II

SOLUTIONS OF $\frac{d^2x}{dt^2}+f'(x)=m\phi'(x, t)$.

21. This is the type of equation which arises in many resonance problems. In general $m\phi'$ will be small compared with f', and solutions can be obtained when $m=0$. A process for approximating to the solution when m is not 0 has been described and illustrated in section 19; it was also pointed out that under certain circumstances this process fails. As a matter of fact it fails in those cases which are particularly under investigation here, so that some other method is required.

The method which will be developed in detail below is known in text-books on differential equations as that of the Variation of Arbitrary Constants—a term which conceals its essential characteristics and not infrequently leads to erroneous interpretations. Fundamentally, it consists of a change of variables which is carried out in such a manner as to have the following properties.

(i) Two new variables replace the variable x.

(ii) One relation between x and the new variables is indicated by the solution of a differential equation which can be solved completely. In the present case it will be the equation given above with $m=0$.

(iii) The replacement of a single variable by two others needs a second relation in order that the change may be definite; this second relation is furnished by the condition that the two differential equations to be satisfied by the new variables shall each be of the first order.

Essentially the method is the same as those known under the names of Jacobi and Hamilton. The canonical forms obtained by the latter are not, however, always useful for the complete solution of a dynamical problem without considerable changes. For this and other reasons, the method will be developed *ab initio*, first with a simple example, and then for the equation which constitutes the heading of this section. Finally a method for the solution of the new equations will be developed.

22. The example referred to at the end of the preceding paragraph is the solution of

$$\frac{d^2x}{dt^2}+\kappa^2x=m\sin t. \tag{22.1}$$

The solution of this equation when $m=0$, that is, of

$$\frac{d^2x}{dt^2}+\kappa^2x=0, \tag{22.2}$$

may be written

$$x=A\cos\kappa t+B\sin\kappa t, \tag{22.3}$$

where A, B are arbitrary constants.

This suggests a change of variables from x to u, v in which

$$x=u\cos\kappa t+v\sin\kappa t. \tag{22.4}$$

Since there are two new variables we need a relation connecting them. Let this relation be

$$\frac{du}{dt}\cos\kappa t+\frac{dv}{dt}\sin\kappa t=0. \tag{22.5}$$

The variable x will now be replaced by u, v in equation (22.1). Differentiating (22.4) we have

$$\begin{aligned}\frac{dx}{dt}&=\frac{du}{dt}\cos\kappa t+\frac{dv}{dt}\sin\kappa t-\kappa u\sin\kappa t+\kappa v\cos\kappa t\\&=-\kappa u\sin\kappa t+\kappa v\cos\kappa t,\end{aligned} \tag{22.6}$$

on account of (22.5).

The relation (22.6) is usually expressed in the form

$$\frac{dx}{dt}=\frac{\partial x}{\partial t}, \tag{22.7}$$

which evidently means that the derivative of x with respect to t has the same form whether we treat u, v as constants or variables.

Differentiating (22.6), we obtain

$$\frac{d^2x}{dt^2}=-\kappa\frac{du}{dt}\sin\kappa t+\frac{dv}{dt}\cos\kappa t-\kappa^2(u\cos\kappa t+v\sin\kappa t),$$

or, on account of (22.4),

$$\frac{d^2x}{dt^2}+\kappa^2x=-\kappa\frac{du}{dt}\sin\kappa t+\kappa\frac{dv}{dt}\cos\kappa t. \tag{22.8}$$

The substitution of this for the left-hand member of (22.1) gives

$$-\kappa\frac{du}{dt}\sin\kappa t+\kappa\frac{dv}{dt}\cos\kappa t=m\sin t. \tag{22.9}$$

In the place of the equation (22.1), we have the two simultaneous equations (22.5), (22.9), each of the first order. The values of du/dt and dv/dt are easily found from them. They are

$$\frac{du}{dt}=-\frac{m}{\kappa}\sin t\sin\kappa t,\quad \frac{dv}{dt}=\frac{m}{\kappa}\sin t\cos\kappa t. \tag{22.10}$$

These are the required equations. It is evident that the change of variables given by (22.4) is equivalent to the assumption that the arbitraries A, B are variable and that (22.5) has the effect of preventing the occurrence of d^2u/dt^2, d^2v/dt^2.

In this simple example, the equations (22.10) are easily integrated if the products are expressed as sums of sines and cosines, we find

$$u=\text{const.}-\frac{m}{2\kappa}\left\{\frac{\sin(t-\kappa t)}{1-\kappa}-\frac{\sin(t+\kappa t)}{1+\kappa}\right\},$$

$$v=\text{const.}-\frac{m}{2\kappa}\left\{\frac{\cos(t-\kappa t)}{1-\kappa}+\frac{\cos(t+\kappa t)}{1+\kappa}\right\}.$$

The value of x is obtained by substituting these in (22.4). If the constants in u, v be denoted by A_0, B_0 we obtain

$$x = A_0 \cos \kappa t + B_0 \sin \kappa t + \frac{m \sin t}{\kappa^2 - 1},$$

a result which may be tested by direct substitution in (22.1).

In this case, the right-hand member of (22.1) does not contain x. It is evident, however, that the process of changing the variables will be the same as far as (22.10) whatever the right-hand member may be. In fact it is only when we arrive at the point where the equations corresponding to (22.10) have to be integrated, that further devices become necessary.

In the general problem the partial derivatives $\partial x/\partial t$, $\partial^2 x/\partial t^2$ will be used. These are to be formed on the assumption that x is expressed as a function of u, v, t. Thus (22.4) gives

$$\frac{\partial^2 x}{\partial t^2} + \kappa^2 x = 0.$$

23. *Transformation of the equation*

$$\frac{d^2 x}{dt^2} + f'(x) = m\phi'(x, t), \tag{23.1}$$

by the use of the solution of

$$\frac{d^2 x}{dt^2} + f'(x) = 0. \tag{23.2}$$

The solution of (23.2) will contain two arbitrary constants and is supposed to have been obtained. Let us denote this solution by

$$x = \psi(c, \epsilon, t). \tag{23.3}$$

Another aspect of the meaning of (23.3), indicated by the last paragraph of section 22, will be useful. Suppose we regard (23.3) as defining x in terms of those variables c, ϵ, t and that we form $\partial^2 x/\partial t^2$, or, what is the same thing, $\partial^2 \psi/\partial t^2$, which means that t is alone varied in forming the partial derivatives. If this and the value of x be substituted in

$$\frac{\partial^2 x}{\partial t^2} + f'(x), \tag{23.4}$$

the "solution" means that c, ϵ will disappear whatever their meaning and that the expression (23.4) will reduce to zero.

The equation for the transformation from x to the new variables c, ϵ is (23.3). Differentiating it with respect to t we obtain

$$\frac{dx}{dt} = \frac{\partial\psi}{\partial c}\frac{dc}{dt} + \frac{\partial\psi}{\partial\epsilon}\frac{d\epsilon}{dt} + \frac{\partial\psi}{\partial t},$$

or, as it is usually written,

$$\frac{dx}{dt} = \frac{\partial x}{\partial c}\frac{dc}{dt} + \frac{\partial x}{\partial\epsilon}\frac{d\epsilon}{dt} + \frac{\partial x}{\partial t}. \tag{23.5}$$

The additional relation connecting c, ϵ, t, needed to define the change of variables, will be taken to be

$$\frac{\partial x}{\partial c}\frac{dc}{dt} + \frac{\partial x}{\partial\epsilon}\frac{d\epsilon}{dt} = 0, \tag{23.6}$$

so that, in virtue of (23.5),

$$\frac{dx}{dt} = \frac{\partial x}{\partial t}. \tag{23.7}$$

Differentiate (23.7), remembering that $\partial x/\partial t$ is a function of c, ϵ, t. We obtain

$$\frac{d^2x}{dt^2} = \frac{\partial^2 x}{\partial c\,\partial t}\cdot\frac{dc}{dt} + \frac{\partial^2 x}{\partial\epsilon\,\partial t}\cdot\frac{d\epsilon}{dt} + \frac{\partial^2 x}{\partial t^2}. \tag{23.8}$$

Substituting the result in (23.1), and making use of the fact that the expression (23.4) is zero, we obtain

$$\frac{\partial^2 x}{\partial c\,\partial t}\cdot\frac{dc}{dt} + \frac{\partial^2 x}{\partial\epsilon\,\partial t}\cdot\frac{d\epsilon}{dt} = m\phi'(x,t). \tag{23.9}$$

Equations (23.6), (23.9) may be regarded as two simultaneous equations for finding dc/dt, $d\epsilon/dt$. On solving them as such we obtain

$$(23.10)\quad \frac{dc}{dt}=\frac{m}{K}\frac{\partial x}{\partial \epsilon}\phi'(x,t),\quad \frac{d\epsilon}{dt}=-\frac{m}{K}\frac{\partial x}{\partial c}\phi'(x,t),$$

where

$$(23.11)\qquad K\equiv\frac{\partial^2 x}{\partial c\,\partial t}\cdot\frac{\partial x}{\partial \epsilon}-\frac{\partial^2 x}{\partial \epsilon\,\partial t}\cdot\frac{\partial x}{\partial c}.$$

When the right-hand members of (23.10) have been expressed in terms of c, ϵ, t by the use of (23.3), these equations become a pair of differential equations of the first order for finding c, ϵ in terms t. The values thus found are substituted in (23.3) and give the value of x.

The divisor K does not contain t explicitly. This important property is deduced from the fact that (23.4) is zero whatever c, ϵ, may be and therefore that its partial derivatives with respect to c, ϵ are also zero. Hence

$$\frac{\partial^3 x}{\partial c\partial t^2}+\frac{\partial f'}{\partial x}\cdot\frac{\partial x}{\partial c}=0,\qquad \frac{\partial^3 x}{\partial \epsilon\partial t^2}+\frac{\partial f'}{\partial x}\cdot\frac{\partial x}{\partial \epsilon}=0.$$

The elimination of $\partial f'/\partial x$ from these gives

$$\frac{\partial^3 x}{\partial c\partial t^2}\cdot\frac{\partial x}{\partial \epsilon}-\frac{\partial^3 x}{\partial \epsilon\partial t^2}\cdot\frac{\partial x}{\partial c}=0,$$

and this may be written

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 x}{\partial c\,\partial t}\cdot\frac{\partial x}{\partial \epsilon}-\frac{\partial^2 x}{\partial \epsilon\,\partial t}\cdot\frac{\partial x}{\partial c}\right)=0,$$

or $\partial K/\partial t=0$, which proves the statement.

It is evident from (23.10) that c, ϵ become constants when $m=0$, and therefore that the solution of (23.10) gives their variations due to the presence of the term $m\phi'$; this point of view, as stated before, is responsible for the term "variation of arbitrary constants."

It is to be noticed also that the relations

$$x=\psi(c,\epsilon,t),\quad \frac{dx}{dt}=\frac{\partial\psi}{\partial t}\equiv\frac{\partial x}{\partial t},$$

may be interpreted as meaning that not only x, but also dx/dt, have the same *form* when expressed as functions of c, ϵ, t, and whether c, ϵ be variable or constant. This is not true of d^2x/dt^2.

24. *Second change of variables.* If the solution of $d^2x/dt^2+f'(x)=0$ has either of the forms given in section 20 difficulties may arise when the variable values of c, ϵ are substituted in the expression for x. For example, if in (20.3), we choose the arbitraries n, ϵ as our new variables, it may turn out that n is a periodic function of t and we thus apparently have terms of the form t multiplied by a periodic function of t, which we know will disappear from the final expression. The presence of such terms can be avoided by a change of variables somewhat different from that used in section 23.

Either of the solutions (20.3) or (20.4) may be expressed in the form

$$x=\text{function of } c,\ l;\ l=nt+\epsilon;\ n=\text{function of } c,$$

where c, ϵ represent the adopted arbitrary constants.

Since t is present in x only through its presence in l, we have

$$\frac{\partial x}{\partial t}=\frac{\partial x}{\partial l}\cdot\frac{\partial l}{\partial t}=n\frac{\partial x}{\partial l},\quad \frac{\partial^2 x}{\partial t^2}=n^2\frac{\partial^2 x}{\partial l^2}.$$

Thus the equation $\partial^2x/\partial t^2+f'(x)=0$ can be written

$$n^2\frac{\partial^2 x}{\partial l^2}+f'(x)=0, \tag{24.1}$$

where x is now a function of c, l and not of t; and n is the function of c previously defined.

The new variables will now be c, l and the additional relation needed will be chosen to be

$$\frac{dx}{dt}=n\frac{\partial x}{\partial l}. \tag{24.2}$$

Since

$$\frac{dx}{dt}=\frac{\partial x}{\partial c}\cdot\frac{\partial c}{dt}+\frac{\partial x}{\partial l}\cdot\frac{dl}{dt}, \tag{24.3}$$

we have

$$\frac{\partial x}{\partial c}\cdot\frac{dc}{dt}+\frac{\partial x}{\partial l}\left(\frac{dl}{dt}-n\right)=0. \tag{24.4}$$

Also since n is a function of c only and $\partial x/\partial l$ a function of c, l, we have, from the differentiation of (24.2),

$$\frac{d^2x}{dt^2}=\frac{\partial}{\partial c}\left(n\frac{\partial x}{\partial l}\right)\cdot\frac{dc}{dt}+n\frac{\partial^2 x}{\partial l^2}\cdot\frac{dl}{dt}.$$

Substituting this in (23.1) and making use of (24.1), we obtain

$$\frac{\partial}{\partial c}\left(n\frac{\partial x}{\partial l}\right)\cdot\frac{dc}{dt}+n\frac{\partial^2 x}{\partial l^2}\left(\frac{dl}{dt}-n\right)=m\phi'(x,t). \tag{24.5}$$

The equations (24.4), (24.5) may be regarded as simultaneous for the determination of dc/dt, $(dl/dt)-n$. Their solution gives

$$\frac{dc}{dt}=\frac{m}{K}\frac{\partial x}{\partial l}\phi',\quad \frac{dl}{dt}=n-\frac{m}{K}\frac{\partial x}{\partial c}\phi', \tag{24.6}$$

where

$$K\equiv\frac{\partial x}{\partial l}\cdot\frac{\partial}{\partial c}\left(n\frac{\partial x}{\partial l}\right)-n\frac{\partial x}{\partial c}\cdot\frac{\partial^2 x}{\partial l^2}. \tag{24.7}$$

The substitution for x of its value in terms of c, l and of n in terms of c, gives us two equations of the first order for the determination of l, c in terms of t.

The proof that K is a function of c only can be obtained as in section 23. Another proof is as follows:—

On account of (24.1) we can write (24.7) in the forms

$$Kn = n\frac{\partial x}{\partial l}\cdot\frac{\partial}{\partial c}\left(n\frac{\partial x}{\partial l}\right) + \frac{\partial x}{\partial c}f'(x)$$

$$= \frac{\partial}{\partial c}\left\{\frac{1}{2}\left(n\frac{\partial x}{\partial l}\right)^2 + f(x)\right\}$$

$$= \frac{\partial}{\partial c}\left\{\frac{1}{2}\left(\frac{\partial x}{\partial t}\right)^2 + f(x)\right\},$$

by (24.2). But the integral of $d^2x/dt^2 + f'(x) = 0$ is

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + f(x) = \frac{1}{2}C, \tag{24.8}$$

where C is evidently a function of c only. Thus

$$K = \frac{1}{2n}\ \frac{\partial C}{\partial c}. \tag{24.9}$$

25. We have supposed that ϕ' was a function of x, t only. There is nothing in the transformation to prevent it from being a function of dx/dt also, since on account of the relation $dx/dt = n\partial x/\partial l$, this derivative can be expressed as a function of l, c. Thus the transformation can be used when frictional forces depending on the velocity are present.

When, however, ϕ' is a function of x, t only we can write

$$\phi' = \frac{\partial \phi}{\partial x},$$

and the equations (24.6) can then be written

$$\frac{dc}{dt} = \frac{m}{K}\frac{\partial \phi}{\partial l},\quad \frac{dl}{dt} = n - \frac{m}{K}\frac{\partial \phi}{\partial c}, \tag{25.1}$$

a form which saves much laborious calculation, since ϕ only has to be expressed in terms of c, l, and K is usually a quite simple function of c. It is recalled that l occurs in ϕ only through its presence in the expression which gives x as a function of c, l.

Although the canonical form of these equations will not be used here, it may be deduced immediately by defining new variables c_1, B, with the equations,

$$dc_1 = K\,dc, \qquad dB = -n\,dc_1 = -nK\,dc.$$

Since B is independent of l, the equations (25.1) can then be written

$$\frac{dc_1}{at} = \frac{\partial}{\partial l}(m\phi + B), \quad \frac{dl}{dt} = -\frac{\partial}{\partial c}(m\phi + B).$$

Equation (24.9) shows that $B = -\frac{1}{2}C$ and that $dc_1 = dC/2n$.

26. *Approximate Solution of the Equations for c, l.* When the equations (25.1) have been formed and a solution is needed, it is important to remember that n is a function of c, and therefore that if an approximation process be adopted, the equation for n must be solved before that for l.

We have seen, however, that in resonance problems it may not be possible to follow the usual processes because developments in powers of m fail to be convergent. The following plan will often be found effective in such cases.

Differentiate the equation for l and substitute in the result the expressions for dc/dt, dl/dt, remembering that ϕ may be a function of t as well as of c, l. We obtain

$$\frac{d^2l}{dt^2} = \frac{\partial}{\partial c}\left(n - \frac{m}{K}\frac{\partial\phi}{\partial c}\right)\cdot\frac{dc}{dt} - \frac{m}{K}\frac{\partial^2\phi}{\partial l\partial c}\cdot\frac{dl}{dt} - \frac{m}{K}\frac{\partial^2\phi}{\partial c\partial t}$$

$$(26.1) \qquad = \frac{m}{K}\left(\frac{\partial n}{\partial c}\cdot\frac{\partial\phi}{\partial l} - n\frac{\partial^2\phi}{\partial l\partial c} - \frac{\partial^2\phi}{\partial c\partial t}\right)$$

$$+\frac{m^2}{K^2}\left\{\frac{\partial^2\phi}{\partial l\partial c}\cdot\frac{\partial\phi}{\partial c} - K\frac{\partial}{\partial c}\left(\frac{1}{K}\frac{\partial\phi}{\partial c}\right)\cdot\frac{\partial\phi}{\partial l}\right\}.$$

In a first approximation, the terms factored by m^2 will be neglected. The equation for d^2c/dt^2 might also be formed but it will not be needed.

The problems to be considered are those in which, after x has been given its value in terms of c, l, we have a term in ϕ of the form

(26.2) $$\phi = -a_i \cos(il - n't - \epsilon'),$$

where a_i is a function of c only; n', ϵ' are given constants, and i is an integer. We then have

$$n\frac{\partial^2\phi}{\partial l \partial c} + \frac{\partial^2\phi}{\partial t \partial c} = \left(n - \frac{n'}{i}\right)\frac{\partial^2\phi}{\partial l \partial c}$$
$$= (in - n')\frac{da_i}{dc} \sin(il - n't - \epsilon').$$

This result enables us to write (26.1) (without its last line) in the form

(26.3) $$\frac{d^2l}{dt^2} + \frac{m}{K}\left\{-ia_i\frac{dn}{dc} + (in - n')\frac{da_i}{dc}\right\}\sin(il - n't - \epsilon') = 0,$$

or

(26.3a) $$\frac{d^2l}{dt^2} + \frac{m}{K}(in - n')^2\frac{\partial}{\partial c}\left(\frac{a_i}{in - n'}\right)\sin(il - n't - \epsilon') = 0.$$

Finally, if we put

$$il - n't - \epsilon' = l_i \text{ or } l_i + \pi$$

according as

(26.4) $$\frac{m}{K}\frac{\partial}{\partial c}\left(\frac{a_i}{in - n'}\right) > 0 \text{ or } < 0,$$

the equation will take the form

(26.5) $$\frac{d^2l_i}{dt^2} + p^2 \sin l_i = 0,$$

where

(26.6) $$p^2 \equiv \left| i(in - n')\frac{m}{K}\frac{da_i}{dc} - i^2\frac{dn}{dc}\cdot\frac{m}{K}a_i \right|,$$

or

(26.6a) $$p^2 \equiv \left| i(in - n')^2\frac{m}{K}\frac{\partial}{\partial c}\left(\frac{a_i}{in - n'}\right) \right|.$$

It will be shown below that a first approximation to the solution of (26.5) may be obtained by putting c equal to a constant c_0; this assumption makes n, K, a_i, constants. Thus p^2 is constant *and* (5) *becomes the same as the characteristic equation for the motion of a simple pendulum.*

27. We can therefore make use of the discussion given in sections 14–18.

If l_i makes complete revolutions with a mean angular velocity $in_0 - n'$, that is, if

$$il = i(n_0 t + \epsilon_0) + \delta l_i, \tag{27.1}$$

in which the arbitrary constant n_0 has been so chosen that $in_0 - n'$ is not 0, and δl_i is an oscillating function, we have the analogue to the case in which the pendulum is making complete revolutions. When the departure of l_i from its mean value is small, the integration of the equation gives approximately (18.1),

$$\delta l_i = \left\{\frac{mi}{K}\frac{\partial}{\partial c}\left(\frac{a_i}{in - n'}\right)\right\}_0 \sin(in_0 t + i\epsilon_0 - n't - \epsilon'). \tag{27.2}$$

This is the non-resonance case. The suffix zero denotes that c has been put equal to c_0 in the inclosed expression.

The resonance case is that in which l_i is an oscillating angle; it corresponds to the case in which the pendulum is oscillating. Accordingly, we must have

$$in_0 - n' = 0, \quad i\epsilon_0 = \epsilon' \text{ or } \epsilon' + \pi, \tag{27.3}$$

since we saw that the pendulum must oscillate about one of the values 0 or π of the variable.

With the use of the form (26.6) and the insertion of $in_0 = n'$, we have

$$p^2 = \left| i^2 \frac{m}{K}\frac{dn}{dc} a_i \right|_0. \tag{27.4}$$

When the amplitude of the oscillation is small, we have (18.2)

(27.5) $$il - n't - \epsilon' \equiv l_i = \lambda \sin(pt + \lambda_0),$$

where λ, λ_0 are arbitrary constants. In certain physical problems this oscillation is called a "libration."

It is to be noticed that the original arbitrary constants c_0, ϵ_0 have become definite, for c_0 is defined by $in_0 - n' = 0$, where n_0 is a known function of c_0, and $\epsilon_0 = \epsilon'$ or $\epsilon' + \pi$. They are replaced as arbitrary constants by λ, λ_0.

Just as in the motion of the pendulum an essential singularity separates the solutions for complete revolution and for oscillation about the vertical, so a similar singularity separates the non-resonance from the resonance case. So long as we confine our work to this first approximation the two problems give analogous results.

28. The solution which gives a first approximation to l has been so carried out that we were able to neglect the variation of c in finding it. This variation has still to be obtained.

Equations (25.1), (26.2) give

(28.1) $$\frac{dc}{dt} = \frac{m}{K}\frac{\partial \phi}{\partial l} = \frac{m}{K} ia_i \sin l_i.$$

Since the right-hand member has the factor m, we shall neglect the variation of c therein. The use of (26.5) then gives

$$\frac{dc}{dt} = -\left(\frac{mia_i}{Kp^2}\right)_0 \frac{d^2 l_i}{dt^2},$$

which, on integration, furnishes

(28.2) $$c = \text{const.} - \left(\frac{mia_i}{Kp^2}\right)_0 \frac{dl_i}{dt}.$$

In this equation one of the values of l_i obtained in section 27 is to be used.

In the non-resonance case, the constant part of dl_i/dt may be supposed to be absorbed into the arbitrary constant.

The oscillating part is given by (27.2) and it has the factor m. But p^2 also has the factor m. Thus the oscillating term in c has the factor m and the earlier assumption that it may be neglected when multiplied by the factor m leads to no contradiction.

In the resonance case, when the libration is small, the substitution of (27.5) in (28.2) gives

$$c = c_0 - \left(\frac{mia_i}{Kp}\right)_0 \lambda \cos (pt + \lambda_0), \tag{28.3}$$

where c_0 is the constant part of c. The insertion of the value (27.4) gives for the coefficient of the periodic term,

$$\left|\frac{ma_i}{K} \div \frac{dn}{dc}\right|_0^{\frac{1}{2}} \lambda. \tag{28.4}$$

This expression has the factor $m^{\frac{1}{2}}$ and therefore, provided no other part of the coefficient tends to become large when the adopted value of c_0 is used, it will be small, and again leads to no contradiction in our earlier assumptions.

We recall that c_0 in this case is not arbitrary but is determined by solving the equation $in_0 = n'$, where n_0 is a known function of c_0. The two arbitrary constants needed in the solution have been already shown to be λ, λ_0.

Attention is again directed to the fact that in non-resonance cases, the expansions proceed according to powers of m, but that in resonance cases they start with $m^{\frac{1}{2}}$, when (28.4) does not tend to become large. If this last condition is not satisfied the approximation may not be valid and some other procedure for the solution of the equations will have to be devised. An example of this will be shown below.

29. The variable l is the phase of the periodic motion when $m = 0$, while c is connected with the amplitude of this motion. We have seen that in non-resonance cases, l, c both

receive variations which have m as a factor, so that these variations will in general be small, although they tend to become large as a resonance range is approached.

In resonance, the circumstances are quite different. The oscillation of l has an arbitrary amplitude, and can apparently be of any magnitude. But the analogy with the motion of the pendulum shows that, in general, the oscillation of l will be between $\pm\pi$ in the limiting case between resonance and non-resonance, or at least will be always finite and of this order of magnitude. The rate of change of this oscillation is however always small.

On the other hand, the oscillation of c has in general a small amplitude in both resonance and non-resonance cases and has $m^{\frac{1}{2}}$ or m as a factor. By (28.2), its amplitude of oscillation is a maximum when that of dl_i/dt is a maximum; according to section 18, this latter maximum is $2p_0$. The maximum amplitude of oscillation of c is therefore

$$(29.1) \qquad 2\left|\frac{mia_i}{Kp}\right|_0 = 2\left|\frac{ma_i}{K\frac{dn}{dc}}\right|_0^{\frac{1}{2}}.$$

The importance of this result lies in the fact that *there is no tendency for c to become infinite*, as often assumed, at least in the types of resonance indicated by the equations we have used above. Thus there is no need of the damping factor often introduced to avoid this difficulty. In fact, as we shall see below, the presence of a damping factor may in certain cases ensure the passage from non-resonance to resonance, and thus actually produce the phenomena which its introduction was intended to avoid.