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THE DIATONIC CHORDS

1. The Problem of Formal Music

MODERN Western music must be regarded as an unparalleled artistic achievement. In every age and civilization music in simple forms has played an important part. But only in Europe since the Renaissance has it broken the bounds of homophonic song and thereby acquired an almost transcendent expressive power.

From the formal mathematical point of view, Western music stands preëminent by virtue of its purity and its extraordinary degree of development. In poetry we have seen that there are formal elements which can be isolated and analyzed. But in poetry the meaning is of such dominant importance and so completely eludes formal analysis, that the field of poetry is not pure in the same sense. Similarly it is obvious that other aesthetic fields are inferior to music, either in purity, or else in degree of development as in the case of polygonal form. For, in the case of music, we have a succession of musical sounds, characterized by pitch and time, replete with relationships and devoid of obvious connotation. Furthermore this music has a deep and almost universal appeal.

All of these considerations indicate that Western music should afford an ideal and perhaps crucial field for the application of our general theory. In fact the complexity, $C$, of such a musical structure should be readily measurable;
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the elements of order, \( O \), which are appreciated by the ear, should be determinable; and then we should expect to find that, with suitable assignment of indices to these elements, the æsthetic measure, \( M \), would be given as the density of the elements of order in the musical structure, that is as \( O/C \).

Music owes much of its æsthetic importance to its peculiar emotional effect. This attribute is readily understood. The human voice is a primary means of human expression, and at the same time is a musical instrument. All of us become accustomed to its musical tones, and simultaneously learn to appreciate that which the voice expresses. Thus musical tones have become intimately associated with emotional feeling.

In this attribute lies not only the most subtle source of musical appeal, but also the final limitation of any purely formal theory. For, according to our general theory, music will be most effective when it unites surpassing beauty of form with effective suggestion of emotional utterance.

This elusive power of suggestion plays a relatively unimportant part in the simpler forms of pure music, such as we shall consider. At the same time it should not be forgotten that, in the last analysis, one musical work is likely to appeal to us more than another, not because of any superiority of abstract form, but rather because of some connotative suggestion of this kind.

We shall attempt to analyze explicitly some of the simpler formal elements in music, and to obtain a suitable definition of æsthetic measure. As far as I am aware, no systematic attempt at quantitative analysis has been made previously. In fact the usual treatments of questions of musical form either formulate empirical rules or fall into the opposite extreme of philosophic discourse.

At every stage of musical development certain specific
limitations in the use of available musical forms have been observed. These forms have, however, undergone a process of gradual elaboration, and new forms have been established. The best composers of each period have been amazingly successful in discovering the latent possibilities nearest at hand and exploiting them thoroughly. In this connection it is significant that composers seldom achieve high success exclusively along lines employed by their predecessors.

Since our theory is based on the felt relationships of order among musical notes, and since our appreciation of such relationships is thus continually expanding and developing, it would be highly absurd to try to formulate a definitive theory of aesthetic measure, valid for the music of the future as of the past.

For the same reason the problem of musical form, in order to be precise, must involve a definite allowance of musical means. A satisfactory theoretic solution of this problem should account for the large body of classical music in which these limitations are observed. If such a solution were successful, it would be anticipated that, by suitable extension of the underlying principles, the same kind of explanation could be applied to more elaborate musical forms.

Thus the main problem of musical form may be stated as follows:

*With a given allowance of musical means, to determine the extent to which the relations of order among the notes of a musical composition constitute the effective basis of musical enjoyment.*

The principal claim of the theory of aesthetic measure, by the aid of which we attack this very difficult problem, is that the aesthetic effect is essentially a summational one, due to the presence of an unusually large number of elements of order. The elements which we shall take account of are
simple in all cases, and for the most part are obvious if not apparently trivial.

2. **Harmony as an Ästhetic Factor**

   It has long been customary to reduce music to harmony, melody, and rhythm. For the present we shall accept this conventional division of the subject.

   Of the three factors harmony is the most elementary, for it is concerned with the ästhetic effect of a complex musical sound (a chord), and of two or even several such sounds heard in succession.

   Despite the fact that harmonic intervals, such as the octave and perfect fifths between successive musical notes, are basic in the older homophonic music, it can scarcely be claimed that harmony plays a rôle in it, although melody and rhythm obviously do. However, with the advent of polyphonic music, harmony became important; and the modern Western ear has grown so accustomed to it that a simple mental harmonization is automatically effected when any tune is heard.

   In order to formulate precisely the problem of harmony, it is desirable that certain facts of fundamental importance be recalled.

3. **Consonance and Dissonance**

   Musical sounds are differentiated from all others, and in particular from noises, by the fact that the vibrations of the air which impinge upon the ear and produce the sensation of sound, are periodic. The number of vibrations per second determine the pitch of the musical sound; thus the note which we shall designate as middle C has a pitch with vibration frequency of about 256 per second.
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The fact that pitch is numerically measurable was known to the early Greek philosopher Pythagoras who observed that if the length of a musical string be divided in the ratio of 1 to 2 then the note of the shorter string is an octave higher. Thus the unmistakable relationship of a note and the note an octave higher, which makes the second note seem to the ear almost identical with the first, is associated with the simple ratio of 2 to 1 between the lengths of the strings which produce the two notes. Furthermore this relationship is agreeable to the ear, and is properly called consonance. From the modern point of view the fact involved is more conveniently expressed in the form that the pitch numbers or frequencies of a note and of its octave are in the ratio of 2 to 1; for, when the length of a musical string (under a specified tension) is halved, its frequency of vibration is doubled, as is that of the attendant vibration of the air.

Similarly it appeared that if the length of a musical string be divided in the ratio of 2 to 3, the higher note is agreeably related to the lower, being a perfect fifth higher in the usual terminology. For example, the perfect fifth above C on the pianoforte keyboard is G, being fifth from C if we start to count the white keys from C as first. Likewise if the lengths are in the ratio 3 to 4, the higher note is a perfect fourth above the lower, and is agreeably related to it. The perfect fourth above C is F on the pianoforte keyboard. The frequency ratios of the perfect fifth and perfect fourth to the lower note are of course 3/2 and 4/3 respectively.

Thus Pythagoras discovered that consonant musical notes are in general produced when the lengths of the corresponding musical strings are in the ratio of small integers; this fact signifies to us that the frequencies are in the reciprocal ratios. His observation was essentially that a certain
element of order (a simple vibration ratio) is correlated with a certain æsthetic value (the consonance of two notes), and so may be regarded as a primitive illustration of our general thesis.

Two notes which are consonant may be sounded simultaneously or in succession with pleasurable effect.

On the other hand if two musical sounds are heard whose frequencies are not thus related, the effect is that of unpleasant dissonance. If, for instance, we strike the notes C and D on the keyboard, with frequency ratio 9 to 8 (nearly), the effect is dissonant; it is even more dissonant when we strike B and C for which the frequency ratio is 16 to 15 (nearly).

4. Musical Notes, Intervals, Triads and Chords

A tuning fork or other resonator produces a pure musical note in the sense that the attendant vibratory motion of the air is not only periodic but is "sinusoidal." On the other hand a musical instrument such as the violin or human voice produces a musical note for which the vibration is periodic but not necessarily sinusoidal. Nevertheless it may be experimentally demonstrated that this note can be reproduced both in pitch and timbre, by exciting simultaneously a number of resonators of the same frequencies as the given note, or exact multiples thereof. Care must be taken to excite each resonator to the right intensity.

These facts show that any musical note is to be regarded as made up of a certain pure fundamental note of the same frequency, and of the first, second, etc., overtones of double, triple, etc., the frequency of the fundamental note. In fact the trained ear can distinguish the various components of a musical note (fundamental note and overtones). For such musical notes, the fundamental note strongly dom-
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inates, and the successive overtones are usually of diminishing intensities. The timbre of the notes produced by a musical instrument depends only on the relative intensities of the dominating fundamental note and of the various overtones; these are usually approximately the same throughout the entire range of a well-constructed instrument.

Suppose now that two pure musical notes are sounded together, the frequency of neither being an exact multiple of that of the other, although the ratio of their frequencies is expressible in whole numbers. For the sake of definiteness suppose that their frequencies are represented by 2 and 3. If then we consider a pure note of frequency expressed by 6, the first of the two notes is the second overtone of this note, and the second note is its first overtone. Therefore the two notes sounded together can be regarded theoretically as the combination of a fictitious fundamental note and its first two overtones, in which, however, only the overtones actually appear. This will obviously produce a musical sound of entirely different character from a natural musical note in which the fundamental note predominates.

In harmony we deal with musical sounds made up of such combinations of natural musical notes whose vibration frequencies are in the ratios of whole numbers. These are called "intervals" if two notes enter, and "triads" if three notes enter. Whenever two or more notes enter, the sound will be called a "chord."

The consonance or dissonance of such combinations of musical notes can be readily explained in two ways. We may either agree with Helmholtz that, when the frequencies are in the ratios of integers, the resulting musical sound is pleasantly consonant because the excitation of the ear is regular in character; in the case of large whole numbers the unpleasant dissonance would then be attributed to the
irregularity of the excitation. Or, we may bear in mind that any note and its overtones are com- present in all musical sounds, inclusive of those of the voice, and for that reason are associated. Hence we may expect that the two notes with a common overtone of low order will also be felt to be associated. Consequently we may look upon the feeling of consonance or dissonance as arising according as there is or is not association of this kind.

According to this second type of explanation, when we hear an interval made up of C and G for instance, this will appear to be consonant because the second overtone of C, with frequency expressed by 3 if that of C is 1, is the same as the first overtone or octave of G, with frequency expressed by 3. On the other hand if we hear C and D together, the overtone of C is (theoretically) the eighth overtone of D; hence if there is any such association, it is very remote, and this fact would explain the obvious dissonance of the interval of a full tone.

Very likely a combination of the two types of explanation is nearer the truth than either separately.

5. The Natural Diatonic Scale

All musical systems employ some set of simple musical notes of definite pitches. Without such limitation, the ear would be lost in a labyrinth of musical sound. Such a set of notes may be said to form a musical "scale." All known scales are much like our own; in particular that of the Greeks was almost the same. The reason for this similarity is not far to seek: any scale may be looked upon as the outcome of an effort to arrive at a set of notes with the closest possible relations of consonance.

On this basis the natural diatonic scale is readily explained, and is highly successful and indeed almost inevitable. All
other scales are either contained in it, like the pentatonic scale, or differ from it slightly, or are further elaborations, like the so-called quarter tone scale.

In fact let us start with an arbitrary note, say C. The notes most closely related to it are its octave c which is the first overtone, and the octave above the perfect fifth G above C, which is the second overtone g. But it is immediately verified that the ear regards any note and its octave as substantially equivalent so far as musical effect is concerned. Thus at the very outset we are led to include in our scale the notes

\[ \ldots c, \quad g, \quad C, \quad G, \quad c, \quad g, \quad c', \quad g', \ldots \]

\[ \ldots 1/2, \quad 3/4, \quad 1, \quad 3/4, \quad 2, \quad 3, \quad 4, \quad 6, \ldots \]

where we have written the frequency ratios underneath the corresponding notes. Of course we have already noted that if the frequency of C is represented by 1, that of G is represented by 3/2. This meaning of the notation used is obvious, e.g., g is an octave below G, g an octave above G, and so on.

Suppose that we inquire next for the note which is related to c as C is to G, that is for the note of which c is the perfect fifth. This will be the perfect fourth F above C, of frequency given by 4/3, since we have the proportion

\[ 4/3 : 2 :: 1 : 3/2 \]

Thus our scale at this stage takes the form

\[ C, \quad F, \quad G, \quad c, \quad 1, \quad 4/3, \quad 3/2, \quad 2, \]

where we restrict attention to the compass of a single octave, since the extension to higher and lower octaves is immediate.
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These three notes C, G, F are the so-called "primary notes" of the scale, and are termed "tonic," "dominant," and "subdominant" respectively.

The remaining notes are then obtained as follows: The third overtone of C, not yet considered, is two octaves above C with a frequency ratio 4, and so yields nothing new. The fourth overtone e', with frequency represented by 5, when reduced two octaves becomes E with frequency represented by 5/4. This note, which is a so-called major third above C, is felt by the ear to be consonant with C. The two further notes A and B are related similarly to F and G respectively, i.e., form major thirds with these notes.

Finally D is obtained as the perfect fifth of G reduced one octave. The same process applied to the two other primary notes yields nothing new, since the perfect fifths of C and F are G and c respectively. As thus completed we have the natural diatonic scale:

C, D, E, F, G, A, B, c, 1, 9/8, 5/4, 4/3, 3/2, 5/3, 15/8, 2

The new notes E, A, B, D are called the "mediant," "submediant," "leading note," and "supertonic," respectively.

The advantage of this scale becomes obvious when it is observed that the scale contains the perfect fifth of any note in it except B, the perfect fourth of any note except F, the major third of C, F, and G, as well as the octaves of all its notes.

6. THE EQUALLY TEMPERED DIATONIC SCALE

Now the natural diatonic scale as thus constructed has a remarkable property which has led to the equally tempered form of the scale which is at the basis of Western music. The eight notes of the octave in the natural scale are so
distributed that, to all practical purposes, the successive notes occur either at the interval of a full tone or of a semitone, with C-D, F-G, G-A, A-B as full tones and E-F, B-c as semitones. More precisely, five further notes C#, D#, F#, G#, A# can be intercalated so that all the musical intervals between successive notes

$$\text{C, C\#}, \text{D, D\#}, \text{E, F, F\#}, \text{G, G\#}, \text{A, A\#}, \text{B, c}$$

are substantially equal; that is, the frequency ratios of successive notes are all nearly equal. As a matter of experience, it is found that the musical relation of two notes depends only on this frequency ratio; for example all notes and their octaves, with frequency ratio of 2, are felt to have the same characteristic relation to one another.

In the equally tempered form of the diatonic scale all of these 12 intervals are made exactly the same. When this is done the change from the natural diatonic scale is so slight that under ordinary circumstances the ear does not detect the modification.

As far as the equally tempered diatonic scale is concerned, it presents a disadvantage and an advantage, when no use is made of the intercalated notes. The disadvantage is that the precise harmonic relationships no longer hold except between octaves, although the precise relationships are more satisfying than the approximate ones. The advantage is that all the pairs of notes are consonant excepting only those which fall at adjacent degrees of the scale (up to octaves) and the pair B-F. In fact these consonant pairs reduced to closest position within an octave have essentially the frequency ratios

$$2/1, 3/2, 4/3, 5/4, 6/5$$

involving the small integral numbers 1 to 6, whereas the
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others have essentially the frequency ratios 9/8, 16/15, 45/32, involving the larger integers, and so are dissonant.

7. THE CHROMATIC SCALE, TONALITY, MODULATION

The decisive superiority of the equally tempered scale first appears when use is made of all 12 notes of the octave. This augmented chromatic scale has the extraordinary property that any note of it may serve as generating tonic of an equally tempered diatonic scale. Each such note defines a major key. Thus far we have considered only the key of C major, in which the notes are those of the diatonic scale with C as tonic.

For the comprehension of the structure of any possible key, it is convenient to regard any intercalated note such as D#, as derived either by heightening the pitch of D, in which case D is called D# (read D sharp) or lowering the pitch of E, in which case D is called Db (read D flat). The white notes and the black notes on the pianoforte keyboard represent respectively the notes of the diatonic scale and the intercalated notes, as indicated in the adjoining series:

\[
\begin{align*}
&C# & D# & F# & G# & A# \\
&C & D & EF & G & A & BC \\
&Db & Eb & Gb & Ab & Bb
\end{align*}
\]

If we start from G of the equally tempered scale as tonic, we obtain the scale of 8 notes

\[G, A, B, c, d, e, f#, g\]

and if we start from F as tonic we obtain the scale of 8 notes

\[F, G, A, Bb, c, d, e, f\]

Thus we have specified the keys of C major, G major and F major in the chromatic scale, of which C, G and F respec-
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tively form the tonic centers. These illustrations show the convenience of such notations as F# and B♭.

The notes of a single key are said to possess a definite "tonality" since they are derived diatonically from a single tonic.

The requirement of tonality has long been fundamental in Western music, and we shall accept this limitation for the general reasons already stated (section 1). From time to time the key may change in accordance with well-defined rules, but this takes place in such a way that the feeling of definite tonality in each changing key is maintained. A change of tonality is called "modulation."

We shall restrict attention to music in a single definite tonality in order to avoid undue complication.

8. Major and Minor Modes

A selection of notes within a scale may be termed a mode. Within the chromatic scale we have the "major mode" defined by any diatonic key. There is also the "minor mode" in the corresponding key. This is obtained by flatting the third and sixth notes of the diatonic scale. Thus the diatonic scale of C in the minor mode is given by

\[ C, D, E♭, F, G, A♭, B, C \]

Any composition in the major mode may be transposed to the minor mode in accordance with simple rules. There are well-known variants of the minor mode here described, to which we can only allude.

While we shall confine attention exclusively to the major mode, it is necessary to bear in mind that there is a parallel minor mode. The interval C to E is called a "major third" and is the fundamental third in the major mode. The corresponding smaller interval in the minor mode is that
from C to Eb and is called a "minor third" on that account. Evidently it is equal to the interval from E to G on the ordinary diatonic scale since both are intervals of three steps in the chromatic scale. On the other hand the intervals from the tonic to the fourth, fifth and octave are the same in either mode. These are the so-called "perfect" intervals.

9. THE PROBLEM OF HARMONY. THE SINGLE CHORD

We have glanced at the well-known reasons, psychological and æsthetic, which have led to the adoption in the West of the equally tempered diatonic scale. This scale, with its possibilities of harmonization and modulation, forms the basis on which Western music has been built. With these facts in mind we may proceed to consider further the fundamental problem of harmony in this scale.

The technical problem of harmony is concerned with the determination of the conditions under which the æsthetic effect of a single chord, and of two (or possibly several) sounded in succession, is pleasing. A partial solution is to be found in the usual empirical rules and their exceptions formulated in treatises on harmony.

Such a purely empirical solution of the problem, if it can be called a solution, can never satisfy the scientific mind. In fact the ordinary person, by merely hearing without analysis a certain amount of music, learns rapidly to appreciate harmony even in its more complicated forms. Furthermore this appreciation develops in a characteristic manner, so that if the two harmonic passages be heard by different persons, the opinions concerning their comparative merits will be almost coincident. This unanimity concerning harmony and music generally must rest on some rational
basis. Otherwise we must agree with Gurney and others who postulate a mystical “musical faculty” which discerns the “ideal movement” which is music. Such a point of view can scarcely be termed rational.

If, however, it can be shown that in harmony, just as in polygonal form for instance, there are simple elements of order 0, having an obvious origin, and that these, on being assigned suitable indices, give rise to a satisfactory aesthetic measure for the chord or succession of chords, the problem of harmony will have been solved in a manner which has some claim to being called scientific.

Furthermore the approach to unanimity in aesthetic judgment will then be explained by the fact that the associations which correspond to these simple elements of order are substantially identical for every one.

The simplest problem of harmony is evidently that of the single diatonic chords, with which we shall deal first.

In a theory of the diatonic chords, the notion of complexity, C, may be laid aside at the outset. This is because a single chord, no matter what its constituents may be, is a unitary fact; the only automatic adjustment involved is an incipient adjustment of the larynx to one of the constituent notes. Hence the complexity C is to be regarded as invariable, so that the aesthetic measure, M, is identified with the elements of order, O, according to the fundamental aesthetic formula: \( M = O \). Thus we have only to discover the various elements of order which enter and their indices, and then to unite them by summation.

10. The Intervals

The intervals of the equally tempered diatonic scale (except for the octave) are not natural intervals, and may

\( ^1 \)The Power of Sound, London (1880).
be completely characterized by the number of steps of the chromatic scale in passing from the lower note to the higher note. The aesthetic effect of any two intervals involving the same number of steps, as two perfect fifths, is the same, at least provided the intervals are heard in isolation and not against the background of an established tonality.

It is convenient to list at the outset the usual names of these intervals, and all instances of them in the key of C major:

0 steps; C-C, D-D, etc. unison (or first)
1 step; E-F, B-c semitone (or minor second)
2 steps; C-D, D-E, F-G, G-A, A-B full tone (or major second)
3 steps; E-G, A-C, B-d minor third
4 steps; C-E, F-A, G-B major third
5 steps; C-F, D-G, E-A, G-C, A-d, B-e perfect fourth
6 steps; F-B augmented fourth
6 steps; B-f diminished fifth
7 steps; C-G, D-A, E-B, F-c, G-d, A-e perfect fifth
8 steps; E-C, A-f, B-g minor sixth
9 steps; C-A, D-B, G-e major sixth
10 steps; D-c, E-d, G-f, A-g, B-a minor seventh
11 steps; C-B, F-e major seventh
12 steps; C-c, D-d, etc. octave

Here we confine attention to intervals formed by pairs of notes within a single octave, but with the understanding that the general effect of an interval is not modified if either note is displaced by an octave. Thus we are led to group together pairs of "complementary intervals" which together form an octave, as follows: unison and octave; minor second and major seventh; major second and minor
seventh; minor third and major sixth; major third and minor sixth; perfect fourth and perfect fifth; augmented fourth and diminished fifth.

Hence there are essentially only seven different types of intervals to be considered, of which the one of 6 steps, F-B, B-f, is to be regarded as peculiar in that it is complementary to itself.

Now, in all probability, a wholly untrained ear would grade these musical intervals according to the degree of consonance of the two constituent notes. In that case the semitone and major seventh would seem to be the most disagreeable of intervals because of their harsh dissonance, and the full tone and minor seventh would be felt as somewhat less disagreeable. On the other hand, the octave, the perfect fourth and fifth would appear to be the most agreeable of intervals because of their consonant quality.

However, those who are accustomed to the diatonic scale are affected somewhat differently. Two cases need to be distinguished: the one in which an interval is heard in isolation, and the other in which it is heard with reference to an established tonality. For the moment we direct attention to the first and simpler case.

Under these circumstances the semitone and major seventh are still the most disagreeable of intervals, and the full tone and minor seventh are somewhat less so. The consonant intervals of the perfect fifth, perfect fourth, and especially of the octave are adjudged, however, to be somewhat insipid. Perhaps this is the case because so many instances of these intervals occur in the diatonic scale, and the two notes are too closely related.

Thus, among the consonant intervals, it proves to be the major and minor thirds and sixths which are most pleasing to the trained ear, for they are found to possess the requisite
degree of consonance, while the two constituent notes do not blend too completely. The thirds are liked best of all. Perhaps this is because the succession of two notes involved is so easily sung.

We have still to consider the interval of 6 steps. This interval of the diminished fifth is definitely dissonant. But it has also the unique property of being complementary to itself; in other words the lower note is related to the upper note just as the upper is to (the octave of) the lower. This property undoubtedly lends a peculiar interest to the interval. We shall assume that its favorable æsthetic effect more than offsets the dissonance, and so we shall classify this interval in the same group as the thirds.

In this way we are led to arrange all possible intervals of the diatonic scale, considered without reference to an established tonality, in the following order of preference: major and minor thirds, diminished fifth or augmented fourth; major and minor sixths; perfect fourth and fifth, unison, octave; major second, minor seventh; minor second, major seventh.

The above partial genetic explanation for the usual order of preference among the intervals is essentially similar to the explanation of all elementary æsthetic phenomena and does not seem to admit of being carried further to advantage.

It will readily be verified that, when the notes of the diatonic scale are actually held in mind, so that the feeling of tonality is established, this classification of intervals no longer suffices. For instance, in the scale of C major, the perfect fifth D-A, is definitely less agreeable than the perfect fifth C-G. Under similar circumstances the major second C-D is found to be definitely less agreeable than the major second F-G. In both cases the cause of the difference lies in the fact that the intervals C-G and F-G, whose values are
enhanced, are “construed” by the ear to lie in major chords, namely in the “tonic chord” C-E-G and the “dominant 7th chord” G-B-d-f respectively.

11. **The Major, Minor and Diminished Triads**

Let us continue the classification of chords by proceeding to the triads. Obviously we can take the three notes of such a triad to be in closest position, within the compass of a single octave. Henceforth we shall consider triads and other chords only in relation to an established tonality, say that of C major.

If two of the three notes differ by a semitone or major seventh, there will be marked dissonance and the triad will be unpleasant.

Even if two of the constituent notes differ by a full tone or minor seventh, the triad will in general be found sufficiently dissonant not to be pleasant. But there are two definite exceptions. Both of the triads D-F-G and F-G-B contain the dissonant interval F-G formed by the sub-dominant and dominant of the scale. Nevertheless these two dissonant triads are not displeasing. However, this is not an intrinsic property of the two triads, for, if we repeat one of them, as D-F-G, in a different position such as E-G-A, the triad is not satisfactory. Evidently the effect observed is due to the fact that these triads are construed in their first position to lie in the dominant 7th chord G-B-d-f of major character mentioned above.

Let us first consider only the types of triads not containing any pair of adjacent dissonant notes. Of course, if we use the octave C-c, then B must be considered as adjacent to C = c in this sense. Thus it is a question of selecting 3 to 7 objects which are arranged in circular order in such a way that no two of them are adjacent.
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Hence the three notes must occur at alternate degrees of the scale, with the consequent possibilities:

(a) C - E - G, F - A - C, G - B - d
(b) D - F - A, E - G - B, A - c - e
(c) B - d - f

Here the notes are written in ascending order.

These are the so-called fundamental positions of the triads. The lowest note will be called the “root” of the triad, so that we obtain seven such triads, with a root for each degree of the scale. In consequence these triads may be identified by their roots. Thus D-F-A may be called the supertonic triad, etc.

When the root of a triad is not in bass position, the triad is said to be “inverted.” Thus in a first inversion the lowest
note is the first above the root; in a second inversion, the lowest note is the second above the root. For example, G-c-e is a second inversion of the tonic triad.

The tonic, subdominant and dominant triads in class (a) are the so-called primary or major triads. They are all of identical structure, consisting of a major third followed by a minor third. The intervals of the major third, minor third and perfect fifth which are involved are all consonant intervals of pleasing type.

Furthermore, the root in these cases is properly called the "generator" of the triad, since the other two notes occur among its overtones of lowest orders. Thus it might be reasonably predicted in advance that these major triads are the best in aesthetic quality, and this fact can be immediately verified. It should be observed that this pleasing quality is independent of any feeling of tonality, and is the same for all three primary triads.

The three triads of class (b), with the supertonic, mediant, and submediant, respectively, as roots, are the so-called minor triads. They are of identical structure, since each of them consists of a minor third followed by a major third. Thus the same consonant intervals are involved in the minor triads as in the major triads, and this fact might seem to warrant the expectation that the minor triads would be as pleasing as the major triads. They obviously are not, although thoroughly consonant. In fact the third of the chord is not an early overtone of the root, so that the root is not a true generator.

By way of partial compensation, however, there is the familiar parallelism of the major and minor modes, which necessitates that the primary tonic triad in the minor mode will be a minor triad. The appropriateness of the terminology "major and minor triads" in this connection is
apparent. Moreover this parallelism renders legitimate the designation of the lowest note as root of such a minor triad.

Thus we find that the minor triads, when used in the major mode, are not displeasing, but are inferior in general to the major triads.

The diminished triad consists of two superimposed minor thirds of consonant type, and involves the characteristic diminished fifth. Despite its obvious minor quality, the triad produces an interesting effect. The root is obviously in no sense a generator. Because of this fact we shall regard the diminished triad as without true fundamental position.

In the case of a first inversion when D is in the bass, and the slightly dissonant B, F fall in the upper parts, the chord is scarcely felt as dissonant by the Western ear.

The diminished triad is also used frequently in such a way as to partake of the major character of the dominant 7th chord G-B-d-f already referred to. We shall not consider this to be an intrinsic property of the triad.

All of these triads are consonant, with the exception of the diminished triad in which the root and fifth are dissonant.

The remaining triads, which involve at least a full tone or semitone dissonance, are in general not pleasing. They will not be considered further at this point.

12. The Corresponding Chords

When the constituent notes of any chord whatever are reduced to closest position within the octave, there will always be marked dissonance if two of these notes differ by a semitone or major seventh. Moreover, even if two of these notes differ by a full tone or minor seventh, there will be dissonance which is generally unpleasant.

Hence, just as in the case of the triads, we are led to
consider first those chords in which no two notes fall at adjacent degrees of the scale.

The case in which only one note enters can be passed over without comment, aside from the statement that such a note will be construed by the ear to lie in some definite triad in accordance with definite laws (section 19).

If two notes enter, the chord is essentially an interval, but will again be construed as in some definite triad.

When three consonant notes enter, there is one and but one corresponding triad, from which the chord is naturally named.

Consequently every such chord may be assigned to a corresponding triad, whose name it bears. Similarly the chord will be said to be in root or fundamental position in case the bass note is the root of the triad. A like terminology will be applied to the inversions of the chord.

If four or more notes enter into the chord, there will be at least two notes on adjacent degrees of the scale. Hence such chords are dissonant and in general not pleasing.

There are, however, certain exceptional cases. For example, the dominant 7th chord, G-B-d-f, is distinctly agreeable despite the dissonant interval G-f. This chord may be regarded as the prototype of the dissonant chords most used in practice. We proceed with the consideration of such chords.

13. The Dominant 7th and 9th Chords

The most important of dissonant chords is the dominant 7th chord, formed by a major third, perfect fifth and minor seventh, placed above the root. In the diatonic scale of C major, this chord appears only in the position G-B-d-f, with the dominant G as root. Here there is a single full tone dissonance involved, namely of the root G and its seventh f.
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It is easy to explain why the dominant 7th chord is the most agreeable of all possible chords which involve four distinct notes. The second and fourth overtones of the root G are the notes d' and b'' respectively. This relationship accounts for the pleasing quality of the primary dominant triad G-B-d. Moreover the sixth overtone of G is practically f'', two octaves above the note f. Consequently the dominant 7th chord, thus regarded, forms a unitary chord of major character of which the root G is the true generator. Musical usage has combined with this natural superiority of the dominant 7th chord to establish it firmly.

It will be noted that the dominant 7th chord in its fundamental position is made up of successive thirds. This suggests a further dominant 9th chord, formed by root, major third, perfect fifth, minor seventh and major ninth, exemplified in the single position G-B-d-f-a in the diatonic scale of C major. The added ninth almost coincides with the eighth overtone of the root lowered two octaves, so that, like the dominant 7th chord, this chord is a unitary chord of major character.

The dominant 7th and 9th chords retain their characteristic major effect even when inverted, except that in the latter case the dissonant ninth should not fall below or just above the root, nor below the third of the chord; the ninth is of course dissonant with both of these notes. If this requirement is not met, the chord is distorted so much as to lose its major character. Such an exceptional dominant 9th chord will be regarded as "irregular," in contradistinction to the other regular forms of the chord.

14. The Higher Dominant Chords

By a higher dominant chord we shall mean any dissonant chord containing not only the dominant G, and certain notes
of the dominant 7th chord perhaps, but also at least one of the three remaining mutually consonant notes A, C, E.

If we proceed from the root G of such a dominant chord upwards by thirds we obtain all of the notes of the octave in the following order:

\[
\begin{array}{cccccccc}
1 & 3 & 5 & 7 & 9 & 11 & 13 \\
G & B & d & f & a & c' & e' \\
\end{array}
\]

The highest note to appear will be used to characterize the dominant chord. Thus if the seventh or the ninth is the highest note, the chord is called the dominant 7th or 9th chord as the case may be, in agreement with our previous usage. Similarly the chord G-B-f-e' is a dominant 13th chord.

Such a dominant chord will be said to have the dominant note itself as root, and to be in fundamental position if the root is in the bass. Its successive inversions are calculated on the basis of the complete series displayed above: e.g., F-G-B-e is a third inversion of the dominant 13th chord.

In case the ninth A of a dominant 11th or 13th chord is below or just above the root G or below the third B, or the eleventh C is below or just above the third B or below the root G, or the thirteenth E is not at the top of the chord, the characteristic effect of the chord is destroyed. Hence, as in the case of a dominant 9th chord, such a chord will be regarded as regular only if these requirements are met. Otherwise it will be considered to be “irregular” and without major character, true root, or corresponding fundamental position.

15. **The Derivatives of Dominant Chords**

Under certain conditions dissonant chords not containing the dominant note itself are felt to have dominant quality.
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This happens in virtue of the particular musical context, which is such that either a preceding dominant note is held over in memory, or a subsequent dominant note is present by anticipation. Such a dissonant chord will be termed a "derivative" of the particular dominant chord suggested.

More precisely, any dissonant chord in which a note of the dominant chord is dominating (section 21) and which becomes a regular dominant chord by the addition of the dominant note in some position will be termed a derivative of this dominant chord.

Such a derivative will never be taken to have fundamental position since the dominant is lacking.

16. REGULAR AND IRREGULAR CHORDS

All of the fundamental chords and the regular dominant chords have a definite fundamental position. Certain of them, namely the primary chords and the regular dominant chords, and their derivative chords possess major character.

All other chords are said to be irregular, and are never regarded as having fundamental position or major character.

We shall not attempt to consider the chords of the minor mode, despite the fact that the classification proceeds along entirely similar lines. Of particular interest is the chord of the diminished seventh, which in the key of A minor is represented by D-F-G♯-B. The chord of the diminished seventh is only slightly dissonant since it is made up of four equal intervals of a minor third to the octave, and involves no dissonances except the diminished fifth. This chord is very important for harmonization in the minor mode. Because of its structure, the various positions are functionally indistinguishable.
17. Final Classification of Chords

Thus we obtain the following list of regular chords:

I  $C^*-E-G$  (tonic)
ii $D^*-F-A$  (supertonic)
iii $E^*-G-B$  (mediant)
IV $F^*-A-C$  (subdominant)
V  $G^*-B-D$  (dominant)
V7 $G^*-B-D-F$  (dominant 7th)
V9 $G^*-B-D-F-A$  (dominant 9th)
    Irregular if $A$ is below $G$ or $B$, or one note above $G$.
V11 $G^*-B-D-F-A-C$  (dominant 11th)
    Irregular if $A$ is below $G$ or $B$, or one note above $G$; or if $C$ is below $G$ or $B$, or one note above $B$.
V13 $G^*-B-D-F-A-C-E$
    Irregular if $A$ is below $G$ or $B$, or one note above $G$; or if $C$ is below $G$ or $B$, or one note above $B$; or if $E$ is not at the top of the chord.
vi $A^*-C-E$  (submediant)
vii $B^*-D-F$  (leading tone)
    Not dissonant if $D$ is in the bass, and $B$, $F$ are not doubled.

In each case the Roman numeral refers to the position of the root, which appears first and is starred. The large Roman numerals are used for those regular chords which have major character.

The notes which are felt as dissonant in a higher dominant chord are in general those not belonging to the primary dominant part of it; such a note is dissonant, however, only in case the notes with which it is dissonant are present. This is because the higher notes $F$, $A$, $C$, $E$ are felt as dissonant against this dominant chord.
All irregular chords may either be thought of as irregular dominant chords or as formal derivatives of such chords, i.e., as inversions in which the root is absent.

18. INCOMPLETE AND AMBIGUOUS REGULAR CHORDS

In musical compositions there appear often certain combinations of notes constituted by some but not all of the notes in one of the regular chords. Such a chord or interval will be said to be “incomplete” in case it contains two notes only, or is part of a higher dominant chord in which the characteristic seventh is absent. In such a case the chord is notably deficient in quality.

Furthermore, in case its notes are consonant and can be ascribed to two consonant chords, it will be said to be “ambiguous.” Thus the interval C-G in the key of C major is incomplete because it lies in the tonic chord but does not contain the mediant; it is not ambiguous however. On the other hand the interval E-G is not only incomplete but is ambiguous also, since it may be ascribed either to the tonic or to the mediant chord.

No other regular chords are to be regarded as ambiguous. For these are assignable to one and only one of the dissonant regular chords.

In the case of an ambiguous consonant chord preceded by a consonant chord, a most important rule of association is readily formulated. Any ambiguous consonant chord is construed to lie in an augmented consonant chord, primary if possible, which contains a maximum number of notes of the preceding chord.

Suppose, for instance, that we have the sequence of two intervals A-c and c-e. If the second interval were heard in isolation it would be construed as in the tonic chord I of course. But, being preceded by A-c, it is clear that the sub-
mediant chord vi is the augmented chord determined by the rule. Hence in such a case the interval c-e is construed to lie in the submediant chord. This is in obvious agreement with the facts.

Our reason for specifying the rule of association is to make it clear that ambiguous regular chords are construed to lie in a definite complete regular chord in all cases.

19. The Chord Value \( C_d \)

With these preliminaries in hand, we are in a position to define the three elements of order in a chord which we regard as determining its æsthetic measure \( m \). The chord will be taken in the tonality of a major diatonic key, say C major, and will be regarded as definitely construed in case it is ambiguous (section 18).

The first of these elements will be called chord value and denoted by \( C_d \). It refers to certain obvious attributes of the chord which are not changed when its upper notes are moved up or down by octaves.

We have seen that there is a fundamental division of chords into two types: namely, the regular chords of major character, and all others. Those of the first type have a brightness of tone which makes them generally superior to those without it. The characteristic differentiation between major and other chords gives rise to the first component in \( C_d \).

Of course this is true only in the tonality of a major key. When a minor mode is used, the superior quality is transferred in part from the major to the minor chords.

Still another classification separates the allowed chords

\footnote{We use the small letter \( m \), instead of \( M \), since the latter will be reserved for the æsthetic measure of a sequence of two chords.}
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into those which do not involve dissonance, and those which do. A dissonance of a full tone always operates unfavorably as far as the isolated chord is concerned. If a dissonance of a semitone enters, or of a ninth (sixteenth, etc.) the dissonance is still more unpleasant. The third component of $Cd$ takes account of the effect of dissonance.

Finally, if the chord is incomplete (section 19), it loses its representative character. The last component of $Cd$ corresponds to this factor.

Hence we are led to define the chord value $Cd$ in the following manner as the sum of four components: a component 1 if the chord is major in character; a component 1 if the chord is in fundamental position; a component $-1$ if the given chord is dissonant, and $-1$ in addition if the dissonance of a semitone or ninth is involved; a component $-1$ if the chord is incomplete.

It is obvious that certain facts whose importance has appeared during our consideration of chords, are taken account of by this element of order $Cd$, namely those concerned with major character, fundamental position, dissonance and in completeness.

20. The Interval Value $I$

If one examines any melody with a simple harmonization it will be found that while other notes of a consonant chord may be lacking, the third above the bass is almost invariably present, and that this is the case to such an extent that a consonant chord seems inferior in case the third above the bass does not occur.

It is interesting to ask the basis of this feeling. The explanation is perhaps the following: The fundamental triads are the most frequently employed chords, and in par-
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ticular are best in fundamental position with their root and generator in the bass. In this event the upper notes are felt to be naturally related to this root. Furthermore such a triad is naturally sung in ascending from the root through the third and fifth. In consequence the third is regarded as the most characteristic note after the root, not only for the major triads but for the minor and diminished triads as well. If the third is actually present, there is a feeling of satisfaction, and, if it is not present, there is dissatisfaction. This feeling is then extended by association to the more complicated regular chords.

The situation is somewhat altered in the case of the dissonant regular chords as follows. In the case of the dominant chords and their derivatives, when the characteristic seventh, F, is in the bass, this note seems to be transferred mentally to the higher parts of the chord; here the note above F plays the rôle of the bass note. If the third above this note occurs in the chord with no dissonant note interposed, the expected third is felt to be present. In the case of irregular chords, if the third above the bass occurs and no dissonant note is interposed, the expected third is again felt to be present.

We shall include all of these possibilities under the heading of the "expected third."

Besides the expected third, another interval which is felt pleasantly is that of the diminished fifth. The pleasantness of this dissonant interval has already been alluded to (section 11).

With these facts in mind we are led to formulate the following definition:

The interval value \( I \) of a chord is the sum of the following two components: 1 if the interval of the expected third appears; 1 if the interval of the diminished fifth appears.
21. **The Value D of the Dominating Note**

Among the notes of a chord there is a lowest one which is repeated at least as often as any other in the chord. This note will be termed the "dominating note" for obvious reasons. In the particular case of a chord of four parts, which is of especial interest, the doubled note is the dominating note, unless there are no doubled notes or two pairs of doubled notes, in which cases the dominating note is clearly the bass.

Generally speaking, it is important that the right note of a chord be dominating if the effect is to be as good as possible. It is this last element $D$ of the dominating note that we proceed to consider.

The quality of a chord is enhanced when a consonant primary note is dominating. In fact the brightness of a major chord is thereby increased, while the sombre effect of a minor chord is lessened. If the dominating note is not only such a primary note but is the root of the chord in fundamental position, the best result is obtained of course.

For a diminished triad the only primary note is the fifth, which is regarded as dissonant when dominating, even if the third is in the bass; here it is of no advantage to have a dominating primary note.

For the dissonant dominant chords of regular type it has been observed that the dominant note G is not felt as a dissonant note of the chord (section 22), and the brightness of these chords is increased if G is dominating.

On the other hand it is very undesirable that a doubled dissonant note or the leading note, or even the fourth above the consonant bass, be dominating. In fact the leading note strongly suggests the dissonant tonic, and the fourth above the consonant bass is dissonant with the expected third.
All of these facts are taken account of in the following definition of the value $D$ of the dominating note:

A component 1 is assigned to $D$ if a consonant primary note is dominating, and 1 in addition if it is root and bass of the chord; a component $-2$ is assigned to $D$ if a doubled dissonant note, or the leading note, or the fourth above the (consonant) bass is dominating. The value of $D$ is the sum of these components.

22. The æsthetic measure $m$ of the single chord

The æsthetic measure $m$ of the single chord in a definite tonality will be taken as the sum of the chord value $Cd$, the interval value $I$ and the value $D$ of the dominating note:

$$m = Cd + I + D.$$ 

The definition of $Cd, I, D$ may be briefly restated as follows:

$Cd$ is the sum of the following four components: 1, if the chord is major; 1, if it is in fundamental position; $-1$, if it is dissonant, and $-1$ in addition if it involves a semitone or ninth dissonance; $-1$ if it is incomplete.

$I$ is the sum of the following two components: 1, if the interval of the expected third appears; 1, if the interval of the diminished fifth appears.

$D$ is the sum of the following two components: 1, if a consonant primary note chord is dominating and 1 in addition if it is the root and bass of the chord; $-2$ if a dissonant note, or the leading note, or a fourth above the consonant bass is dominating and doubled.

In the application of the formula it is taken for granted that the notes of the chord are not too widely separated.

1A note of the diminished fifth in the upper parts is not regarded as dissonant in the application of this rule.
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For definiteness let it be assumed once and for all that the interval between the bass and the first note above it is less than two octaves, and that no other interval between adjacent notes exceeds an octave. If these limits are exceeded, the chord becomes of dubious value, because of its obvious lack of unity.¹

It is assumed also that the chord does not consist of only a single note or of the same chord repeated in different octaves.

The attached diagram not only illustrates the formula, but serves to recall its associative basis according to the general theory.

A cursory examination of the diagram shows that the

¹ In general "close positions" are superior to "extended positions," but we shall not attempt to evaluate this factor.
measure of no chord can exceed 5, this limit being attained only by the primary and dominant 7th chords in fundamental position, with dominating root. The least possible measure is −4. We shall find that a chord with measure 4 or 5 may be considered as good, with measure 3 as fair, with measure 2 as passable, and others as in general bad.

It may happen that a dissonant chord rated as passable does not admit of satisfactory resolution. Such a chord is not usable in sequence, and so is to be regarded as unsatisfactory from this point of view. On the other hand, occasionally a chord not even passable must be introduced in sequence owing to the rigid requirements of the pattern used.

23. The Complete Triads in Fundamental Position

It is interesting to consider the usual rules concerning the single chord in the light of the theory proposed above. Here we shall make the natural assumption that, other things being equal, the chords most employed are best in themselves. Frequently used chords will be expected to rate as good or at least fair, with \( m \) not less than 3. Likewise, for chords which are used but seldom, we shall expect to find \( m \) equal to 2.

In effecting such a comparison with the usual rules of harmony we shall limit attention to harmony in four parts, which is the case usually treated, and shall consider these rules as they are formulated by Prout (referred to as P) in his *Harmony; its Theory and Practice* (31st edition).\(^1\)

The first general statement is to the effect that “the relative positions of the upper notes of a chord make no dif-

---

\(^{1}\) We have selected this practical text *because* it is thoroughly conventional in its treatment, and representative of a crystallized “classical” point of view.
ference to its nature, *provided the same note of the chord is in the bass*” (italics as in P., p. 36).

This conclusion is obviously in complete agreement with our formula, in the sense that the values of the $Cd$, $I$, $D$, and of their component parts, are not altered by such rearrangement of the upper notes of any regular chord.

Next (P., p. 37) we may quote the statements “with one exception, . . . it is possible to double any of the roots of a triad: but they are by no means all equally good to double. . . . *Double a primary rather than a secondary note.*” This statement is of course directed towards the element $D$ in our formula.

Inasmuch as only the primary triads in fundamental position are being considered by Prout at this stage, we have to inspect the following tabulation of values of $m$:

$$\begin{align*}
C &- E - G, & F &- A - C, & G &- B^* - D \\
5 & 3 & 4 & 5 & 3 & 4 & 5 & 1 & 3
\end{align*}$$

Here the corresponding value of $m$ is listed underneath each possible dominating note in these triads. Here and subsequently the asterisk is used to indicate that the corresponding chord is not even “passable.” It appears then that all of these chords are rated as good when a primary note is dominating, and as fair otherwise, with one exception. This exception is the one referred to, namely the doubling of the leading note in the dominant chord which is never allowed “excepting in the repetition of a sequence” (P., p. 38).

The second rule concerning doubling is formulated as follows: “*In the root position of a chord, it is seldom good to double the fifth*” (P., p. 37). Evidently this conclusion is justified by reference to the above table in which $m$ is greater by 1 or 2 if the root is doubled, than if the fifth is doubled.
Although these two rules have reference in the first place only to the primary triads in fundamental position, they are represented as equally applicable to the minor triads in fundamental position, inasmuch as explicit exception is made of the diminished triad vii. But for these minor triads the corresponding tabulation is

\[ D - F - A, \quad E - G - B^*, \quad A - C - C \]

From these values it appears that this extension of the rules is also in accordance with the formula.

Concerning the remaining diminished triad vii it is stated (P., pp. 37–38): “a diminished triad is very seldom found in root position, except in a sequence”; “in the chord vii the fifth is the only primary note. Here, however, it cannot be doubled in root position because the fifth of this chord is not a perfect, but a diminished fifth, and we shall learn later that it is not generally good to double a dissonant note”; “excepting in the repetition of a sequence . . . the leading note must never be doubled.”

The most favorable tabulation of this chord of the diminished fifth arises on the assumption that the chord is a derivative of V8, when we get

\[ B^* - D^* - F^* \]

These results show that the root and fifth are not to be doubled in any case, and that the chord with doubled third, \( D \), can only be used in sequence. This agrees with Prout’s statements quoted above.

The fact that the primary or major triads are in general of higher rating than the minor and diminished triads agrees of course with the characterization of the “primary chords, as the most important in the key” (P., p. 49).
Thus so far as the complete fundamental triads in fundamental position are concerned, the ratings given by the formula are in complete agreement with the usual rules.

24. THE INCOMPLETE TRIADS IN FUNDAMENTAL POSITION

In the case of incomplete triads, the ratings of the fundamental positions are as follows:

\[
\begin{align*}
C & - E, C - G; F - A, F - C; G - B, G - D; \\
D & - F, D - A; E - G, E - B; A - C, A - E
\end{align*}
\]

Here we have put the major triads in the first line, the minor triads in the second line. The diminished triad is omitted since it is not passable even when complete.

These ratings indicate that the best note to omit is the fifth, in conformity with the rule (P., p. 39): “One note of a triad is sometimes omitted. This is mostly the fifth of the chord—very rarely the third . . . ” It will be noted that all of the triads have passable forms if the fifth is omitted. This latter fact justifies the use of such chords as stated (P., p. 39); “But it not infrequently becomes necessary to omit the fifth, in order to secure a better progression of the parts.”

25. THE FIRST INVERSIONS OF THE TRIADS

Let us turn next to examine the first inversions of the fundamental triads. The tabulation with all three notes present is as follows:

\[
\begin{align*}
E & - G - C, A - C - F, B - D - G, \\
F & - A - D, G - B - E, C - E - A, \\
D & - F - B
\end{align*}
\]
where we have put the major triads on the first line, the minor triads on the second line, and the diminished triad, taken as a derivative of V7, on the third line.

Evidently from this tabulation we are led to expect that "all the triads . . . can be used in their first inversion" (P., p. 63). As for doubling it clearly remains advisable to double a primary note instead of a secondary excepting in the case of the diminished triad in accordance with the statement (P., p. 63): "The rule . . . to double a primary, rather than a secondary note, applies to first inversions as well as to root positions."

The exceptional case of the diminished triad is treated separately (P., p. 63): "The root, being the leading note, must on no account be doubled except in the repetition of a sequence. As a general rule, in the first inversion of a diminished triad, the best note to double is the bass note. But it is not forbidden to double the fifth (the primary note of the chord), if a better progression is obtained thereby."

It will be noted, moreover, that for the major triads there is no longer any especial advantage in doubling the primary root instead of the primary fifth, as indicated in the following statement (P., p. 63): "But the objection to doubling the fifth of the chord no longer holds good."

If we tabulate the incomplete forms of the triads in their first inversions, we obtain the following:

\[
\begin{align*}
\hat{E} - G, \quad \hat{E} - \hat{C}; \quad \hat{A} - \hat{C}, \quad \hat{A} - \hat{F}; \quad \hat{B} - \hat{D}, \quad \hat{B} - \hat{G}; \\
1 & 2 & 0 & 1 & 1 & 2 & 0 & 1 & -1 & 2 & -2 & 1 \\
\hat{F} - \hat{A}, \quad \hat{F} - \hat{d}; \quad \hat{G} - \hat{B}, \quad \hat{G} - \hat{E}; \quad \hat{C} - \hat{E}, \quad \hat{C} - \hat{A} \\
1 & 0 & 0 & -1 & 1 & -2 & 0 & -1 & 1 & 0 & 0 & -1
\end{align*}
\]

Here the major and minor triads appear on successive lines.

From this table it appears that the fifth of the chord cannot be omitted in a first inversion except possibly in se-
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quence, although in certain cases the root may be; the third is of course the bass in a first inversion. These conclusions are to be compared with the statement (P., p. 63) "while the fifth is often omitted in a root position, it is hardly ever good to omit it in a first inversion."

In concluding our remarks about the first inversions of the fundamental triads, it may be observed that our formula contains a positive component $Cd$ corresponding to the fundamental position of a chord, and so recognizes a certain natural superiority of the fundamental positions as compared with first inversions.

26. The Second Inversions of the Triads

The importance of securing a naturally moving bass note in a melody outweighs that of using the chords in their fundamental position, so that first and higher inversions are often employed.

Concerning the second inversion of a triad it is stated that "its effect is much more unsatisfying than that of either the root position or the first inversion" (P., p. 65). From our point of view, the inferior effect is due to the lack of an expected third above the bass, although the fourth, dissonant with the third, is present. This is the gist of the italicized statement (P., p. 65) "a fourth with the bass produces the effect of a dissonance."

Let us tabulate the second inversions of the triads in complete form:

\[
\begin{align*}
G & \cdot C & \cdot E, & C & \cdot F & \cdot A, & D & \cdot G & \cdot B, \\
2 & 0 & 1 & 2 & 0 & 1 & 1 & 0 & -1 \\
A & \cdot D & \cdot F, & B & \cdot E & \cdot G, & E & \cdot A & \cdot C, \\
0 & -2 & 1 & -2 & -2 & -2 & 1 & 0 & -2 \\
F & \cdot B & \cdot D \\
0 & -1
\end{align*}
\]
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It appears, therefore, that the second inversions of the minor and diminished triads (the latter taken as a derivative of V7), are not even passable, even if the best note is doubled and no note is omitted. This is in agreement with the observation (P., p. 66): "Though it is possible to take any triad in its second inversion, the employment of any but primary triads in this position is extremely rare."

This same tabulation is in entire agreement with the following rule (P., p. 71): "The best note to double in a second inversion is not the root of the chord, even when this is a primary note. . . . The bass note itself is almost always the best note to double; but it is possible, and occasionally even advisable, to double either the third or the root of the chord. In the extremely rare cases iiic, the second inversion of iii, the bass note, being the leading note, must of course not be doubled; here the third of the chord (the primary note) is, as in other positions of the same chord, the best to double."

27. The Dominant Seventh Chord

Among the regular chords there remain for consideration all of the dissonant chords, aside from the diminished triad. We consider first the dominant 7th chord. The tabulation for the chord in four parts, complete or incomplete, is as follows:

\[
\begin{align*}
G - B - D - F, & \quad G - B - F, \quad G - D - F, \quad G - F, \\
B - D - F - G, & \quad B - F - G, \\
D - F - G - B, & \quad D - F - G, \\
F - G - B - d, & \quad F - G - B, \quad F - G - D, \quad F - G.
\end{align*}
\]
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Here the first line contains the fundamental position, and the three following lines contain the three successive inversions respectively.

The first statement to be considered is the following (P., p. 96): "The chord of the dominant seventh is very frequently found with all of its notes present: but the fifth is sometimes omitted, and in this case the root is doubled. Neither the third nor the seventh of the chord can be doubled . . ."

This statement is in entire accord with the results derived from our formula tabulated above. It will be observed that the fundamental position is rated as fair \((m = 3)\) when the third is omitted provided the root is doubled, and as passable even if the fifth is also omitted, provided the root is tripled. However when the third is present the chord has its maximum rating \((m = 5)\), so that naturally the third is to be included if possible.

It will be seen from the tabulation that the three inversions of the dominant 7th chord have a number of satisfactory forms. They are used frequently (P., pp. 103–109).

28. The Dominant 9th Chord

The dominant 9th chord has next to be considered. Since it involves five notes all of them cannot be present in four-part harmony. However the root \(G\) and the dissonant ninth, \(A\), are present of course.

It is to be pointed out first of all that the requirements of regularity which have been imposed are in accordance with the usual rules and also with the theory elaborated above. To this end we quote (P., p. 164): "As the major ninth if placed below the third of the chord will be a major second below that note, it will frequently sound harsh in that position. It is therefore generally better to put the ninth
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above the third” (P., p. 161). Furthermore we observe that for an irregular dominant 9th chord $Cd$ is negative, $I$ is at most 2, and $D$ is at most 0, so that no irregular dominant 9th chord can rate as even passable. Consequently we need only consider the regular dominant 9th chords.

It is easy to determine the only passable forms of the dominant 9th chord, in which the characteristic seventh, $F$, is absent. Evidently, since the dissonant ninth falls above the root, $Cd$ is at most $-1$. Also since $F$ is absent, the interval of the diminished fifth is not present, and $I$ is at most 1. Hence $D$ must be at least 2 and $I$ must be 1, if the chord is to be passable. This means that the root $G$ must be in the bass and dominating, and that the third $B$ is also present. Hence the only passable possibilities are

\[ G - B - D - A, \quad G - B - A. \]

Only the forms of the regular dominant 9th chords with the seventh present remain to be considered. The tabulation of these is as follows:

\[
\begin{align*}
G - B - F - A, & \quad G - D - F - A, & \quad G - \tilde{F} - \tilde{A}, \\
\hat{B} - F - G - A, & \quad \hat{D} - F - G - A, \\
\tilde{F} - G - B - A, & \quad \tilde{F} - G - D - A, & \quad \hat{F} - G - A
\end{align*}
\]

Here the root position and the three possible inversions appear in successive lines.

The statement to be considered in the light of all these results is the following (P., p. 161): “The seventh is almost always either present in the chord or, if not, it is added
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when the ninth is resolved. In the root position it is generally the fifth that is omitted; but if the root be not in the bass, it is seldom present at all. Inversions of the chord of the ninth are therefore very rare. ” Evidently this statement corroborates the tabulation, which indicates that the only good form of the chord in four-part harmony \((m = 4)\) is the fundamental position with the fifth absent.

29. The Dominant 11th Chord

Let us proceed similarly with the dominant 11th chord, in which the position of the ninth and eleventh from the root are restricted as in the dominant 9th chord, it is at once found that there are no irregular cases which are even rated as passable. Also if the chord is regular but incomplete, the possibilities in fundamental position are seen to be rated as follows:

\[
\begin{align*}
G - B - D - C, & \quad G - B - A - C, \quad G - D - A - C, \\
G - B - C, & \quad G - D - C, \quad G - A - C.
\end{align*}
\]

No inversions need be considered since they cannot rate higher than 1 in the most favorable case.

Thus it remains only to tabulate the complete forms of the regular dominant 11th chord with root, seventh, and eleventh present:

\[
\begin{align*}
G - B - F - C, & \quad G - D - F - C, \quad G - F - A - C, \quad G - F - C, \\
B - F - G - C & \quad , \\
D - F - G - C & \quad , \\
F - G - B - C, & \quad F - G - D - C, \quad F - G - A - C, \quad F - G - C
\end{align*}
\]
Here the fundamental position, the first, second and third, inversions appear on successive lines; the fourth inversion is not possible in the regular case.

These results must be compared with the following statements: "As the eleventh is a dissonance, the usual resolution of which is by descent of a second [i.e., descent to the third,] the third is mostly omitted in accordance with the general principle [of resolution] . . . Either the fifth or ninth of the chord is also generally omitted; but the seventh is usually present . . . " (P., p. 172). "Owing to the harsh dissonance of the eleventh against the third, the first inversion of this chord is very rare" (P., p. 174). "The second inversion of the chord is much more common than the first" (P., p. 174). "The third inversion of the chord is so rare that we are unable to give an example of it." (P., p. 175). "The fourth and fifth inversions are also very seldom met with" (P., p. 175).

In making such a comparison let us recall the fundamental fact that the aesthetic measure proposed is a measure of the single chord taken in a definite tonality. Hence according to the theory the first chord listed G-B-F-C is the best of all the dominant 11th chords, when taken singly. In my judgment this is true. It seems to me also to be true that, of the other forms, the fundamental positions rated as passable are precisely the ones which are tolerable to the ear, and that the single second inversion D-F-G-C, which has highest rating \((m = 1)\) among the others, comes next in order.

If we accept this as a fact, the following explanation of the first statement quoted seems to be the correct one: It is not possible to use this best form, as it stands, in a sequence of chords, for, as stated by Prout in the first quotation, there would then be a difficulty in the resolution
to the dominant chord; in fact the thirteenth C must fall to the third B, which cannot then be present in the chord. However, if the third is omitted from the chord, while the eleventh drops to the third in the resolution, the third may be regarded as present by anticipation.

With this interpretation, it is also reasonable to think of the second inversion D-F-G-C as D-F-G-(B-)-C with \( m = 2 \). Here we count the interval value \( I \) as though B were actually present, but do not admit the semitone dissonance B-C as actual. Hence the comparative commonness of the second inversion seems to be what is to be expected.

30. **The Dominant 13th Chord**

In dealing with the dominant 13th chords, we are immediately reduced to those of regular type, as the only ones which can be passable, just as in the case of the dominant 9th and 11th chords. Of the incomplete chords, which lack the seventh, it is again clear that none can be passable which are not in fundamental position; and that even then the root must be dominating and the third must be present. Thus the only passable incomplete forms are found to be the following:

\[
\begin{align*}
\text{G - B - D - E, G - B - A - E, G - B - C - E, G - B - E} & \\
\text{G - B - D - E, G - B - A - E, G - B - C - E, G - B - E} & \\
\text{G - B - D - E, G - B - A - E, G - B - C - E, G - B - E} & \\
\text{G - B - D - E, G - B - A - E, G - B - C - E, G - B - E} & \\
\end{align*}
\]

In the complete case, when G, F, E are all present in the chord, the following possibilities arise:

\[
\begin{align*}
\text{G-B-F-E, G-D-F-E, G-F-A-E, G-F-C-E, G-F-E} & \\
\text{G-B-F-E, G-D-F-E, G-F-A-E, G-F-C-E, G-F-E} & \\
\text{G-B-F-E, G-D-F-E, G-F-A-E, G-F-C-E, G-F-E} & \\
\text{G-B-F-E, G-D-F-E, G-F-A-E, G-F-C-E, G-F-E} & \\
\end{align*}
\]
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Here the fundamental positions and the three successive inversions are given on the successive lines.

Concerning the dominant 13th chord, Prout says (p. 186): "Though an enormous number of possibilities are possible, comparatively few are in actual use. . . . We begin with those most frequently used." He lists the three principal types as follows: I. Root, third and thirteenth; II. Root, third, seventh and thirteenth; III. Root, third, fifth and thirteenth.

Now the first of these chords is merely the mediant chord although it may function differently in a sequence of chords (Cf. IV). This chord need not occupy us further since it has already been rated as passable in itself \( (m = 2) \). The second form is "the commonest and most useful form of the chord" (P., p. 188) and is the only one rated above as good \( (m = 4) \). The third form is "much rarer than the preceding" (P., p. 190) and is rated as fair \( (m = 3) \).

On the basis of the preceding tabulation we can at once determine all usable forms of derivative chords.

V7.

The only possible derivative here is the dissonant diminished fifth B-D-F already considered (section 23).

V9.

The only possible derivative here is made up of some or all of the notes of B-D-F-A, the so-called "leading tone seventh chord." The notes B and A must actually occur, with A above B.

An examination of the possibilities yields only the following passable cases

\[
\begin{align*}
B - D - F - A, & \quad D - F - A - B, & \quad F - A - B - D
\end{align*}
\]

The first and third of these can only occur in five or more parts since the note D must be doubled. Thus in four-part
harmony only the second form is usable. This is a "first inversion" of the leading tone 7th chord, in complete form.

Of the three examples given by Prout of the use of this chord in a major key (P., pp. 167–168), the first, taken from Graun’s *Te Deum*, illustrates the "root position" of the leading tone note seventh, with doubled leading tone. This chord is extremely harsh as one would expect; its rating in this form is 0. The second example, taken from Mendelssohn’s *Variations*, Op. 82, is precisely the first inversion in the four-part form, rated above as passable. The third example, taken from his *St. Paul*, illustrates the second inversion above but with no note doubled: this chord rates as 1 and so may be regarded as usable in sequence.

As Prout notes (p. 169), it is the leading tone seventh chord in a minor key that is of first importance; this is indeed the chord of the diminished 7th chord (section 17), "the chief derivation of the chord of the ninth and its most frequently used form." Indeed this chord in the form B-D-F-G is useful for chromatic purposes and for modulation in major harmony also.

Since we are confining attention strictly to diatonic chords, the diminished 7th chord does not enter into consideration.

V11.

The only possibilities in question here are made up of a selection of the notes B-D-F-A-C in which C is present. If B is also present, there is the harsh discord of a major seventh, and the æsthetic measure \( m \) cannot exceed 0, since there is no consonant primary note to place in dominating position. Hence "the derivative of the first inversion [of the dominant 9th chord] is as rare as the inversion itself, and for the same reason the harshness of its dissonance" (P., p. 175).
Thus there remains to consider only the notes of the “supertonic 7th chord” D-F-A-C, in which D as well as C must be present. Taken as a derivative and in complete form, this chord rates 2 if the consonant primary note F is dominating, and at most 1 otherwise. In four-part harmony then there is only one passable form, namely that of the first inversion of the supertonic 7th chord with all of its notes present. This seems to explain why the first inversion is “one of the most frequently used of the derivatives of the chords of the eleventh” (P., p. 178).

In connection with all derivative chords, it should be noted that the æsthetic measure depends on the intensity and definiteness with which the lacking dominant note G and its third are suggested by the musical context. Hence under certain circumstances the derivatives may have an æsthetic measure hardly less than that of the best corresponding dominant chords.

V13.

Here we have to make a selection of the notes B-D-F-A-C-E in which E is to be included.

If F is present there is a semitone dissonance and it is seen at once that no form can rate as passable unless there is a dominating consonant primary note, which would then have to be C. Hence neither B nor D can be present, and the only notes possible are F-A-C-E. These form the “subdominant 7th chord.” Moreover since B is not present, there is no interval of the diminished fifth in this chord. Under these circumstances the chord cannot rate as high as 2 and so is not passable.

On the other hand when F is absent, if B is present, there will still be a semitone dissonance of B and C if C occurs so that there is no consonant primary note in the chord; on the other hand if C does not occur there is no primary note at all.
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Hence there is no passable form in which B is present. We infer then that B as well as F is absent in any passable form.

Consequently the selection has to be made from the notes D-A-C-E. But here the single primary note C is dissonant if D occurs. Thus we are reduced to the consonant form A-C-E, which is not a true derivative according to our definition.

Consequently, according to our theory, there is no passable derivative of the chord of the dominant thirteenth in a major key.

With this result is to be compared the statement (P., p. 191) that the derivatives of the chord of the thirteenth in common use are "nearly all chromatic" and so not diatonic, and that "the only important diatonic derivative is that in which the root, third, fifth of the chord are all wanting."

Apparently the second statement concerning the derivative F-A-C-E, i.e., the "subdominant 7th chord" does not agree with our result; for this chord rates 0 in four-part form with all notes present.

It is to be observed, however, that because of the unquestionable harsh dissonance of this seventh chord, it is usual to prepare the dissonant seventh or root, that is to prolong one of these notes from a consonant note of the preceding choral. By this device the harsh dissonance is somewhat masked. If we regard this device as diminishing the index of dissonance from 2 to 1 the chord thereby becomes usable in sequence.

Such preparation is made in the single example in a major key given by Prout (P., p. 191, ex. 363).

31. Irregular Chords

It is easily shown that no irregular diatonic chord can rate even as passable in its æsthetic measure $m$. 
In fact the chord value $Cd$ of such a chord is necessarily negative because of its dissonance, while the value $D$ of the dominating note is at most 1. Hence $Cd + D$ is at best 0. Thus $I$ must be at least 2 if the chord is to be passable, and so both the expected third and the diminished fifth must be present. Furthermore a consonant primary note, E or C, must occur if $D$ is to be 1. But C is dissonant with B, and E is dissonant with F. Hence there is no passable irregular chord.

On the other hand such irregular chords fall under the head of "secondary discords" which are not considered to be derivatives. According to Prout (P., p. 193) "much more importance was formerly attached [to these discords] than is the case at the present day" (P., p. 193).

In the first of the three examples given by Prout, (P., pp. 194, 195, 368, 369, 370) taken from Handel's Joshua, "secondary 7th discords" are used in the sequence of a pattern and there is a rapidly moving soprano over slowly changing lower notes. Thus the lower notes function more or less as drone notes, and this usage steps outside the domain of a succession of chords at equal intervals to which we are confining attention.

In the second example taken from Bach's Fugue in E Minor there is preparation of the secondary 7th discords, so that the chords are irregular only because of a sustained note of preparation continued from the preceding chord. This usage also lies outside of the restricted province under consideration.

The third example shows the "much rarer" secondary 9th discords used in sequence. The strong contrapuntal play between bass and soprano parts justifies an unusually harsh irregular sequence of dissonant chords.
32. **Summary**

In this chapter, then, we have recalled the origin and development of the diatonic scale and its various chords. The genetic account makes clear how the simple elements of order which appeared in our formula for the æsthetic measure $m$ of a diatonic chord arose.

These elements of order are usually regarded as so obvious that they are mentioned only incidentally in textbooks on harmony. According to the quantitative theory, the associative apparatus of the mind casts an automatic balance of these elements, positive and negative, and the resultant tone of feeling determines the sense of æsthetic value, measured by $m$. The diagram of section 22 may be regarded as a schematization of this process.

The ratings of chords so obtained has been found to be in agreement with the usual empirical rules of harmony concerning the use of chords. But of course the theory goes much further than these rules since it affords the means of comparing any two diatonic chords whatever.

It should not be forgotten that in the use of diatonic chords the musical quality of individual chords may become a matter of minor importance relative to the other valid musical effects.