## METHOD OF APPROXIMATION

19. In the majority of physical problems, defined by differential equations, the only practicable method for obtaining a solution is some process of approximation. On account of either mathematical or physical reasoning, we suppose that some portion of the equation or of its solution may be neglected as a first step, when by this neglect we are able to deduce a result by known methods. Various devices are then available for correcting the result.

One method frequently adopted is a process of continued approximation which will be illustrated by a simple example which has been chosen to show how the method may sometimes fail.

Consider the equation

$$(19.1) \qquad \frac{d^2y}{dt^2} = m \sin(y - n't),$$

where $m$ is supposed to be small. If $m$ be neglected, the solution is $y = nt + \epsilon$, where $n$, $\epsilon$ are arbitrary constants. The ordinary meaning attached to this result is that it represents approximately the result we desire to obtain.

The usual procedure is the substitution of this approximate value of $y$ in the right-hand member of (19.1), as giving an approximate value of this term. If we do so and solve again, we obtain

$$(19.2) \qquad y = nt + \epsilon - \frac{m}{(n-n')^2} \sin\{(n-n')t + \epsilon\}.$$

For a further approximation we substitute the more accurate result (19.2) in the right-hand member of (19.1) and solve again. Evidently the process may be continued indefinitely.

It is not difficult to show that the mathematical implication of the process is the possibility of development of the solution in powers of $m$. If, however, $n-n'$ is very small, the process will evidently not be convergent. This example was chosen because in this case the substitution $y=n't+x+\pi$, transforms the equation to

$$\frac{d^2x}{dt^2}+m\sin x=0, \tag{19.3}$$

which is the same as the pendulum equation previously treated if we put $m=\kappa^2$. (If $m$ be negative, the substitution $y=n't+x$ should be used.)

The result shows that when $x$ is an oscillating quantity, $y$ oscillates about $n't+\epsilon$, that is, we must assume $n=n'$. In this case the solution depends on $\kappa$ or on $m^{\frac{1}{2}}$ and is not developable in positive powers of $m$. This example illustrates the manner in which certain approximation processes fail under resonance conditions.