XI

SINGULARITIES AND ARITHMETICAL PROPERTIES OF THE COEFFICIENTS

49. At the beginning of Chapter V, we remarked that some general facts about the singularities of a function may be inferred from certain arithmetical properties of the coefficients, particularly such properties as remain invariant with respect, for example, to differentiation. In that chapter the property considered was the number and the distribution of the lacunae.

We proceed now to a group of theorems which show that with a knowledge of certain other arithmetical properties of the coefficients we may predict the form of the function. The theorems of the present chapter, while therefore related to earlier theorems, differ from those theorems in that the hypotheses on the coefficients are more specialized, and accordingly the determination of the nature of the function is more specific.

50. THEOREM 1 (Eisenstein[1]): *A series*

$$\sum_{n=0}^{\infty} A_n x^n, \quad R = 1, \tag{1}$$

where the A_n are rational, has the property that if the function defined by the series is algebraic, there will exist an integer N such that the quantities $A_n N^n$ are integers.

The converse is not true, for we have seen that in any series with a finite radius of convergence it suffices to change the

[1] Monatshefte d. Akad. d. Wiss. zu Berlin (1852), p. 441.

signs of an infinity of coefficients in order that the circle of convergence of the resulting series shall be a cut. Consequently the function represented by the altered series, although its coefficients have the required property, can not be algebraic.

For the case in which all the A_n are integers, we have the theorem of Carlson[1]:

THEOREM 2: *If the circle of convergence of* (1) *is not a cut, the series represents a rational function of the form*

$$\frac{P(x)}{(1 - x^p)^q},$$

where $P(x)$ *is a polynomial, and* p, q *are positive integers.*

51. The author[2] has proved a theorem which may in certain cases be regarded as the converse of the theorem of Eisenstein.

THEOREM 3: *If the series* (1) *has rational coefficients such that there exists an integer* N *for which the quantities*

$$A_0,\ A_1N, \cdots,\ A_nN^n, \cdots$$

are integers, and if the function represented by the series is regular exterior to and on the circumference of the circle of radius $\dfrac{N}{N^2 - 1}$ *and center* $\dfrac{N^2}{N^2 - 1}$, *then the function is of the form*

$$\frac{P(x)}{(1 - x)^h},$$

where $P(x)$ *is a polynomial, and* h *is an integer.*

We remark that if the above condition on the distribution of the singularities is dispensed with, the function defined

[1] Math. Zeitschrift, bd. 9 (1921), p. 1.

[2] Comptes Rendus, t. 178 (1924), p. 985.

by (1) may have an almost arbitrary distribution of singularities. The theorem is therefore of a character quite different from that of Carlson.

For the proof of Theorem 3, let

$$\left.\begin{aligned} \phi(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n N^n}{z^{n+1}}, \\ f(z) &= \sum_{n=0}^{\infty} \frac{A_{n+1} N^{n+1}}{z^{n+1}}. \end{aligned}\right\} \tag{2}$$

The singularities of $f(z)$ are the points which correspond by the transformation

$$z = \frac{N}{x} \tag{3}$$

to the singularities of $\sum_{n=1}^{\infty} A_n x^n$. A direct calculation, on the basis of (3), shows that to the circle C with center at $\frac{N^2}{N^2-1}$ and radius $\frac{N}{N^2-1}$ corresponds in the z-plane the circle C_1 with center at N and radius 1. For the transformation carries real values of x into real values of z; the circle C has its center on the axis of reals, and intersects this axis at the points

$$x_1 = \frac{N^2}{N^2-1} + \frac{N}{N^2-1} = \frac{N}{N-1},$$

$$x_2 = \frac{N^2}{N^2-1} - \frac{N}{N^2-1} = \frac{N}{N+1},$$

to which correspond the real points

$$z_1 = N - 1, \quad z_2 = N + 1.$$

Hence C_1 has its center at $\frac{1}{2}(z_1 + z_2) = N$, and radius 1.

The theorem of Hurwitz applied to the series (2) shows that the function

$$\psi(z) = \sum_{n=0}^{\infty} \frac{\gamma_n}{z^{n+1}},$$

where

$$\begin{aligned}\gamma_n &= (A_1N)N^n - C_n^1(A_2N^2)N^{n-1} + \cdots \pm (A_nN^n)N \\ &= \delta\, N^{n+1}\Delta_n A_1,\end{aligned} \tag{4}$$

$\Delta_n A_1$ denoting the nth difference of the sequence $\{A_n\}$ with respect to A_1, and δ being equal to 1 or -1, is regular exterior to and on the circumference of the unit circle, since the only singularity of $\phi(z)$ is the point $-N$. Hence

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|\gamma_n|} = \theta < 1.$$

But from the hypothesis and (4), the γ_n are integers. Hence $\gamma_n = 0$ for $n > n_0$.

In the series $\sum_{k=1}^{\infty} A_k x^{k-1}$, replace x by $\frac{y}{1+y}$. We have

$$\sum_{k=1}^{\infty} A_k \left(\frac{y}{1+y}\right)^{k-1} = (1+y) \sum_{k=1}^{\infty} A_k \frac{y^{k-1}}{(1+y)^k}.$$

The general term of this series may be written

$$A_k \frac{y^{k-1}}{(1+y)^k} = \sum_{r=k-1}^{\infty} (-1)^{r-k+1} A_k C_r^{k-1} y^r.$$

Hence

$$\begin{aligned}\sum_{k=1}^{\infty} A_k \frac{y^{k-1}}{(1+y)^k} &= \sum_{k=1}^{\infty} \sum_{r=k-1}^{\infty} A_k C_r^{k-1} y^r (-1)^{r-k+1} \\ &= \sum_{r=0}^{\infty} y^r \sum_{k=1}^{r+1} A_k C_r^{k-1} (-1)^{r-k+1}.\end{aligned}$$

Then

$$\sum_{k=1}^{\infty} A_k x^{k-1} = \frac{1}{1-x}\sum_{r=0}^{\infty}\left(\frac{x}{1-x}\right)^r (-1)^r \sum_{k=1}^{r+1}(-1)^{k-1}C_r^{k-1}A_k$$

$$= \sum_{r=0}^{\infty}\left(\frac{x}{1-x}\right)^r \frac{1}{1-x}(-1)^r \Delta_r A_1$$

$$= \sum_{i=0}^{n_0}\left(\frac{x}{1-x}\right)^r \frac{1}{1-x}(-1)^r \Delta_r A_1$$

$$= \frac{P_1(x)}{(1-x)^h},$$

so that

$$A_0 + A_1 x + A_2 x^2 + \cdots = A_0 + \frac{xP_1(x)}{(1-x)^h}$$

$$= \frac{P(x)}{(1-x)^h},$$

and the theorem is proved.

Achyeser [1] has shown that the preceding theorem may be proved by means of Euler's transformation.

We remark an interesting consequence of this theorem. Given the series (1). If N is a positive integer such that

$$\sum A_n N^n x^n$$

has integral coefficients, then unless $\Sigma A_n x^n$ represents a function of the form $\frac{P(x)}{(1-x)^h}$ we must have

$$N \geq \frac{1}{\rho} - 1, \tag{5}$$

where ρ is the greatest distance from the point 1 to a singularity.

[1] Bull. de l'Acad. des Sciences de l'Oucraïne, t. 1 (1925), p. 32.

We may suppose that ρ is less than 1, for otherwise the statement is trivial. Write (5) in the form

$$\frac{1}{N+1} \leq \rho.$$

If this inequality were not true, all the singularities would lie within the circle of center $\frac{N^2}{N^2-1}$ and radius $\frac{N}{N^2-1}$. The function defined by $\Sigma A_n x^n$ would therefore be of the form $\frac{P(x)}{(1-x)^h}$, with only one singularity.

We thus obtain a lower bound for N in order that a function represented by a series with rational coefficients be, for instance, algebraic, not rational.

The author has applied the foregoing type of research to the theory of transcendental numbers.