

I

THE SOLUTIONS OF THE EQUATION $\frac{d^2x}{dt^2} + \kappa^2 \sin x = 0$.

14. It has been mentioned above that the equation which in many cases gives the resonance phenomena is the same as that of the motion of a simple pendulum. Since this motion is easily visualized, it will assist in the comprehension of the phenomena if the solutions and their physical meanings are carried out in detail. Certain features of the motion of a pendulum, usually left aside, will be emphasized here on account of their importance in the later applications.

Let us suppose that a particle be attached to one end of a light (weightless) rod of length l , the other end being attached to a horizontal axis about which the rod can turn without friction in a vertical plane. If g be the acceleration produced by gravity and x the angle which the rod makes with a downward drawn vertical line at time t , it is well known that the motion is given by the differential equation

$$(14.1) \quad \frac{d^2x}{dt^2} + \kappa^2 \sin x = 0,$$

where $\kappa^2 = g/l$. According to the manner in which it is started, the rod may make complete revolutions or it may oscillate to and fro on each side of the vertical through any angle up to 180° .

The immediate object in view is the discovery of the different types of solution of (14.1), considered merely as a differential equation. The physical illustration aids in giving a concrete idea as to the nature of these solutions.

The equation

$$\frac{d^2x}{dt^2} - \kappa^2 \sin x = 0$$

can be reduced to (14.1) by the substitution $x + \pi$ for x . A more general type,

$$\frac{d^2x}{dt^2} + f'(x) = 0,$$

will be discussed later.

15. On multiplication of (14.1) by $2dx/dt$, we can integrate and obtain

$$(15.1) \quad \left(\frac{dx}{dt}\right)^2 = C + 2\kappa^2 \cos x,$$

where C is an arbitrary constant to be determined from the initial value of x and of the angular velocity dx/dt . The different types of solution depend on the value of C . Since $\cos x \geq -1$, we must have $C \geq -2\kappa^2$, in order that the velocity may be real.

There are three cases.

(i) $C > 2\kappa^2$. The velocity never vanishes and as it is always finite, it must be always positive or always negative. The rod is making complete revolutions clockwise or counter-clockwise.

The integral of (15.1) is

$$(15.2) \quad t + \text{const.} = \int \frac{dx}{(C + 2\kappa^2 \cos x)^{\frac{1}{2}}},$$

where the square root may have either sign. Since $|2\kappa^2/C| < 1$, the integrand may be expanded into a series of the form

$$(15.3) \quad \frac{1}{n} + \sum_{i=1}^{\infty} a_i \cos ix,$$

where, by Fourier's theorem,

$$(15.4) \quad \frac{1}{n} = \frac{1}{2\pi} \int_0^{2\pi} \frac{dx}{(C + 2\kappa^2 \cos x)^{\frac{1}{2}}}$$

This last equation gives a relation between n , C ; either may be used as an arbitrary constant.

The value of x in terms of t may be deduced from (15.2), (15.3) after integration. But it is more easily obtained by assuming

$$(15.5) \quad x = nt + \epsilon + x_1 \equiv nt + \epsilon + \sum_{i=1}^{i=\infty} b_i \sin i(nt + \epsilon),$$

and substituting in (14.1). We have

$$\begin{aligned} \sin x &= \sin(nt + \epsilon) \cos x_1 + \cos(nt + \epsilon) \sin x_1 \\ &= \sin(nt + \epsilon) \cdot (1 - \frac{1}{2}x_1^2 + \dots) \\ &\quad + \cos(nt + \epsilon) \cdot (x_1 - \frac{1}{6}x_1^3 + \dots). \end{aligned}$$

On substituting the series (15.5) for x_1 and equating to zero the coefficients of $\sin i(nt + \epsilon)$, we obtain by continued approximation

$$(15.6) \quad x = nt + \epsilon + \frac{\kappa^2}{n^2} \sin(nt + \epsilon) + \frac{1}{8} \frac{\kappa^4}{n^4} \sin 2(nt + \epsilon) + \dots,$$

in which n , ϵ are the arbitrary constants.

The mean angular velocity of the rod is n . The periodic terms in (15.6) may be regarded as constituting a periodic oscillation about the mean phase $nt + \epsilon$. The physical illustration shows that *the half amplitude of the oscillation is always less than π* , and it evidently diminishes as n increases.

16. (ii) $C < 2\kappa^2$. Here $dx/dt = 0$ and changes sign when $\cos x = -C/2\kappa^2$. If we put $C = -2\kappa^2 \cos c$, x evidently oscillates between the values $x = \pm c$. The equation (15.1) may then be written

$$\left(\frac{dx}{dt}\right)^2 = 4\kappa^2(\sin^2 \frac{1}{2}c - \sin^2 \frac{1}{2}x),$$

giving

$$(16.1) \quad t + \text{const.} = \int \frac{dx}{2\kappa(\sin^2 \frac{1}{2}c - \sin^2 \frac{1}{2}x)^{\frac{1}{2}}}.$$

The transformation

$$\sin \frac{1}{2}x = \sin \frac{1}{2}c \sin \psi,$$

turns (16.1) into

$$(16.2) \quad t + \text{const.} = \int \frac{d\psi}{\kappa(1 - \sin^2 \frac{1}{2}c \cdot \sin^2 \psi)^{\frac{1}{2}}}.$$

The integrand can be expanded by the binomial theorem and expressed in a Fourier series with argument ψ . If we put

$$(16.3) \quad \frac{1}{p} = \frac{1}{\pi} \int_0^\pi \frac{d\psi}{\kappa(1 - \sin^2 \frac{1}{2}c \sin^2 \psi)^{\frac{1}{2}}} \\ = \frac{1}{\kappa} \left\{ 1 + \left(\frac{1}{2}\right)^2 \sin^2 \frac{1}{2}c + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \sin^4 \frac{1}{2}c + \dots \right\},$$

we have the period $2\pi/p$ of a complete oscillation expressed in terms of the semi-amplitude c of x . The frequency p diminishes as c increases. It has a lower limit κ corresponding to $c=0$, but this limit is never reached since $c=0$ corresponds to equilibrium and we are supposing that the pendulum is in motion.

When c is small we have approximately

$$(16.4) \quad p = \kappa \left(1 - \frac{1}{16}c^2 + \dots\right).$$

To obtain x in terms of t when c is small, it is simplest to expand (15.1) in the form

$$(16.5) \quad \frac{d^2x}{dt^2} + \kappa^2 \left(x - \frac{1}{8}x^3 + \dots\right) = 0,$$

and to assume as the solution

$$x = \sum_{i=1}^{i=\infty} c_i \sin i(p t + \epsilon).$$

This is substituted in (16.5) and the powers of the series are expressed as sines of multiples of $p t + \epsilon$; evidently only odd multiples will be present. The coefficients of the various sines are then equated to zero and the resulting equations solved by approximation. The coefficient of $\sin(p t + \epsilon)$ equated to zero gives

$$(16.6) \quad p = \kappa \left(1 - \frac{c_1^2}{16} + \dots \right),$$

and c_1 is arbitrary. The solution is found to be

$$(16.7) \quad x = c_1 \sin(pt + \epsilon) + \frac{c_1^3}{192} \sin 3(pt + \epsilon) + \dots$$

The maximum value c of x is given by $pt + \epsilon = 90^\circ$. Hence $c = c_1 - c_1^3/192 + \dots$.

In the applications to resonance problems it will usually be found sufficient to confine (16.7) to its first term. The only part of the higher approximation which we shall need is the second term of (16.6) and in this we can put $c = c_1$. The presence of this second term is fundamental for the development of the resonance phenomena. The maximum value of c is evidently less than π .

It is still possible to use ϵ and the frequency p as the arbitrary constants instead of ϵ, c . But it is inconvenient, for c is then defined approximately by the expression $4(1 - p/\kappa)^{\frac{1}{2}}$ which gives complicated results when derivatives with respect to p are needed.

17. (iii). $C = 2\kappa^2$. This is the limiting case separating cases (i), (ii). Here

$$\left(\frac{dx}{dt} \right)^2 = 4\kappa^2 \cos^2 \frac{1}{2}x,$$

the complete solution, containing one arbitrary constant ϵ only, being

$$\kappa t + \epsilon = \log \tan \frac{1}{2}(x + \pi),$$

or

$$(17.1) \quad x + \pi = 4 \tan^{-1} \exp.(\kappa t + \epsilon).$$

In this case dx/dt oscillates between $\pm 2\kappa$. We have $x \rightarrow \pi$ when $t \rightarrow \infty$ and $x \rightarrow -\pi$ when $t \rightarrow -\infty$. At both limiting positions $dx/dt = 0$, $d^2x/dt^2 = 0$. If we form the higher derivatives of x in succession from (15.1), we see that they will all be

zero at these limiting places. From the analytical point of view these are singularities of (17.1) and no expansions in powers of t about these points exist. They are evidently the singular points of (15.6) as $n \rightarrow 0$ and of (16.7) as $c \rightarrow \pi$.

Hence infinitesimal changes in the initial values of x , dx/dt will give different types of motion according to the nature of these changes. If they are within the probable errors of their determination by observation, we must therefore regard the future motion as indeterminate or non-calculable.

18. *Summary.* In the applications of the solutions of (15.1) to resonance problems, emphasis on the following results will be needed.

There are two principal types of solution in one of which the mean value of dx/dt is always positive or always negative but not zero, and in the other of which the mean value of dx/dt is zero.

When the mean value of dx/dt is zero there is a range of solutions in which x oscillates about the value zero, the range being characterized by the half amplitude c which can have any value between $0, \pi$.

The constant ϵ merely gives the origin of reckoning of t . If we suppose this to be settled and then attempt to classify the solutions according to the mean value of dx/dt we find a single solution for each non-zero value and an infinite number for the zero value.

A discontinuity separates the zero from the non-zero values, such that we cannot pass from one set of solutions to the other set by mere changes in the constants.

When the mean value of dx/dt is not zero, the solution has the form

$$(18.1) \quad x = nt + \epsilon + \frac{\kappa^2}{n^2} \sin(nt + \epsilon) + \frac{1}{8} \frac{\kappa^4}{n^4} \sin 2(nt + \epsilon) + \dots$$

where n, ϵ are the arbitrary constants.

When the mean value of dx/dt is zero, it takes the form

$$(18.2) \quad x = c \sin(pt + \epsilon) + \frac{c^3}{192} \sin 3(pt + \epsilon) + \dots;$$

with

$$(18.3) \quad p = \kappa \left(1 - \frac{c^2}{16} + \dots \right),$$

and with c, ϵ as the arbitrary constants.

In (18.1), the development proceeds according to powers of κ^2 . In (18.2), it depends on the first power of κ in a quite different manner.

The maximum oscillation of dx/dt occurs at the limit between resonance and non-resonance and its amplitude is 2κ .