

## IX

### FATOU'S THEOREM

42. In the theorems of the preceding chapters, the hypotheses, insofar as they have involved the coefficients in a Taylor's series, have referred mainly to their absolute values. The object of this chapter is to show that the algebraic signs of the  $a_n$  affect the number and the character of the singularities. Theorems of this kind originated in an important theorem of Fatou.<sup>1</sup>

THEOREM 1: *Given the series*

$$\sum a_n x^n, \quad 0 < R < \infty, \quad (1)$$

*there exists a sequence  $\{\lambda_n\}$  such that the series obtained by changing the signs of the  $a_{\lambda_n}$  has the circle of convergence as a cut.*

Write  $a_n = \alpha_n + i\beta_n$ . Fatou's proof of the theorem required that  $\lim \alpha_n = 0$ ,  $\lim \beta_n = 0$ , and  $\sum \sqrt{\alpha_n^2 + \beta_n^2}$  diverges. He remarked, however, that it is probable that the conclusion is valid without any restriction on the  $\sum \sqrt{a_n^2 + b_n^2}$ . That this is indeed the case was proved by Hurwitz and Polya. It is a modification of Hurwitz's proof that we give here, not only because of its elegance, but also because the method can be applied to the proof of other theorems, in particular to Theorem 2 below.

<sup>1</sup> Acta Math., t. 30 (1906), p. 335. See also Hurwitz and Polya, Acta Math., t. 40 (1916), p. 179, and Mandelbrojt, Ann. de l'École Norm. Sup., t. 40 (1923), p. 413.

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Assume  $R = 1$ . If the circle of convergence is not a cut there will be at least one rational<sup>1</sup> value of  $\phi$  for which  $f(e^{i\phi})$  is holomorphic.

Select a sequence  $\{n_i\}$  for which

$$\lim_{i \rightarrow \infty} (n_{i+1} - n_i) = \infty, \tag{2}$$

$$\lim_{i \rightarrow \infty} \sqrt[n_i]{|a_{n_i}|} = 1, \tag{3}$$

and such that none of the  $a_i$  vanishes. Let

$$\begin{aligned} Q(x) &= \sum_{i=1}^{\infty} a_{n_i} x^{n_i}, \\ P(x) &= \sum_{n=0}^{\infty} a_n x^n - Q(x) \\ &= \sum_{j=1}^{\infty} a_{n'_j} x^{n'_j}. \end{aligned}$$

Arrange the sequence  $\{a_{n_i}\}$  in the following form:

$$\begin{array}{cccccccc} a_{n_1} & & & & & & & \\ a_{n_2} & & a_{n_3} & & & & & \\ a_{n_4} & & a_{n_5} & & a_{n_6} & & & \\ \dots & & \dots & & \dots & & \dots & \end{array}$$

Denote by  $A_k$  the set of elements in the  $k$ -th column, and by  $(-1)A_k$  the set obtained by changing the sign of each element of  $A_k$ . Consider the sets

$$(-1)^{\alpha_1} A_1, (-1)^{\alpha_2} A_2, \dots, \tag{4}$$

where  $\alpha_1, \alpha_2, \dots$  take on the values 0 and 1. The totality of these sets has the power of the continuum. Let  $\eta = 0.\alpha_1\alpha_2\dots$ ,  $0 \leq \eta \leq 1$ . Denote the corresponding set (4) by

$$(-1)^{\binom{\eta}{1}} A_1, (-1)^{\binom{\eta}{2}} A_2, \dots \tag{4'}$$

<sup>1</sup> Hurwitz considers, instead of rational points, arcs on which  $f(e^{i\phi})$  is holomorphic.

If a number  $\eta$  can be written in two different ways, we shall retain in (4) only one of them. The set will still have the power of the continuum. Let  $Q_\eta(x) = \sum c_{n_i}^{(\eta)} x^{n_i}$ , where the  $c_{n_i}^{(\eta)}$  are the elements of the set (4'), and let  $f_\eta(x) = Q_\eta(x) + P(x)$ .

Suppose that for each  $\eta$ , none of the  $f_\eta(x)$  has the circle of convergence as a cut. Then, since (a), each  $f_\eta(x)$  has at least one regular point  $e^{i\phi_\eta}$ , where  $\phi_\eta$  is rational, and the set of such  $\phi_\eta$  accordingly denumerable, and (b), the  $f_\eta(x)$  constitute a set having the power of the continuum, there will be at least two different numbers  $\eta_1, \eta_2$  such that  $f_{\eta_1}(x)$  and  $f_{\eta_2}(x)$  are regular at the same point  $e^{i\phi} = e^{i\phi_{\eta_1}} = e^{i\phi_{\eta_2}}$ . Then the function

$$f_{\eta_1}(x) - f_{\eta_2}(x) = Q_{\eta_1}(x) - Q_{\eta_2}(x)$$

is regular at that point. This, however, is impossible. In fact, since  $\eta_1$  and  $\eta_2$  are distinct, there will be some  $k$  such that if the  $\alpha_k$  of  $\eta_1$  is 0, the  $\alpha_k$  of  $\eta_2$  is 1, or *vice versa*. Denote by  $c_{n_i}^{(\eta_1)}$  the  $c_{n_i}^{(\eta_1)}$  belonging to the partial set  $(-1)^{\alpha_k} A_k$  which differ in sign from the  $c_{n_i}^{(\eta_2)}$  belonging evidently to  $(-1)^{\alpha_k} A_k$ . Then

$$Q_{\eta_1}(x) - Q_{\eta_2}(x) = \sum d_{n_i} x^{n_i}, \tag{5}$$

where at least the coefficients  $d_{n_i} = 2 c_{n_i}^{(\eta_1)}$  are not zero.

Hence

$$|d_{n_{i_1}}| = 2 |a_{n_{i_1}}|. \tag{6}$$

The series for  $Q_{\eta_1}(x)$  and  $Q_{\eta_2}(x)$  both have unit radius of convergence. Hence the radius of convergence  $r$  of (5) is at least 1. Consequently, from (3) and (6),  $r = 1$ . From the way in which the  $n_i$  were chosen, the series (5) has its circle of convergence as a cut. Hence the assumption that no  $f_\eta(x)$  has its circle of convergence as a cut leads to a contradiction. But any  $f_\eta(x)$  for which the circle of conver-

gence is a cut has the same coefficients as the given function except for changes in sign of an infinity of coefficients. The theorem is therefore proved.

43. THEOREM 2:<sup>1</sup> *Given  $f(x) = \sum a_n x^n$ , there exists a sequence  $\{n_i\}$  such that it suffices to change the signs of the coefficients  $a_{n_i}$  in order that the circle of convergence of the resulting series is a cut, and every point on the circle of convergence has the maximum order, viz., the order  $\omega$  of  $f(x)$  on the circle of convergence:*

$$\omega = \overline{\lim}_{n \rightarrow \infty} \frac{\log |a_n|}{\log n} + 1.$$

If a function represented by a series with unit radius of convergence has on the circle of convergence a singularity of order  $\omega_1 < \omega$ , then, by definition, this point may be enclosed in an arc in which the order of the function is less than  $\omega$ . It follows that if the order of the function is  $\omega' < \omega$  in  $e^{i\phi}$ , there is a rational number  $\phi_1$  such that the order in  $e^{i\phi_1}$  is less than  $\omega$ . If two functions have in the point  $x_0$  the orders  $\omega', \omega'', < \omega$ , respectively, the sum of the two functions has in  $x_0$  the order  $\omega''' < \omega$ , where  $\omega'''$  is the greater of  $\omega'$  and  $\omega''$ .

The proof of Theorem 2 is similar to that of Theorem 1. We now choose the sequence  $\{n_i\}$  so that

$$n_{i+1} - n_i > \sqrt{n_i \log n_i} \text{ for } i > i_0, \text{ and also}$$

$$\lim_{i \rightarrow \infty} \frac{\log |a_{n_i}|}{\log n_i} + 1 = \omega = \overline{\lim}_{n \rightarrow \infty} \frac{\log |a_n|}{\log n} + 1.$$

Instead of Hadamard's theorem, we apply Fabry's theorem<sup>2</sup> on the order of singularities. Replace "regular point" by "point of order  $\omega' < \omega$ ," and the proof is the same as that of Fatou's theorem.

<sup>1</sup> Mandelbrojt, *Comptes Rendus*, t. 184 (1927), p. 509.

<sup>2</sup> Chapter VII, Theorem 8.

44. In conclusion, we state the following theorem:

THEOREM 3<sup>1</sup>: Given a sequence  $\{n_i\}$  such that

$$\lim_{i \rightarrow \infty} (n_{i+1} - n_i) = \infty, \quad \lim_{i \rightarrow \infty} \sqrt[n_i]{|a_{n_i}|} = 1.$$

Denote by

$$\sum a_n^{(\phi)} x^n, \quad 0 \leq \phi \leq 2\pi, \tag{7}$$

all series such that

$$\begin{aligned} a_{n_i}^{(\phi)} &= e^{i\phi} a_{n_i}, \\ a_n^{(\phi)} &= a_n, \quad n \neq n_i. \end{aligned}$$

There exists a set of values of  $\phi$  having the power of the continuum such that the corresponding series (7) has the circle of convergence as a cut, and a denumerable set of values of  $\phi$  for which the corresponding series does not have the circle of convergence as a cut.

<sup>1</sup> Mandelbrojt, Ann. de l'École Norm. Sup., t. 40 (1923), p. 413.