V

SOLUTIONS OF $\frac{d^2x}{dt^2}+\kappa^2x=m\phi'(x, t)$.

42. When $m=0$, the solution is

$$x=c\cos(\kappa t+\epsilon),$$

so that the frequency κ is independent of the amplitude c and phase ϵ. Since the existence of the standard resonance form developed above depended on the non-vanishing of dn/dc, that theory is not applicable to the case $f'(x)=\kappa^2x$. If ϕ' contains a periodic term with frequency κ, we have seen in section 9 that the solution contains a term whose amplitude increases continually with the time. Since infinite amplitudes do not occur in actual mechanical problems, some discussion is necessary.

In certain problems, ϕ' contains a non-linear portion $\psi'(x)$, independent of t. If this be included in $f'(x)$, the solution with the remaining portion of ϕ' neglected has no longer a constant frequency, so that dn/dc is no longer zero. It evidently contains the factor m, however, and the presence of this factor invalidates the approximate solution given in section 26, where terms factored by m^2 were neglected. No general theory for these cases appears to be at hand. Those which actually arise can usually be shown to give finite oscillations, except in quite special cases.

Another set of problems is included in the cases in which ϕ' contains terms of the form $\kappa^2x\psi(t)$, where $\psi(t)$ is a periodic function of t. If these be included in the right-hand member, the discussion starts with the solution of

$$\frac{d^2x}{dt^2}+\kappa^2x\{1-m\psi(t)\}=0,$$

a well-known type, occurring frequently in the problems of celestial mechanics. When ϕ' contains further terms which have resonance relations with those in the solution of this equation, the problems become very difficult and quite outside the limits of these lectures. They are mentioned here mainly because those which have been discussed do not give infinite amplitudes in general.

43. There are, however, certain mechanical problems which are usually classified under the equations of section 42 but to which the preceding theory does appear to be applicable. Consider, for example, the oscillations of a tightly stretched wire, under an external force which has nearly the same frequency as the principal natural frequency of the wire. Our actual experience shows that the amplitudes of the oscillations are so small that one hesitates to invoke the change of period with change of amplitude which maintains finite oscillations in the foregoing resonance theory. Further, such wires are not usually stretched in the motion sufficiently to enable us to assume a non-linear law of extensibility. Frictional damping does not appear to be a complete explanation, since it is possible to suggest experimental conditions in which this could be made very small.

The answer probably lies in a change of "natural" period due to another cause. Under the high tensions which are momentarily produced when the amplitudes are near their maxima, the framework to which the wire is attached yields to some extent. It therefore becomes part of the oscillating system and thus provides the change of period with change of amplitude which the foregoing theory shows is sufficient to prevent the development of large amplitudes. The same explanation would seem to be applicable to the oscillations under resonance conditions of any member of a frame which is under high tension when at rest.

It is known that too great rigidity of a frame is to be avoided if the frame is to be subjected to external forces which have the same periods as any combination of members of the frame. If the chief reason for this avoidance is the danger of producing additional strains due to large oscillations under resonance conditions, the foregoing theory suggests methods of preventing these additional strains other than those usually adopted. What is needed is a change in the "natural" frequency as soon as the amplitude of oscillation begins to increase. For example, if, instead of a single wire connecting two portions of a frame, we have three wires at *different* tensions and nearly in contact with one another, any increase of amplitude of any one of them will bring it into contact with one or both of the other two, causing a change of period which either destroys the resonance relation or alters the phase sufficiently rapidly so that a large amplitude cannot be built up. It is necessary that they be able to vibrate independently through very small amplitudes; if fastened together along their lengths in any way, they become equivalent to a single vibrating system with a new period of vibration which is subject to resonance in the same manner as a single wire. The same principle can evidently be applied to rods or plates. Another method is a device which shall substantially add to the mass of the vibrating body when the amplitudes exceed a certain amount, since this addition changes the natural period. These suggestions involve, not the avoidance of the resonance, but the control of the amplitude under resonance conditions.