ELEMENTS OF THE THEORY OF RESONANCE ILLUSTRATED BY THE MOTION OF A PENDULUM[1]

INTRODUCTION

1. The original observations which gave rise to the word "resonance" were audible sounds which are familiar to most of us. A note is struck on a piano or other musical instrument and some body—a wall or another musical instrument—will take up the same note and will continue to sound it even after the original source is stopped. The megaphone takes up the vibrations of the air produced by the voice and gives them out again, concentrated in a particular direction. A vibrating body, for example, the cylinders in the engine of an automobile, will set up vibrations in other parts of the car, certain vibrations being noticeable at certain speeds and others at other speeds. To all these phenomena the term resonance is applied.

But the actual nature of the phenomena is not always the same. The megaphone and loud speaker are designed to take up and emit any vibration within a certain range of frequency. On the other hand, the audible vibrations of a stretched wire are confined to a limited number of sounds as long as we keep the tension unaltered, and these bear definite relations to one another; to produce resonance in such a wire, it is necessary to sound a note with a frequency

[1]A course of lectures, given at the Rice Institute, April 22, 23, and 24, 1931, by Ernest William Brown, D.Sc., F.R.S., Josiah W. Gibbs Professor of Mathematics in Yale University

very near that of one of these modes of vibration. These illustrations bring up the necessity for a definition of what are usually called the *natural* frequencies or periods of vibration of any system.

2. A stretched wire when made to vibrate appears to give out the same note, in general, however it may be struck or when the bow of the violin is drawn across it. An ordinary drinking glass exhibits the same phenomenon: it gives out the same note when struck as when the finger is wetted and run around the top edge of the glass. The periods of these sounds are called "natural periods" of vibration: they depend only on the construction of the mechanism and not on the manner in which its vibrations are started.

But this last statement is not exact. Actually the note given by a vibrating wire is not quite the same when the sound is loud as when it is soft, although the difference is not easily detected even by a sensitive ear. In the former the excursions of the wire—usually called the amplitude of vibration—are greater than in the latter, and the period of vibration which gives the pitch of the note depends to some extent on this amplitude. The "natural" period is, in fact, a mathematical fiction since it is only present when the vibrations have infinitely small amplitudes, which amounts to saying that the wire is not vibrating at all. More properly a natural frequency should be defined as the lower limit of the frequency of that particular mode of vibration. It is necessary to insist on this change of frequency with change of amplitude because the existence of the *phenomena* of resonance depends on the existence of this change.

3. A mechanical system which is free to move at all can, however, usually be made to vibrate with any frequency whatever. When any such vibration is impressed on the system, it is called a forced vibration. It may be present

at the same time as the "natural" vibration so that the actual motion is compounded of the two types. It is usual to treat the frequency of the forced vibration as unchanging and in many problems this procedure can be justified, but this is not always the case. If two wires be tuned so as to give out notes with nearly the same pitch, each will force its frequency on the other and there will be a reaction of this effect which must sometimes be taken into account. The procedure fails altogether when the pitches of the two notes are sufficiently nearly the same.

4. The usual definition of resonance is the state of motion which is present when the natural and forced frequencies are the same, but it is evident from the remarks made above concerning the change of frequency with change of amplitude, that the definition lacks precision. It will be shown that the state of motion in which the frequencies are the same is fundamentally different from that in which they are not the same. Not only is it impossible to represent the two states by the same mathematical formula, but there is a fundamental physical discontinuity separating them. It is the exploitation of this discontinuity which is one of my chief objects in these lectures, partly because it is a feature of many mechanical problems in which resonance takes place and partly because I believe that it has far-reaching effects in gravitational problems and particularly in the past and future history of the solar system.

5. To illustrate the phenomena, consider two piano wires, one of which is being "tuned" so as to give the same note as the other. As the pitch of the second note approaches that of the first with the change in tension, the phenomenon known as "beats" occurs. Apparently, the two notes have the same pitch but the volume of sound rises and falls at regular intervals; this interval is called the "beat" frequency.

Actually the periods are not the same, but the ear is unable to distinguish the difference because its function is mainly to integrate the impulses which it receives. When the two vibrations have the same phases, the two sets of impulses are added together and increase the volume of sound; when the phases are opposite, the impulses nearly cancel one another and the sound is much diminished. The frequency of the "beat" is nothing else than the difference of the frequencies of the two notes.

As the tuning proceeds, the frequency of the beat becomes smaller, that is, the interval between the maxima of the sound becomes longer, until a stage is reached in which it can sometimes be heard and sometimes not. According to the theory, as developed in a later section, the audibility of the beat will depend on the magnitudes of the impulses given to the notes; the beat may be heard when the notes are struck gently but not when they are struck hard. With very slight further tuning, the beats disappear altogether, and we have the phenomenon known as resonance in which the two periods are exactly the same.

6. The last statement may give rise to the impression that to produce resonance, it is necessary to adjust the tuning with a degree of exactness which anyone familiar with experimental work would say cannot be achieved. It is not so. Resonance takes place when the two frequencies differ by a small but measurable amount. In fact, a certain degree of "tuning" can be made without destroying the resonance. What has actually happened is that we have produced a common average period of oscillation. About this common period, however, a new oscillation has been set up, and when we "tune" slightly, the alteration which takes place is one not in the common period but in the amplitude of this new oscillation. In astronomy, it is known as a physical "libra-

tion." The frequency of this oscillation is quite unconnected with the beat frequency; it depends mainly on the construction of and tension of the wires and on the magnitude of the blows which are given to set them vibrating. Sometimes it has a minimum period which is approached as the amplitude dies down. It is, however, doubtful whether it could be detected without the use of elaborate apparatus designed for the measurement of very small differences in the intensities of sounds.

7. Vibrating wires will not be used further for the illustration of the phenomena of resonance, partly because the oscillations are so rapid that observation is difficult, but mainly because of mathematical and physical difficulties which make the analysis very complicated.[1] We have, however, in the simple pendulum a vibrating system in which experiment can approach a simple mathematical theory very closely. The pendulum will be used in two ways. In later examples, it is an oscillating system on which various types of external forces can act. But its more important function here is due to the fact that the equation which gives the motion of the pendulum will be shown to be the same as the fundamental equation which we reach when considering resonance in a wide variety of types of mechanical systems. If then we have a complete solution of the motion of a pendulum, it is necessary only to reinterpret it for any case in which the same differential equation arises.

8. The pendulum in this latter case is, however, one which can perform complete revolutions as well as oscillate about a horizontal axis. A more practical form of illustration is that of a bicycle wheel loaded at one point of its rim by a weight clamped on; it is mounted on a fixed axis, so that it can turn freely in its bearings. Such a wheel can be made

[1]See section 44.

to oscillate about its position of stable equilibrium or can make complete revolutions in either sense according to the manner in which it is started, and the differential equation reduces at once to that of the pendulum, in the absence of frictional forces. It will be shown that the two types of motion—complete revolution or oscillation—correspond respectively to non-resonance and resonance conditions, and that the fundamental phenomena of the latter can be exhibited without difficulty. With the results for the simple pendulum in mind, the equations can be generalized and an approach made to more complicated problems such as those which are presented by the "problem of three bodies."

9. Certain mathematical features in the investigation of resonance should be mentioned at the outset. The most fundamental of these is due to the fact that *the phenomena cannot be investigated by linear differential equations alone.* In the ordinary theory of small oscillations, the periods are considered as independent of the amplitudes. This is an approximation which is usually sufficient as long as the amplitudes remain small. But the approximation is no longer valid when the amplitudes increase, and this usually happens when two of the frequencies in the system are nearly the same.

Take, for example, the equation

$$\frac{d^2x}{dt^2}+n^2x=m\sin n't. \tag{9.1}$$

When $m=0$, this gives a harmonic oscillation with frequency n. When m is not 0, there is, in addition, a similar oscillation with frequency n'. As long as n, n' are unequal we have the solution

$$x=\lambda\sin(nt+\lambda_0)+\frac{m}{n^2-n'^2}\sin n't, \tag{9.2}$$

where λ, λ_0 are arbitrary constants.

When, however, $n = n'$, the solution is

$$x = \lambda \sin(n't + \lambda_0) - \tfrac{1}{2}\frac{mt}{n'}\cos n't. \tag{9.3}$$

This contains an oscillation with a coefficient which increases continually with t.

A common practice is the insertion of a "damping" factor $\mu dx/dt$, on the ground that all mechanical systems are subject to friction. It is easily seen that this prevents the coefficient from becoming infinitely great with t, but it will be shown below that it fails to give even an approximation to the motion of certain mechanical systems under resonance or near-resonance conditions. The frictional force acting on the pendulums used for geodetic surveys, for example, is so small that it may be neglected in comparison with certain types of disturbance which produce resonance, and the motion in the latter case is entirely different from that which would be produced by a frictional force. In the problem of three bodies, this device cannot be used; in most of the applications to the solar system, there is no evidence of any frictional effect.

10. The real defect lies in the assumption that the forces in a vibrating system are proportional to the displacements, so that the equations of motion are linear. This is not true in any mechanical system we know, although many systems approach it very closely. As long as no two of the frequencies in the system are very nearly the same, the assumption gives a good approximation to the motion, mainly because the amplitudes of the oscillations remain small. As soon, however, as the amplitude begins to become large, as it does when two of the frequencies are nearly the same, the approximation fails. It is necessary to take into account the effect of finite amplitude and this demands the use of non-linear equations.

Other cases common to many gravitational problems, are those in which the disturbing forces depend partly on the displacements of the system. Some examples of this will be given in which it will be shown not only that non-linear equations must be used, but that the methods of approximation usual in such cases fail. The nature of this failure will be easily seen by the mathematician when it is caused by the attempt to expand a function in positive integral powers of a certain constant instead of expansion in powers of the square root of this constant.

11. A feature of motion under resonance conditions already hinted at in section 6 will be brought out in detail by the examples given below. This feature, which cannot be too strongly emphasized, is that resonance is not a *single* special case of motion but is a *group of cases extending over a finite range of values of the constants.* It is true that one of the constants, previously arbitrary, is given a particular value; this constant, the frequency, becomes the same as that of the disturbing force (we shall show that another constant, the phase, must also have a special value).

Suppose, however, we regard this case as a particular solution of the equations of motion and then proceed to find the possible deviations from it. If the resonance motion is stable, we find that small oscillations about the resonance configuration are possible. In other words, we find that on the average the resonance relation is maintained, but that there are periodic deviations from it, and that the period of these deviations has no direct relation to the resonance period.

We find also that the amplitude and phase of these deviations are arbitrary constants of the solution. Thus though two arbitrary constants have been given special values when we set up the resonance relation, two new arbitraries appear in the deviations from this resonance relation. The chief period of oscillation is "locked" to that of the disturbing force, but the locking does not prevent deviations to and fro if the system be slightly disturbed.

The mathematical and physical characters of the motion are quite different when there is resonance, from those when resonance does not exist. Neither can be described in terms of the other.

12. These differences between the two groups of cases suggest that there must be at least a mathematical discontinuity separating them. This will be shown to be the actual fact. The differential equations for each group are the same, but the solutions of these equations are not continuous functions of the arbitrary constants. The formulae which give the motion have to be completely changed. At one place there will be found a solution which belongs neither to the one case nor to the other. The place corresponds to a discontinuity of an analytic function.

In a certain sense, this discontinuity is physical as well as mathematical. Suppose that we have been able to deduce the constants in a certain type of motion from observation. Usually small changes in the constants within the errors of observation produce only small changes in the subsequent calculated motion. At the point of discontinuity, however, such small changes in the observed values are found to change fundamentally the character of the subsequent motion. There is a certain degree of analogy between this case and that of equilibrium at an unstable position where the subsequent motion often depends on the nature of a minute and perhaps not measurable disturbance.

The quantities which we ordinarily associate with motion, namely, position, velocity, acceleration are continuous in the sense that there are no sudden changes in their values with a change in the time. The discontinuity is that of indeterminateness. Not only is there only a small change in them with a small change in the time, but the change may be still small with a finite change in the time, while the particle is passing through this particular stage. Later it may move in a manner which is easily calculable. Thus we might be unable to distinguish between the position at a given future time and that at a time later by a finite interval. In other words, we are unable to predict the position after a given interval of time.

13. The existence of this discontinuity in most cases prevents calculation of the motion of a system if it passes from non-resonance to resonance conditions. Calculations are made of the motions as the system approaches resonance and also of the motion when the resonance is fully established. For the stages in between appeal is made to some simple physical case like that of the pendulum where experiment can give a qualitative description of the probable changes which take place. In the simple examples worked out below there will not be much doubt that this method of procedure gives the principal features of the motion.

For logical completeness, proofs of the existence and convergence of the series used should be given. Owing partly to limitations of space and partly to avoid the introduction of developments which are of more interest to the mathematician than to the physicist, these have been omitted. It is necessary to know the forms which the series must take for purposes of calculation; that these forms will give approximate numerical results is assumed on the basis of past experience, or is capable of proof by known methods which are not developed here.