce. But that is due to the fact that mathematics and mathematical proof are a farce as long as the transfinite axioms do not enter into the game. The formula $\mathfrak{b} \rightarrow \mathfrak{c}$ is always evaluated *after* $\mathfrak{b}$ and $\mathfrak{c}$ have been assigned truth-values. In the figure of the syllogism, this relation is inverted. We see from this that the syllogism is powerless without the transfinite axioms; the results it produces are much more easily reached by direct insight, that is, by the calculation of the value of the final formula as determined by its construction according to our evaluation rule. The syllogism would not save mathematics from being an immense tautology, but the transfinite is the vehicle which carries us beyond the domain of what is immediately conceivable.

Let us now add the transfinite logical axioms. It is convenient to use our four axioms separately in building up mathematics, but for the consistency proof it is more suitable to replace them by a single one in which their entire force is concentrated:

$$(\infty) \qquad \mathfrak{A}(\mathfrak{b}) \rightarrow \mathfrak{A}(\rho_x \mathfrak{A})$$

and to use only the transfinite symbol ρ_x instead of our three, ρ_x, Σ_x, Π_x. It is surely out of the question to arrange the descriptive valuation rule in such a manner that we can be assured that all possible axioms formed according to the rule (∞) receive the value T; this signifies that the insight into true and false has here come to an end. However, less than this is sufficient for the consistency proof. We assume that two proofs are concretely given which lead to a formula $\mathfrak{b}$ and its negation $\backsim\mathfrak{b}$. Let the axioms, given explicitly and containing nothing undetermined, which are used in these two proofs, be briefly designated by

$$(\dagger) \qquad \mathfrak{S}: \quad \mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n.$$

In order to convince oneself that the situation assumed above cannot possibly come to pass it suffices to find an evaluation which is "correct within $\mathfrak{S}$." I mean by this that the rule of evaluation shall satisfy the conditions (2) and (3) as before, but instead of (1) we shall only require that *the* *few axioms of our set* $\mathfrak{S}$ *shall have the truth value T.* This "artificial evaluation" will naturally depend on that system $\mathfrak{S}$. Stated more precisely, our problem is *to construct by means of a definite universal method an evaluation for any set* $\mathfrak{S}$ *of arbitrary, but explicitly given, axioms* (†), *which is correct within* $\mathfrak{S}$.

I here consider the simplest case of all: wherever the transfinite symbol ρ occurs in those particular cases of the rule (∞) which are contained in our given set $\mathfrak{S}$ it shall contain the same index x and it shall always govern the *same* definite *finite* formula $\mathfrak{C}$ ($\mathfrak{C}$ of course contains only x as a free variable). If you like you may call this property $\mathfrak{C}$ "honesty." Let $\mathfrak{b}$ be a given finite normal formula; we try whether $\mathfrak{b}$ can be used as representative of $\mathfrak{C}$—as "Aristides;" i.e., we replace $\rho_x\mathfrak{C}$, wherever it occurs, by $\mathfrak{b}$. By this process there arises from an arbitrary formula $\mathfrak{c}$ the "reduced formula" $\hat{\mathfrak{c}}$. We call this "the reduction $\mathfrak{b}$" and give to $\mathfrak{c}$ the value T or F according as the reduced formula $\hat{\mathfrak{c}}$ is given the value T or F by the rational evaluation. The only point which is to be tested is whether the special cases of the transfinite axiom rule (∞) which occur in the system $\mathfrak{S}$ are given the value T. They contain, by assumption, ρ only in the same combination $\rho_x\mathfrak{C}$; let them be the following

(A) $\quad \mathfrak{C}(\mathfrak{b}_i) \rightarrow \mathfrak{C}(\rho_x\mathfrak{C}) \qquad (i=1, 2, \cdots, h)$

$\mathfrak{b}_1, \mathfrak{b}_2, \cdots, \mathfrak{b}_h$ need not be finite, but can themselves contain the transfinite symbol ρ in the combination $\rho_x\mathfrak{C}$.

We try for luck the reduction 1 and the corresponding evaluation; the reduction symbol ^ may now refer to this case. The original mathematical object 1, Adam, shall be our Aristides. The formulas (A) are changed into (Â) by this reduction: $\mathfrak{C}(\hat{\mathfrak{b}}_i) \rightarrow \mathfrak{C}(1)$. We distinguish between several cases. The most favorable is that in which Adam stands the test, in which the finite formula $\mathfrak{C}(1)$ is true, i.e., is given the truth-value T in the rational evaluation. If, however, $\mathfrak{C}(1)$ is false, if Adam is a scoundrel, our first unhappy choice does no harm provided none of the finite formulas

(B) $$\mathfrak{C}(\hat{\mathfrak{b}}_1),\quad \mathfrak{C}(\hat{\mathfrak{b}}_2),\ \cdots,\ \mathfrak{C}(\hat{\mathfrak{b}}_h)$$

have the value T: both sides of these relations have then the value F, and then themselves have therefore the value T. It may have been absurd to choose 1 as the representative of $\mathfrak{C}$ from the standpoint of absolute truth; Adam is dishonest and in spite of that there may be honest men. But it need not disturb us if none of the few definite people $\hat{\mathfrak{b}}_1, \hat{\mathfrak{b}}_2, \cdots, \hat{\mathfrak{b}}_h$ with whom we are now dealing in these axioms, are not scoundrels. The only case in which our evaluation 1 does not lead to our goal is that in which $\mathfrak{C}(1)$ is false but one of the finite formulas (B), e.g. $\mathfrak{C}(\hat{\mathfrak{b}}_1)$ is true. In this case, however, we possess a legitimate representative of the property $\mathfrak{C}$ in $\hat{\mathfrak{b}}_1$. Thereupon we reject the first process of reduction and instead of it perform a second one $\hat{\mathfrak{b}}_1$, i.e., we replace $\rho_x\mathfrak{C}$ wherever it occurs by $\hat{\mathfrak{b}}_1$. So if the first reduction chosen at random fails, its very failure gives us automatically another reduction which leads us to our goal.

But we only stand at the very beginning of the complications that can arise. ρ can combine with different $\mathfrak{A}$'s to $\rho_x\mathfrak{A}$, the $\mathfrak{A}$'s can again contain this transfinite symbol, so that the ρ's are heaped onto one another. When a certain reduction fails, the failure does automatically furnish a new reduction which is successful *there* where the original one failed. But in return for that it will in general go wrong with regard to those axioms of the set $\mathfrak{s}$, for which the first one worked; so that on the whole we are not at all sure that this auto-correction really produces an improvement. The point to be proved is that one does not turn about in a circle, but that after a number of steps of successive corrections which can be indicated at the outset, a reduction is arrived at which does not fail anywhere within the given finite set of axioms $\mathfrak{S}$. The successive corrections can be considered as steps in a certain combinatorial game; and we assert that the game comes necessarily to an end after a finite number of steps, however we play, our freedom being restricted only by the rules of the game. J. v. Neumann has proved this theorem of finiteness. We are here dealing with a concrete mathematical problem which is not trivial, but at the same time is solvable, and I cannot imagine that any mathematician can find the courage to elude its honest solution by means of a metaphysical dogma.

Thus Mr. v. Neumann succeeded in proving that one does not encounter any contradictions in a "restricted analysis," where one handles the sequence of natural numbers as though it were a closed set of objects existing in themselves. The justification of the same procedure for the continuum of real numbers would require the proof that the restricted transfinite axiom concerning sets does not introduce a contradiction into the system of axioms. This problem is surely of a much deeper nature. As I hear, Mr. Ackermann, a pupil of Hilbert's, has reduced it to a form similar to that which Neumann obtained for the restricted analysis. But he has not yet obtained, and it is perhaps even permitted to doubt whether his combinatorial game, which is of course of a rather complicated character, really possesses, the requisite property of finiteness. Thus the situation remains serious but not entirely hopeless for classical analysis.